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Total coloring of planar graphs without adjacent short cycles

Huijuan Wang[1] · Bin Liu[2] · Yan Gu[1] · Xin Zhang[3] · Weili Wu[4,5] · Hongwei Gao[1]

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Abstract In the study of computer science, optimization, computation of Hessians matrix, graph coloring is an important tool. In this paper, we consider a classical coloring, total coloring. Let $G = (V, E)$ be a graph. Total coloring is a coloring of $V \cup E$ such that no two adjacent or incident elements (vertex/edge) receive the same color. Let G be a planar graph with $\Delta \geq 8$. We proved that if for every vertex $v \in V$, there exists two integers $i_v, j_v \in \{3, 4, 5, 6, 7\}$ such that v is not incident with adjacent i_v-cycles and j_v-cycles, then the total chromatic number of graph G is $\Delta + 1$.

✉ Bin Liu
sduliubin@163.com

Huijuan Wang
sduwhj@163.com

Yan Gu
guyan1913@163.com

Xin Zhang
xzhang@xidian.edu.cn

Weili Wu
weiliwu@utdallas.edu

Hongwei Gao
gaohongwei@qdu.edu.cn

1 College of Mathematics, Qingdao University, Qingdao 266071, China

2 Department of Mathematics, Ocean University of China, Qingdao 266100, China

3 School of Mathematics and Statistics, Xidian University, Xi'an 710071, China

4 College of Computer Science and Technology, TaiYuan University of Technology, Taiyuan 030024, China

5 Department of Computer Science, University of Texas at Dallas, Richardson, TX 75080, USA

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1 Introduction

In graph theory, coloring is an important and classical problem. It has many applications in pattern matching, frequency assignment in optical communication networks, and so on. A *vertex coloring* of a graph is a coloring such that every vertex receives color, and every two adjacent vertices have different colors. A *total coloring* assigns each vertex or edge with a color such that no adjacent vertices receive the same color, no adjacent edges receive the same color, and no incident vertex and edge receive the same color. For a graph $G = (V, E)$, its *total graph* $T(G)$ is defined to have vertex set $V \cup E$ and edge set consisting of all pairs of adjacent or incident elements in $V \cup E$. A total coloring of graph G is equivalent to a vertex coloring of $T(G)$.

In total coloring problem, the target of total coloring is to find the minimum number of colors to do this coloring. For simplicity of speaking, if G has a total coloring with k colors, then G is said to be *total-k-colorable*. The *total chromatic number* $\chi''(G)$ is the smallest number k such that G is total-k-colorable. Clearly, $\chi''(G) \geq \Delta + 1$. In 1964, Behzad and Vizing gave a popular conjecture that every graph G is total-$(\Delta + 2)$-colorable, where Δ is the maximum vertex degree of G. For short, this conjecture is referred as TCC which has attracted many researchers' attention. Yap (1996) proved $\chi''(K_n) = n$ if n is odd and $\chi''(K_n) = n + 1$ otherwise. Moreover, TCC is also proved for interval graph Bojarshinov (2001), series-parallel graphs Wu (2004), and so on. For a general graph, TCC hold for graphs with $\Delta \leq 5$ (see Yap 1996; Kowalik et al. 2008). Snchez-Arroyo (1989) proved that it is NP-complete to decide whether $\chi''(G) = \Delta + 1$ for a given graph. McDiarmid and Snchez-Arroyo (1994) further proved that for every fixed $k \geq 3$, it is even NP-complete to decide whether $\chi''(G) = k + 1$ for a given k-regular bipartite graph G.

In our paper, we consider planar graph. For planar graphs, the only open case of TCC is $\Delta = 6$ (see Kostochka 1996; Sanders and Zhao 1999). Interestingly, the total chromatic number of planar graphs with large maximum degree equals the lower bound, i.e., $\chi''(G) = \Delta + 1$. This result was proved for $\Delta \geq 9$ in Kowalik et al. (2008). In the following, we consider the planar graph with $\Delta \geq 8$. Some related results can be found in Du et al. (2009), Liu et al. (2009), Roussel and Zhu (2010), Shen and Wang (2009), Wang et al. (2013), and Yap (1996). In this paper we get the following theorem.

Theorem 1 *Suppose G is a planar graph with $\Delta \geq 8$. If for every vertex v, there exists two integers $i_v, j_v \in \{3, 4, 5, 6, 7\}$ such that v is not incident with adjacent i_v-cycles and j_v-cycles, then $\chi''(G) = \Delta + 1$.*

Here, we say that two cycles are *adjacent* if they share at least one common edge. Note that in Theorem 1, i_v and j_v are related to v, respectively. If $i_v = i$, $j_v = j$ for each vertex $v \in V$, then we can easily get one corollary.

Corollary 1 *For two fixed integers i and j ($i, j \in \{3, 4, 5, 6, 7\}$), if G is a planar graph with $\Delta \geq 8$ and without adjacent i-cycles and j-cycles, then $\chi''(G) = \Delta + 1$.*

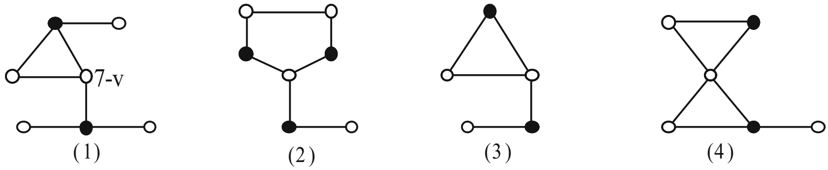

Fig. 1 Reducible configurations of Lemma 1

This corollary generalizes the result for $\Delta = 8$, which has been recently proved (see Wang et al. 2014), that is $\chi''(G) = \Delta + 1$ if G is a planar graph without adjacent cycles of size i and j, for some $i, j \in \{3, 4, 5\}$. Clearly, it also generalizes the result in Du et al. (2009) that if G is a planar graph with $\Delta \geq 8$ and without adjacent triangles, then $\chi''(G) = 9$. In addition, it strengthens the results in Shen and Wang (2009), Wu and Wang (2008) and Tan et al. (2009).

In this paper, all graphs are finite, simple and undirected. Most of the notions are standard and we refer the readers to Bondy and Murty (1976). Let G be a graph. A *k-vertex*, *k^--vertex* or a *k^+-vertex* is a vertex of degree k, at most k or at least k, respectively. Similarly, we can define a *k-face*, *k^--face* and a *k^+-face*. We use $(v_1, v_2, \cdots, v_n)$ to denote a cycle whose vertices are consecutively $v_1, v_2, \cdots, v_n$. If the boundary of a face f is $(v_1, v_2, \cdots, v_n)$, then f is simply referred to as a $(d(v_1), d(v_2), \cdots, d(v_n))$-face. We use $n_k(v)$ to denote the number of k-vertices adjacent to v, use $n_k(f)$ to denote the number of k-vertices incident with f, and use $f_k(v)$ to denote the number of k-faces incident with v.

2 Reducible configurations

In Kowalik et al. (2008), Theorem 1 was proved for $\Delta \geq 9$. So in the following we assume that $\Delta = 8$. Let $G = (V, E, F)$ be a minimal counterexample to Theorem 1 with $|V| + |E|$ as small as possible.

In this section we start the proof of Theorem 1 by obtaining structural informations about our minimal counterexample G, which shows that certain configurations are *reducible*, that is, they cannot occur in G. First, we shown some known properties.

(a) G is 2-connected.
(b) If uv is an edge of G with $d(u) \leq 4$, then $d(u) + d(v) \geq \Delta + 2 = 10$ (see Wang et al. 2014).
(c) The subgraph G_{28} of G induced by all edges joining 2-vertices to Δ-vertices is a forest (see Wang et al. 2014).

Lemma 1 (Du et al. 2009) *G has no configurations depicted in Fig. 1, where the vertices marked by $\bullet$ have no other neighbors in G and 7-v denotes the vertex with degree of 7.*

Lemma 2 (Wang et al. 2014) *Suppose $d(v) = d \geq 6$, whose adjacent vertices are consecutively $v_1, v_2, \cdots, v_d$ and whose incident faces are consecutively $f_1, f_2, \cdots, f_d$, where v_i is incident with f_{i-1} and f_i $(i = 1, 2, \cdots, d)$. Note that*

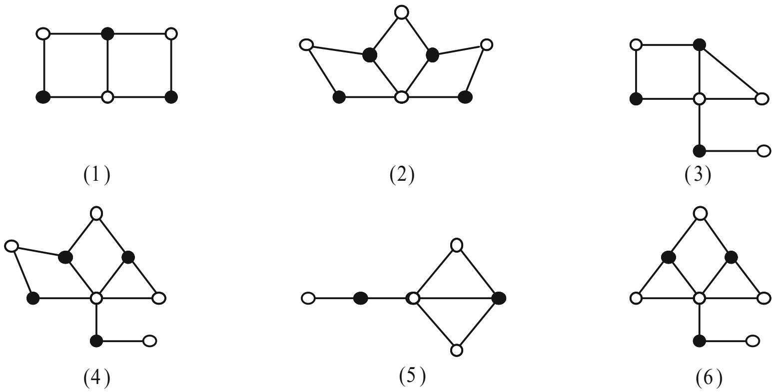

Fig. 2 Reducible configurations of Corollary 2

f_0 and f_d are the same face. Let $d(v_1) = 2$ and $N(v_1) = \{v, u_1\}$. Then G does not satisfy one of the following conditions, where

(1) *there exists an integer k $(2 \le k \le d-1)$ such that $d(v_{k+1}) = 2$, $d(v_i) = 3$ $(2 \le i \le k)$ and $d(f_j) = 4$ $(1 \le j \le k)$.*
(2) *there exist two integers k and t $(2 \le k < t \le d-1)$ such that $d(v_k) = 2$, $d(v_i) = 3$ $(k+1 \le i \le t)$, $d(f_t) = 3$ and $d(f_j) = 4$ $(k \le j \le t-1)$.*
(3) *there exist two integers k and t $(3 \le k \le t \le d-1)$ such that $d(v_i) = 3$ $(k \le i \le t)$, $d(f_{k-1}) = d(f_t) = 3$ and $d(f_j) = 4$ $(k \le j \le t-1)$.*
(4) *there exists an integer k $(2 \le k \le d-2)$ such that $d(v_d) = d(v_i) = 3$ $(2 \le i \le k)$, $d(f_k) = 3$ and $d(f_j) = 4$ $(0 \le j \le k-1)$.*

By Lemma 2, we can easily get the corollary as follows.

Corollary 2 (Wang et al. 2014) *G does not contain the configurations depicted in Fig. 2.*

Lemma 3 *If a 6-vertex u is adjacent to one 4-vertex v and incident with one 3-cycle (u, v, s), then u is adjacent to no other 4-vertex.*

Proof Suppose u is adjacent to another 4-vertex w. By the minimality of G, $G' = G - uv$ has a total-9-coloring ϕ. Erase the colors on v and w. Let $C(x) = \{\phi(xy) : y \in N(x)\} \cup \{\phi(x)\}$, then the forbidden colors for uv is at most 9. Without loss of generality, let $C(u) = \{1, 2, 3, 4, 5, 6\}$, $C(v) = \{7, 8, 9\}$, $\phi(u) = 1$, $\phi(us) = 5$, $\phi(vs) = 7$, $\phi(uw) = 6$. Then $C(w) \setminus \phi(uw) = \{7, 8, 9\}$. Otherwise, without loss of generality, if $7 \notin C(w)$, we can recolor uw with 7 and color uv with 6. By proper coloring the 4-vertices v and w, we get a total-9-coloring of G, a contradiction. In the following, change the colors of us and vs, that is, recolor us with 7, recolor vs with 5. Then color 5 does not appear in $C(u)$. So recolor uw with 5, color uv with 6, by proper coloring the 4-vertices v and w, we get a total-9-coloring of G, a contradiction. □

3 Discharging

We shall complete the proof of Theorem 1 by using the *discharging* method. This is an important and interesting tool in the proof of the coloring of planar graphs. By Euler's formula $|V| - |E| + |F| = 2$, we have

$$\sum_{v\in V}(2d(v) - 6) + \sum_{f\in F}(d(f) - 6) = -6(|V| - |E| + |F|) = -12 < 0$$

We define $c(x)$ to be the initial charge. Let $c(v) = 2d(v) - 6$ for each $v \in V$ and $c(f) = d(f) - 6$ for each $f \in F$. So $\sum_{x\in V\cup F} c(x) = -12 < 0$. Then we apply the following rules to redistribute the initial charge that leads to a new charge $c'(x)$.

(R1) From each 8-vertex to each of its adjacent 2-vertices, transfer 1.
(R2) From each 4-vertex to each of the k-faces incident with it, where $3 \le k \le 5$, transfer $\frac{1}{2}$.
(R3) From each 5-vertex v to each of the k-faces incident with it, where $3 \le k \le 5$, transfer
$\frac{4}{5}$, if $k = 3$ and $f_3(v) = 5$;
$\frac{7}{8}$, if $k = 3$ and $f_3(v) = 4$;
$\frac{7}{6}$, if $k = 3$ and $f_3(v) = 3$;
$\frac{5}{4}$, if $k = 3$ and $f_3(v) \le 2$;
$\frac{1}{2}$, if $k = 4$;
$\frac{1}{5}$, if $k = 5$.
(R4) From each 6-vertex v to each of the k-faces f incident with it, transfer
$\frac{5}{4}$, if $k = 3$ and $n_4(f) = 1$;
$\frac{11}{10}$, if $k = 3$ and $n_5(f) = 1$ or $n_5(f) = 2$;
1, if $k = 3$ and $n_{6^+}(f) \ge 3$;
$\frac{1}{2}$, if $k = 4$;
$\frac{1}{3}$, if $k = 5$.
(R5) From each 7^+-vertex to each of the k-faces f incident with it, transfer
$\frac{3}{2}$, if $k = 3$ and $n_3(f) = 1$;
$\frac{5}{4}$, if $k = 3$ and $n_3(f) = 0$;
1, if $k = 4$ and $n_{3^-}(f) = 2$;
$\frac{3}{4}$, if $k = 4$, $n_{3^-}(f) = 1$ and $n_4(f) = 1$ or $n_5(f) = 1$;
$\frac{2}{3}$, if $k = 4$, $n_{3^-}(f) = 1$ and $n_{6^+}(f) = 3$;
$\frac{1}{2}$, if $k = 4$ and $n_{3^-}(f) = 0$;
$\frac{1}{3}$, if $k = 5$.

The rest of this paper is to check that $c'(x) \ge 0$ for all $x \in V \cup F$ which will be the desired contradiction.

Final charge of faces. Let $f \in F$. Suppose $d(f) = 3$. Then $c(f) = d(f) - 6 = -3$. If $n_{3^-}(f) = 1$, then $n_{7^+}(f) = 2$ by (b), and $c'(f) = -3 + \frac{3}{2} \times 2 = 0$ by (R5). If $n_4(f) = 1$, then $c'(f) = -3 + \frac{5}{4} \times 2 + \frac{1}{2} = 0$ by (R2), (R4) and (R5). If $n_5(f) = 1$, then $n_{6^+}(f) = 2$ and $c'(f) \ge -3 + \frac{11}{10} \times 2 + \frac{4}{5} = 0$. If $n_5(f) = 2$ and one 5-vertex incident with f is incident with at least four 3-faces, then the other 5-vertex incident with f is incident with at most three 3-faces. So $c'(f) = -3+\min\{\frac{4}{5}+\frac{7}{6}+\frac{11}{10}, \frac{7}{8}+\frac{7}{6}+\frac{11}{10}\} > 0$. If $n_5(f) = 2$ and no 5-vertex incident with f is incident with at least four 3-faces, then $c'(f) \geq -3+\frac{7}{6}\times 2+\frac{11}{10} = \frac{13}{30} > 0$. If $n_5(f) = 3$, then the number of 5-vertices incident with f and incident with five 3-faces is at most one, so $c'(f) = -3+\min\{\frac{4}{5}+\frac{7}{6}\times 2, \frac{7}{8}+\frac{7}{6}\times 2, \frac{7}{6}\times 3\} = \frac{2}{15} > 0$. If $n_{6^+}(f) = 3$, then $c'(f) = -3+1\times 3 = 0$. Suppose $d(f) = 4$. Then $c(f) = d(f)-6 = -2$. If $n_{3^-}(f) = 2$, then $n_{7^+}(f) = 2$ by (b), and $c'(f) = -2+1\times 2 = 0$ by (R5). If $n_{3^-}(f) = 1$ and $n_4(f) = 1$, then $c'(f) = -2+\frac{1}{2}+\frac{3}{4}\times 2 = 0$. If $n_{3^-}(f) = 1$ and $n_5(f) = 1$, then $c'(f) = -2+\frac{1}{2}+\frac{3}{4}\times 2 = 0$. If $n_{3^-}(f) = 1$ and $n_{6^+}(f) = 3$, then $c'(f) = -2+\frac{2}{3}\times 3 = 0$. If $n_{3^-}(f) = 0$, then $c'(f) = -2+\frac{1}{2}\times 4 = 0$. Suppose $d(f) = 5$. Then $c(f) = d(f)-6 = -1$ and $n_{5^+}(f) \geq 3$. If $n_5(f) = 0$, then $c'(f) \geq -1+\frac{1}{3}\times 3 = 0$. Otherwise, $n_5(f) \geq 1$ and $c'(f) \geq -1+\min\{\frac{1}{5}\times 5, \frac{1}{5}+\frac{1}{3}\times 3, \frac{1}{5}\times 4+\frac{1}{3}\} = 0$. Suppose $d(f) \geq 6$. Then $c(f) = d(f)-6 \geq 0$.

Final charge of vertices. Let $v \in V$. Note that G has no vertex of degree one. Suppose $d(v) = 2$. Then $c(v) = 2d(v)-6 = -2$ and $n_8(v) = 2$. So $c'(v) = -2+1\times 2 = 0$ by (R1). Suppose $d(v) = 3$. Then clearly $c'(v) = c(v) = 0$. Suppose $d(v) = 4$. Then $c(v) = 2$ and v sends at most $\frac{1}{2}$ to each of its incident faces. So $c'(v) \geq 2-\frac{1}{2}\times 4 = 0$ by (R2). Suppose $d(v) = 5$. Then $c(v) = 4$, and $n_{4^-}(v) = 0$. If $f_3(v) = 5$, then $c'(v) = 4-\frac{4}{5}\times 5 = 0$ by (R3). If $f_3(v) = 4$, then $c'(v) \geq 4-\frac{7}{8}\times 4-\frac{1}{2} = 0$. Suppose $f_3(v) = 3$. Then v is incident with at most one 4-face. If $f_4(v) = 1$, then $f_5(v) = 0$ and so $c'(v) \geq 4-\frac{7}{6}\times 3-\frac{1}{2} = 0$. If $f_4(v) = 0$, then $c'(v) \geq 4-\frac{7}{6}\times 3-\frac{1}{5}\times 2 = \frac{1}{10} > 0$. If $f_3(v) \leq 2$, then $c'(v) \geq 4-\frac{5}{4}\times 2-\frac{1}{2}\times 3 = 0$. Suppose $d(v) = 6$. Then we have $c(v) = 6$, $n_{3^-}(v) = 0$, and $f_3(v) \leq 5$. By lemma 2, v is incident with at most two 3-faces each of which receives $\frac{5}{4}$ from v. If $f_3(v) = 5$, then $f_{6^+}(v) = 1$ and $c'(v) \geq 6-\frac{5}{4}\times 2-\frac{11}{10}\times 3 = \frac{1}{5} > 0$. If $f_3(v) \leq 4$, then $c'(v) \geq 6-\frac{5}{4}\times 2-\frac{11}{10}\times 2-\frac{1}{2}\times 2 = \frac{3}{10} > 0$.

Suppose $d(v) = 7$. Then $c(v) = 8$, $n_2(v) = 0$, and $f_3(v) \leq 5$. We also known v is incident with at most two 3-faces each of which receives $\frac{3}{2}$ from v. Moreover, if v is incident with one 3-face which incident with a 3-vertex, then v is adjacent to no other 3-vertex. If $f_3(v) = 5$, then $c'(v) \geq 8-\frac{3}{2}\times 2-\frac{5}{4}\times 3-\frac{1}{2}\times 2 = \frac{1}{4} > 0$. If $f_3(v) = 4$, then $c'(v) \geq 8-\frac{3}{2}\times 2-\frac{5}{4}\times 2-\frac{3}{4}\times 3 = \frac{1}{4} > 0$. If $f_3(v) = 3$, then $c'(v) \geq 8-\frac{3}{2}\times 2-\frac{5}{4}-\frac{3}{4}\times 4 = \frac{3}{4} > 0$. If $f_3(v) \leq 2$, then $c'(v) \geq 8-\frac{3}{2}\times 2-1\times 5 = 0$.

In the following, we consider the vertex of degree 8. Suppose $d(v) = 8$, whose adjacent vertices are consecutively $v_1, v_2, \cdots, v_8$ and whose incident faces are consecutively $f_1, f_2, \cdots, f_8$, where v_i is incident with f_{i-1} and f_i $(i = 1, 2, \cdots, 8)$. Note that f_0 and f_8 are the same face. We also have $c(v) = 2\times 8-6 = 10$.

Lemma 4 *Suppose* $d(v) = 8$ *and* $v_1, v_2, \cdots, v_{k-1}, v_k$ *be the consecutively adjacent vertices of* v *for* $k \geq 3$. *If* $d(v_1) = d(v_k) = 2$, $d(v_j) \geq 3$ *for all* $j = 2, 3, \cdots, k-1$, *and* $\min\{d(f_2), d(f_3), \cdots, d(f_{k-2})\} \geq 3$, *then* v *sends at most* $\frac{3}{2}+(k-3)\times\frac{5}{4}$ *(in total) to* $f_1, f_2, \cdots, f_{k-1}$.

Proof Firstly, we have $\min\{d(f_1), d(f_{k-1})\} \geq 4$ by Lemma 1. Suppose $\min\{d(f_1), d(f_{k-1})\} \geq 5$. Then v is not incident with two 3-faces which are incident with a common 3-vertex, and the 3-face incident with v needs more charge than the 4-face incident with v. So v sends at most $\frac{1}{3} \times 2 + (k-3) \times \frac{5}{4}$(in total) to $f_1, f_2, \cdots, f_{k-1}$, which is less than $\frac{3}{2} + (k-3) \times \frac{5}{4}$. Suppose $\min\{d(f_1), d(f_{k-1})\} = 4$ and $\max\{d(f_1), d(f_{k-1})\} \geq 5$. Then v sends at most $\frac{1}{3} + 1 + (k-3) \times \frac{5}{4}$(in total) to $f_1, f_2, \cdots, f_{k-1}$, which is less than $\frac{3}{2} + (k-3) \times \frac{5}{4}$. Suppose $d(f_1) = d(f_{k-1}) = 4$. If $\min\{d(f_2), d(f_3), \cdots, d(f_{k-2})\} \geq 5$, then v sends at most $1 \times 2 + (k-3) \times \frac{1}{3}$(in total) to $f_1, f_2, \cdots, f_{k-1}$, which is less than $\frac{3}{2} + (k-3) \times \frac{5}{4}$. If $\min\{d(f_2), d(f_3), \cdots, d(f_{k-2})\} = 4$ and $\max\{d(f_2), d(f_3), \cdots, d(f_{k-2})\} \geq 5$, then v sends at most $1 \times 2 + (k-5) \times 1 + \frac{3}{4} + \frac{1}{3}$(in total) to $f_1, f_2, \cdots, f_{k-1}$, which is less than $\frac{3}{2} + (k-3) \times \frac{5}{4}$ for $k \geq 3$. If $d(f_2) = d(f_3) = \cdots = d(f_{k-2}) = 4$, then v sends at most $1 \times 2 + (k-5) \times 1 + \frac{3}{4} \times 2$(in total) to $f_1, f_2, \cdots, f_{k-1}$, which is less than $\frac{3}{2} + (k-3) \times \frac{5}{4}$ for $k \geq 3$. In the following, suppose $\min\{d(f_2), d(f_3), \cdots, d(f_{k-2})\} = 3$. Since v needs to sends at least $\frac{5}{4}$ to each of its incident 3-faces and sends at most 1 to each of its incident 4-faces, we suppose $d(f_2) = d(f_3) = \cdots = d(f_{k-2}) = 3$. Then there are at most two 3-faces, that is f_2, f_{k-2}, which may be need receive $\frac{3}{2}$ from v. Suppose there is exactly one 3-face from f_2, f_{k-2} which needs receive $\frac{3}{2}$ from v. Then $\max\{d(f_1), d(f_{k-1})\} \geq 5$. So v sends at most $\frac{3}{2} + (k-4) \times \frac{5}{4} + \frac{3}{4} + \frac{1}{3}$(in total) to $f_1, f_2, \cdots, f_{k-1}$. Suppose each of f_2 and f_{k-2} needs receive $\frac{3}{2}$ from v. Then $\min\{d(f_1), d(f_{k-1})\} \geq 5$. So v sends at most $\frac{3}{2} \times 2 + (k-5) \times \frac{5}{4} + \frac{1}{3} \times 2$(in total) to $f_1, f_2, \cdots, f_{k-1}$. Suppose none of f_2 and f_{k-2} needs receive $\frac{3}{2}$ from v. Then v sends at most $\frac{3}{2} + (k-3) \times \frac{5}{4}$(in total) to $f_1, f_2, \cdots, f_{k-1}$. □

Suppose $n_2(v) = 8$. Then clearly $f_{6^+}(v) = 8$ and v sends each of its adjacent 2-vertices at most 1 by R1. So $c'(v) = 10 - 1 \times 8 = 2$. Suppose $n_2(v) = 7$. Then $f_{6^+}(v) \geq 6$, $f_4(v) \leq 2$ and $f_3(v) = 0$. So $c'(v) \geq 10 - 1 \times 7 - 1 \times 2 = 1$. Suppose $n_2(v) = 6$. Then $f_3(v) \leq 1$. If $f_3(v) = 1$, then $f_{6^+}(v) \geq 5$ and $c'(v) \geq 10 - 1 \times 6 - \frac{3}{2} - 1 \times 2 = \frac{1}{2} > 0$. Otherwise, $f_3(v) = 0$, $f_{6^+}(v) \geq 4$ and $c'(v) \geq 10 - 1 \times 6 - 1 \times 4 = 0$. Suppose $n_2(v) = 5$. Then $f_3(v) \leq 2$. If $f_3(v) = 2$, then $f_{6^+}(v) \geq 4$ and $c'(v) \geq 10 - 1 \times 5 - \frac{3}{2} \times 2 - 1 \times 2 = 0$. If $f_3(v) = 1$, then $f_{6^+}(v) \geq 3$ and $f_4(v) \leq 4$. We also known there are at least two 4-faces each of which needs receive $\frac{3}{4}$ from v. So $c'(v) \geq 10 - 1 \times 5 - \frac{3}{2} - 1 - \frac{3}{4} \times 2 = 0$. If $f_3(v) = 0$, then $f_{6^+}(v) \geq 2$, $f_4(v) \leq 6$ and none of 4-faces incident with v needs receive 1 from v. So $c'(v) \geq 10 - 1 \times 5 - \frac{3}{4} \times 6 = \frac{1}{2} > 0$.

Suppose $n_2(v) = 4$. Then there are eight possibilities in which 2-vertices are located. They are shown as configurations in Fig. 3. In Fig. 3(1), by Lemma 4, $c'(v) \geq 10 - 4 - \frac{3}{2} - \frac{5}{4} \times 3 = \frac{3}{4} > 0$. In Fig. 3(2), $c'(v) \geq 10 - 4 - \frac{3}{2} - \frac{5}{4} \times 2 - \frac{3}{2} = \frac{1}{2} > 0$. In Fig. 3(3) and (7), $c'(v) \geq 10 - 4 - (\frac{3}{2} + \frac{5}{4}) \times 2 = \frac{1}{2} > 0$. In Fig. 3(4), $c'(v) \geq 10 - 4 - \frac{3}{2} - \frac{5}{4} \times 2 - \frac{3}{2} = \frac{1}{2} > 0$. In Fig. 3(5-6), $c'(v) \geq 10 - 4 - \frac{3}{2} - \frac{5}{4} - \frac{3}{2} \times 2 = \frac{1}{4} > 0$. In Fig. 3(8), $c'(v) \geq 10 - 4 - \frac{3}{2} \times 4 = 0$.

Suppose $n_2(v) = 3$. Then there are five possibilities in which 2-vertices are located. They are shown as configurations in Fig. 4. In Fig. 4(1), by Lemma 4, $c'(v) \geq 10 - 3 - \frac{3}{2} - \frac{5}{4} \times 4 = \frac{1}{2} > 0$. In Fig. 4(2), $c'(v) \geq 10 - 3 - \frac{3}{2} - \frac{5}{4} \times 3 - \frac{3}{2} = \frac{1}{4} > 0$.

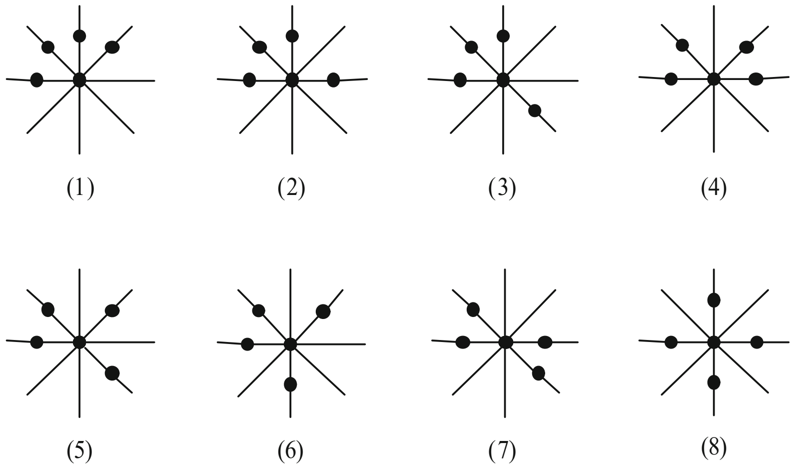

Fig. 3 $n_2(v) = 4$

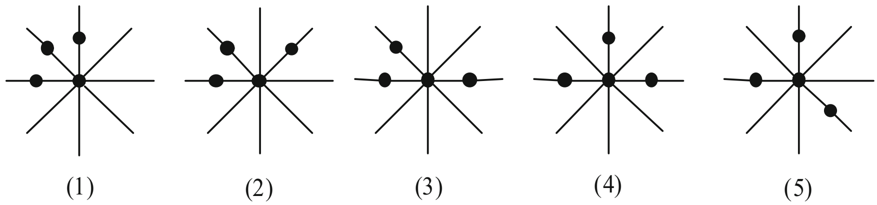

Fig. 4 $n_2(v) = 3$

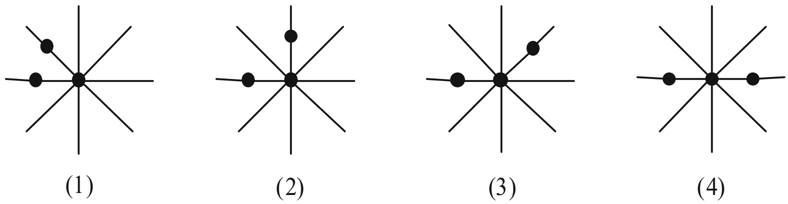

Fig. 5 $n_2(v) = 2$

In Fig. 4(3), $c'(v) \geq 10 - 3 - \frac{3}{2} - \frac{5}{4} \times 2 - \frac{3}{2} - \frac{5}{4} = \frac{1}{4} > 0$. In Fig. 4(4), $c'(v) \geq 10 - 3 - \frac{3}{2} - \frac{5}{4} \times 2 - \frac{3}{2} \times 2 = 0$. In Fig. 4(5), $c'(v) \geq 10 - 3 - (\frac{3}{2} + \frac{5}{4}) \times 2 - \frac{3}{2} = 0$.

Suppose $n_2(v) = 2$. Then there are four possibilities in which 2-vertices are located. They are shown as configurations in Fig. 5. In Fig. 5(1), by Lemma 4, $c'(v) \geq 10 - 2 - \frac{3}{2} - \frac{5}{4} \times 5 = \frac{1}{4} > 0$. In Fig. 5(2), $c'(v) \geq 10 - 2 - \frac{3}{2} - \frac{5}{4} \times 4 - \frac{3}{2} = 0$. In Fig. 5(3), $c'(v) \geq 10 - 2 - \frac{3}{2} - \frac{5}{4} \times 3 - \frac{3}{2} - \frac{5}{4} = 0$. In Fig. 5(4), $c'(v) \geq 10 - 2 - (\frac{3}{2} + \frac{5}{4} \times 2) \times 2 = 0$.

Suppose $n_2(v) = 1$, let u be a 2-vertex adjacent to v and uv is not incident with a 3-cycle. Then $f_3(v) \leq 5$. Suppose $f_3(v) = 5$. Then v is incident with at least two 6^+-faces or at least one 7^+-face. So $c'(v) \geq 10 - 1 - \frac{3}{2} \times 3 - \frac{5}{4} \times 2 - 1 \times 2 = 0$. Suppose $f_3(v) = 4$. Then v is incident with at least two 5^+-faces or at least one 6^+-face. So $c'(v) \geq 10 - 1 - \frac{3}{2} \times 4 - 1 \times 3 = 0$. Suppose $f_3(v) = 3$. Then v is incident with at least one 5^+-face. So $c'(v) \geq 10 - 1 - \frac{3}{2} \times 3 - 1 \times 4 - \frac{1}{3} = \frac{1}{6} > 0$. Suppose $f_3(v) \leq 2$. Then $c'(v) \geq 10 - 1 - \frac{3}{2} \times 2 - 1 \times 6 = 0$.

Suppose uv is incident with a 3-cycle. Then $f_3(v) \leq 6$, v is incident with at most one 3-face which need receive $\frac{3}{2}$ from v and other 3-faces each of which receive $\frac{5}{4}$ from v. Suppose $f_3(v) = 6$. Then v is incident with at least one 6^+-face and $c'(v) \geq 10 - 1 - \frac{3}{2} - \frac{5}{4} \times 5 - 1 = \frac{3}{4} > 0$. Suppose $f_3(v) = 5$. Then $c'(v) \geq 10 - 1 - \frac{3}{2} - \max\{\frac{5}{4} \times 4 + \frac{3}{4} \times 2 + 1, \frac{5}{4} \times 4 + 2 \times 2 + \frac{1}{3}\} = 0$. Suppose $f_3(v) = 4$. Then v is incident with at least one 5^+-face and $c'(v) \geq 10 - 1 - \frac{3}{2} - \frac{5}{4} \times 3 - 1 \times 3 - \frac{1}{3} = \frac{5}{12} > 0$. Suppose $f_3(v) \leq 3$. Then $c'(v) \geq 10 - 1 - \frac{3}{2} - \frac{5}{4} \times 2 - 1 \times 5 = 0$.

Suppose $n_2(v) = 0$. Then $f_3(v) \leq 6$. Suppose $f_3(v) = 6$. Then v is incident with at least one 6^+-face and $c'(v) \geq 10 - \frac{3}{2} \times 6 - 1 = 0$. Suppose $f_3(v) = 5$. Then v is incident with at least one 5^+-face and $c'(v) \geq 10 - \frac{3}{2} \times 5 - 1 \times 2 - \frac{1}{3} = \frac{1}{6} > 0$. Suppose $f_3(v) \leq 4$. Then $c'(v) \geq 10 - \frac{3}{2} \times 4 - 1 \times 4 = 0$.

Hence we complete the proof of Theorem 1.

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References

Bojarshinov VA (2001) Edge and total coloring of interal graphs. Discret Appl Math 114:23–28

Bondy JA, Murty USR (1976) Graph theory with applications. MacMillan, London

Du DZ, Shen L, Wang YQ (2009) Planar graphs with maximum degree 8 and without adjacent triangles are 9-totally-colorable. Discret Appl Math 157:2778–2784

Kostochka AV (1996) The total chromatic number of any multigraph with maximum degree five is at most seven. Discret Math 162:199–214

Kowalik L, Sereni J-S, Škrekovski R (2008) Total-coloring of plane graphs with maximum degree nine. SIAM J Discret Math 22:1462–1479

Liu B, Hou JF, Wu JL, Liu GZ (2009) Total colorings and list total colorings of planar graphs without intersecting 4-cycles. Discret Math 309:6035–6043

McDiarmid CJH, Snchez-Arroyo A (1994) Total colorings regular bipartite graphs is NP-hard. Discret Math 124:155–162

Roussel N, Zhu X (2010) Total coloring of planar graphs of maximum degree eight. Inf Process Lett 110:321–324

Sanchez-Arroyo A (1989) Determing the total colouring number is NP-hard. Discret Math 78:315–319

Sanders DP, Zhao Y (1999) On total 9-coloring planar graphs of maximum degree seven. J Graph Theory 31:67–73

Shen L, Wang YQ (2009) Total colorings of planar graphs with maximum degree at least 8. Sci China Ser A 52:1733–1742

Tan X, Chen HY, Wu JL (2009) Total colorings of planar graphs without adjacent 4-cycles. In: The eighth international symposium on operations research and its applications (ISORA'09), 167–173

Wang GH, Yan GY, Yu JG, Zhang X (2013) The r-acyclic chromatic number of planar graphs. J Comb Optim. doi:10.1007/s10878-013-9680-2

Wang HJ, Wu LD, Wu WL (2014) Total coloring of planar graphs with maximum degree 8. Theor Comput Sci 522:54–61

Wang HJ, Wu LD, Wu WL, Pardalos PM, Wu JL (2014) Minimum total coloring of planar graph. J Global Optim 60:777–791

Wu JL (2004) Total coloring of series-parallel graphs. Ars Comb 73:215–217
Wu JL, Wang P (2008) List-edge and list-total colorings of graphs embedded on hyperbolic surfaces. Discret Math 308:6210–6215
Yap HP (1996) Total colorings of graphs. Lecture notes in mathematics, vol 1623. Springer-Verlag, Berlin