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A linear-time algorithm for clique-coloring problem in circular-arc graphs

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Abstract A *maximal clique* of G is a clique not properly contained in any other clique. A k-clique-coloring of a graph G is an assignment of k colors to the vertices of G such that no maximal clique with at least two vertices is monochromatic. The smallest integer k admitting a k-clique-coloring of G is called clique-coloring number of G. Cerioli and Korenchendler (Electron Notes Discret Math 35:287–292, 2009) showed that there is a polynomial-time algorithm to solve the clique-coloring problem in circular-arc graphs and asked whether there exists a linear-time algorithm to find an optimal clique-coloring in circular-arc graphs or not. In this paper we present a linear-time algorithm of the optimal clique-coloring in circular-arc graphs.

Keywords Clique-coloring · Circular-arc graph · Linear time algorithm

Mathematics Subject Classification 05C15 · 05C69

1 Introduction

All graphs considered here are finite, simple and nonempty. Let G be a graph with *vertex set* $V(G)$ and *edge set* $E(G)$. The number of vertices of G is called the *order* of G and let n and m denote $|V(G)|$ and $|E(G)|$, respectively. For a vertex $v \in V(G)$, the *open neighborhood* $N(v)$ of v is defined as the set of vertices adjacent to v, i.e.,

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$N(v) = \{u\colon uv \in E(G)\}$. The *closed neighborhood* of v is $N[v] = N(v) \cup \{v\}$. Every vertex in $N(v)$ is also called a *neighbor* of v. The *degree* of v is equal to $|N(v)|$, denoted by $d_G(v)$ or simply $d(v)$. For a subset $S \subseteq V(G)$, the subgraph induced by S is denoted by $G[S]$. As usual, K_n, P_n and C_n denote the *complete graph*, *path* and *cycle* on n vertices, respectively. For standard terminology not given here we refer the reader to Bondy and Murty (2008).

A *hypergraph* $\mathcal{H}$ is a pair $(V, \mathcal{E})$ where V is a finite set of vertices and $\mathcal{E}$ is a family of non-empty subsets of V called *hyperedges*. A k-colouring of $\mathcal{H}$ is a mapping $\phi : V \to \{1, 2, \ldots, k\}$ such that for each $S \in \mathcal{E}$, with $|S| \geq 2$, there exist $u, v \in S$ with $\phi(u) \neq \phi(v)$, that is, there is no monochromatic hyperedge of size at least two. If such a function exists we say that $\mathcal{H}$ is *k-colorable*. The *chromatic number* $\chi(\mathcal{H})$ of $\mathcal{H}$ is the smallest k for which $\mathcal{H}$ admits a k-colouring.

A *clique* C is defined as a complete subgraph of a graph G, or equivalently a subset of $V(G)$, which induces a complete subgraph of G. A clique is said to be *maximal* if it is not properly contained in any other clique of G. We call *clique-hypergraph* of G the hypergraph $\mathcal{H}(G) = (V, \mathcal{E})$ that has the same vertices as G and whose set of hyperedges is the set of maximal cliques of G of cardinality at least 2.

A k-coloring of the clique-hypergraph $\mathcal{H}(G)$ is also called a *k-clique-coloring* of G, and the chromatic number $\chi(\mathcal{H}(G))$ of $\mathcal{H}(G)$ is called the *clique-chromatic number* of G, denoted by $\chi_C(G)$. If $\mathcal{H}(G)$ is k-colorable we say that G is *k-clique-colorable*. The k-clique coloring problem consists in deciding, for a given graph, if it admits a k-clique coloring.

Note that what we call k-clique-coloration here is also called *weak k-coloring* by Andreae et al. (1991), Bacsó and Zs (2009) or *strong k-division* by Hoàng and McDiarmid (2002). Clique-coloring has some similarities with usual vertex coloring, for example, any (vertex) k-coloring of G is also a k-clique-coloring of G, and optimal (vertex) colorings and clique-colorings coincide in the case of triangle-free graphs. But there are also essential differences, for example, a clique-coloring of a graph may not be a clique-coloring for its induced subgraphs. Induced subgraphs may even have a greater clique-chromatic number than the original graph. Let G be a graph with $\chi_C(G) > 2$ and G' be obtained from G by adding a vertex of full degree. Clearly, $\chi(G') = 2$ while $\chi_C(G) > 2$.

The clique-hypergraph coloring problem was posed by Duffus et al. (1991). Clique-coloring is harder than ordinary vertex coloring. Bacsó et al. (2004) proved that the decision problem of clique-coloring on general graphs is coNP-complete and it is NP-complete on graphs with maximum degree 3. Kratochvíl and Zs (2002) proved that testing $\chi_C = 2$ is still NP-hard for perfect graph and it is NP-complete on 3-chromatic perfect graphs. Défossez (2009) proved that testing $\chi_C = 2$ is Σ_2^p-complete on odd-hole-free graphs. In 2011, Marx (2011) proved that testing $\chi_C = k$ is Σ_2^p-complete on general graphs.

Many classes of special graphs have been studied and turned out to have a bounded clique-chromatic number. Bacsó et al. (2004) proved that almost all perfect graphs are 3-clique-colorable. Défossez (2006) conjectured that every odd-hole-free graph is 3-clique-colorable. For several subclasses of odd-hole-free graphs, we have a positive answer. {Odd hole, claw}-free graphs in Bacsó et al. (2004), {odd hole, co-diamond}-free graphs in Défossez (2009), {odd hole, bull}-free graphs in Défossez (2006) and {odd hole, P_5}-free graphs in Défossez (2006) are 2-clique-colorable, and {diamond, odd hole}-free graphs in Défossez (2006) are all 4-clique-colorable. Furthermore, all planar graphs in Mohar and Škrekovski (1999), Circular-arc graphs in Cerioli and Korenchendler (2009) and UEH graphs in Cerioli and Priscila (2008) have been proved to be 3-clique-colorable. Claw-free planar graphs in Shan et al. (2014), planar graphs without maximal 2-cliques in Thomassen (2008), claw-free graphs of maximum degree at most four in Bacsó and Zs (2009) and powers of cycles in Campos et al. (2013), other than odd cycles longer than three, are 2-clique-colorable.

Circular-arc graphs are natural generalizations of interval graphs to the circle. They possess interesting structures (see, Durán et al. 2014; Tucker 1970). Some of the motivations for studying circular-arc graphs are their rich structure, in addition to their applications in cyclic scheduling problems, such as those that arise in traffic light scheduling, in assignment of variables to registers in loops, and in other areas (see Golumbic 2004; Roberts 1978).

Circular-arc graphs and interval graphs are frequently studied in algorithmic graph theory (see, Golumbic 2004). For the optimal clique-coloring problem on circular-arc graphs, Cerioli and Korenchendler (2009) provided a polynomial-time algorithm and they asked whether there exists a linear-time algorithm. In this paper we present a linear-time algorithm to find an optimal clique-coloring of circular-arc graphs.

2 Preliminaries

A *circular-arc graph* is the intersection graph of a set of arcs on a circle C. Formally, let $\mathcal{A} = \{I_1, I_2, \ldots, I_n\}$ be a set of arcs on a circle C. Then the corresponding circular-arc graph is $G = (V, E)$ where $V = \{I_1, I_2, \ldots, I_n\}$ and $I_i I_j \in E$ if and only if $I_i \cap I_j \neq \emptyset$. The set $\mathcal{A}$ of arcs is called *circular-arc model* of G. A family of sets S is said to satisfy the *Helly property* if every subfamily of it, consisting of pairwise intersecting sets, has a common element (Butzer et al. 1984). A *Helly circular-arc* (HCA) *graph* is a circular-arc graph admitting a circular-arc model whose arcs satisfy the Helly property.

Let $\mathcal{A}$ be an arc model of a circular-arc graph G. We say that each arc $A_i = (s_i, t_i) \in \mathcal{A}$ traverses the circle C, in clockwise direction, from the point s_i to the point t_i, called the *extreme points* of A_i. We may assume, without loss of generality, that no two extreme points coincide. For $\mathcal{A}' \subset \mathcal{A}$, the arcs in $\mathcal{A}'$ are *removable* (or $\mathcal{A}'$ is a *removable arc set*) if there exists an arc $A_i \in \mathcal{A} \setminus \mathcal{A}'$ satisfying the following conditions:

(i) Every arc $A' \in \mathcal{A}'$ is properly contained in A_i.
(ii) For every $A \in \mathcal{A} \setminus (\mathcal{A}' \cup \{A_i\})$ and $A' \in \mathcal{A}'$, $A \cap A' = \emptyset$.
(iii) The vertices corresponding to arcs in $\mathcal{A}'$ induce a connected graph.

Specially, if $\mathcal{A} \setminus \mathcal{A}' = \{A_i\}$, then the corresponding vertex of the arc A_i is a vertex of full degree and the circular-arc graph G is 2-clique-colorable and 2-connected. If $\mathcal{A} \setminus \mathcal{A}' \neq \{A_i\}$ and $\mathcal{A}'$ is a removable arc set, then the vertex v_i corresponding to the arc A_i is a cut vertex of G and $\mathcal{A}' \cup \{A_i\}$ induce a maximal 2-connected subgraph of G. Note that, given a cut vertex v_i and a maximal 2-connected subgraph G' including v_i, we can easily check that whether the arc set corresponding to $G' - v_i$ is a removable arc set in $O(|V(G' - v_i)|)$ time. If an arc model $\mathcal{A}$ has no removable arc set, we call $\mathcal{A}$ an *irreducible arc model* and the corresponding circular-arc graph G of $\mathcal{A}$ is an *irreducible circular-arc graph*. Note that all the cut vertices and maximal 2-connected subgraphs of a circular-arc graph can be listed in linear time (see, Hopcroft and Tarjan 1973), so we immediately have the following result.

Lemma 2.1 *Given an arc model $\mathcal{A}$ and the corresponding circular-arc graph G of $\mathcal{A}$, we can get the irreducible arc model $\mathcal{A}^*$ of $\mathcal{A}$ in linear time.*

A *simplicial vertex* of a graph G is a vertex whose neighbors induce a clique. A *simplicial order* of G is an enumeration $v_1, v_2, \ldots, v_n$ of its vertices such that v_i is a simplicial vertex of induced subgraph $G[\{v_i, v_{i+1}, \ldots, v_n\}]$, $1 \leq i \leq n$. In other words, each set $X_i = \{v_j \in N[v_i] : j > i\}$ induce a clique. A *chordal graph* is a simple graph in which every cycle of length greater than three has a chord. Equivalently, the graph contains no induced cycle of length four or more. For chordal graphs, we have the following lemma.

Lemma 2.2 (Bondy and Murty 2008; Rose et al. 1976) *A graph is chordal if and only if it has a simplicial order.*

There is a linear-time algorithm due to Rose et al. (1976), and known as *lexicographic breadth-first search*, for finding a simplicial order of a graph if one exists. In this algorithm, a simplicial order of a graph G is found if G is recognized to be chordal; otherwise an induced cycle of length four or more is found. By using the algorithm, we can easily give a 2-clique-coloring of chordal graphs as follows.

Algorithm 1 Find a 2-clique-coloring of a chordal graph.
Input. A chordal graph $G = (V, E)$.
Output. A 2-clique-coloring $\phi : V \to \{1, 2\}$ of G.
Step 1: Give a simplicial order $\sigma = \{v_1, v_2, \cdots, v_n\}$ of G by running the algorithm 'lexicographic breadth-first search'.
Step 2: Set $S := \emptyset$, $i := n$.
Step 3: If $i = 0$, then turn to Step 4 directly. Otherwise, let $X_i := \{v_j \in N[v_i] : j \geq i\}$. If $X_i \cap S = \emptyset$, then set $S := S \cup \{v_i\}$; if not, set $S := S$. Set $i := i - 1$, turn to Step 3 again.
Step 4: For every $v \in S$, let $\phi(v) = 1$. For every $v \in V - S$, let $\phi(v) = 2$. Stop.

Theorem 2.1 *Algorithm 1 gives a 2-clique-coloring of a chordal graph G in linear time $O(n + m)$.*

Proof We first show, by induction on $|V(G)|$, that the set S in the end of Algorithm 1 is an independent set of G and it contains one vertex of every maximal clique of G. This is clearly true if $|V(G)| \leq 2$. Suppose, then, that $|V(G)| = n \geq 3$. By Lemma 2.2, G has a simplicial order. Let $\sigma = \{v_1, v_2, \ldots, v_n\}$ be a simplicial order of G. Then clearly $\sigma' = \{v_2, \ldots, v_n\}$ is also a simplicial order of $G' = G - v_1$. Note that, when $i = 2$ and Step 3 is carried out, the set S is the set S' obtained by Algorithm 1 for G'. By the inductive hypothesis, S' is an independent set of G' and includes one vertex of every maximal clique of G'. Note that when $i = 1$ and Step 3 is carried, the set S

equals either S' or $S' \cup \{v_1\}$. By Step 3, it is easy to see that S is an independent set, and all maximal cliques of G except the maximal clique $G[N[v_1]]$ are also maximal cliques of G'. This implies that S contains exactly one vertex of every maximal clique of G.

By Step 4, we see that ϕ is a 2-clique-coloring of G. In step 1 the time is $O(n+m)$ by the algorithm 'lexicographic breadth-first search'. Obviously, it needs $O(m)$ in step 3 and $O(n)$ in step 4. So the running time of Algorithm 1 is $O(n+m)$. □

By Theorem 2.1, we have the following result on the optimal clique-coloring of circular-arc graphs.

Lemma 2.3 *Let G be a circular-arc graph with arc model $\mathcal{A}$ and $\mathcal{A}'$ a removable arc set of $\mathcal{A}$. Let G' and H be the circular-arc graphs whose arc models are $\mathcal{A} \setminus \mathcal{A}'$ and $\mathcal{A}' \cup \{A_i\}$, respectively, where A_i is the arc in the definition of the removable set. Then $\chi_C(G) = \chi_C(G')$, and we can obtain an optimal clique-coloring of G in $O(n_1 + m_1)$ time from any given optimal clique-coloring of G', where $n_1 = |V(H)|$ and $m_1 = |E(H)|$.*

Proof Since $\mathcal{A}'$ is a removable arc set, the graph H is an interval graph. Note that an interval graph is also a chordal graph. By Theorem 2.1, we can give a 2-clique-coloring of H in $O(n_1 + m_1)$, where $n_1 = |V(H)|$ and $m_1 = |E(H)|$. Obviously, a maximal clique of G' is also a maximal clique of G, so $\chi_C(G) \geq \chi_C(G')$. Let ϕ' be a $\chi_C(G')$-clique-coloring of G' and ϕ'' be a 2-clique-coloring of H. Let v be the vertex corresponding to the arc A_i. As mentioned earlier, v is a cut vertex of G. Without loss of generality, we may assume that $\phi''(v) = \phi'(v)$. Hence $\phi' \cup \phi''$ is a $\chi_C(G')$-clique-coloring of G. Thus $\chi_C(G) = \chi_C(G')$ and, if given an optimal clique-coloring of G', we can obtain an optimal clique-coloring of G in $O(n_1 + m_1)$ time. □

By Lemmas 2.1 and 2.3, we have the following result on optimal clique-coloring of circular-arc graphs.

Theorem 2.2 *Let $\mathcal{A}$ be the arc model of a circular-arc graph G and $\mathcal{A}^*$ be the arc model of the irreducible circular-arc graph G^* from G. Given an optimal clique-coloring of G^*, we can obtain an optimal clique-coloring of G in linear time.*

3 The optimal clique-coloring of circular-arc graphs

For Helly circular-arc graphs, Cerioli and Korenchendler (2009) obtained the following result.

Theorem 3.1 (Cerioli and Korenchendler 2009) *For an irreducible Helly circular-arc graph G, $\chi_C(G) = 2$ if and only if G is not an odd cycle of length at least* 5.

By using the ideas involved in the proof of Theorem 3.1, we can get an optimal clique-coloring of an irreducible Helly circular-arc graph G in linear time.

Theorem 3.2 *If G is an irreducible Helly circular-arc graph, then we can give an optimal clique-coloring of G in linear time.*

Proof First we can easily decide whether G is an odd cycle of length at least 5 in linear time. If it is, then clearly $\chi_C(G) = 3$ and we can give a 3-clique-coloring of G in linear time. If not, then by using the algorithm 'lexicographic breadth-first search' in Rose et al. (1976), we decide whether G is a chordal graph, which needs linear time. If G is a chordal graph, then $\chi_C(G) = 2$ and we can give an optimal clique-coloring of G by using Algorithm 1. Otherwise, we can get a hole C_k ($k \geq 4$) (see Bondy and Murty 2008; Rose et al. 1976). Since C_k is an induced cycle, the arcs corresponding to the vertices of C_k cover the whole circle. Note that, if an edge $e = v_i v_j$ is a maximal 2-clique, this edge must be on the cycle C_k by the Helly-property of G. This implies that we can easily check whether G has no maximal 2-cliques in linear time. We consider the following two cases.

Case 1 G has no maximal 2-clique. In this case, every maximal clique has size at least 3. By the Helly-property of G, every maximal clique of G contains at least one vertex of C_k and one vertex of $V \setminus C_k$. Hence we can obtain a 2-clique-coloring of G by simply coloring the vertices on C_k with color 1 and the others with color 2. Clearly, the running time is linear.

Case 2 G has maximal 2-cliques. As mentioned above, every maximal 2-clique of G is on C_k. Let $\mathcal{P} = \{P_1, P_2, \ldots, P_p\}$ be the set of all maximal paths of G such that every edge in these paths is a maximal clique of G and $\mathcal{Q} = \{Q_1, Q_2, \ldots, Q_q\}$ be the set of all connected components of the subgraph induced by $E(G) - \{e : e \text{ is a maximal 2-clique in } G\}$. Then each Q_i in $\mathcal{Q}$ is an interval graph. Note that some component Q_i may consist of only one vertex. If Q_i consists of an isolated vertex, we don't need to consider the clique-coloring of Q_i. If Q_i has at least two vertices, then every maximal clique of Q_i is also a maximal clique of G. Moreover, every maximal clique of Q_i has more than two vertices, and at least one of them is a vertex of C_k.

Denote by v_i' and v_i'' the two vertices whose corresponding arcs have the smallest starting extreme point and maximum ending extreme point, respectively, among those arcs corresponding to vertices of Q_i. Since G is an irreducible circular-arc graph, v_i' and v_i'' are different and are vertices on C_k.

Now we can give two 2-clique-colorings f_1 and f_2 of Q_i such that $f_1(v_i') = f_1(v_i'')$ and $f_2(v_i') \neq f_2(v_i'')$. For the first case, assign color 1 to the vertices of $Q_i \cap C_k$ and color 2 to the others. Note that every maximal clique of Q_i contains at least three vertices and has at least a vertex is not on C_k, so no maximal clique of Q_i will be monochromatic. To obtain a color f_2, just assign the color 1 to vertices in $V(Q_i) \cap (V(C_k) \setminus \{v_i''\})$ and to vertices in $V(Q_i) \setminus V(C_k)$ whose only neighbor on C_k is v_i'' (maybe there is no such a vertex); the other vertices should be colored with color 2. It is not hard to show that no maximal clique of Q_i is monochromatic.

Now we give a 2-clique-coloring f of G as follows. First, color the paths in $\mathcal{P} = \{P_1, P_2, \ldots, P_p\}$ such that each maximal 2-clique has two colors. Then the two vertices v_i' and v_i'' of each Q_i have been colored. According to the colors of v_i' and v_i'', we give a 2-clique-colorings f_1 or f_2 of each Q_i as above. So we get a 2-clique-coloring of G.

Note that coloring all paths in $\mathcal{P}$ needs at most $O(n)$. In addition, the time of coloring each Q_i is $O(|Q_i|)$, thus the total time of coloring all Q_i's is also $O(n)$. Hence the running time is linear. □

By Algorithm 1 and Theorem 3.2, we get the following algorithm in linear time for the optimal clique-coloring of irreducible Helly circular-arc graphs.

Next we assume that, a circle and the set of arcs with given extreme points for circular-arc graphs are given explicitly as part of the input in the following algorithms.

Algorithm 2 The optimal clique-coloring of an irreducible Helly circular-arc graph.
Input. An irreducible Helly circular-arc graph $G = (V, E)$ and an arc model of G.
Output. A k-clique-coloring, where $k = \chi_C(G)$.
Step 1: If G is a odd cycle of order at least 5, give a 3-clique-coloring of G directly, stop. If not, turn to Step 2.
Step 2: Check whether G is a chordal graph by using 'lexicographic breadth-first search'. If G is a chordal graph, then perform Algorithm 1 on G, stop. If not, then we get a hole C_k ($k \geq 4$), turn to Step 3.
Step 3: If every edge of C_k lies in a triangle of G (i.e., G has no maximal 2-clique), then give a 2-clique-coloring of G by using the method in Case 1 of Theorem 3.2, stop. If not, then find out $\mathcal{P} = \{P_1, P_2, \ldots, P_p\}$, $\mathcal{Q} = \{Q_1, Q_2, \ldots, Q_q\}$ and, v_i' and v_i'' for each Q_i (see Case 2 in Theorem 3.2), turn to Step 4.
Step 4: Give a 2-coloring of each path P_i. According to the colors of v_i' and v_i'', further give a 2-clique-coloring of each Q_i, stop.

A circular-arc graph G is called a *non-Helly circular-arc graph* if G has no model with the Helly property.

Lemma 3.1 (Joeris et al. 2011) *If G is a non-Helly circular-arc graph with arc model $\mathcal{A}$, then $\mathcal{A}$ contains two or three arcs that cover the whole circle C.*

Theorem 3.3 *If G is a non-Helly circular-arc graph, then we can give a 2-clique-coloring of G in linear time.*

Proof Let $\mathcal{A}$ be the arc model of G. By Lemma 3.1, $\mathcal{A}$ contains either two or three arcs that cover the whole circle C. We first check whether or not $\mathcal{A}$ contains two arcs that cover the whole circle C in at most $O(m)$ time, where $m = |E(G)|$.

If we find two arcs A_i and A_j of $\mathcal{A}$ that cover the whole circle C, let v_i and v_j be the vertices corresponding to A_i and A_j, respectively. Clearly $v_i v_j \in E(G)$ and $N(v_i) \cup N(v_j) = V(G)$. We now give a 2-clique-coloring of G by assigning color 1 to the vertices of $N(v_i)$ and color 2 to all the other vertices of G in time $O(n)$.

If there exist no such two arcs above, then $\mathcal{A}$ must contain three arcs A_i, A_j, A_k that cover the whole circle by Lemma 3.1. Joeris et al. (2011) provided a recognition algorithm with complexity $O(m)$ for HCA graphs. This algorithm can search directly for the existence of such three arcs in time $O(m)$. (see p. 223). By using the algorithm, we find three arcs $A_i, A_j, A_k \in \mathcal{A}$ which cover the whole circle. Let v_i, v_j and v_k be the three vertices of G corresponding to the three arcs A_i, A_j, and A_k. Obviously, every vertex of $V(G) \setminus \{v_i, v_j, v_k\}$ is adjacent to at least one of $\{v_i, v_j, v_k\}$. We obtain a 2-clique-coloring of G in time $O(n)$ as follows. Assign color 1 to v if $v = v_i$, $v = v_j$, $v \in N(v_k) \setminus (N(v_i) \cup N(v_j))$ or $v \in (N(v_k) \cap (N(v_i)) \setminus N(v_j)$ and color 2 to the others. We claim that every maximal clique is not monochromatic. Let $V_1 = N(v_k) \setminus (N(v_i) \cup N(v_j))$, $V_2 = (N(v_k) \cap (N(v_i)) \setminus N(v_j)$, $V_3 = N(v_i) \cap N(v_j)$,

$V_4 = N(v_i)\backslash(N(v_j) \cup N(v_k))$ and $V_5 = N(v_j)\backslash N(v_i)$. According to our coloring, the vertices with color 2 are in $V_3 \cup V_4 \cup V_5$. Suppose that S is an arbitrary monochromatic clique of G. If S has color 1, then we have that $V(S) \subseteq V_1 \cup V_2$, $V(S) \subseteq V_2 \cup \{v_i\}$ or $V(S) = \{v_i, v_j\}$. Hence $S \cup \{v_k\}$ induces a bigger clique than S, but v_k is assigned color 2. If S has color 2, then we have that $V(S) \subseteq V_3 \cup V_4$ or $V(S) \subseteq V_5 \cup \{v_k\}$. But $V(S) \cup \{v_i\}$ or $V(S) \cup \{v_j\}$ will induce a bigger clique than S, while both v_i and v_j have color 1. Hence, no monochromatic clique is maximal, as claimed. □

Finally, we obtain an algorithm for the optimal clique-coloring of circular-arc graphs.

Algorithm 3 The optimal clique-coloring of circular-arc graphs.
Input. A circular-arc graph $G = (V, E)$ and its arc model $\mathcal{A}$.
Output. A k-clique-coloring, where $k = \chi_C(G)$.
Step 1: Find out all the removable arc sets of $\mathcal{A}$ and, give the irreducible arc model $\mathcal{A}^*$ and irreducible circular-arc graph G^* with arc model $\mathcal{A}^*$.
Step 2: Recognize whether G^* is a Helly circular-arc graph (see Joeris et al. 2011). If it is, then perform Algorithm 2 for G^*. If not, then give a 2-clique-coloring of G by using the method in Theorem 3.3.
Step 3: Extend the optimal clique-coloring of G^* into an optimal clique-coloring of G by using Lemma 2.3 and Theorem 2.2.

Theorem 3.4 *Algorithm 3 gives an optimal clique-coloring of G in linear time.*

Proof By Theorems 2.2, 3.2 and 3.3, Algorithm 3 give an optimal clique-coloring of G. By Lemma 2.1, the time is linear in Step 1. By Theorems 3.2 and 3.3, the time of Step 2 is linear. By Theorem 2.2, the time of Step 3 is also linear. Thus Algorithm 3 runs in linear time. □

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