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More bounds for the Grundy number of graphs

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Abstract A coloring of a graph $G = (V, E)$ is a partition $\{V_1, V_2, \ldots, V_k\}$ of V into independent sets or color classes. A vertex $v \in V_i$ is a Grundy vertex if it is adjacent to at least one vertex in each color class V_j for every $j < i$. A coloring is a Grundy coloring if every vertex is a Grundy vertex, and the Grundy number $\Gamma(G)$ of a graph G is the maximum number of colors in a Grundy coloring. We provide two new upper bounds on Grundy number of a graph and a stronger version of the well-known Nordhaus-Gaddum theorem. In addition, we give a new characterization for a $\{P_4, C_4\}$-free graph by supporting a conjecture of Zaker, which says that $\Gamma(G) \geq \delta(G) + 1$ for any C_4-free graph G.

Keywords Grundy number · Chromatic number · Clique number · Coloring number · Randić index

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1 Introduction

All graphs considered in this paper are finite and simple. Let $G = (V, E)$ be a finite simple graph with vertex set V and edge set E, $|V|$ and $|E|$ are its *order* and *size*, respectively. As usual, $\delta(G)$ and $\Delta(G)$ denote the minimum degree and the maximum degree of G, respectively. A set $S \subseteq V$ is called a *clique* of G if any two vertices of S are adjacent in G. Moreover, S is *maximal* if there exists no clique properly contain it. The clique number of G, denoted by $\omega(G)$, is the cardinality of a maximum clique of G. Conversely, S is called an *independent* set of G if no two vertices of S are adjacent in G. The independence number of G, denoted by $\alpha(G)$, is the cardinality of a maximum independent set of G. The *induced subgraph* of G induced by S, denoted by $G[S]$, is subgraph of G whose vertex set is S and whose edge set consists of all edges of G which have both ends in S.

Let C be a set of k colors. A *k-coloring* of G is a mapping $c : V \to C$, such that $c(u) \neq c(v)$ for any adjacent vertices u and v in G. The *chromatic number* of G, denoted by $\chi(G)$, is the minimum integer k for which G has a k-coloring. Alternately, a k-coloring may be viewed as a partition $\{V_1, \ldots, V_k\}$ of V into independent sets, where V_i is the set of vertices assigned color i. The sets V_i are called the color classes of the coloring.

The *coloring number* $col(G)$ of a graph G is the least integer k such that G has a vertex ordering in which each vertex is preceded by fewer than k of its neighbors. The *degeneracy* of G, denoted by $deg(G)$, is defined as $deg(G) = max\{\delta(H) : H \subseteq G\}$. It is well known (see Page 8 in Jensen and Toft 1995) that for any graph G,

$$col(G) = deg(G) + 1. \tag{1}$$

It is clear that for a graph G,

$$\omega(G) \leq \chi(G) \leq col(G) \leq \Delta(G) + 1. \tag{2}$$

Let c be a k-coloring of G with the color classes $\{V_1, \ldots, V_k\}$. A vertex $v \in V_i$ is called a *Grundy vertex* if it has a neighbor in each V_j with $j < i$. Moreover, c is called a *Grundy k-coloring* of G if each vertex v of G is a Grundy vertex. The *Grundy number* $\Gamma(G)$ is the largest integer k, for which there exists a Grundy k-coloring for G. It is clear that for any graph G,

$$\chi(G) \leq \Gamma(G) \leq \Delta(G) + 1. \tag{3}$$

A stronger upper bound for Grundy number in terms of vertex degree was obtained in Zaker (2008) as follows. For any graph G and $u \in V(G)$ we denote $\{v \in V(G) : uv \in E(G), d(v) \leq d(u)\}$ by $N_{\leq}(u)$, where $d(v)$ is the degree of v. We define $\Delta_2(G) = \max_{u \in V(G)} \max_{v \in N_{\leq}(u)} d(v)$. It was proved in Zaker (2008) that $\Gamma(G) \leq \Delta_2(G) + 1$. Since $\Delta_2(G) \leq \Delta(G)$, we have

$$\Gamma(G) \leq \Delta_2(G) + 1 \leq \Delta(G) + 1. \tag{4}$$

The study of Grundy number dates back to 1930s when Grundy (1939) used it to study kernels of directed graphs. The Grundy number was first named and studied by Christen and Selkow (1979). Zaker (2005) proved that determining the Grundy number of the complement of a bipartite graph is NP-hard.

A *complete k-coloring c* of a graph G is a k-coloring of the graph such that for each pair of different colors there are adjacent vertices with these colors. The *achromaic number* of G, denoted by $\psi(G)$, is the maximum number k for which the graph has a complete k-coloring. It is trivial to see that for any graph G,

$$\Gamma(G) \leq \psi(G). \tag{5}$$

Note that $col(G)$ and $\Gamma(G)$ (or $\psi(G)$) is not comparable for a general graph G. For instacne, $col(C_4) = 3$, $\Gamma(C_4) = 2 = \psi(C_4)$, while $col(P_4) = 2$, $\Gamma(P_4) = 3 = \psi(P_4)$. Grundy number was also studied under the name of first-fit chromatic number, see Kierstead et al. (1995) for instance.

2 New bounds on Grundy number

In this section, we give two upper bounds on the Grundy number of a graph in terms of its Randić index, and the order and clique number, respectively.

2.1 Randić index

The Randić index $R(G)$ of a (molecular) graph G was introduced by Randić (1975) as the sum of $\frac{1}{\sqrt{d(u)d(v)}}$ over all edges uv of G, where $d(u)$ denotes the degree of a vertex u in G, i.e, $R(G) = \sum\limits_{uv \in E(G)} 1/\sqrt{d(u)d(v)}$. This index is quite useful in mathematical chemistry and has been extensively studied, see Li and Gutman (2006). For some recent results on Randić index, we refer to Divnić and Pavlović (2013), Li and Shi (2010), Liu et al. (2011, 2013).

Theorem 2.1 (Bollobás and Erdős 1998) *For a graph G of size m,*

$$R(G) \geq \frac{\sqrt{8m+1}+1}{4},$$

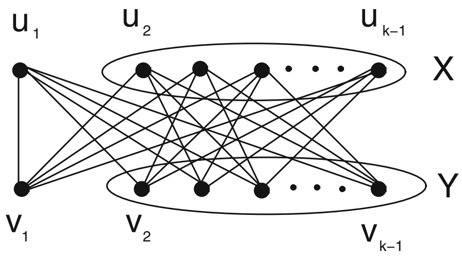
with equality if and only if G is consists of a complete graph and some isolated vertices.

Theorem 2.2 *For a connected graph G of order $n \geq 2$, $\psi(G) \leq 2R(G)$, with equality if and only if $G \cong K_n$.*

Proof Let $\psi(G) = k$. Then $m = e(G) \geq \frac{k(k-1)}{2}$. By Theorem 2.1,

$$R(G) \geq \frac{\sqrt{8m+1}+1}{4} \geq \frac{\sqrt{4k(k-1)+1}+1}{4}$$

Fig. 1 The graph B_k for $k \geq 2$



$$= \frac{\sqrt{(2k-1)^2}+1}{4} = \frac{2k-1+1}{4} = \frac{k}{2}.$$

This shows that $\psi(G) \leq 2R(G)$. If $k = 2R(G)$, then $m = \frac{k(k-1)}{2}$ and the equality is satisfied. Since G is connected, by Theorem 2.1, $G = K_k = K_n$.

If $G = K_n$, then $\psi(G) = n = 2R(G)$. □

Corollary 2.3 *For a connected graph G of order n, $\Gamma(G) \leq 2R(G)$, with equality if and only if $G \cong K_n$.*

Theorem 2.4 (Wu et al. 2014) *If G is a connected graph of order $n \geq 2$, then $col(G) \leq 2R(G)$, with equality if and only if $G \cong K_n$*

So, combining the above two results, we have

Corollary 2.5 *For a connected graph G of order $n \geq 2$, $\max\{\psi(G), col(G)\} \leq 2R(G)$, with equality if and only if $G \cong K_n$.*

2.2 Clique number

Zaker (2006) showed that for a graph G, $\Gamma(G) = 2$ if and only if G is a complete bipartite (see also the page 351 in Chartrand and Zhang 2008). Zaker and Soltani (2015) showed that for any integer $k \geq 2$, the smallest triangle-free graph of Grundy number k has $2k-2$ vertices. Let B_k be the graph obtained from $K_{k-1,k-1}$ by deleting a matching of cardinality $k-2$, see B_k for an illustration in Fig. 1. The authors showed that $\Gamma(B_k) = k$.

One may formulate the above result of Zaker and Soltani: $\Gamma(G) \leq \frac{n+2}{2}$ for any triangle-free graph G of order n.

Theorem 2.6 (i) *For a graph G of order $n \geq 1$, $\Gamma(G) \leq \frac{n+\omega(G)}{2}$.*

(ii) *Let k and n be any two integers such that $k+n$ is even and $k \leq n$. Then there exists a graph $G_{k,n}$ on n vertices such that $\omega(G_{k,n}) = k$ and $\Gamma(G_{k,n}) = \frac{n+k}{2}$.*

(iii) *In particular, if G is a connected triangle-free graph of order $n \geq 2$, then $\Gamma(G) = \frac{n+2}{2}$ if and only if $G \cong B_{\frac{n+2}{2}}$.*

Proof We prove part (i) of the assertion by induction on $\Gamma(G)$. Let $k = \Gamma(G)$. If $k = 1$, then $G = \overline{K_n}$. The result is trivially true. Next we assume that $k \geq 2$.

Let $V_1, V_2, \ldots, V_k$ be the color classes of a Grundy coloring of G. Set $H = G \setminus V_1$. Then $\Gamma(H) = k - 1$. By the induction hypothesis, $\Gamma(H) \le \frac{n-|V_1|+\omega(H)}{2}$. Hence,

$$\Gamma(G) = \Gamma(H) + 1 \le \frac{n - |V_1| + \omega(H)}{2} + 1 = \frac{n + \omega(H) + 2 - |V_1|}{2}.$$

We consider two cases. If $|V_1| \ge 2$, then

$$\Gamma(G) \le \frac{n + \omega(H) + 2 - |V_1|}{2} \le \frac{n + \omega(G) + 2 - 2}{2} = \frac{n + \omega(G)}{2}.$$

Now assume that $|V_1| = 1$. Since V_1 is a maximal independent set of G, every vertex in H is adjacent to the vertex in V_1, and thus $\omega(H) = \omega(G) - 1$. So,

$$\Gamma(G) = \frac{n + \omega(G) + 1 - |V_1|}{2} = \frac{n + \omega(G)}{2}.$$

To prove part (ii), we construct $G_{k,n}$ as follows. First consider a complete graph on k vertices and partition its vertex set into two subsets A and B such that $||A|-|B|| \le 1$. Let t be an integer such that $t = \frac{n-k}{2}$. Let also H_t be the graph obtained from $K_{t,t}$ by deleting a perfect matching of size t. It is easily observed that $\Gamma(H_t) = t$, where any Grundy coloring with t colors consists of t color classes $C_1, \ldots, C_t$ such that for each i, $|C_i| = 2$. Let $C_i = \{a_i, b_i\}$. Now for each $i \in \{1, \ldots, t\}$, join all vertices of A to a_i and join all vertices of B to b_i. We denote the resulting graph by $G_{k,n}$. We note by our construction that $\omega(G_{k,n}) = k$ and $\Gamma(G_{k,n}) \ge k + t$. Also, clearly $\Delta(G_{k,n}) = t + k - 1$. Then $\Gamma(G_{k,n}) = k + t = (n + k)/2$. This completes the proof of part (ii).

Now we show part (iii) of the statement. It is straightforward to check that $\Gamma(B_k) = k$. Next, we assume that G is connected triangle-free graph of order $n \ge 2$ with $\Gamma(G) = \frac{n+2}{2}$. Let $k = \frac{n+2}{2}$. To show $G \cong B_k$, let $V_1, \ldots, V_k$ be a Grundy coloring of G.

Claim 1 *(a)* $|V_k| = 1$; *(b)* $|V_{k-1}| = 1$; *(c)* $|V_i| = 2$ *for each* $i \le k - 2$.

Since G is triangle-free, there are at most two color classes with cardinality 1 among $V_1, \ldots, V_k$. Since

$$2k - 2 = |V_1| + \cdots + |V_k| \ge 1 + 1 + 2 + \cdots + 2 = 2(k - 1),$$

there are exactly two color classes with cardinality 1, and all others have cardinality two. Let u and v the two vertices lying the color classes of cardinality 1. Observe that u and v are adjacent.

We show (a) by contradiction. Suppose that $|V_k| = 2$, and let $V_k = \{u_k, v_k\}$. Since u is adjacent to v and both u_k and v_k are adjacent to u and v, we have a contradiction with the fact that G is triangle-free. This shows $|V_k| = 1$.

To complete the proof of the claim, it suffices to show (b). Toward a contradiction, suppose $|V_{k-1}| = 2$, and let $V_{k-1} = \{u_{k-1}, v_{k-1}\}$. By (a), let $|V_i| = 1$ for an integer

$i < k-1$. Without loss of generality, let $V_i = \{u\}$ and $V_k = \{v\}$. Since $u_{k-1}u \in E(G)$, $v_{k-1}u \in E(G)$, and at least one of u_{k-1} and v_{k-1} is adjacent to v, it follows that there must be a triangle in G, a contradiction.

So, the proof of the claim is completed.

Note that $uv \in E(G)$. Let $V_i = \{u_i, v_i\}$ for each $i \in \{1, \ldots, k-2\}$. Since G is triangle-free, exactly one of u_i and v_i is adjacent to u and the other one is adjacent to v. Without loss of generality, let $u_i v \in E(G)$ and $v_i u \in E(G)$ for each i. Since G is triangle-free, both $\{u_1, \ldots, u_{k-1}, u\}$ and $\{v_1, \ldots, v_{k-1}, v\}$ are independent sets of G, implying that G is a bipartite graph.

To complete the proof for $G \cong B_k$, it remains to show that $u_i v_j \in E(G)$ for any i and j with $i \neq j$. Without loss of generality, let $i < j$. Since $v_i v_j \notin E(G)$, $u_i v_j \in E(G)$.

So, the proof is completed. □

Since for any graph G of order n, $\chi(\overline{G})\omega(G) = \chi(\overline{G})\alpha(\overline{G}) \geq n$, by Theorem 2.6, the following result is immediate.

Corollary 2.7 (Zaker 2007) *For any graph G of order n, $\Gamma(G) \leq \frac{\chi(\overline{G})+1}{2}\omega(G)$.*

Corollary 2.8 (Zaker 2005) *Let G be the complement of a bipartite graph. Then $\Gamma(G) \leq \frac{3\omega(G)}{2}$.*

Proof Let n be the order of G and (X, Y) be the bipartition of $V(\overline{G})$. Since X and Y are cliques of G, $\max\{|X|, |Y|\} \leq \omega(G)$. By Theorem 2.6,

$$\Gamma(G) \leq \frac{n+\omega(G)}{2} = \frac{|X|+|Y|+\omega(G)}{2} \leq \frac{3\omega(G)}{2}.$$

□

The following result is immediate from by the inequality (2) and Theorem 2.6.

Corollary 2.9 *For any graph G of order n, $\Gamma(G) \leq \frac{n+\chi(G)}{2} \leq \frac{n+col(G)}{2}$.*

Chang and Hsu (2012) proved that $\Gamma(G) \leq log_{\frac{col(G)}{col(G)-1}} n+2$ for a nonempty graph G of order n. Note that this bound is not comparable to that given in the above corollary.

3 Nordhaus-Gaddum type inequality

Nordhaus and Gaddum (1956) proved that for any graph G of order n,

$$\chi(G) + \chi(\overline{G}) \leq n+1.$$

Since then, relations of a similar type have been proposed for many other graph invariants, in several hundred papers, see the survey paper of Aouchiche and Hansen (2013). Cockayne and Thomason (1982) proved that

$$\Gamma(G) + \Gamma(\overline{G}) \leq \lfloor \frac{5n+2}{4} \rfloor$$

for a graph G of order $n \geq 10$, and this is sharp. Füredi et al. (2008) rediscovered the above theorem. Harary and Hedetniemi (1970) established that $\psi(G)+\chi(\overline{G}) \leq n+1$ for any graph G of order n extending the Nordhaus-Gaddum theorem.

Next, we give a theorem, which is stronger than the Nordhaus-Gaddum theorem, but is weaker than Harary-Hedetniemi's theorem. Our proof is turned out to be much simpler than that of Harary-Hedetniemi's theorem.

It is well known that $\chi(G-S) \geq \chi(G)-|S|$ for a set $S \subseteq V(G)$ of a graph G. The following result assures that a stronger assertion holds when S is a maximal clique of a graph G.

Lemma 3.1 *Let G be a graph of order at least two which is not a complete graph. For a maximal clique S of G, $\chi(G-S) \geq \chi(G)-|S|+1$.*

Proof Let $V_1, V_2, \ldots, V_k$ be the color classes of a k-coloring of $G-S$, where $k = \chi(G-S)$. Since S is a maximal clique of G, for each vertex $v \in V(G) \setminus S$, there exists a vertex v' which is not adjacent to v. Hence $G[S \cup V_k]$ is s-colorable, where $s = |S|$. Let $U_1, \ldots, U_s$ be the color classes of an s-coloring of $G[S \cup V_k]$. Thus, we can obtain a $(k+s-1)$-coloring of G with the color classes $V_1, V_2, \ldots, V_{k-1}, U_1, \ldots, U_s$. So,

$$\chi(G) \leq k+s-1 = \chi(G-S)+|S|-1.$$

□

Theorem 3.2 *For a graph G of order n, $\Gamma(G)+\chi(\overline{G}) \leq n+1$, and this is sharp.*

Proof We prove by induction on $\Gamma(G)$. If $\Gamma(G) = 1$, then $G = \overline{K_n}$. The result is trivially holds, because $\Gamma(G) = 1$ and $\chi(\overline{G}) = n$. The result also clearly true when $G = K_n$.

Now assume that G is not a complete graph and $\Gamma(G) \geq 2$. Let $V_1, V_2, \ldots, V_k$ be a Grundy coloring of G. Set $H = G \setminus V_1$. Then $\Gamma(H) = \Gamma(G)-1$. By the induction hypothesis,

$$\Gamma(H)+\chi(\overline{H}) \leq n-|V_1|+1.$$

Since V_1 is a maximal independent set of G, it is a maximal clique of $\overline{G}$. By Lemma 3.1, we have

$$\chi(\overline{H}) \geq \chi(\overline{G})-|V_1|+1.$$

Therefore

$$\begin{aligned}\Gamma(G)+\chi(\overline{G}) &\leq (\Gamma(H)+1)+(\chi(\overline{H})+|V_1|-1)\\ &\leq (\Gamma(H)+\chi(\overline{H}))+|V_1|\\ &\leq n-|V_1|+1+|V_1|\\ &= n+1.\end{aligned}$$

To see the sharpness of the bound, let us consider $G_{n,k}$, which is the graph obtained from the joining each vertex of K_k to all vertices of $\overline{K_{n-k}}$, where $1 \leq k \leq n-1$. It can be checked that $\Gamma(G_{n,k}) = k+1$ and $\chi(\overline{G_{n,k}}) = n-k$. So $\Gamma(G_{n,k})+\chi(\overline{G_{n,k}}) = n+1$. The proof is completed. □

Finck (1966) characterized all graphs G of order n such that $\chi(G) + \chi(\overline{G}) = n + 1$. It is an interesting problem to characterize all graphs G attaining the bound in Theorem 3.2. Since $\alpha(G) = \omega(\overline{G}) \leq \chi(\overline{G})$ for any graph G, the following corollary is a direct consequence of Theorem 3.2.

Corollary 3.3 (Effantin and Kheddouci 2007) *For a graph G of order n,*

$$\Gamma(G) + \alpha(G) \leq n + 1.$$

4 Perfectness

Let $\mathcal{H}$ be a family of graphs. A graph G is called *$\mathcal{H}$-free* if no induced subgraph of G is isomorphic to any $H \in \mathcal{H}$. In particular, we simply write H-free instead of $\{H\}$-free if $\mathcal{H} = \{H\}$. A graph G is called *perfect*, if $\chi(H) = \omega(H)$ for each induced subgraph H of G. It is well known that every P_4-free graph is perfect.

A *chordal* graph is a simple graph which contains no induced cycle of length four or more. Berge (1961) showed that every chordal graph is perfect. A *simplicial* vertex of a graph is vertex whose neighbors induce a clique.

Theorem 4.1 (Dirac 1961) *Every chordal graph has a simplicial vertex.*

Corollary 4.2 *If G is a chordal graph, then $\delta(G) \leq \omega(G) - 1$.*

Proof Let v be a simplicial vertex. By Theorem 4.1, $N(v)$ is a clique, and thus $d(v) \leq \omega(G) - 1$. □

Markossian et al. (1996) remarked that for a chordal graph G, $col(H) = \omega(H)$ for any induced subgraph H of G. Indeed, its converse is also true. For convenience, we give the proof here.

Theorem 4.3 *A graph G is chordal if and only if $col(H) = \omega(H)$ for any induced subgraph H of G.*

Proof The sufficiency is immediate from the fact that any cycle C_k with $k \geq 4$, $col(C_k) = 3 \neq 2 = \omega(C_k)$.

Since every induced subgraph of a chordal graph is still a chordal graph, to prove the necessity of the theorem, it suffices to show that $col(G) = \omega(G)$. Recall that $col(G) = deg(G) + 1$ and $deg(G) = \max\{\delta(H) : H \subseteq G\}$. Observe that

$$\max\{\delta(H) : H \subseteq G\} = \max\{\delta(H) : H \text{ is an induced subgraph of } G\}.$$

Since $\omega(G) - 1 \leq \max\{\delta(H) : H \subseteq G\}$ and $\delta(H) \leq \omega(H) - 1$ for any induced subgraph H of G, $\max\{\delta(H) : H$ is an induced subgraph of $G\} \leq \omega(G)-1$, we have $deg(G) = \max\{\delta(H) : H \subseteq G\} = \omega(G) - 1$. Thus, $col(G) = \omega(G)$. □

Let $\alpha, \beta \in \{\omega, \chi, \Gamma, \psi\}$. A graph G is called *$\alpha\beta$-perfect* if for each induced subgraph H of G, $\alpha(H) = \beta(H)$. Among other things, Christen and Selkow proved that

Theorem 4.4 (Christen and Selkow 1979) *For any graph G, the following statements are equivalent:*

(1) G is $\Gamma\omega$-perfect.
(2) G is $\Gamma\chi$-perfect.
(3) G is P_4-free.

For a graph G, $m(G)$ denotes the number of of maximal cliques of G. Clearly, $\alpha(G) \leq m(G)$. A graph G is called *trivially perfect* if for every induced subgraph H of G, $\alpha(H) = m(H)$. A partially order set $(V, <)$ is an arborescence order if for all $x \in V$, $\{y : y < x\}$ is a totally ordered set.

Theorem 4.5 (Wolks 1962; Golumbic 1978) *Let G be a graph. The following conditions are equivalent:*

(i) *G is the comparability graph of an arborescence order.*
(ii) *G is $\{P_4, C_4\}$-free.*
(iii) *G is trivially perfect.*

Next we provide another characterization of $\{P_4, C_4\}$-free graphs.

Theorem 4.6 *Let G be a graph. Then G is $\{P_4, C_4\}$-free if and only if $\Gamma(H) = col(H)$ for any induced subgraph H of G.*

Proof To show its sufficiency, we assume that $\Gamma(H) = col(H)$ for any induced subgraph H of G. Since $col(C_4) = 3$ while $\Gamma(C_4) = 2$, and $col(P_4) = 2$ while $\Gamma(P_4) = 3$, it follows that G is C_4-free and P_4-free.

To show its necessity, let G be a $\{P_4, C_4\}$-free graph. Let H be an induced subgraph of G. Since G is P_4-free, by Theorem 4.4, $\Gamma(H) = \omega(H)$. On the other hand, by Theorem 4.3, $col(H) = \omega(H)$. The result then follows. □

Gastineau et al. (2014) posed the following conjecture.

Conjecture 1 (Gastineau et al. 2014). *For any integer $r \geq 1$, every C_4-free r-regular graph has Grundy number $r + 1$.*

So, Theorem 4.6 asserts that the above conjecture is true for every regular $\{P_4, C_4\}$-free graph. However, it is not hard to show that a regular graph is $\{P_4, C_4\}$-free if and only if it is a complete graph. Indeed, Zaker (2011) *made the following beautiful conjecture, which implies Conjecture 1.*

Conjecture 2 (Zaker 2011) *If G* is *C_4-free graph, then $\Gamma(G) \geq \delta(G) + 1$.*

Note that Theorem 4.6 shows that Conjecture 2 is valid for all $\{P_4, C_4\}$-free graphs.

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