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A PTAS for minimum weighted connected vertex cover P_3 problem in 3-dimensional wireless sensor networks

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Abstract Given a connected and weighted graph $G = (V, E)$ with each vertex v having a nonnegative weight $w(v)$, the minimum weighted connected vertex cover P_3 problem ($MWCVCP_3$) is required to find a subset C of vertices of the graph with minimum total weight, such that each path with length 2 has at least one vertex in C, and moreover, the induced subgraph $G[C]$ is connected. This kind of problem has many applications concerning wireless sensor networks and ad hoc networks. When homogeneous sensors are deployed into a three-dimensional space instead of a plane, the mathematical model for the sensor network is a unit ball graph instead of a unit disk graph. In this paper, we propose a new concept called weak c-local and give the first

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polynomial time approximation scheme (PTAS) for $MWCVCP_3$ in unit ball graphs when the weight is smooth and weak c-local.

Keywords PTAS · Connected vertex cover P_3 · Smooth weights · Weak c-local · Unit ball graph

1 Introduction

In practice, when the sensors have omnidirectional antennas with the same transmission range, the topology of the three-dimensional wireless sensor network can be modeled as a unit ball graph. An undirected graph G is called a *unit ball graph* if its vertices can be represented as points in three-dimensional space so that the Euclidean distance between two points corresponding to an edge in G is not greater than one. In this paper, we consider the minimum weighted connected vertex cover P_3 problem in unit ball graph which plays an important role in three-dimensional wireless sensor networks. We use Bondy and Murty (2008), Du et al. (2012) and Garey and Johnson (1979), for standard graph theory, approximation algorithms and computational complexity terminology and notations.

Given an undirected and simple graph $G = (V, E)$, a vertex cover (VC) of G is a subset of vertices $C \subseteq V$ which covers all edges, i.e., $\forall e = (u, v) \in E$, either $u \in C$ or $v \in C$. Let $w : V \to \mathbb{R}^+$ be a vertex weight function. The minimum weighted vertex cover (MWVC) problem consists of finding a vertex cover of minimum total weight. Many problems have been shown to be NP-complete by a transformation from VC, including well-known problems as the Hamilton cycle problem and the clique problem. Vertex covers are fundamental within graph theory, one reason being that vertex covers can be considered as the duals of matchings. The VC problem also represents a large class of related vertex deletion problems, in which one is interested in finding a minimum subset $C \subseteq V$ whose deletion gives the graph induced by $V - C$ satisfying a desired property, e.g., such that the graph $G[V - C]$ is edgeless (C is a vertex cover) or $G[V - C]$ has no cycle (C is a feedback vertex set). There are many other examples, including weighted variants (in which one is interested in a set C with a small total weight) and variants in which one imposes structural conditions on C (like being connected).

Here we study a variation of the vertex cover problem in which the set $C \subseteq V$ should intersect every path of length two in G with the graph induced by C in G being connected, and be of minimum total weight subject to these conditions. So the degree of each vertex in the graph induced by $V - C$ in G is at most one. For describing more background and related recent results on this topic, we first define the more general problem.

The concept of vertex cover P_k problem is a generalization of the vertex cover problem. Given a graph $G = (V, E)$ with vertex weight function $w : V \to \mathbb{R}^+$, the minimum weight vertex cover P_k problem is the problem of finding a minimum weight vertex cover set $C \subseteq V$ such that the graph $G[V - C]$ has no P_k as a subgraph, where P_k is the path on k vertices. Or, equivalently, the problem is to find a minimum weight vertex cover set $C \subseteq V$ such that for any P_k in G, $V(P_k) \cap C \neq \emptyset$. Here, the weight of a vertex set $C \subseteq V$ is $w(C) = \sum_{v \in C} w(v)$.

If furthermore, the subgraph $G[C]$ induced by a vertex cover P_k set C is required to be connected, then we call C a connected vertex cover P_k set. In this paper, we restrict our attention to the case of $k = 3$. The Minimum Weight Connected Vertex Cover P_3 problem (MWCVCP3 for short) is the optimisation problem of finding such a set with minimum total weight.

Connectivity constraints come in naturally, e.g., in many applications concerning wireless sensor networks, in which it is usually important to ensure connectivity if the sensor devices have limited capabilities of computation, energy and communication. Moreover, they are often deployed in accessible areas, where they can be rather easily captured by attackers. Therefore, the design of security protocols has become a challenge. One such protocols, known as the Canvas protocol, was designed in Menezes et al. (1996), Novotny (2010) to provide data integrity or data origin authentication (Menezes et al. 1996) in sensor networks. The k-generalised Canvas scheme (Novotny 2010) guarantees data integrity if at least one vertex is not captured on each path of length $k - 1$ in the communication graph. Thus, during the deployment and initialisation of a sensor network, it should be ensured that at least one protected vertex exists on each path of length $k - 1$ in the communication graph, and the problem of minimising the cost of the network by minimising the number of protected vertices arises naturally in Novotny (2010).

In the field of wireless sensor networks, unit ball graphs are widely used. A wireless sensor network is an ad hoc wireless network which consists of a huge amount of static or mobile sensors. The sensors collaborate to sense, collect, and process the raw information of the phenomenon in the sensing area and transmit the processed information to the observers. Suppose the sensors of the network have omnidirectional antennas with the same transmission range, two sensors can communicate if and only if they fall into the transmission ranges of each other, in other words, if and only if the Euclidean distance between them is at most one. There are cases in which three-dimensional models are needed, such as under-water sensor systems, outer-space sensor systems, notebooks in a multi-layered buildings, etc. For example, in a mountain area or underwater (Akyildiz et al. 2005), environment is often not flat. In such kind of underwater networks, in order to observe a given phenomenon, sensor nodes float at different depths. Then deployed sensors would form a three-dimensional wireless sensor network, which has a mathematical model, the unit ball graph. One can use under-water sensor systems to detect and observe phenomena that cannot be adequately observed by means of ocean bottom sensor nodes, i.e., to perform cooperative sampling of the 3D ocean environment.

Given a graph without its geometric representation, it is NP-hard to determine whether it can be represented as a unit ball graph (Breu and Kirkpatrickz 1998). Thus, in our paper we assume the geometric representation of the unit ball graph is given. This is usually the case in applications.

In practice, it is natural to assume that the vertices of the graph have some positive weights. In the context of wireless ad-hoc networks, these weights usually reflect residual energy, power capabilities, and information loads of a node for a specific task. The assumption of the smoothness of the weights (Wang et al. 2005) is reasonable in many applications such as homogeneous wireless sensor or ad hoc networks, where the weights of neighboring nodes do not vary significantly.

1.1 Related work

The case $k = 2$ of the minimum weight vertex cover P_k problem without the condition of connectivity is the well-studied minimum weight vertex cover problem. Boštjan (Brešar et al. 2011) proved that minimum vertex cover P_k problem is NP-complete for any fixed integer $k \geq 2$, Tu and Zhou (2011) gave a 2-approximation minimum weight vertex cover P_3 set by using the primal–dual method or the technique of layering. With the condition of connectivity, Liu et al. (2013) present a PTAS for the minimum k-path connected vertex cover problem in unit disk graphs. Notice that their work only studied the problem *without weight*.

Wang et al. (2015) have proved that the minimum weighted connected vertex cover P_3 problem remains NP-hard when restricted to unit disk graphs, or even to the more specific subclass of grid graphs. Since the unit disk graphs can be viewed as a subclass of the unit ball graphs, we can draw the conclusion that the MWCVCP3 problem is NP-hard on unit ball graphs.

A polynomial-time approximation scheme (PTAS) is a family of approximation algorithms with performance ratio $1 + \varepsilon$ (for any positive real number ε) that can be executed in polynomial time (depending on the size of the input and ε).

Wang and Jiang (1996) first introduced the concept of c-local and gave a PTAS for the Steiner tree problems in the plane with Euclidean and rectilinear metrics under the assumption of c-local. They call a Steiner tree problem c-local (for some positive constant c) if in a minimum spanning tree for the terminals, the length of a longest edge is at most c times the length of a shortest edge. Fan et al. (2011) presented a PTAS for the minimum weight connected vertex cover problem in unit disk graphs, provided the problem satisfies a c-local condition where in the solution C_0 obtained by some constant-approximation algorithm, the maximum weight of the vertices in C_0 is at most c, (assume, without loss of generality, that every vertex has weight at least one). It should be noted that maybe some vertices in G which will never be selected into the approximation solution, might have very large weights, since they do not play any important role in connection. So the assumption of c-local is different from the requirement that the maximum weight of the vertices of G is at most c. Inspired by these works, Wang et al. (2015) define a similar concept of c-local for MWCVCP3 and obtained a PTAS for MWCVCP3 problem on unit disk graphs under c-local assumption, when the unit disk graphs have minimum degree at least two.

1.2 Our contribution

In this paper, we study the problem of constructing $MWCVCP_3$ in unit ball graphs. The contributions of this paper can be summarized as follows:

1. In this paper we introduce a new concept of weak c-local which is weaker than the original concept of c-local as in Wang et al. (2015). We define an instance of MWCVCP3 to be weak c-local (for some positive constant c) if in a solution C_0 obtained by some approximation algorithm, the maximum weight of the vertices in $F \subseteq C_0$ which is a vertex cover P_3 set, is at most c, assuming without loss of generality that every vertex has weight at least one. Note that F may not be connected. Hence, if

an instance I of MWCVCP3 is c-local, it must be weak c-local as $F \subseteq C_0$. However, an instance I of MWCVCP3 of weak c-local is not necessary to be c-local.

2. In this paper, we present a PTAS for MWCVCP3 problem with smooth weights in unit ball graphs under the condition of weak c-local. It should be noticed that our result is not a direct generalization of Wang et al. (2015) to higher dimensional space. The technique used in Wang et al. (2015) is only valued when the minimum degree $\delta(G) \geq 2$, and it assumes a stronger condition of c-local. To obtain the results in this paper, new ideas of utilizing and combining the primal-dual approximation algorithm (Tu and Zhou 2011) and the techniques in high dimensional spaces (Zhang and Wu 2013) have to be explored.

1.3 Organisation of the paper

In Sect. 2, we introduce some preliminaries which will be needed later. In Sect. 3, we present our PTAS for MWCVCP3 problem on unit ball graphs. The approximation solution, the proof of the correctness of our algorithm, analysis of the time complexity and the performance ratio are given in Sect. 4. Finally, the concluding remarks and future research are drawn in Sect. 5. In the appendix we show that the minimum weighted connected vertex cover $P_k (k \geq 4)$ problem is NP-complete for grid graphs.

2 Preliminaries

In this section, we introduce some useful definitions and denotations that will be used in the partition and shifting strategy.

Given a geometric representation of a connected unit ball graph $G = (V, E)$ with $|V| = n$, we initially find a minimal three-dimensional cube Q to contain all the unit balls in G. Without loss of generality, assume $Q = \{(x, y, z)|0 \leq x \leq q, 0 \leq y \leq q, 0 \leq z \leq q\}$, where q is related to n. Let m be a large integer that will be determined later. Set $p = \lfloor \frac{q}{m} \rfloor + 1$, and $\widetilde{Q} = \{(x, y, z)| -m \leq x \leq pm, -m \leq y \leq pm, -m \leq z \leq pm\}$. Using partition strategy, we divide $\widetilde{Q}$ into $(p+1) \times (p+1) \times (p+1)$ smaller cubes (called cells) such that each cell is an $m \times m \times m$ cube (each cube is half closed and half open, including the back, left, and bottom sides, excluding the front, right, and top sides). Define this partition as $P(0)$. For $d = 0, 1, \ldots, m-1$, let $P(d)$ be the partition obtained from $P(0)$ by shifting the left-bottom-hind corner of $P(0)$ from $(-m, -m, -m)$ to $(-m+d, -m+d, -m+d)$. It should be noted that the enlargement of Q guarantees that after shifting, all the unit balls to be covered are still enclosed by every partition $P(d)$.

For each cell e, we define the *boundary region* B_e and *inner region* I_e as follows. The *boundary region* B_e of e is the region contained in e such that each point in this region is at most distance 3 from the boundary of e. The *inner region* I_e of e is the region of e such that each point at least distance 1 away from the boundary of e. Note that I_e and B_e have an overlap of width 2 (see Fig. 1). This ensures the output of our algorithm is a vertex cover P_3 set. If we add some additional vertices in the algorithm, the connection of the output computed by our algorithm can be ensured.

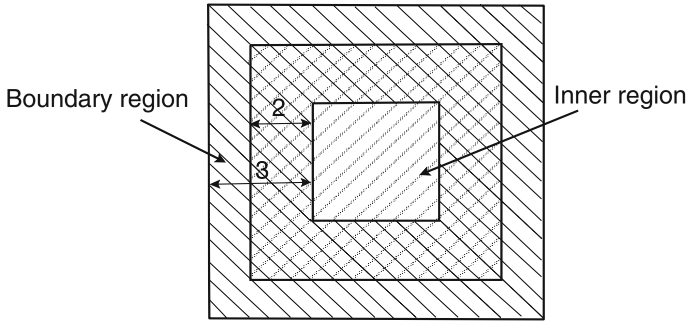

Fig. 1 *Shaded area* marked by *solid lines* indicates the *boundary region* and *shaded area* marked by *dotted lines* indicates the *inner region*. *Inner region* and *boundary region* have an overlap of width 2

Next, we give a useful definition of smoothness (Wang et al. 2005; Zhu et al. 2010) which will be used to devise polynomial algorithms that solve $MWCVCP_3$ problem.

Definition 1 Given an undirected and weighted graph $G = (V, E)$ with each vertex v assigned with a nonnegative weight $w(v)$, the weight function $w : V \to R^+$ is called smooth if there exists a constant $\beta \geq 1$ such that $max_{(uv)\in E} \frac{w(u)}{w(v)} \leq \beta$.

3 The algorithm for the PTAS

We describe the algorithm in this section. We denote the boundary region of a partition $P(d)$ as $B(P(d)) = \bigcup_{e\in P(d)} B_e$. The algorithm is executed in three phases.

Phase 1 Given the geometric representation of the unit ball graph G, adopt the primal-dual approximation algorithm of Tu and Zhou (2011), we can obtain a vertex cover P_3 set F of G with $w(F) \leq 2w(F^*)$, where F^* is an optimum vertex cover P_3 set of G.

If the induced subgraph $G[F]$ is connected, we can obtain a connected vertex cover P_3 set $S_0 = F$; If $G[F]$ is not connected, for two closest components R_1 and R_2 of $G[F]$, we can make R_1 and R_2 connect by finding a path T with the minimum total weight. Continue this procedure until $G[F]$ is connected. Then we can obtain a connected vertex cover P_3 set $S_0 = F$.

Denote by $S_0(d) = S_0 \cap B(P(d))$ the set of vertices of S_0 lying in the boundary region of partition $P(d)$. Using the shifting strategy to select a partition $P(d^*)$ such that $w(S_0(d^*)) = \min\{w(S_0(d)) \mid 0 \leq d \leq m-1\}$.

Phase 2 For every small cube $e \in P(d^*)$, denote by G_e the subgraph of G induced by the vertices in I_e, and $Comp(G_e)$ the set of connected components in G_e. For each small cube e and each component $H \in Comp(G_e)$, use exhaust search to find a minimum weighted connected vertex cover P_3 set S_H of H. Set $S_e = \bigcup_{H\in Comp(G_e)} S_H$.

Phase 3 If there exists a connected component $H \in Comp(G_e)$ such that $S_H \cap S_0 = \emptyset$ and there is no vertex of S_H adjacent with other vertex in $S_0(d^*)$, we can make S_H

and $S_0(d^*)$ connect by finding a path P_H with the minimum total weight (Here, we enumerate all the paths which connect S_H and $S_0(d^*)$, and then choose a path P_H with minimum total weight which can be executed in polynomial time). Set $S[e] = \bigcup_{H \in Comp(G_e)} P_H$, else, Set $S[e] = \emptyset$ (in this case, $S_H \cap S_0 = \emptyset$, and there is a vertex of S_H adjacent with other vertex in $S_0(d^*)$).
Final result Output $S = S_0(d^*) \cup \left(\bigcup_{e \in P(d^*)} S_e\right) \cup \left(\bigcup_{e \in P(d^*)} S[e]\right)$.

4 Analysis of the algorithm

In this section, we first show that we can obtain a constant approximation solution. Secondly, we prove the correctness of the algorithm. Furthermore, we analyze the time complexity, and show that the algorithm can be executed in polynomial time. Finally, we prove the main conclusions that the performance ratio of the algorithm is $(1+\varepsilon)$ for any arbitrarily small positive constant ε.

4.1 Approximation solution

In this section, with the help of the primal-dual approximation algorithm of Tu and Zhou (2011), we can obtain a constant ratio approximation solution of the $MWCVCP_3$ problem in the following.

Lemma 1 *There exists a ρ-approximation algorithm (with $\rho = 2(1+\beta c+\beta^2 c)$) to obtain a minimum weight connected vertex cover P_3 set in a smooth weighted unit ball graph G, if we assume that the problem is weak c-local.*

Proof Let S^* be an optimal solution for $MWCVCP_3$ of a weighted unit ball graph G. By using the primal-dual approximation algorithm of Tu and Zhou (2011) in Phase 1, we can obtain a vertex cover P_3 set F of G such that

$$w(F) \le 2w(F^*) \le 2w(S^*) \tag{*}$$

where F^* is an optimum vertex cover P_3 set of G. If we assume that J is the maximum weight of the vertices in F, we can obtain a constant ratio approximation solution of the $MWCVCP_3$ problem as following.

We assert that if the induced subgraph $G[F]$ is not connected, we can reduce the number of connected components of $G[F]$ by one through adding at most 2 vertices into F. Since the given unit ball graph is connected, we assume R_1 and R_2 are two closest components of $G[F]$, and denote by $T = (v_1, v_2, \ldots, v_t)$ the shortest path between R_1 and R_2, where $v_1 \in V(R_1)$, $v_t \in V(R_2)$. Firstly, $v_2 \notin F$, otherwise this path can be reduced to $(v_2, \ldots, v_t)$. If $v_3 \in F$, we can get $t = 3$, since R_1 and R_2 are two closest components. Then adding v_2 to F, we can connect R_1 and R_2. If $v_3 \notin F$, then v_4 must be in F, otherwise there must exist a path P_3 in $G[V - F]$. Hence, $t = 4$. We add v_2, v_3 to F so that we can make R_1 and R_2 connect. Therefore, our assertion is correct.

So if the induced subgraph $G[F]$ is not connected, we need to add at most $2(a-1)$ vertices into F to get a connected vertex cover P_3 set S_0 of G, where $a(a \leq |F|)$ is the number of connected components of $G[F]$ in graph G. Since the weights is smooth, and by the assumption that J is the maximum weight of the vertices in F, we can connect R_1 and R_2 by adding at most two vertices whose total weights are no more than $\beta J + \beta^2 J$. Hence, we can get a connected vertex cover P_3 set S_0 of G such that

$$\begin{aligned} w(S_0) &\leq w(F) + (a-1)(\beta J + \beta^2 J) \\ &\leq w(F) + w(F)(\beta J + \beta^2 J) \\ &\leq 2(1 + \beta J + \beta^2 J)w(S^*) \\ &\leq 2(1 + \beta c + \beta^2 c)w(S^*), \end{aligned}$$

where the second inequality follows that every vertex has weight at least one, the third inequality follows from $(*)$, and the last inequality follows the assumption of weak c-local. □

4.2 Correctness

In this subsection, we prove that the output S of our algorithm is a connected vertex cover P_3 set for the graph $G = (V, E)$.

Theorem 1 *The output S of our algorithm is a connected vertex cover P_3 set for the graph G.*

Proof Firstly, we prove that the induced graph $G[S]$ is connected. We prove this by three steps. In step 1, we show that distinct connected components in $G[S_0(d^*)]$ (if they exist) can be connected through vertices in $\bigcup_{e \in P(d^*)} S_e$. In step 2, we show that S_H is connected with $S_0(d^*)$ if there exists a connected component $H \in Comp(G_e)$ such that $S_H \cap S_0 \neq \emptyset$. In step 3, we can find a path P_H which connects S_H and $S_0(d^*)$ if there exists a connected component $H \in Comp(G_e)$ such that $S_H \cap S_0 = \emptyset$.

Step 1. Let H_1 and H_2 be the two distinct connected components in the induced subgraph $G[S_0(d^*)]$ which are 'closest' in $G[S_0]$ with each other. Since the induced subgraph $G[S_0]$ is connected, there exists a path $P = (v_1, v_2, \ldots, v_{t-1}, v_t)$ of $G[S_0]$ connecting H_1 and H_2 through the inner region of one small cube e. Note that the inner region I_e and the boundary region B_e of each small cube have an overlap with width 2, without loss of generally, we may assume that $\{v_1, v_2\} \subseteq V(H_1)$, $\{v_{t-1}, v_t\} \subseteq V(H_2)$ and $\{v_3, \ldots, v_{t-2}\} \subseteq I_e \setminus B_e$. Then we can observe that $\{v_1, v_2, v_{t-1}, v_t\} \subseteq B_e \cap I_e$, so the path $P = (v_1, v_2, \ldots, v_t)$ is in a connected component H of G_e. Based on Phase 2 of our algorithm, the path $(v_1, \ldots, v_t)$ is covered by S_H. It follows that at least one vertex of $\{v_1, v_2, v_3\}$ belongs to S_H, and at least one vertex of $\{v_{t-2}, v_{t-1}, v_t\}$ belongs to S_H. Since the induced subgraph $G[S_H]$ is connected, we can observe that H_1 and H_2 are connected through $G[S_H]$, $S_H \in S_e$.

Step 2. For each small cube e of $P(d^*)$, if there exists a connected component $H \in Comp(G_e)$ such that $S_H \cap S_0 \neq \emptyset$, there must exist a vertex $x \in S_H \cap S_0$. Since the induced subgraph $G[S_0]$ is connected in G, there exists a path L in $G[S_0]$

connecting x to another vertex $y \in S_0(d^*)$ which belongs to the other parts of G outside of e. We assume that the path $L = (v_0, v_1, \ldots, v_t)$, where $v_0 = x, v_t = y$ and $\{v_0, \ldots, v_{t-1}\} \subseteq e$. Let i be the index such that v_i is the first vertex on L with $v_i \in B_e$. Then we can see that $v_{i-1} \in I_e \setminus B_e, v_i, v_{i+1} \in I_e$, so there must exist a vertex in $\{v_{i-1}, v_i, v_{i+1}\}$ belongs to S_H. Therefore, S_H is connected with $S_0(d^*)$.

Step 3. For each small cube e of $P(d^*)$, if there exists a connected component $H \in Comp(G_e)$ such that $S_H \cap S_0 = \emptyset$, we can make S_H and $S_0(d^*)$ connect by finding a path P_H with the minimum total weight by Phase 3 of our algorithm. Or there exists a vertex of S_H adjacent with other vertex in $S_0(d^*)$. So after adding the vertices of the path P_H to S, we can get S_H is connected with $S_0(d^*)$.

For each case, we can draw the conclusion that S_H is connected with $S_0(d^*)$. Therefore, we have proved that $G[S]$ is connected.

Secondly, we prove that S is a vertex cover P_3 set for $G = (V, E)$. For any path (u, v, w) with length 2 in G, the Euclidean distance of the edge uv or vw is no more than one. Suppose (u, v, w) lies completely in a small cube e. Since the *inner* region I_e and the *boundary* region B_e have an overlap of width 2, there are two cases to be considered. The first case is that the path (u, v, w) belongs to the boundary region B_e. According to Phase 1 of our algorithm, S_0 is a connected vertex cover P_3 set of the graph G, and $S_0(d^*)$ is the set of vertices of S_0 lying in the boundary region of partition $P(d^*)$. So we can get $\{u, v, w\} \cap S_0(d^*) \neq \emptyset$. The second case is that the path (u, v, w) is in the inner region I_e of a small cube e, and thus the path (u, v, w) belong to a connected component H in G_e. Based on Phase 2 of our algorithm, S_H is a connected vertex cover P_3 set of H, hence the path $\{u, v, w\} \cap S_H \neq \emptyset$. In any case, the path (u, v, w) is covered by $S_e \cup S_0(d^*) \subseteq S$. The case that (u, v, w) crosses two adjacent cubes can be considered similarly to the first case. Hence, we have proved that S is a vertex cover P_3 set for $G = (V, E)$.

Based on the above analysis, we complete the proof of the theorem. □

4.3 Time complexity

In this subsection, we show that our algorithm runs in polynomial time. Phase 1 of our algorithm can be executed in polynomial time to obtain a ρ-approximation solution. Phase 3 can also be executed in polynomial time to find a path which connects S_H and $S_0(d^*)$. However, Phase 2 uses exhaustive search to achieve the desired solution. This is the most time consuming part, so we need to prove that this phase can also be completed within polynomial time.

Theorem 2 *The running time of our algorithm is no more than $n^{O(1/\varepsilon^3)}$, where n is the number of vertices in the graph.*

Proof In the Phase 2 of our algorithm, for each small cube e of $P(d^*)$ and a connected component $H \in Comp(G_e)$, we can use the exhaust search to obtain a subset $S_H \subseteq V(H)$ such that S_H is a minimum weight connected vertex cover P_3 set of H, i.e., the induced graph $G[V(H) - S_H]$ will consist of isolated vertices and isolated edges only. Let t_0 and t_1 denote the number of isolated vertices and isolated edges in $G[V(H) -$

S_H] respectively. It is easily seen that $t_0 + t_1$ is less than the maximum number of independent unit balls in the cube e.

For each small cube e, in order to make all the whole balls whose centers are in the $m \times m \times m$ cube lie completely in a specific cube, we need to enlarge the side length m of a cube to $m + 1$. Since each unit ball occupies volume $\pi/6$, we can get that the number of independent unit balls in an $m \times m \times m$ cube e is at most $\lceil \frac{6(m+1)^3}{\pi} \rceil$. So we have

$$|V(H) - S_H| = t_0 + 2t_1 \leq 2t_0 + 2t_1 \leq 2\left\lceil \frac{6(m+1)^3}{\pi} \right\rceil.$$

Now, in the following we show how to compute S_H. Firstly, we enumerate all the induced subgraphs of H with no more than $2\lceil \frac{6(m+1)^3}{\pi} \rceil$ vertices. Then, we find all induced subgraphs whose components do not contain a P_3. Finally, we take complements and find the one which is connected with minimum total weight.

The above exhaustive search for S_H takes time at most

$$\sum_{i=0}^{2\left\lceil \frac{6(m+1)^3}{\pi} \right\rceil} \binom{n_H}{i} = n_H^{O(m^3)},$$

where n_H is the number of vertices in H, and the total running time for Phase 2 is at most

$$\sum_{e,H} n_H^{O(m^3)} = \left(\sum_{e,H} n_H\right)^{O(m^3)} = n^{O(m^3)} = n^{O(1/\varepsilon^3)}.$$

(As we will see at the end of the proof of Theorem 3, $m = \lceil \frac{[12+144(2c\beta^3+c\beta^4)]\rho}{\varepsilon} \rceil$ is a suitable choice for our purposes.) So we prove the conclusion. □

4.4 Performance analysis

In this section, we prove that our algorithm has performance ratio $(1 + \varepsilon)$. Firstly, we introduce a lemma that will be used in this section, then give the performance ratio.

The following property for unit ball graph plays an important role in the approximation analysis.

Lemma 2 *For any vertex u in a unit ball graph G, the neighborhood $N_G(u)$ contains at most 12 independent vertices.*

The proof of the Lemma 2 has been given in Zhang et al. (2009). For convenience of the readers, we also present the proof in the appendix.

Based on Definition 1, Lemmas 1, 2 and Theorem 1, we then show that our algorithm is a PTAS as follows.

Theorem 3 *Suppose S^* is an optimal solution to the minimum weight connected vertex cover P_3 set in unit ball graph G, and S is the output of our algorithm. Then we have*

$$w(S) \leq (1+\varepsilon)w(S^*).$$

Proof We prove our conclusion in three steps.

Firstly, we prove that

$$w(S_0(d^*)) \leq \frac{12\rho}{m} w(S^*), \tag{4.1}$$

where m only depends on ε and c.

When we adopt the shifting strategy, it can be easily observed that a vertex of S_0 appears at most 12 times in the boundary area of $B(P(d))$s (see Fig. 2). Therefore, we have

$$w(S_0(0)) + w(S_0(1)) + \cdots + w(S_0(m-1)) \leq 12w(S_0).$$

Combining this with $w(S_0) \leq \rho w(S^*)$, we have

$$w(S_0(d^*)) \leq \frac{12\rho}{m} w(S^*).$$

Secondly, we are to add some vertices to S^* such that the resulting vertex set $\widetilde{S}$ satisfies the following two requirements:

(a) For each small cube e and each component $H \in Comp(G_e)$, $\widetilde{S} \cap V(H)$ is a connected vertex cover P_3 set of H.
(b) $w(\widetilde{S} \cap I_e) \leq w(S^* \cap I_e) + 12(c\beta^3 + c\beta^4)|S_0(d^*) \cap e|$.

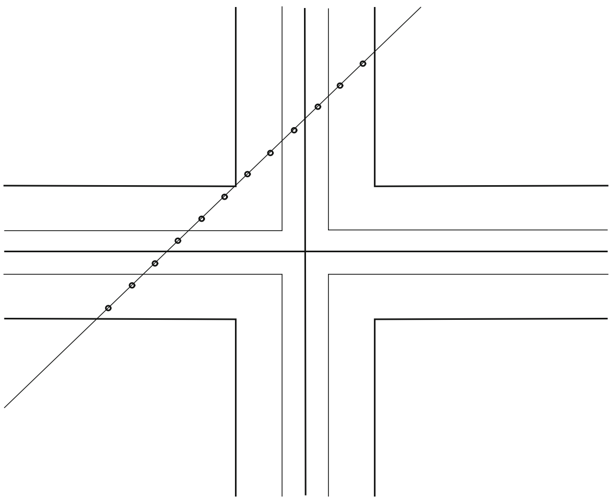

Fig. 2 The trace of a vertex has at most 12 points lying in the boundary area

Then, we show that the path P_H in Phase 3 of the algorithm satisfies $|P_H| \leq 1$, and

$$w(S[e]) \leq 12c\beta^3 |S_0(d^*) \cap e| \tag{4.2}$$

Before showing how to construct $\widetilde{S}$, and proving $|P_H| \leq 1$, we first show the theorem can be proved as long as the above two requirements are satisfied. In fact, it is easy to observe that in Phase 2 of our algorithm, S_e is a minimum weight connected vertex cover P_3 set satisfying requirement (a), so we have $w(S_e) \leq w(\widetilde{S} \cap I_e)$.

Combining this with inequalities (4.1), (4.2) and requirement (b), and the assumption that every vertex has weight at least one in the Introduction, we have

$$\begin{aligned} w(S) &\leq w(S_0(d^*)) + \sum_{e \in P(d^*)} w(S_e) + \sum_{e \in P(d^*)} w(S[e]) \\ &\leq w(S_0(d^*)) + \sum_{e \in P(d^*)} \left[w(S^* \cap I_e) + 12\left(c\beta^3 + c\beta^4\right) |S_0(d^*) \cap e| \right] \\ &\quad + 12c\beta^3 |S_0(d^*)| \\ &\leq w(S_0(d^*)) + w(S^*) + 12\left(2c\beta^3 + c\beta^4\right) |S_0(d^*)| \\ &\leq w(S_0(d^*)) + w(S^*) + 12\left(2c\beta^3 + c\beta^4\right) w(S_0(d^*)) \\ &\leq \left(1 + \frac{[12 + 144\left(2c\beta^3 + c\beta^4\right)]\rho}{m}\right) w(S^*). \end{aligned}$$

If we let $m = \lceil \frac{[12+144(2c\beta^3+c\beta^4)]\rho}{\varepsilon} \rceil$ in our algorithm, $w(S) \leq (1+\varepsilon)w(S^*)$. Then, the theorem is proved.

In the following we show how to construct $\widetilde{S}$ satisfying the requirements (a) and (b), and then prove $|P_H| \leq 1$.

For convenience, let $S_e^* = S^* \cap I_e$ for a small cube e. It is easy to see that for each component $H \in Comp(G_e)$, $S_e^* \cap V(H)$ is a vertex cover P_3 of H. Suppose there exists a component $H \in Comp(G_e)$ such that condition (a) is not satisfied. According to the proof of Lemma 1, there are two components R_1, R_2 of $G[S_e^* \cap V(H)]$ such that R_1 and R_2 can be connected through one vertex or two vertices in $V(H) \backslash S_e^*$. Add this vertex or two vertices into S_e^* to merge R_1 and R_2. It is easy to see that the new additional vertices are at most distance 2 away from some vertex of S_0. By using the assumption that the weight is smooth and the condition of weak c-local, we obtain S_0 by adding some additional vertices to F, so we can get that any $u \in S_0$, $w(u) \leq c\beta^2$, the total weight of the two additional vertices are at most $c\beta^3 + c\beta^4$. Continue this procedure until S_e^* satisfies requirement (a). Suppose this is done by k times, we get the new set $\widetilde{S_e^*}$ satisfies (a), then

$$w(\widetilde{S_e^*}) \leq w(S_e^*) + k\left(c\beta^3 + c\beta^4\right). \tag{4.3}$$

On the other hand, we can prove that

$$|S_0(d^*) \cap e| \geq \frac{k}{12} \tag{4.4}$$

For this purpose, we assume that the components merged are in the order of R_1 with R_2, R_3 with R_4, ..., and R_{2k-1} with R_{2k}.

We assume, without loss of generality, that these components are all different. For each $i = 1, 2, \ldots, k$, R_{2i-1} and R_{2i} are not connected in I_e. Since R_{2i-1} is connected to the outer parts of e through S^*, there are two cases to be distinguished.

The first case is that there only exists one vertex u_i lying in $V(R_{2i-1}) \cap B_e \cap I_e$, such that u_i is adjacent to a vertex $w_i \in B_e \backslash I_e$. Since R_{2i-1} belongs to H, and the *inner* region I_e and the *boundary* region B_e have an overlap of width 2, there exists a vertex $v_i \in V(H)$ adjacent to u_i, where $v_i \in B_e \cap I_e$. Hence, there exists a P_3 path $w_i u_i v_i$, where $w_i, u_i, v_i \in B_e$, and at least one of the vertices of the path belongs to S_0, denoted by z_i. Therefore, $z_i \in S_0(d^*) \cap e$.

The second case is that there exists two vertices u_i, v_i lying in $V(R_{2i-1}) \cap B_e \cap I_e$, where $u_i v_i$ is an edge of G such that u_i is adjacent to a vertex $w_i \in B_e \backslash I_e$. Since $w_i u_i v_i$ is a P_3 path, we have $z_i \in S_0(d^*) \cap e$ by the same argument as the first case. Note that a vertex may be used more than once as z_i. For instance, there may be two independent vertices x_i, y_i covered by the same vertex of S_0, as they belong to different components of $G[S^* \cap I_e]$. Then it follows from Lemma 2 that the number of times is no more than 12 when such a vertex serves as z_i. Hence inequality (4.4) holds.

According to inequalities (4.3) and (4.4), we have

$$w(\widetilde{S_e^*}) \leq w(S_e^*) + 12\left(c\beta^3 + c\beta^4\right)|S_0(d^*) \cap e|.$$

Let $\widetilde{S}$ be the union of the modified $\widetilde{S_e^*}$'s, we can get

$$w(\widetilde{S} \cap I_e) \leq w(S^* \cap I_e) + 12\left(c\beta^3 + c\beta^4\right)|S_0(d^*) \cap e|.$$

Finally, we consider each path P_H. From our algorithm we know that P_H connects S_H and $S_0(d^*)$ with minimum total weight. Since the inner region I_e and the boundary region B_e of each small cube have an overlap area with width 2. So there exists a path $P = (v_0, v_1, \ldots, v_t)$ connecting S_H and $B_e \setminus I_e$. Let $v_0 \in S_H$ and $v_t \in B_e \setminus I_e$. Then there must exist a vertex $x \in \{v_0, v_1, v_2\}$ belonging to $S_0(d^*)$. If $v_1 \in S_0(d^*)$, then S_H is connected with $S_0(d^*)$. If $v_2 \in S_0(d^*)$, then by the minimality of the path P_H, we have $|P_H| \leq 1$.

A vertex may be used more than once as x. So, by using the similar argument as above, we have

$$w(S[e]) \leq 12c\beta^3|S_0 \cap B_e \cap I_e| \leq 12c\beta^3|S_0(d^*) \cap e|$$

Based on the above conclusions, we complete the proof. □

5 Conclusion

We presented a polynomial time approximation scheme for this problem on unit ball graphs with a given geometric representation, under the conditions that the problem is weak c-local and the weights is smooth. This problem is an extension of the minimum weight connected vertex cover problem for which, as far as we know, there is no constant-factor approximation algorithm or PTAS known in unit disk graphs or unit ball graphs without any additional restriction. We strongly believe that a completely different new approach is needed in the case the assumption of the weak c-local and smooth are dropped.

And we can easily extend the complexity result in Wang et al. (2015) and show that the decision version of the minimum weighted connected vertex cover P_k problem ($k \geq 4$) is NP-complete for grid graphs. This is done in the Appendix. However, to our best knowledge, the hardness results on the “unweighted” version of this problem remains to be discovered. Furthermore, it is interesting to design and analyze approximation algorithms for the minimum weight connect vertex cover P_3 problem in grid graphs. It is still open whether our results can be generalized to the minimum weight connect vertex cover P_k problem under the same assumption. These are problems of our future interest.

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Appendix

In this section we present the proof of Lemma 2 and the complexity result given in the concluding section.

The proof of Lemma 2 is given as follows:

Proof The result can be obtained by transforming the problem into the famous Gregory–Newton Problem concerning about kissing number (Zong 1999). The *kissing number* is the maximum number of unit balls that can simultaneously touch the surface of a unit ball ('touch' means two balls have exactly one point in common). Let $S(u)$ be the unit ball with center u, and $\{x_1, \ldots, x_t\}$ be a maximum set of independent vertices in $S(u)$. For each $i = 1, \ldots, t$, draw a radial r_i with origin u which goes through x_i. Suppose r_i intersects the surface of $S(u)$ at point $\widetilde{x_i}$. Let S_i be a unit ball touching $S(u)$ at $\widetilde{x_i}$. Since $x_i^{\prime}s$ are independent, the angle between any two radials is at least $\frac{\pi}{3}$. Hence $S_i^{\prime}s$ are non-intersecting. It follows that t is at most the kissing number, which is 12. $\square$

Next, we introduce a crucial lemma due to Valiant (1981) and some preliminaries for proving the complexity result.

Lemma 3 (Valiant 1981) *A planar graph $G = (V, E)$ with maximum degree 4 can be embedded in the plane using $O(|V|)$ area in such a way that its vertices are at integer coordinates (x, y) and its edges are drawn so that they are made up of a number of line segments of the form $x = i$ (we refer to this as horizontal line segments) or $y = j$ (vertical line segments), for integers i and j.*

The above lemma plays an important role in the construction of a grid graph $G^{'}$ corresponding to the planar graph instance G of the following decision version of the minimum connected vertex cover problem for planar graphs with maximum degree 4.
Instance: Given a planar graph $G = (V, E)$ with maximum degree 4 and a positive integer l.
Question: Does G have a connected vertex cover with at most l vertices?
The decision version of the minimum weighted connected vertex cover P_k problem for grid graphs is stated as follows.
Instance: Given a grid graph $G^{'} = (V^{'}, E^{'})$, two positive integer $l^{'}$ and k, and a nonnegative weight for every vertex $v^{'} \in V^{'}$, where $k \geq 4$.
Question: Does $G^{'}$ have a connected vertex cover P_k set with weight at most $l^{'}$?
We now have all the ingredients to present and prove our complexity result.

Theorem 4 *The decision version of the minimum weighted connected vertex cover P_k problem is NP-complete for grid graphs, where $k \geq 4$.*

Proof It is not hard to see that this decision problem is in NP. To complete the complexity proof we use a reduction from the minimum connected vertex cover problem in planar graphs with maximum degree 4, which was shown to be NP-complete in Garey and Johnson (1977). We transform a planar graph $G = (V, E)$ with maximum degree 4 into a grid graph $G^{'}$ such that G has a connected vertex cover C with $|C| \leq l$ if and only if $G^{'}$ has a connected vertex cover P_k set $C^{'}$ with weight at most $l^{'}$. We assume without loss of generality that G is connected.

Using Lemma 3, we first embed G in a 2-dimensional grid with edges represented by horizontal and vertical line segments of length at least k, where $k \geq 4$, and with parallel lines at least a grid square apart. The set $V^{'}$ of vertices of our grid graph $G^{'}$ will be made up of two sets: V_1, the set of all the grid points corresponding to the vertices of G, and V_2, consists of all the grid points that are internal to the paths corresponding to the line segments that represent the edges of G. Note that all these paths representing the edges of G have at least one internal P_{k-1} (i.e., not containing a vertex corresponding to the original graph G). For each vertex $u \in V$, we denote the vertex of V_1 corresponding to u by $f(u)$. Figure 3 shows what a subgraph of $G^{'}$ corresponding to an edge uv of G might look like. Now, we define a weight function $w : V^{'} \rightarrow \{1, 2, ...\}$ by $w(f(v)) = |V_2|$ for every $v \in V$, and $w(v) = 1$ for every $v \in V_2$.

The construction of $G^{'}$ can clearly be accomplished in polynomial time. To complete our proof, we claim that there exists a connected vertex cover C in G with $|C| \leq l$ if and only if there exists a connected vertex cover P_k set $C^{'}$ in $G^{'}$ with weight at most $(l + 1)|V_2|$.

First suppose that the desired connected vertex cover C exists in G. If we take $C^{'}$ to be the union of $f(C) = \{f(v) \mid v \in C\}$ and V_2, then, since C is a connected vertex

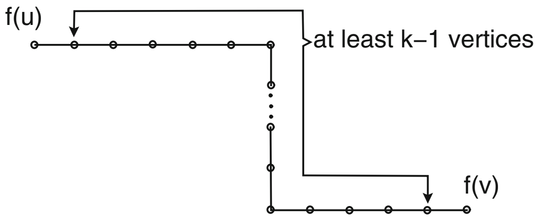

Fig. 3 A possible subgraph of $G^{'}$ corresponding to an edge uv of G

cover in G, the subgraph induced by $f(C)$ and V_2 in $G^{'}$ is connected. Moreover, all the neighbors of a vertex $f(u)$ for $u \in V \setminus C$ are in V_2, and hence in $C^{'}$, so $C^{'}$ is indeed a connected vertex cover P_k set in $G^{'}$, with $w(C^{'})$ at most $|C||V_2| + |V_2| \leq (l+1)|V_2|$.

For the converse, suppose that there is a connected vertex cover P_k set $C^{'}$ with $w(C^{'})$ at most $(l+1)|V_2|$ in $G^{'}$. We are going to show that G has a connected vertex cover of cardinality at most l. We may assume that E is not empty; otherwise there is nothing to prove. Since E is not empty, there is at least one path between vertices of $f(V)$ in $G^{'}$, so with an internal P_k that is covered by $C^{'}$. So, $C^{'}$ contains at least one vertex of V_2. Since $w(C^{'}) \leq (l+1)|V_2|$, this implies that $C^{'}$ contains at most l vertices of $f(V)$. In the following, it remains to show that $C = \{v \in V | f(v) \in C^{'}\}$ is a connected vertex cover of G. Clearly, C induces a connected subgraph of G because $f(C)$ together with paths of V_2-vertices induces a connected subgraph of $G^{'}$. Suppose there is an edge $e = uv$ of E that is not covered by C. Then neither $f(u)$ nor $f(v)$ is in C. Since $G'[C']$ is connected, this implies that no V_2-vertices on the path P of G' between $f(u)$ and $f(v)$ representing $e = uv$ belong to C'. But then P contains an internal P_k that is not covered by C', a contradiction. This completes the proof. □

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