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Cardinality constraints and systems of restricted representatives

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Abstract Cardinality constraints have received considerable attention from the Constraint Programming community as (so-called) global constraints that appear in the formulation of several real-life problems, while also having an interesting combinatorial structure. After discussing the relation of cardinality constraints with well-known combinatorial problems (e.g., systems of restricted representatives), we study the polytope defined by the convex hull of vectors satisfying two such constraints, in the case where all variables share a common domain. We provide families of facet-defining inequalities that are polytime separable, together with a condition for when these families of inequalities define a convex hull relaxation. Our results also hold for the case of a single such constraint.

Keywords Global cardinality constraint · Polyhedral combinatorics · Constraint programming

1 Motivation

The *cardinality constraint,* initially introduced as the *global cardinality constraint* (Regin 1996), restricts the number of occurrences of values assigned to a set of variables. Each value is given a lower bound and an upper bound, and the constraint requires that the number of occurrences of each of the values falls within these bounds in this set of variables. This constraint can be written as (Hooker 2012, Sect. 7.10.1)

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$$cardinality(x, J; l, u),$$
$$x_j \in D_j, j \in J.$$

Let $\bigcup_{j \in J} D_j = D = \{d_0, \ldots, d_{|D|-1}\}$ be the set of all values and $K = \{0, \ldots, |D| - 1\}$ the set indexing it. The constraint states that each value d_k ($k \in K$) must occur at least l_k and at most u_k times among the variables $\{x_j : j \in J\}$, where $l \leq u$ (i.e., $l_k \leq u_k$ for all $k \in K$). Without loss of generality, let $d_0 < \cdots < d_{|K|-1}$.

This constraint has several applications (Bulatov and Marx 2010), including instruction scheduling (van Beek and Wilken 2001) and car sequencing (Quimper et al. 2005). This has motivated an extensive literature within the Constraint Programming (CP) community in the form of procedures that reduce the set of solutions through tightening the domain of each variable. In CP terms, this research effort aims at accomplishing various forms of consistency (see Hooker 2012 for related definitions) including arc-consistency (Regin 1996), cost-based arc consistency (Regin 2002) and bounds consistency (Katriel and Thiel 2005; Quimper et al. 2005). A generalization of the cardinality constraint, discussed in Samer and Szeider (2011), is the *extended global cardinality constraint,* in which the number of occurrences of each value $d_k \in D$ must belong to a set of (not necessarily subsequent) integers, called the *cardinality set* of d_k. Another interesting extension of the cardinality constraint to set, multiset and tuple variables appears in Quimper and Walsh (2006).

In contrast, the literature from an Integer Programming (IP) perspective remains limited. Two families of valid inequalities for the single cardinality constraint appear in (Hooker, 2012, Sect. 7.10.1), together with a claim that these inequalities describe the convex hull of vectors satisfying a cardinality constraint. Preliminary results are presented in Mourtos (2013) regarding two cardinality constraints in the special case where all variables share a common domain $D = \{0, \ldots, |D| - 1\} = K$. These results include a condition for the associated polytope to be full-dimensional, and, given that the polytope is full-dimensional, conditions for the inequalities of Hooker (2012) to be facet-defining. The polyhedral study of multiple cardinality constraints, e.g., the dimension and the facets of the associated polytope, is also limited. Notably, results of this kind exist for the multiple *alldifferent* constraints (Bergman and Hooker 2014; Magos and Mourtos 2011; Magos et al. 2012) (the alldifferent constraint is a special case of the cardinality constraint, in which $l_k = 0$ and $u_k = 1$ for all $k \in K$).

Multiple cardinality constraints give rise to a *cardinality system,* i.e., a set C of cardinality constraints in which all variables share the same domain (i.e., $x_j \in D, \forall j \in J$) and all constraints admit the same lower and upper bounds on the occurrence of each value. The formulation

$$cardinality(x, J_c; l, u), c \in C, \tag{1}$$

dictates that each value $d_k \in D$ must occur at least l_k and at most u_k times ($l \leq u$) in each constraint $c \in C$, where $J = \bigcup_{c \in C} J_c$ is the set of all variables.

The polyhedral study of global constraints has been the common theme of several recent papers. A typical such paper considers a constraint

$$constraint(x, J; ...) \subseteq \bigotimes_{j \in J} D_j, \text{ where } x_j \in D_j, j \in J,$$

where J is the set of variables and $\bigotimes_{j \in J} D_j$ is the external product of the domains of all the variables. Then, such a paper considers the polytope defined by the convex hull of vectors satisfying such a constraint, assuming that $D_j = D$ ($j \in J$); i.e., the polytope $P = conv\{x \in D^{|J|} : x$ satisfies $constraint(x, J; ...)\}$. Obtaining valid inequalities for P allows the formulation of LP-relaxations for the constraint in hand, while using (in the formulation) only the facet-defining among these inequalities results in a 'tight' relaxation. A formulation containing all facet-defining inequalities provides a *convex hull relaxation*. There are several arguments in support of such relaxations regarding the effective integration of CP and IP methods (Hooker 2012; Milano et al. 2002).

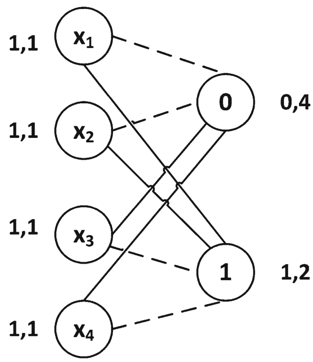
Furthermore, it remains important to obtain these relaxations using just the variables appearing originally, without including any additional $0-1$ variables (as in Hooker 2012, Sect 7.10.2). At the very least, this approach results in a significant reduction in the number of variables, since the standard approach replaces each variable $x_j \in D$ with $|D|$ binary variables z_{ij} through setting $x_j = \sum_{i \in D} i \cdot z_{ij}$ (see, for example, Williams and Yan 2001). Constraints for which this has been accomplished via finding some (or all) of the facets of P include *alldifferent* (Williams and Yan 2001), *cumulative scheduling* (Hooker and Yan 2002), *cardinality rules* (Balas et al. 2004; Yan and Hooker 1999) and *circuit* (Kaya and Hooker 2011). Such polytopes are also of interest from a technical perspective, since the polyhedral combinatorics literature focuses on examining polytopes defined on binary, rather than n-ary, variables. The facet-defining inequalities of the polytopes arising from n-ary formulations are typically quite different from the facet-defining inequalities of the associated binary polytopes (e.g., Kaya and Hooker 2011). Another benefit is that valid inequalities defined in the original n-ary space can be 'translated' into a binary model in order to further strengthen it (Bergman and Hooker 2014).

The cardinality constraint is of particular interest also because of its relation to fundamental combinatorial problems. To better illustrate this relation, consider the 'variable-value' graph $G(V_G, E_G)$, defined by $V_G = \{x_j : j \in J\} \cup D$ and $E_G = \{(x_j, d) : j \in J, d \in D_j\}$. An example is shown at Fig. 1. For each $v \in V_G$, define $\delta(v) = \{e \in E_G : e$ is incident to $v\}$. Given a vector $b \in Z_+^{|V_G|}$, a subset S of edges (i.e., $S \subseteq E_G$) is a *simple b-edge cover* if $|S \cap \delta(v)| \geq b_v$ (Schrijver, 2004, Sect. 21.9) and a *simple b-matching* if $|S \cap \delta(v)| \leq b_v$ (Schrijver, 2004, Sect. 21.3). In other words, a subset of edges is a simple b-edge cover if the subset includes at least b_v edges incident to node v (for each $v \in V$) and a simple b-matching if the subset includes at most b_v such edges.

For each $e \in E_G$ let $y_e \in \{0, 1\}$ and, for each $v \in V_G$, define $y(v) = \sum\{y_e : e \in \delta(v)\}$. It is clear that there is a $1-1$ correspondence between vectors x satisfying $cardinality(x, J; l, u)$ and subsets of E_G satisfying

$$y(x_j) = 1, j \in J,$$
$$l_d \leq y(d) \leq u_d, d \in D.$$

Fig. 1 Graph G for $cardinality(x, \{1, 2, 3, 4\}; [0, 1], [4, 2])$, $x_j \in D_j = \{0, 1\}$.



Therefore, $cardinality(x, J; l, u)$ is an n-ary representation of the subsets of edges that are simple $(\dot{e}, l)$-edge covers *and* $(\dot{e}, u)$-matchings of graph G, where $\dot{e} = [1 \cdots 1] \in R^{|J|}$. Considering the example of Fig. 1, notice that the ordered pair appearing besides each node denotes the smallest and the largest number of edges to be incident to it, e.g., node 'x_1' must have exactly one edge incident to it (since variable x_1 must receive a value) whereas node '1' must have at least 1 and at most 2 edges incident to it (since value 0 must be received by at least $l_1 = 1$ and at most $u_1 = 2$ variables). The 'broken' edges at Fig. 1 correspond to the solution $x_1 = x_2 = 0, x_3 = x_4 = 1$.

As another example, consider the collection of sets $\{D_j, j \in J\}$ where $\bigcup_{j \in J} D_j = D$ and $l, u \in Z_+^{|D|}$ with $l \leq u$. According to Ford and Fulkerson (1958), a *system of restricted representatives (SRR)* with respect to l and u is a sequence $(x_1, \ldots x_{|J|})$ such that

$$x_j \in D_j, l_d \leq |\{j \in J : x_j = d\}| \leq u_d \text{ for all } d \in D = \bigcup_{j \in J} D_j. \tag{2}$$

In this way, each vector x satisfying $cardinality(x, J; l, u)$ is an SRR and vice-versa. Equivalently, a cardinality constraint models transversals (Mirsky 1971) with lower and upper bounds (Schrijver (2004),Theorems 22.17 and 22.18]. We recall that a *transversal* with respect to the collection of sets $\{D_j, j \in J\}$ is a sequence of pairwise different elements $(d_1, \ldots, d_{|J|})$ such that $d_j \in D_j$, while a transversal with lower and upper bounds $l, u \in Z_+^{|D|}$ $(l \leq u)$ is a sequence $(x_1, \ldots x_{|J|})$ satisfying (2).

The 'variable-value' graph can be generalized for a cardinality system (1), the only difference being that V_G now includes one node for each variable occurring in any $J_c, c \in C$. For that graph, the problem of *simultaneous matchings* (Kutz et al. 2008) asks for "a subset of edges that is simultaneously a perfect matching for each constraint set in C" (i.e., for each $\{x_j, j \in J_c\}, c \in C$). As observed in Kutz et al. (2008), a solution to this problem corresponds to a solution for multiple *alldifferent* constraints and vice-versa. Accordingly, a solution of a cardinality system corresponds to a solution to the problem of *simultaneous* $(\dot{e}, l)$-edge covers and $(\dot{e}, u)$-matchings (and simultaneous SRRs/transversals) and vice-versa. Since the cardinality system generalizes simultaneous matchings (Kutz et al. 2008, Theorem 1) implies that it remains *NP-complete* to find a feasible solution to a cardinality system having $|C| \geq 2$ if the domains of the variables are not the same.

In this paper, we pursue a polyhedral study of the cardinality system (and SRRs) by examining the polytope defined by the convex hull of vectors satisfying two cardinality constraints, in the case where all variables share a common domain D of arbitrary, yet pairwise different, real numbers. We establish the dimension of this polytope (Sect. 2) and examine which inequalities of two known families are facet-defining (Sect. 3). Then, we provide a condition for these families to define a convex hull relaxation (Sect. 4).

2 The polytope and its dimension

Let $|C| = 2$ and let $J = J_1 \cup J_2$ be the set indexing the variables belonging to at least one of the cardinality constraints. The cardinality system is written as

$$cardinality(x, J_1; l, u), \tag{3}$$

$$cardinality(x, J_2; l, u), \tag{4}$$

$$x_j \in D, \forall j \in J. \tag{5}$$

Note that the above system assumes that the bounds on the number of occurrences of each domain value are the same for both constraints. However, these bounds can differ from one constraint to the other. Therefore, let us emphasize that this paper concentrates on the case where the bounds are the same for both constraints, since several of the results presented here do not generalize when these bounds differ from one constraint to the other.

Let $J_1 \cap J_2 = T$ and $I_1 = J_1 \backslash T$, $I_2 = J_2 \backslash T$, i.e., the set T indexes the variables that are common to both constraints, while I_1 and I_2 are the sets of variables appearing exclusively in the first and the second constraint, respectively. Notice that $T = \emptyset$ implies that the constraints are variable-wise disjoint while $I_1 = \emptyset$ or $I_2 = \emptyset$ yields $|C| = 1$, since one constraint would be 'dominated' by the other. Thus, we only consider cardinality systems for which none of I_1, I_2 or T is empty. The polytope of two cardinality constraints, namely P_I, is defined by

$$P_I = conv\{x \in D^{|J|} : \text{(3), (4) are satisfied}\}.$$

A point of $P_I \cap D^{|J|}$ satisfying (3) and (4) is hereafter called a *vertex*; i.e., P_I is the convex hull of its vertices. We use the term 'vertex' with a slight abuse of terminology, meaning that a vertex here is not necessarily a face of P_I with dimension 0. That is, a *vertex* is defined here as a point of P_I, whose coordinates are all from the set D, that is feasible with respect to (3) and (4).

To facilitate our presentation, we denote as $o(x; c, k)$ the number of occurrences of value d_k in constraint c at vertex x, i.e., $o(x; c, k) = \left|\{j \in J_c : x_j = d_k\}\right|$. In addition, we define the following notation to enable us to compactly represent changes to vertices.

Table 1 A *cardinality* system of two constraints

Predicate	I_1			T		I_2		
1	x_1	x_2	x_3	x_4	x_5			
2				x_4	x_5	x_6	x_7	x_8
$l=[0,0,1,0], u=[1,1,2,1]$								
x'	2	2	1	3	0	2	2	1
$\tilde{x}=x'(5\leftrightarrow\{1,6\})$	0	2	1	3	2	0	2	1
$\bar{x}=\tilde{x}(1\leftrightarrow 2)$	2	0	1	3	2	0	2	1
$l=[0,0,2,0], u=[2,2,3,1]$								
x'	2	2	2	0	0	2	2	2
$\tilde{x}=x'(5\leftrightarrow\{1,6\})$	0	2	2	0	2	0	2	2
$\bar{x}=x'(1;0\rightarrow 3)$	3	2	2	0	0	2	2	2
$\check{x}$	0	0	1	2	2	1	0	0

Notation For two vertices $x', \tilde{x} \in P_I$:

(i) $\tilde{x} = x'(j_1; d_k \rightarrow d_{k'})$ denotes that $\tilde{x}$ is derived from x' by only changing the value of variable x_{j_1} ($j_1 \in J$) from d_k to $d_{k'}$ ($\{k, k'\} \subseteq K$); i.e., $x'_{j_1} = d_k \neq d_{k'}$ while $\tilde{x}_{j_1} = d_{k'}$ and $\tilde{x}_j = x'_j$ for all $j \in J\backslash\{j_1\}$.

(ii) $\tilde{x} = x'(j_1 \leftrightarrow j_2)$ denotes that $\tilde{x}$ is derived from x' by only swapping the values of variables x_{j_1} and x_{j_2} ($\{j_1, j_2\} \subseteq J$); i.e., $x'_{j_1} \neq x'_{j_2}$ while $\tilde{x}_{j_1} = x'_{j_2}, \tilde{x}_{j_2} = x'_{j_1}$ and $\tilde{x}_j = x'_j$ for all $j \in J\backslash\{j_1, j_2\}$.

(iii) $\tilde{x} = x'(t \leftrightarrow \{i_1, i_2\})$ denotes that $\tilde{x}$ is derived from x' by only swapping the value of variable x_t ($t \in T$) with the common value of variables x_{i_1} ($i_1 \in I_1$) and x_{i_2} ($i_2 \in I_2$); formally, $x'_{i_1} = x'_{i_2} \neq x'_t$ while $\tilde{x}_t = x'_{i_1}, \tilde{x}_{i_1} = \tilde{x}_{i_2} = x'_t$ and $\tilde{x}_j = x'_j$ for all $j \in J\backslash\{i_1, i_2, t\}$. Note that this operation will always preserve feasibility.

Examples appear in Table 1, which assumes $D = \{0, 1, 2, 3\}$.

Occasionally, it becomes convenient, and unambiguous, to say that a value d_k ($k \in K$) *appears* in S ($S \subseteq J$) at a vertex $x \in P_I$ to denote that $x_j = d_k$ for some $j \in S$. Accordingly, we may say that a value d_k ($k \in K$) *appears* at least (or at most) o times in S ($S \subseteq J$) in predicate c ($c \in C$) at a vertex $x \in P_I$ to denote that $o(x; c, k) \geq o$ ($\leq o$).

We assume without loss of generality that $|J_1| \leq |J_2|$. Then, one may establish that

$$P_I \neq \emptyset \text{ if and only if } \sum_{k\in K} l_k \leq |J_1| \leq |J_2| \leq \sum_{k\in K} u_k. \quad (6)$$

Let us also stipulate that $0 \leq l_k \leq u_k$ and $u_k \geq 1$ for all $k \in K$.

A degenerate case arises if $l_k = |J_1|$ for some value d_k. In this case, all the variables in the first constraint must receive the same value at all vertices. Thus, the study of P_I reduces to the study of a single cardinality constraint. Hence, assume hereafter that $l_k < |J_1|$ for all $k \in K$. Since $|J_1| \leq |J_2|$, this assumption implies also that $l_k < |J_2|$ for all $k \in K$.

Let us provide a vertex of P_I, which will be used in the following proofs. Let the variables in any $x \in P_I$ be indexed, starting with the variables in I_1, then T, then I_2; i.e.,

$$x = \left(x_1, \ldots, x_{|I_1|}, x_{|I_1|+1}, \ldots, x_{|I_1|+|T|}, x_{|I_1|+|T|+1}, \ldots, x_{|I_1|+|T|+|I_2|}\right). \tag{7}$$

Also, assuming $|D| \geq 2$, let $l_{k_0} = \max\{l_k : k \in K\}$ and $l_{k_1} = \max\{l_k : k \in K\backslash\{k_0\}\}$; i.e., d_{k_0} and d_{k_1} are the values with the largest and the second largest lower bounds. For simplicity let $k_0 = 0$ and $k_1 = 1$; i.e., $l_0 \geq l_1 \geq l_k$ for all $k \in K\backslash\{0, 1\}$. The same argument works if the domain values d_{k_0} and d_{k_1} are not the domain values with the lower index.

Now, consider the vertex x' with $x'_1 = d_0$, $x'_{|I_1|+|T|} = d_1$, variables $x'_2, \ldots, x'_{l_0}$ assigned value d_0 only if $l_0 > 1$, variables $x'_{l_0+1}, \ldots, x'_{l_0+l_1-1}$ assigned value d_1 only if $l_1 > 1$, and, for each $k = 2, \ldots, |K| - 1$, the next l_k variables assigned value d_k. Up to this point, the number of variables assigned in J_1 is

- 2 (i.e., x'_1 and $x'_{|I_1|+|T|}$), if $l_0 = 0$ thus $l_k = 0$ for all $k \in K\backslash\{0\}$, or
- $l_0 + 1$, if $l_0 \geq 1$ but $l_1 = 0$ thus $l_k = 0$ for all $k \in K\backslash\{0, 1\}$, or
- $\sum_{k\in K} l_k$, if $l_0 \geq l_1 \geq 1$.

These variables exist in J_1, since $|J_1| \geq 2$ because I_1 and T are both non-empty, $|J_1| > l_0$ by assumption and $|J_1| \geq \sum_{k\in K} l_k$ by (6). Hence, the number of $m = \max\{\sum_{k\in K} l_k, l_0+1, 2\}$ variables assigned so far in J_1 (i.e., $x'_1, \ldots, x'_{m-1}$ and $x'_{|I_1|+|T|}$) suffices to satisfy all lower bounds.

Since $|J_1| \leq \sum_{k\in K} u_k$, there are sufficient occurrences of values to be assigned to the remaining variables in J_1. Hence the first among the variables $x'_m, \ldots, x'_{|I_1|+|T|-1}$ are assigned value d_0 until d_0 occurs u_0 times, the next variables are assigned value d_1 until d_1 occurs u_1 times, and so on, until variable $x'_{|I_1|+|T|-1}$ is assigned. This assigns values to all variables in $J_1 = I_1 \cup T$.

In addition, let $x'_{|I_1|+|T|+j} = x'_j$, $j = 1, \ldots, |I_1|$ (this is possible since $|I_1| \leq |I_2|$) and all variables $x'_{|I_1|+|T|+j}$, $j = |I_1| + 1, \ldots, |I_2|$, assigned any value d_k whose upper bound u_k has not been reached. This assignment of values to variables in J_2 is feasible since setting $x'_{|I_1|+|T|+j} = x'_j$, $j = 2, \ldots, |I_1|$ satisfies the lower bounds, while $|J_2| \leq \sum_{k\in K} u_k$ implies that there are sufficient occurrences of values in I_2 to satisfy the constraint. Therefore, x' satisfies (3) and (4), thus it is a vertex of P_I.

Regarding the example of Table 1 for $l = [0, 0, 1, 0]$ and $u = [1, 1, 2, 1]$, vertex x' is obtained by assigning $x'_1 = 2$ (since l_2 is the maximum lower bound), $x'_5 = 0$ (i.e., value 0 plays the role of d_1), then assigning $x'_2 = 2$ to reach the upper bound for value 2 and $x'_3 = 1$, $x'_4 = 3$ to reach the upper bounds for the remaining values; lastly, $x'_{5+j} = x'_j$, $j = 1, 2, 3$. For $l = [0, 0, 2, 0]$, $u = [2, 2, 3, 1]$, x' has $x'_1 = x'_2 = 2$, $x'_5 = 0$, then $x'_3 = 2$ and $x'_4 = 0$ to reach the upper bounds for values 2 and 0 and, lastly, $x'_{5+j} = x'_j$, $j = 1, 2, 3$.

Studying P_I becomes meaningful if it contains more than one vertex.

Proposition 1 *P_I has more than one vertex if and only if $|D| \geq 2$.*

Proof If $D = \{d_0\}$, P_I has a single vertex x' with $x'_j = d_0$ for all $j \in J$.

To show the 'if' part, consider the vertex x' described above. Observe that $x'_1 = x'_{|I_1|+|T|+1} = d_0$ and $x'_{|I_1|+|T|} = d_1 \neq d_0$. The point $\tilde{x} = x'(|I_1| + |T| \leftrightarrow \{1, |I_1| + |T| + 1\})$ satisfies (3) and (4) (i.e., $\tilde{x}$ is a vertex of P_I), since the number of times each value appears in each constraint at $\tilde{x}$ remains as in x'. Vertex x' is different from $\tilde{x}$, since $x'_1 = d_0$ whereas $\tilde{x}_1 = d_1$. □

It is known (e.g., Schrijver 2004) that P_I is full-dimensional if and only if no equality $\alpha x = \alpha_0$ is satisfied by all $x \in P_I$. If, for some $c \in C$, $|J_c| = \sum_{k \in K} l_k$, then each value d_k appears exactly l_k times in J_c at all vertices of P_I, thus the equality

$$\sum_{j \in J_c} x_j = \sum_{k \in K} l_k \cdot d_k \tag{8}$$

is satisfied by all vertices of P_I (and hence by all $x \in P_I$). In a similar manner, $|J_c| = \sum_{k \in K} u_k$ for some $c \in C$, implies that all $x \in P_I$ satisfy

$$\sum_{j \in J_c} x_j = \sum_{k \in K} u_k \cdot d_k. \tag{9}$$

On the other hand, if, for all $c \in C$,

$$\sum_{k \in K} l_k < |J_c| < \sum_{k \in K} u_k, \tag{10}$$

one would expect that no equality having $\sum_{j \in J_c} x_j$ as its left-hand side is satisfied by all $x \in P_I$. This may not be true, i.e., it may be that, although (10) holds, an equality having $\sum_{j \in J_c} x_j$ as its left-hand side, but with a right-hand side different from (8) and (9), is satisfied by all $x \in P_I$. Let us provide an example.

Example 1 Let $I_1 = \{1, 2\}, T = \{3, 4\}$ and $I_2 = \{5, 6\}, D = \{d_0, d_1\}, l = [2, 1]$ and $u = [2, 6]$. Since $|J_1| = 4 > l_0 + l_1 = 3$, the equality $x_1 + x_2 + x_3 + x_4 = l_0 d_0 + l_1 d_1 = 2d_0 + d_1$ is satisfied by no $x \in P_I$. However, value d_0 occurring at most twice in J_1 forces value d_1 to occur at least twice (despite $l_1 = 1$). Thus, the equality $x_1 + x_2 + x_3 + x_4 = 2(d_0 + d_1)$ is satisfied by all $x \in P_I$.

Let us add the conditions that, for all $k \in K$,

$$l_k \geq |J_2| - \sum_{k' \in K \setminus \{k\}} u_{k'}, \tag{11}$$

$$u_k \leq |J_1| - \sum_{k' \in K \setminus \{k\}} l_{k'}. \tag{12}$$

These conditions can be adopted without loss of generality. If, for example, $l_k < |J_2| - \sum_{k' \in K \setminus \{k\}} u_{k'}$, value k must appear at least $l'_k = |J_2| - \sum_{k' \in K \setminus \{k\}} u_{k'}$ times, hence replacing l_k with l'_k yields a cardinality system with an identical set of solutions. For Example 1, $l_1 = 1$ is replaced by $l'_1 = 2$. Given (11) and (12), (10) becomes a necessary and sufficient condition for P_I to be full-dimensional.

Theorem 2

$$\dim P_I = \begin{cases} |J| & \textit{if } \sum_{k\in K} l_k < |J_1| \le |J_2| < \sum_{k\in K} u_k, \\ |J|-1 & \textit{if } \sum_{k\in K} l_k = |J_1| < |J_2| < \sum_{k\in K} u_k, \\ |J|-1 & \textit{if } \sum_{k\in K} l_k < |J_1| < |J_2| = \sum_{k\in K} u_k, \\ |J|-2 & \textit{if } |J_1| = |J_2| = \sum_{k\in K} l_k \\ |J|-2 & \textit{if } |J_1| = |J_2| = \sum_{k\in K} u_k \\ |J|-2 & \textit{if } |J_1| = \sum_{k\in K} l_k \textit{ and } |J_2| = \sum_{k\in K} u_k. \end{cases}$$

We first provide some preliminary results, with proof, that will be used in the proof of the theorem.

Proposition 3 *If equality $\alpha x = \alpha_0$ is satisfied by all $x \in P_I$ then* ***(i)*** *$\alpha_i = \lambda_c$ for all $i \in I_c, c \in C$ and* ***(ii)*** *$\alpha_t = \lambda_1 + \lambda_2$ for all $t \in T$.*

Proof To show (i), let $c = 1$. For $|I_1| = 1$, the proposition trivially holds. Therefore, assume $|I_1| \geq 2$. Let the variables be indexed as in (7).

Consider the vertices x' and $\tilde{x}$ from the proof of Proposition 1 and recall that $x'_1 = d_0$ and $\tilde{x}_1 = d_1$ while $x'_2 = \tilde{x}_2$. Since variable x_1 receives a different value at vertices x' and $\tilde{x}$, while variable x_2 receives the same value at both x' and $\tilde{x}$, $x'_1 \neq x'_2$ or $\tilde{x}_1 \neq \tilde{x}_2$. Therefore there exists some vertex $\hat{x}$ with $\hat{x}_1 = d_k \neq \hat{x}_2 = d_{k'}$. It is clear that the vertex $\bar{x} = \hat{x}(1 \leftrightarrow 2) \in P_I$. Since $\alpha x = \alpha_0$ for all $x \in P_I$, equation $\alpha\hat{x} = \alpha\bar{x}$ yields, after deleting identical terms,

$$\alpha_1(d_k - d_{k'}) = \alpha_2(d_k - d_{k'}).$$

Therefore, $\alpha_1 = \alpha_2$ since $d_{k'} \neq d_k$. Since index 2 could have been assigned to any variable in $I_1\backslash\{1\}$, all of the coefficients of the variables in I_1 must be equal to α_1. The same reasoning holds for the coefficients associated with the variables in I_2. Therefore, let λ_1 and λ_2 be the coefficients of the variables in I_1 and I_2, respectively.

To show (ii), take the vertices x' and $\tilde{x}$ from the proof of Proposition 1. Then, equation $\alpha x' = \alpha\tilde{x}$ yields, after deleting identical terms,

$$(\lambda_1 + \lambda_2)(d_0 - d_1) = \alpha_{|I_1|+|T|}(d_0 - d_1),$$

so that $\alpha_{|I_1|+|T|} = \lambda_1 + \lambda_2$. As any variable with index in T could have been chosen as $x_{|I_1|+|T|}$, each such variable's coefficient must be $\lambda_1 + \lambda_2$, finishing the proof. □

For the system of Table 1 and $l = [0, 0, 1, 0]$, $u = [1, 1, 2, 1]$, $\alpha\bar{x} = \alpha\tilde{x}$ yields $\alpha_1 = \alpha_2 = \lambda_1$, while $\alpha x' = \alpha\tilde{x}$ yields $\alpha_5 = \alpha_1 + \alpha_6 = \lambda_1 + \lambda_2$.

Lemma 4 *For any vertex $x \in P_I$ and $c \in C$, if $o(x; c, k') > l_{k'}$ for some $k' \in K$, there is $\tilde{k} \in K\backslash\{k'\}$ such that $o(x; c, \tilde{k}) < u_{\tilde{k}}$.*

Proof Assuming that $o(x; c, k) = u_k$ for all $k \in K \setminus \{k'\}$ yields

$$
\begin{aligned}
|J_c| &= \sum_{k \in K \setminus \{k'\}} o(x; c, k) + o(x; c, k') \\
&= \sum_{k \in K \setminus \{k'\}} u_k + o(x; c, k') > \sum_{k \in K \setminus \{k'\}} u_k + l_{k'} \geq |J_c|,
\end{aligned}
$$

a contradiction (notice that the last '$\geq$' follows from (11)). □

Proposition 5 *For $c \in C$, if $\sum_{k \in K} l_k < |J_c| < \sum_{k \in K} u_k$ and $\alpha x = \alpha_0$ is satisfied by all $x \in P_I$ then $\alpha_i = 0$ for all $i \in I_c$.*

Proof Let $c = 1$ and take x' from the proof of Proposition 1. Since $\sum_{k \in K} l_k < |J_1|$, some value $d_{k'}$ appears more that $l_{k'}$ times in J_1; i.e., $o(x; c, k') > l_{k'}$. Then, by Lemma 4, $o(x'; c, \tilde{k}) < u_{\tilde{k}}$ for some value $d_{\tilde{k}} \neq d_{k'}$.

If value $d_{k'}$ appears in I_1, let $i_1 \in I_1$ be an index where this occurs; i.e., $x'_{i_1} = d_{k'}, i_1 \in I_1$. Observe that $\bar{x} = x'(i_1; d_{k'} \to d_{\tilde{k}})$ is a vertex of P_I since $o(\bar{x}; 1, \tilde{k}) = o(x'; 1, \tilde{k}) + 1 \leq u_{\tilde{k}}$ and $o(\bar{x}; 1, k') = o(x'; 1, k') - 1 \geq l_{k'}$. Since $\alpha x = \alpha_0$ is satisfied by all $x \in P_I$, $\alpha x' = \alpha \bar{x}$ yields

$$
\alpha_{i_1} d_{k'} = \alpha_{i_1} d_{\tilde{k}}. \tag{13}
$$

Otherwise, value $d_{k'}$ appears only in T (i.e., $x'_{i_1} \neq d_{k'}$) hence let $t \in T$ be an index where this occurs; i.e., $x'_t = d_{k'}, t \in T$. By construction of x', $x'_{|I_1|+|T|+i_1} = x'_{i_1} \neq d_{k'}$, hence the vertex $\tilde{x} = x'(t \leftrightarrow \{i_1, |I_1| + |T| + i_1\}) \in P_I$ has $\tilde{x}_{i_1} = d_{k'}$. As in the argument above, the vertex $\bar{x} = \tilde{x}(i_1; d_{k'} \to d_{\tilde{k}}) \in P_I$, thus $\alpha \tilde{x} = \alpha \bar{x}$ yields (13).

Equation (13) yields $\alpha_{i_1} = 0$. By Proposition 3i, $\lambda_1 = 0$ hence $\alpha_i = 0$ for all $i \in I_1$. □

For the system of Table 1 and $l = [0, 0, 2, 0]$ and $u = [2, 2, 3, 1]$, $\alpha x' = \alpha \bar{x}$ yields $\alpha_1 = 0$.

Let us now show the proof of Theorem 2.

Proof (Theorem 2) Consider some equality $\alpha x = \alpha_0$ that is satisfied by all $x \in P_I$. Then, α_j, $j \in J$, can be expressed in terms of scalars λ_1 and λ_2 as in Proposition 3i–ii.

If $\sum_{k \in K} l_k < |J_1| \leq |J_2| < \sum_{k \in K} u_k$, Proposition 5 yields that $\alpha_i = 0$ for all $i \in I_1 \cup I_2$. Hence, Proposition 3i implies $\lambda_1 = \lambda_2 = 0$ and, then, Proposition 3ii yields $\alpha_t = 0$ for all $t \in T$; i.e., $\alpha_j = 0$ for all $j \in J$. Thus, no equality is satisfied by all $x \in P_I$, therefore P_I is full-dimensional.

For all other cases, an equality (8) or (9) holds for J_1 or J_2 (or both).

Regarding the second and the third case of Theorem 2, for which $\dim P_I = |J| - 1$, let us show the proof only for the third case. Thus, let $\sum_{k \in K} l_k < |J_1| < |J_2| = \sum_{k \in K} u_k$. Since $\sum_{k \in K} l_k < |J_1| < \sum_{k \in K} u_k$, Proposition 5 yields $a_i = 0$ for all $i \in I_1$, thus Proposition 3i implies $\lambda_1 = 0$. Given that, Proposition 3i-ii yield $\alpha_j = \lambda_2$, $j \in I_2 \cup T = J_2$. Notice also that, at any vertex x of P_I, $|J_2| = \sum_{k \in K} u_k$ implies that each value d_k appears u_k times at J_2. Therefore,

$$\alpha_0 = \alpha x = \sum\nolimits_{j \in I_1} \alpha_j x_j + \sum\nolimits_{j \in J_2} \alpha_j x_j = \sum\nolimits_{j \in J_2} \lambda_2 x_j = \lambda_2 \sum\nolimits_{k \in K} u_k d_k.$$

Thus, any $\alpha x = \alpha_0$ satisfied by all $x \in P_I$ is a multiple of $\sum_{j \in J_2} x_j = \sum_{k \in K} u_k d_k$. Hence, the latter is the only equality (up to scalar multiplication) satisfied by all $x \in P_I$, thus $\dim P_I = |J| - 1$. Regarding the second case of Theorem 2, an analogous argument shows that $\alpha_0 = \lambda_1 \sum_{k \in K} l_k d_k$.

Regarding the last three cases of Theorem 2, for which $\dim P_I = |J| - 2$, we prove that

$$\alpha_0 = (\lambda_1 + \lambda_2) \sum\nolimits_{k \in K} l_k d_k$$

for the fourth case,

$$\alpha_0 = (\lambda_1 + \lambda_2) \sum\nolimits_{k \in K} u_k d_k$$

for the fifth case, and,

$$\alpha_0 = \lambda_1 \sum\nolimits_{k \in K} l_k d_k + \lambda_2 \sum\nolimits_{k \in K} u_k d_k \tag{14}$$

for the last case. Let us illustrate the proof only for the last case of Theorem 2, since the remaining cases can be shown in a similar manner. For that case, $|J_1| = \sum_{k \in K} l_k$ and $|J_2| = \sum_{k \in K} u_k$ imply that, at any vertex $x \in P_I$, each value d_k appears exactly l_k times in J_1 and u_k times in J_2. Then, using Proposition 3, (14) is written as

$$\begin{aligned}
\alpha_0 = \alpha x &= \sum\nolimits_{j \in I_1} \alpha_j x_j + \sum\nolimits_{j \in T} \alpha_j x + \sum\nolimits_{j \in I_2} \alpha_j x_j \\
&= \lambda_1 \sum\nolimits_{j \in I_1} x_j + (\lambda_1 + \lambda_2) \sum\nolimits_{j \in T} x_j + \lambda_2 \sum\nolimits_{j \in I_2} x_j \\
&= \lambda_1 \sum\nolimits_{j \in I_1 \cup T} x_j + \lambda_2 \sum\nolimits_{j \in I_2 \cup T} x_j = \lambda_1 \sum\nolimits_{j \in J_1} x_j + \lambda_2 \sum\nolimits_{j \in J_2} x_j \\
&= \lambda_1 \sum\nolimits_{k \in K} l_k d_k + \lambda_2 \sum\nolimits_{k \in K} u_k d_k.
\end{aligned}$$

Therefore, any $\alpha x = \alpha_0$ satisfied by all $x \in P_I$ is a linear combination of the equalities $\sum_{j \in J_1} x_j = \sum_{k \in K} l_k d_k$ and $\sum_{j \in J_2} x_j = \sum_{k \in K} u_k d_k$. Thus, these two are the only equalities (up to scalar multiplication) satisfied by all $x \in P_I$. In addition, they are linearly independent, since the sets of variables in their left-hand side are different. That is, since the variables indexed by I_1 appear only in the first equality, whereas the variables indexed by I_2 appear only in the second one, no equality is linearly dependent on the other. Hence, these equalities form a minimum equation system of rank 2 for P_I, thus $\dim P_I = |J| - 2$. □

3 Facet-defining inequalities

To identify the facets of P_I, let us recall some definitions from (Hooker, 2012, Sect. 7.10.1). For $c \in C$,

$$p_c(k) = \min\left\{u_k, |J_c| - \sum_{i=0}^{k-1} p_c(i) - \sum_{i=k+1}^{|K|-1} l_i\right\}, \quad k = 0, \ldots, |K|-1, \tag{15}$$

$$q_c(k) = \min\left\{u_k, |J_c| - \sum_{i=k+1}^{|K|-1} q_c(i) - \sum_{i=0}^{k-1} l_i\right\}, \quad k = |K|-1, \ldots, 0. \tag{16}$$

As also discussed in (Hooker, 2012, Sect. 7.10.1), $p_c(k)$ and $q_c(k)$ are the number of occurrences of value d_k if $\sum_{j\in J_c} x_j$ is minimized and maximized, respectively.

In more detail, the sum of all $|J_c|$ variables of constraint $c \in C$ is minimized once the smallest value d_0 has the largest number of occurrences, denoted as $p_c(0)$. That number cannot be larger than u_0 and, in addition, it cannot exceed $|J_c| - \sum_{i=1}^{|K|-1} l_i$, since each value $d_k, k \in K\backslash\{0\}$ must also occur at least l_k times. Then, the number of occurrences of value d_1 is bounded not only by u_1 but also by $|J_c| - p_c(0) - \sum_{i=2}^{|K|-1} l_i$, since value 0 occurs $p_c(0)$ times and each value $d_k, k \in K\backslash\{0, 1\}$ must occur at least l_k times. Repeating this procedure for $k = 0, \ldots, |K|-1$ (in that order) yields $p_c(k)$ as the number of occurrences of value $d_k, k \in K$ when $\sum_{j\in J_c} x_j$ is minimized. The, in a sense, inverse procedure computes $q_c(k), k = |K|-1, \ldots, 0$; i.e., the number of occurrences of value $d_{|K|-1}$ in a maximum sum of all variables is computed first, followed by the number of occurrences of value $d_{|K|-2}$ and so on.

For $S \subseteq J_c$, let $p_c(|S|, k)$ or $q_c(|S|, k)$ denote the number of occurrences of value d_k once the sum of $|S|$ variables is minimized or maximized, respectively. Hence, for $c \in C$ and $S \subseteq J_c$, we define

$$p_c(|S|, k) = \min\left\{p_c(k), |S| - \sum_{i=0}^{k-1} p_c(|S|, i)\right\}, k = 0, \ldots, |K|-1, \tag{17}$$

$$q_c(|S|, k) = \min\left\{q_c(k), |S| - \sum_{i=k+1}^{|K|-1} q_c(|S|, i)\right\}, k = |K|-1, \ldots, 0. \tag{18}$$

That is, the sum of variables indexed by S is minimized by selecting value d_0 $\min\{p_c(0), |S|\}$ times, value d_1 $\min\{p_c(1), |S| - p_c(|S|, 0)\}$ times (since value d_0 already occurs $p_c(|S|, 0)$ times in S) and so on. Accordingly, this sum is maximized by selecting value $d_{|K|-1}$ $\min\{q_c(|K|-1), |S|\}$, value $d_{|K|-2}$ $\min\{q_c(|K|-2), |S| - q_c(|S|, |K|-1)\}$ times (since value $d_{|K|-1}$ already occurs $q_c(|S|, |K|-1)$ times in S) and so on.

Notice that $p_c(|S|, k)$ and $q_c(|S|, k)$ vary not only with respect to $|S|$ but also with respect to c, i.e., to $|J_c|$. Hence, to provide a concise presentation hereafter, let us assume that $|J_1| = |J_2| = n$ and omit 'c' from the definitions (15), (16), (17) and (18). The results hold even if $|J_1| \neq |J_2|$.

Example 2 With respect to the system of Table 1, for $l = [0, 0, 2, 0]$ and $u = [2, 2, 3, 1]$, notice that $p(0) = 2, p(1) = 1, p(2) = 2, p(3) = 0$. Also, $p(3, 0) = p(0), p(3, 1) = p(1), p(3, 2) = p(3, 3) = 0$, while $p(4, 0) = p(0), p(4, 1) = p(1), p(4, 2) = 1 < p(2), p(4, 3) = 0$. Moreover, $q(0) = 0, q(1) = 1, q(2) = 3$ and $q(3) = 1$; also, $q(4, 3) = 1, q(4, 2) = 3, q(4, 1) = q(4, 0) = 0$, while $q(3, 3) = 1, q(3, 2) = 2, q(3, 1) = q(3, 0) = 0$.

It now becomes apparent that the inequalities

$$\sum\nolimits_{j\in S} x_j \geq \sum\nolimits_{k\in K} p(|S|, k) \cdot d_k,\ S \subseteq J_c, c \in C, \tag{19}$$

$$\sum\nolimits_{j\in S} x_j \leq \sum\nolimits_{k\in K} q(|S|, k) \cdot d_k,\ S \subseteq J_c, c \in C, \tag{20}$$

are valid for P_I. Although the number of inequalities (19) or (20) per constraint is $2^n - 1$ (i.e., the number of all subsets of an n-set except for the empty subset), both families of inequalities are separable in $O(n \log n)$ steps (Hooker, 2012, Sect. 7.10.1). For $|S| = 1$, (19) and (20) reduce to the trivial inequalities $d_0 \leq x_j \leq d_{|K|-1},\ j \in J$. Also, $p(n, k) = p(k), q(n, k) = q(k)$, hence for $|S| = n$ (19) and (20) can be written as

$$\sum\nolimits_{j\in J_c} x_j \geq \sum\nolimits_{k\in K} p(k) \cdot d_k,\ c \in C, \tag{21}$$

$$\sum\nolimits_{j\in J_c} x_j \leq \sum\nolimits_{k\in K} q(k) \cdot d_k,\ c \in C. \tag{22}$$

Let us summarize some direct consequences of (15)-(18), to be utilized in the following proofs.

Corollary 6 *(i)* $l_k \leq p(|S|, k) \leq p(k) \leq u_k,\ k \in K, S \subseteq J_c, c \in C$

(ii) $l_k \leq q(|S|, k) \leq q(k) \leq u_k,\ k \in K, S \subseteq J_c, c \in C$
(iii) $\sum_{k\in K} p(k) = \sum_{k\in K} q(k) = n$
(iv) $\sum_{k\in K} p(|S|, k) = \sum_{k\in K} q(|S|, k) = |S|,\ S \subseteq J_c, c \in C$

The following two lemmas, showing which of inequalities (19) and (20) cannot be facet-defining for P_I, appear in Mourtos (2013) for the special case where $D = \{0, \ldots, |D| - 1\} = K$. The proofs for arbitrary D, following analogous arguments, are presented here for completeness.

Lemma 7 *(i) Inequalities (19) are redundant for* $2 \leq |S| \leq p(0)$ *and for* $n - p(|K| - 1) \leq |S| \leq n - 1$.

(ii) Inequalities (20) are redundant for $2 \leq |S| \leq q(|K| - 1)$ *and for* $n - q(0) \leq |S| \leq n - 1$.

Proof We only show (i), since (ii) can be shown in a similar manner. For $2 \leq |S| \leq p(0)$, (17) implies $p(|S|, 0) = |S|$ but $p(|S|, k) = 0$ for $k \in K\backslash\{0\}$. Thus, (19), reduces to

$$\sum_{j \in S} x_j \geq |S| \cdot d_0,$$

hence equals the sum of inequalities $x_j \geq d_0,\ j \in S$.

For $n - p(|K| - 1) \leq |S| \leq n - 1$, (17) yields $p(|S|, k) = p(k)$, for all $k \in K\backslash\{|K| - 1\}$, since $n - p(|K| - 1) \leq |S|$. Hence,

$$\begin{aligned} p(|S|, |K| - 1) &= \min\{p(|K| - 1), |S| - \sum_{k \in K\backslash\{|K|-1\}} p(|S|, k)\} \\ &= \min\{p(|K| - 1), |S| - \sum_{k \in K\backslash\{|K|-1\}} p(k)\}. \end{aligned} \tag{23}$$

By Corollary 6iii, $\sum_{k \in K\backslash\{|K|-1\}} p(k) = n - p(|K| - 1)$ thus (23) becomes

$$p(|S|, |K| - 1) = \min\{p(|K| - 1), |S| - n + p(|K| - 1)\},$$

implying $p(|S|, |K| - 1) = |S| - n + p(|K| - 1)$, since $|S| \leq n - 1$. But then, (19), written as

$$\sum_{j \in S} x_j \geq (|S| - n + p(|K| - 1)) \cdot d_{|K|-1} + \sum_{k \in K\backslash\{|K|-1\}} p(k) \cdot d_k,$$

can be obtained by adding (21) and inequalities $-x_j \geq -d_{|K|-1},\ j \in J_c\backslash S$. □

Let $P^l(S) = \{x \in P_I : x$ satisfies (19) at equality$\}$ and $P^u(S) = \{x \in P_I : x$ satisfies (20) at equality$\}$ be the face defined by (19) and (20), respectively ($S \subseteq J_c, c \in C$). The faces $P^l(J_c)$ and $P^u(J_c)$ are defined accordingly. Define also $K^l(S) = \{k \in K : p(|S|, k) = p(k)\}$ as the subset of values that appear in S at any vertex of $P^l(S)$ as many times as they appear in J_c at any vertex of $P^l(J_c)$. The set $K^u(S) = \{k \in K : q(|S|, k) = q(k)\}$ has an analogous interpretation.

Lemma 8 *(i)* $\sum_{k \in K^l(S)} p(k) + \sum_{k \in K\backslash K^l(S)} l_k \geq n$ *implies* $P^l(S) \subseteq P^l(J_c)$

(ii) $\sum_{k \in K^u(S)} q(k) + \sum_{k \in K\backslash K^u(S)} l_k \geq n$ *implies* $P^u(S) \subseteq P^u(J_c)$

Proof We only show (i), since the proof of (ii) is almost identical. At an arbitrary vertex $x \in P^l(S)$, $o(x; c, k) = p(k)$ for any $k \in K^l(S)$, by the definition of $K^l(S)$. Also, $o(x; c, k) = l_k$ for any $k \in K\backslash K^l(S)$, since the opposite, i.e., $o(x; c, k') > l_{k'}$ for some $k' \in K\backslash K^l(S)$, yields

$$\begin{aligned} n = \sum_{k \in K} o(x; c, k) &= \sum_{k \in K^l(S)} o(x; c, k) + \sum_{k \in K\backslash K^l(S)} o(x; c, k) \\ &> \sum_{k \in K^l(S)} p(k) + \sum_{k \in K\backslash K^l(S)} l_k \geq n, \end{aligned}$$

a contradiction. In addition, $p(k) = l_k$ for all $k \in K \setminus K^l(S)$, since the opposite, i.e., $p(k') > l_{k'}$ for some $k' \in K \setminus K^l(S)$, yields (through Corollary 6iii)

$$n = \sum_{k \in K} p(k) = \sum_{k \in K^l(S)} p(k) + \sum_{k \in K \setminus K^l(S)} p(k) > \sum_{k \in K^l(S)} p(k) + \sum_{k \in K \setminus K^l(S)} l_k \geq n,$$

a contradiction. But then, at any vertex $x \in P^l(S)$, each value d_k appears $p(k)$ times thus $x \in P^l(J_c)$. Since a face of P_I is the convex hull of vertices in it, $P^l(S) \subseteq P^l(J_c)$. □

Example 2 (*cont.*) Since $p(0) = 2$, Lemma 7i yields that (19) is redundant for $|S| = 2$. By Lemma 8i, since $|S| = 3$ implies $K^l(S) = \{0, 1\}$ and $l_2 = 2$, $p(3, 0) + p(3, 1) + l_2 = 5 = n$ hence $P^l(S) \subseteq P^l(J_c)$, $c \in C$. In the same manner, Lemma 8i yields that $P^l(S) \subseteq P^l(J_c)$ for $|S| = 4$.

Since Lemma 7ii does not apply (because $q(3) = 1$ and $q(0) = 0$), (19) is not redundant for $|S| \in \{4, 3, 2\}$. Also, $K^u(S) = \{2, 3\}$ for $|S| = 4$ and $K^u(S) = \{3\}$ for $|S| = 3, 2$, thus no $P^l(S)$ for $|S| \in \{4, 3, 2\}$ is contained in $P^l(J_c)$, $c \in C$ by Lemma 8ii (observe that $q(3) + l_2 = 3 < n$).

Note that Lemmas 7 and 8 hold irrespectively of whether $|C| = 2$ or $|J_1| = |J_2| = n$, since neither condition is assumed within the corresponding proofs.

Let us discuss the consequences of $|J_1| = |J_2| = n$. Given $|J_1| = |J_2| = n$, if $|J_1| = \sum_{k \in K} l_k$ then $|J_2| = \sum_{k \in K} l_k$, hence each value d_k appears exactly l_k times in both J_1 and J_2 at any vertex of P_I. Then, it can only be that $l_k = u_k$ for all $k \in K$ (this follows also from (11) and (12)). Hence, $|J_1| = \sum_{k \in K} l_k$, given $|J_1| = |J_2| = n$, implies $\sum_{k \in K} l_k = n = \sum_{k \in K} u_k$, in which case (8) coincides with (9) for both J_1 and J_2.

In addition, $|J_1| = |J_2|$ yields that (8) holds for J_1 if and only if it holds also for J_2. Thus, $\dim P_I \in \{|J|, |J| - 2\}$. Notice that, if $\dim P_I = |J| - 2$ (i.e., if $\sum_{k \in K} l_k = n = \sum_{k \in K} u_k$), equalities (8) for J_1 and J_2 are part of the minimum equation system of P_I, thus not defining proper faces of P_I (i.e., $P^l(J_1) = P^l(J_2) = P_I$).

Now, $P^l(S)$ cannot be a facet of P_I if (19) is redundant or $P^l(S)$ is contained in another proper face of P_I. Thus, Lemmas 7 and 8 provide conditions under which $P^l(S)$ is not a facet of P_I. Specifically, Lemma 8 provides a condition under which $P^l(S)$ is contained in the face $P^l(J_1)$ or $P^l(J_2)$. Therefore, if $P^l(J_1)$ and $P^l(J_2)$ are proper faces of P_I, Lemma 8 (if its condition holds) yields that $P^l(S)$ is not a facet. However, if $\sum_{k \in K} l_k = n = \sum_{k \in K} u_k$, both $P^l(J_1)$ and $P^l(J_2)$ are not proper faces of P_I (because $P^l(J_1) = P^l(J_2) = P_I$), hence Lemma 8, even if its condition holds, does not yield that $P^l(S)$ is not a facet. That is, Lemma 8 may yield that $P^l(S)$ is not a facet only if $\sum_{k \in K} l_k < n < \sum_{k \in K} u_k$. Following the same reasoning, we add the condition that, if $\sum_{k \in K} l_k = n = \sum_{k \in K} u_k$, then S is not equal to J_1 or J_2. Let us introduce a definition encompassing all these conditions, listed here only for $P^l(S)$.

Definition 9 For $c \in C$, an $S \subseteq J_c$ is *non-dominated* with respect to (19) if S satisfies the following:

(i) $p(0) < |S| < n - p(|K| - 1)$ or $|S| = 1$ or $|S| = n$;
(ii) if $\sum_{k\in K} l_k < n < \sum_{k\in K} u_k$, then $P^l(S) \not\subset P^l(J_c)$;
(iii) if $\sum_{k\in K} l_k = n = \sum_{k\in K} u_k$, then $S \subset J_c$.

Theorem 10 *For $c \in C$ if $S \subseteq J_c$ is* non-dominated *then $P^l(S)$ is a facet of P_I; i.e.,*

$$\dim P^l(S) = \begin{cases} |J| - 1 & \text{if } \sum_{k\in K} l_k < n < \sum_{k\in K} u_k, \\ |J| - 3 & \text{otherwise.} \end{cases}$$

We prove Theorem 10 only for $P^l(S)$, after proving some intermediate results.

Lemma 11 *For $c \in C$, let $S \subseteq J_c$ be non-dominated. If $\sum_{k\in K} l_k < n < \sum_{k\in K} u_k$, there is a vertex $x \in P^l(S)$ such that $o(x; c, k) > l_k$ for some value d_k $(k \in K)$ appearing in $J_c \backslash S$.*

Proof Let $c = 1$ and $S \subseteq J_1$. Assume to the contrary that, at any vertex $x \in P^l(S)$, $o(x; 1, k) = l_k$ for any value d_k appearing in $J_1 \backslash S$. This implies that, at any vertex $x \in P^l(S)$, any value d_k such that $o(x; 1, k) > l_k$ appears only in S. Then, since by (17) a value d_k appears $p(|S|, k)$ times in S, it follows that $o(x; 1, k) = p(|S|, k)$ for any value d_k such that $o(x; 1, k) > l_k$. Let $K_S = \{k \in K : d_k \text{ appears only in } S\}$ and notice that the above imply

$$\begin{aligned} n &= \sum_{k\in K} o(x; 1, k) = \sum_{k\in K_S} o(x; 1, k) + \sum_{k\in K\backslash K_S} o(x; 1, k) \\ &= \sum_{k\in K_S} p(|S|, k) + \sum_{k\in K\backslash K_S} l_k. \end{aligned} \tag{24}$$

By Corollary 6i, $p(k) \geq p(|S|, k) \geq l_k$ for all $k \in K$. It must be that $p(k) = p(|S|, k)$ for all $k \in K_S$ and $p(k) = l_k$ for all $k \in K \backslash K_S$ since the opposite (i.e., $p(k') > p(|S|, k')$ for some $k' \in K_S$ or $p(k') > l_{k'}$ for some $k' \in K \backslash K_S$) yields through (24) and Corollary 6iii,

$$n = \sum_{k\in K} p(k) = \sum_{k\in K_S} p(k) + \sum_{k\in K\backslash K_S} p(k) > \sum_{k\in K_S} p(|S|, k) + \sum_{k\in K\backslash K_S} l_k = n,$$

a contradiction. But then, at any vertex $x \in P^l(S)$, each value d_k appears $p(k)$ times in J_1 thus $x \in P^l(J_1)$. It follows that $P^l(S) \subseteq P^l(J_1)$, which, given $\sum_{k\in K} l_k < n < \sum_{k\in K} u_k$, contradicts Definition 9ii. □

Proposition 12 *For $c \in C$, let $S \subseteq J_c$ be non-dominated. If $|J_c \backslash S| \geq 2$ (resp. $|I_c \backslash S| \geq 2$), there is a vertex of $P^l(S)$ at which two different values appear in $J_c \backslash S$ (resp. $I_c \backslash S$).*

Proof Let $c = 1$, $S \subseteq J_1$, $|J_1 \backslash S| \geq 2$ and $|I_1 \backslash S| \geq 2$. We first show that there is a vertex of $P^l(S)$ at which two different values appear in $J_1 \backslash S$.

For $\sum_{k\in K} l_k = n = \sum_{k\in K} u_k$, Corollary 6i yields $l_k = p(k) = u_k$ for all $k \in K$. That is, at any vertex $x \in P^l(S)$, each value d_k appears exactly $p_k = u_k \geq 1$ times in

J_1, hence value $d_{|K|-1}$ appears at least once in J_1. Since $P^l(S)$ contains the vertices minimizing the sum of variables indexed by S, value $d_{|K|-1}$ appears in $J_1\backslash S$ because it is the largest domain value (hence the last one considered by (17)). Also, since value $d_{|K|-1}$ appears $p(|K|-1)$ times in J_1, while the set $J_1\backslash S$ contains $|J_1|-|S| = n-|S|$ variables and $n-|S| > p(|K|-1)$ (by Definition 9i), not all variables in $J_1\backslash S$ can receive value $d_{|K|-1}$. Thus, $d_{|K|-1}$ and some other value appear in $J_1\backslash S$.

For $\sum_{k\in K} l_k < n < \sum_{k\in K} u_k$, Lemma 11 implies that there is a vertex $x \in P^l(S)$ such that $o(x; 1, k') > l_{k'}$ for some value $d_{k'}$ appearing in $J_1\backslash S$. Then, Lemma 4 implies that there is a $\tilde{k} \in K\backslash\{k'\}$ such that $o(x; 1, \tilde{k}) < u_{\tilde{k}}$. The vertex $\check{x}$ with $\check{x}_j = x_j, j \in J_1$, and $\check{x}_{|I_1|+|T|+j} = x_j, j = 1,\ldots,|I_1|$, also belongs to $P^l(S)$ since all variables indexed by $S \subseteq J_1$ are as in $x \in P^l(S)$. By construction of $\check{x}$, $o(\check{x}; 1, k) = o(\check{x}; 2, k)$ for all $k \in K$, hence $o(\check{x}; 1, k') = o(\check{x}; 2, k') > l_{k'}$ and $o(\check{x}; 1, \tilde{k}) = o(\check{x}; 2, \tilde{k}) < u_{\tilde{k}}$. If a single value appears in $J_1\backslash S$ at $\check{x}$ then $\check{x}_{j_1} = \check{x}_{j_2} = d_{k'}$ for some $j_1, j_2 \in J_1\backslash S$ (j_1, j_2 exist because $|J_1\backslash S| \geq 2$). The vertex $\bar{x} = \check{x}(j_1; d_{k'} \to d_{\tilde{k}})$ belongs to $P^l(S)$ since all variables indexed by S are as in $\check{x} \in P^l(S)$. Moreover, $\bar{x}_{j_1} = d_{\tilde{k}}$ and $\bar{x}_{j_1} = d_{k'}$ thus two different values appear in $J_1\backslash S$ at vertex $\bar{x} \in P^l(S)$.

Hence, let $\bar{x} \in P^l(S)$ denote a vertex at which two different values appear in $J_1\backslash S$. It remains to show that there is a vertex of $P^l(S)$ at which two different values appear in $I_1\backslash S$. Assume that this does not hold for vertex $\bar{x}$. Consider the vertex $x' \in P^l(S)$ with $x'_j = \bar{x}_j, j \in J_1$, and $x'_{|I_1|+|T|+j} = \bar{x}_j, j = 1,\ldots,|I_1|$; i.e., $o(x'; 1, k) = o(x'; 2, k)$ for all $k \in K$. Since at vertex x' two different values appear in $J_1\backslash S$ but not in $I_1\backslash S$, a value appearing in $T\backslash S$ differs from a value appearing in $I_1\backslash S$. Hence let $x'_{i_1} = x'_{i_2} = d_{k'}$ and $x'_t = d_{\tilde{k}} \neq d_{k'}$, where $i_1, i_2 \in I_1\backslash S$ (since $|I_1\backslash S| \geq 2$) and $t \in T\backslash S$. By construction of x', $x'_{|I_1|+|T|+i_1} = d_{k'}$ hence the vertex $\tilde{x} = x'(t \leftrightarrow \{i_1, |I_1|+|T|+i_1\})$ belongs to $P^l(S)$ and has $\tilde{x}_{i_1} = d_{\tilde{k}}$ and $\tilde{x}_{i_2} = d_{k'}$. □

Proposition 13 *For $c \in C$, let $S \subseteq J_c$ be non-dominated. If $\alpha x = \alpha_0$ for all $x \in P^l(S)$ then*

- (i) $\alpha_i = \lambda_c, i \in I_c\backslash S$,
- (ii) $\alpha_t = \lambda_1 + \lambda_2, t \in T\backslash S$,
- (iii) $\alpha_i = \lambda_c + \mu, i \in I_c \cap S$,
- (iv) $\alpha_t = \lambda_1 + \lambda_2 + \mu, t \in T \cap S$.

Proof Let $S \subseteq J_1$.

(i) $\alpha_i = \lambda_1, i \in I_1\backslash S$

Take vertex $\tilde{x}$ from the proof of Proposition 12. It is clear that the vertex $\bar{x} = \tilde{x}(i_1 \leftrightarrow i_2) \in P^l(S)$. Since $\alpha x = \alpha_0$ for all $x \in P^l(S), \alpha x' = \alpha\tilde{x}$ yields, after deleting identical terms,

$$\alpha_{i_1}(d_{k'} - d_{\tilde{k}}) = \alpha_{i_2}(d_{k'} - d_{\tilde{k}}).$$

Therefore, $\alpha_{i_1} = \alpha_{i_2}$ since $d_{\tilde{k}} \neq d_{k'}$. Since index i_2 could have been assigned to any variable in $(I_1\backslash S)\backslash\{i_1\}$, all of the coefficients of the variables in $I_1\backslash S$ must be equal to α_{i_1}.

Regarding the coefficients associated with the variables in I_2, let $i_3 = |I_1|+|T|+i_1$ and $i_4 = |I_1| + |T| + i_2$. Recall that, by construction of $\tilde{x}$, $\tilde{x}_{i_3} = \tilde{x}_{i_1} = d_{\tilde{k}}$ and $\tilde{x}_{i_4} = \tilde{x}_{i_2} = d_{k'}$. Since the vertex $\bar{x} = \tilde{x}(i_3 \leftrightarrow i_4) \in P^l(S)$, $\alpha x = \alpha_0$ yields $\alpha_{i_3} = \alpha_{i_4}$. Since i_4 could have been assigned to any variable in $I_2 \backslash \{i_3\}$, all of the coefficients of the variables in I_2 must be equal to α_{i_3}.

Therefore, let λ_1 and λ_2 be the coefficients of the variables in I_1 and I_2, respectively.

(ii) $\alpha_t = \lambda_1 + \lambda_2, t \in T\backslash S$

Take vertices $x', \tilde{x} \in P^l(S)$ from the proof of Proposition 12. Since $\alpha x = \alpha_0$ for all $x \in P^l(S)$, $\alpha x' = \alpha \tilde{x}$ yields, after deleting identical terms,

$$\alpha_t(d_{k'} - d_{\tilde{k}}) = \left(\alpha_{i_1} + \alpha_{|I_1|+|T|+i_1}\right)(d_{k'} - d_{\tilde{k}}).$$

Therefore, $d_{\tilde{k}} \neq d_{k'}$ implies $\alpha_t = \lambda_1 + \lambda_2$. As any variable with index in $T\backslash S$ could be chosen as x_t, each such variable's coefficient must be $\lambda_1 + \lambda_2$.

(iii) $\alpha_i = \lambda_1 + \mu, i \in I_1 \cap S$

The proof for $|I_1 \cap S| = 1$ is trivial, thus assume $|I_1 \cap S| \geq 2$. Take vertex $x' \in P^l(S)$ from the proof of Proposition 12. Definition 9i (i.e., $p(0) < |S|$) yields that d_0 and some other value, say d_1, appear in $J_1 \cap S$ at x'. Thus let $x'_{i_1} = d_0$, $i_1 \in I_1 \cap S$ and $i_2 \in (I_1 \cap S)\backslash\{i_1\}$.

If value d_1 appears in $I_1 \cap S$, consider x'_{i_2}. If $x'_{i_2} = d_1$, the vertex $\bar{x} = x'(i_1 \leftrightarrow i_2) \in P^l(S)$, since the values appearing in S are as in x'. Since $\alpha x = \alpha_0$ for all $x \in P^l(S)$, $\alpha x' = \alpha \bar{x}$ yields

$$\alpha_{i_1}(d_0 - d_1) = \alpha_{i_2}(d_0 - d_1) \tag{25}$$

Otherwise, let $i_3 \in (I_1 \cap S)\backslash\{i_1, i_2\}$ such that $x'_{i_3} = d_1$. The vertex $\check{x} = x'(i_2 \leftrightarrow i_3) \in P^l(S)$ has $\check{x}_{i_1} = d_0$ and $\check{x}_{i_2} = d_1$. After deriving the vertex $\bar{x} = \check{x}(i_1 \leftrightarrow i_2) \in P^l(S)$, $\alpha \check{x} = \alpha \bar{x}$ yields (25).

If value d_1 appears in $T \cap S$, let $x'_t = d_1$, $t \in (T \cap S)$. Recall that, by construction of x', $x'_{|I_1|+|T|+i_1} = x'_{i_1} = d_0$. Thus, the vertex $\tilde{x} = x'(t \leftrightarrow \{i_1, |I_1|+|T|+i_1\})$ belongs to $P^l(S)$ and has $\tilde{x}_{i_1} = d_1$ and $\tilde{x}_{i_2} = d_0$. After deriving the vertex $\bar{x} = \tilde{x}(i_1 \leftrightarrow i_2) \in P^l(S)$, $\alpha \tilde{x} = \alpha \bar{x}$ yields (25).

For all cases, (25) implies $\alpha_{i_1} = \alpha_{i_2}$. Since index i_2 could have been assigned to any variable in $(I_1 \cap S)\backslash\{i_1\}$, all of the coefficients of the variables in $I_1 \cap S$ must be equal to α_{i_1}.

(iv) $\alpha_t = \lambda_1 + \lambda_2 + \mu, t \in T \cap S$

Take the vertices x' and $\tilde{x}$ from the proof of (iii). Since $\alpha x = \alpha_0$ for all $x \in P^l(S)$, $\alpha x' = \alpha \tilde{x}$ yields $\alpha_t = \alpha_{i_1} + \alpha_{|I_1|+|T|+i_1}$. Because $i_1 \in I_1 \cap S$ and $|I_1|+|T|+i_1 \in I_2$, the result follows from (i) and (iii). □

Example 2 (*cont.*) Considering Table 1, for $l = [0, 0, 2, 0]$ and $u = [2, 2, 3, 1]$, recall that no inequality (19) for $|S| = 2, 3, 4$ is facet-defining. Hence let $S = J_1$. The vertex

$\check{x}$, appearing last in Table 1, belongs to $P^l(S)$ thus let $\check{x}$ play the role of x' in the proof of Proposition 13. After deriving vertex $\bar{x} = \check{x}(1 \leftrightarrow 3)$, $\alpha\check{x} = \alpha\bar{x}$ yields $\alpha_1 = \alpha_3$. After deriving vertex $\tilde{x} = \check{x}(4 \leftrightarrow \{1, 6\})$, $\alpha\check{x} = \alpha\tilde{x}$ yields $\alpha_4 = \alpha_1 + \alpha_6$. Notice that all vertices derived belong to $P^l(S)$.

Proposition 14 *For $c \in C$, let $S \subseteq J_c$ be non-dominated. If $\sum_{k\in K} l_k < n < \sum_{k\in K} u_k$ and $\alpha x = \alpha_0$ is satisfied by all $x \in P^l(S)$ then $\alpha_i = 0$ for all $i \in I_c \backslash S$.*

Proof Let $c = 1$ and $S \subseteq J_1$. Since $\sum_{k\in K} l_k < n < \sum_{k\in K} u_k$, Lemma 11 implies that there is a vertex $x \in P^l(S)$ such that some value $d_{k'}$ appears in $J_1\backslash S$ and $o(x; 1, k') < l_{k'}$, $k' \in K$. Then, Lemma 4 implies that that $o(x; 1, \tilde{k}) < u_{\tilde{k}}$ for some value $d_{\tilde{k}} \neq d_{k'}$.

Consider the vertex $x' \in P^l(S)$ with $x'_j = x_j$, $j \in J_1$ and $x'_{|I_1|+|T|+j} = x'_j$, $j = 1, \ldots, |I_1|$. If value $d_{k'}$ appears in $I_1\backslash S$ let $x'_{i_1} = d_{k'}$, $i_1 \in I_1\backslash S$, and observe that $\tilde{x} = x'(i_1; d_{k'} \to d_{\tilde{k}}) \in P^l(S)$ since no variable indexed by S changes its value. Since $\alpha x = \alpha_0$ is satisfied by all $x \in P^l(S)$, $\alpha x' = \alpha\tilde{x}$ yields, after deleting identical terms,

$$\alpha_{i_1} d_{k'} = \alpha_{i_1} d_{\tilde{k}}. \tag{26}$$

Otherwise, value $d_{k'}$ appears in $T\backslash S$ hence $x'_t = d_{k'}$, $t \in T\backslash S$ and $x'_{i_1} \neq d_{k'}$. By construction of x', $x'_{|I_1|+|T|+i_1} = x'_{i_1} \neq d_{k'}$ hence the vertex $\bar{x} = x'(t \leftrightarrow \{i_1, |I_1| + |T| + i_1\}) \in P^l(S)$ has $\bar{x}_{i_1} = d_{k'}$. It is clear that the vertex $\tilde{x} = \bar{x}(i_1; d_{k'} \to d_{\tilde{k}}) \in P^l(S)$, thus $\alpha\bar{x} = \alpha\tilde{x}$ yields (26).

Thus, (26) yields $\alpha_{i_1} = 0$. By Proposition 13i, $\lambda_1 = 0$ hence $\alpha_i = 0$ for all $i \in I_1\backslash S$. □

Let us now show the proof of Theorem 10.

Proof (Theorem 10) Assume that $\alpha x = \alpha_0$ holds for all $x \in P^l(S)$.

For $\sum_{k\in K} l_k < n < \sum_{k\in K} u_k$, Proposition 14 yields that $\alpha_i = 0$, $i \in (I_1 \cup I_2)\backslash S$, hence $\lambda_1 = \lambda_2 = 0$ by Proposition 13i and $\alpha_t = 0$, $t \in T\backslash S$ by Proposition 13ii; i.e., $\alpha_j = 0$ for all $j \in J\backslash S$. Also, Proposition 13iii–iv yield that $\alpha_j = \mu$ for all $j \in S$. Therefore, (19) is the only equality (up to scalar multiplication) satisfied by all $x \in P^l(S)$, thus $\dim P^l(S) = \dim P_I - 1$.

For $\sum_{k\in K} l_k = n = \sum_{k\in K} u_k$, recall that equality (8) (which coincides with (9)) holds for both J_1 and J_2. We show that $\alpha x = \alpha_0$ is a linear combination of (8) for J_1, J_2 and (19) taken as equality. Since α_j, $j \in J$, can be expressed in terms of scalars λ_1, λ_2 and μ as in Proposition 13i–iv, it remains to show that, using these scalars, α_0 is a linear combination of the right-hand sides of equality (19) and equalities (8) for J_1, J_2. Notice that $S = J_1$ and $S = J_2$ are dominated in this case, since falsifying Definition 9iii. Thus, let without loss of generality $S \subset J_1$ hence $I_2 \cap S = \emptyset$. Then, given that each value d_k appears exactly l_k times in both J_1 in J_2,

$$\begin{aligned}
\alpha_0 = \alpha x &= \sum\nolimits_{j\in I_1} \alpha_j x_j + \sum\nolimits_{j\in T} \alpha_j x + \sum\nolimits_{j\in I_2} \alpha_j x_j \\
&= \sum\nolimits_{j\in I_1\cap S} \alpha_j x_j + \sum\nolimits_{j\in I_1\backslash S} \alpha_j x_j + \sum\nolimits_{j\in T\cap S} \alpha_j x_j \\
&\quad + \sum\nolimits_{j\in T\backslash S} \alpha_j x + \sum\nolimits_{j\in I_2} \alpha_j x_j
\end{aligned}$$

$$= (\lambda_1 + \mu) \sum_{j \in I_1 \cap S} x_j + \lambda_1 \sum_{j \in I_1 \setminus S} x_j + (\lambda_1 + \lambda_2 + \mu) \sum_{j \in T \cap S} x_j + (\lambda_1 + \lambda_2) \sum_{j \in T \setminus S} x_j + \lambda_2 \sum_{j \in I_2} x_j$$

$$= \lambda_1 \sum_{j \in I_1} x_j + \mu \sum_{j \in I_1 \cap S} x_j + \lambda_2 \sum_{j \in I_2} x_j + (\lambda_1 + \lambda_2) \sum_{j \in T} x_j + \mu \sum_{j \in T \cap S} x_j$$

$$= \lambda_1 \sum_{j \in I_1 \cup T} x_j + \lambda_2 \sum_{j \in I_2 \cup T} x_j + \mu \sum_{j \in S} x_j = \lambda_1 \sum_{j \in J_1} x_j + \lambda_2 \sum_{j \in J_2} x_j + \mu \sum_{j \in S} x_j$$

$$= \lambda_1 \sum_{k \in K} l_k d_k + \lambda_2 \sum_{k \in K} l_k d_k + \mu \sum_{k \in K} p(|S|, k) \cdot d_k.$$

It follows that $\alpha x = \alpha_0$ is a linear combination of $\sum_{j \in J_1} x_j = \sum_{k \in K} l_k d_k$, $\sum_{j \in J_2} x_j = \sum_{k \in K} l_k d_k$, and (19) taken as equality. Thus, these are the only three equalities (up to scalar multiplication) satisfied by all $x \in P^l(S)$. Since they are also linearly independent, the minimum equation system for $P^l(S)$ is of rank 3 and $\dim P^l(S) = |J| - 3 = \dim P_I - 1$, by Theorem 2. □

4 A convex hull relaxation

In this section we show the following.

Theorem 15 *If $l_k = 0$ for all $k \in K$, P_I is described by inequalities (19) and (20).*

To prove this, we show that the face of P_I defined by any inequality, which is valid for P_I, is contained in a face defined by (19) or (20). We illustrate the result with respect to an inequality $\alpha x \geq \beta$ ($\alpha \in \mathbb{R}^{|J|}$, $\beta \in \mathbb{R}^+$), since the proof for an inequality $\alpha x \leq \beta$ ($\alpha \in \mathbb{R}^{|J|}$, $\beta \in \mathbb{R}^+$) is analogous. Let $P^{(\alpha,\beta)} = \{x \in P_I : \alpha x = \beta\}$ be the face of P_I induced by $\alpha x \geq \beta$. To prove Theorem 15, it suffices to show that all vertices in $P^{(\alpha,\beta)}$ also satisfy at equality one of (19) or (20), since any face of P_I is the convex hull of vertices appearing in it.

Let J_1^*, J_2^*, T^* and J_2' be subsets of variables that, as shown later in this section, are defined in terms of vector α (for simplicity, we omit α from the notation).

Theorem 16 *Assume $l_k = 0$ for all $k \in K$. If $J_2^* \not\subset J_1^*$ or $|J_1^*| + |J_2' \setminus T^*| \leq \sum_{k \in K} u_k - 1$, all vertices of $P^{(\alpha,\beta)}$ satisfy at equality (19) for $S = J_1^*$.*

Notice that the negation of the condition in Theorem 16 reads 'if $J_2^* \subset J_1^*$ and $|J_1^*| + |J_2' \setminus T^*| \geq \sum_{k \in K} u_k$'. Thus, the following result completes the proof of Theorem 15.

Theorem 17 *Assume $l_k = 0$ for all $k \in K$. If $J_2^* \subset J_1^*$ and $|J_1^*| + |J_2' \setminus T^*| \geq \sum_{k \in K} u_k$, all vertices of $P^{(\alpha,\beta)}$ satisfy (20) for $S = J_2' \setminus T^*$ at equality.*

That is, given any $\alpha x \geq \beta$, all vertices in $P^{(\alpha,\beta)}$ satisfy Eq. (19) with $S = J_1^*$ or Eq. (20) with $S = J_2' \backslash T^*$, based on the conditions of Theorems 16 and 17, respectively.

To introduce the notation that is necessary for showing these theorems, define

$$\gamma_T(\alpha) = \max\{0, \max\{\alpha_t : t \in T\}\}, \tag{27}$$

$$\gamma_c(\alpha) = \max\{0, \max\{\alpha_i : i \in I_c\}\}, c \in C, \tag{28}$$

$$\gamma(\alpha) = \max\{\gamma_T(\alpha), \gamma_1(\alpha) + \gamma_2(\alpha)\}. \tag{29}$$

Notice that at least one entry of vector α must be positive, since otherwise the inequality is of the '$\leq$' type. This implies that at least one of $\gamma_T(\alpha)$, $\gamma_1(\alpha)$ or $\gamma_2(\alpha)$ is positive, thus $\gamma(\alpha) > 0$. Since $\gamma_T(\alpha)$ and $\gamma_c(\alpha)$ denote the maximum coefficient in T and I_c, respectively, $\gamma(\alpha)$ is the maximum sum of coefficients of any $J' \subseteq J$ that includes at most one variable per constraint; i.e.,

$$\gamma(\alpha) = \max\{\sum\nolimits_{j \in J' \subseteq J} \alpha_j : \left|J' \cap J_c\right| \leq 1\}.$$

Since all the following proofs examine an arbitrary, yet given, face $P^{(\alpha,\beta)}$, we omit α from our notation hereafter, e.g., we write γ instead of $\gamma(\alpha)$. Let us define J^* as the set of variables that appear in some $J' \subseteq J$ such that $\left|J' \cap J_c\right| \leq 1$ and $\sum_{j \in J'} \alpha_j = \gamma$. Formally,

$$J^* = \bigcup\nolimits_{c \in C} \{i \in I_c : \alpha_i > 0, \alpha_i + \gamma_{C \backslash \{c\}} = \gamma\} \cup \{t \in T : \alpha_t = \gamma\}. \tag{30}$$

Notice that $J^* \neq \emptyset$ follows from $\gamma > 0$. For $c \in C$, define $J_c^* = J^* \cap J_c$ and, accordingly, $I_c^* = J^* \cap I_c$, $T^* = J^* \cap T$. Thus, $J_1^* \not\subset J_2^*$ implies $J_1^* \backslash J_2^* = I_1^* \neq \emptyset$ or $J_1^* = J_2^* = T^*$.

Since $J_1^* \subset J_2^*$ and $J_2^* \subset J_1^*$ cannot hold simultaneously, we adopt the convention that $J_1^* \not\subset J_2^*$ and either $J_2^* \subset J_1^*$ or $J_2^* \not\subset J_1^*$. In addition:

- for $J_2^* \subset J_1^*$, define the set $I_2' = I_2 \backslash \{i \in I_2 : \alpha_i = 0\}$ and $J_2' = I_2' \cup T$, i.e., I_2' and J_2' exclude variables in I_2 with a zero coefficient in $\alpha x \geq \beta$;
- for $J_2^* \not\subset J_1^*$, assume that $|J_1^*| \leq |J_2^*|$.

That is, if none of the J_1^*, J_2^* is a proper subset the other, J_1^* is the set with the smallest cardinality, whereas if $J_2^* \subset J_1^*$ clearly J_1^* is the set with the largest cardinality. All the above can be made without loss of generality since the roles of the two constraints can be interchanged.

Example 3 For the system of Table 1 and $l = [0, 0, 2, 0]$, $u = [2, 2, 3, 1]$, Table 2 shows the vector α of a valid inequality $\alpha x \geq \beta$. Based on (27)–(30), $\gamma = \gamma_T = \gamma_1 = 2$ and $\gamma_2 = 0$. Also, $J^* = \{1, 2, 4\}$ thus $J_2^* \subset J_1^*$. Observe that $\alpha_i \leq 0$ for all $i \in I_2$. For this vector α, Theorem 16 yields $P^{(\alpha,\beta)} \subseteq P^l(J_1^*)$.

Lemma 18 (i) $j \in J^*$ *implies* $\alpha_j > 0$;

(ii) *for* $c \in C$, $i_1 \in I_c^*$ *and* $i_2 \in I_c \backslash I_c^*$ *imply* $\alpha_{i_2} < \alpha_{i_1}$;

Table 2 A *cardinality* system and a vector α

Predicate	I_1			T		I_2		
1	x_1	x_2	x_3	x_4	x_5			
2				x_4	x_5	x_6	x_7	x_8
α	2	2	1	2	1	0	-1	-1
x'	0	1	3	2	2	3	1	1
$\tilde{x} = x'(4 \leftrightarrow 7)$	0	1	3	1	2	3	2	1
$\bar{x}$	0	1	2	1	2	2	3	1
$\breve{x} = \bar{x}(8; 1 \to 2)$	0	1	2	1	2	2	3	2
$\hat{x}$	1	1	2	0	0	1	3	2
$\dot{x} = \hat{x}(5 \leftrightarrow \{1, 6\})$	0	1	2	0	1	0	1	2

(iii) $t_1 \in T^*, t_2 \in T \backslash T^*$ *imply* $\alpha_{t_1} > \alpha_{t_2}$;
(iv) $t \in T^*, i_1 \in I_1 \backslash I_1^*$ *and* $i_2 \in I_2 \backslash I_2^*$ *imply* $\alpha_t > \alpha_{i_1} + \alpha_{i_2}$;
(v) $t \in T \backslash T^*, i_1 \in I_1^*$ *and* $i_2 \in I_2^*$ *imply* $\alpha_t < \alpha_{i_1} + \alpha_{i_2}$;
(vi) $I_1^* \neq \emptyset = I_2^*$ *implies* $\alpha_i \leq 0$ *for all* $i \in I_2$ *and* $\alpha_t < \alpha_{i_1}$ *for all* $t \in T \backslash T^*, i_1 \in I_1^*$;
(vii) $t \in T^*$ *implies* $\alpha_t > \gamma_1$ *if* $J_2^* \not\subset J_1^*$ *and* $\alpha_t = \gamma_1$ *if* $J_2^* \subset J_1^*$.

Proof Notice that (i) follows from (30) for $j \in I_c^*, c \in C$. For $j \in T^*$, assuming to the contrary that $\alpha_j \leq 0$ yields $\gamma \leq 0$ by (30) hence a contradiction to $\gamma > 0$.

To show (ii), let $c = 1$. Observe that $i_1 \in I_1^*$ yields $\alpha_{i_1} + \gamma_2 = \gamma$ by (30). Then, assuming to the contrary that $\alpha_{i_2} \geq \alpha_{i_1}$ yields $\alpha_{i_2} + \gamma_2 \geq \alpha_{i_1} + \gamma_2 = \gamma$, thus implying $i_2 \in I_1^*$ by (30), a contradiction. In an similar manner one can prove (iii).

To prove (iv), observe that $t \in T^*$ yields $\alpha_t = \gamma$ by (30), while $\gamma_2 \geq \alpha_{i_2}$ by (28). Thus, assuming to the contrary that $\alpha_{i_1} + \alpha_{i_2} \geq \alpha_t$ yields $\alpha_{i_1} + \gamma_2 \geq \gamma$ hence $i_1 \in I_1^*$, a contradiction to $i_1 \in I_1 \backslash I_1^*$.

To prove (v), notice that $i_1 \in I_1^*$ and $i_2 \in I_2^*$ imply, respectively, $\alpha_{i_1} = \gamma_1$ and $\alpha_{i_2} = \gamma_2$. Hence $\alpha_{i_1} + \alpha_{i_2} = \gamma$. Thus, assuming to the contrary that $\alpha_t \geq \alpha_{i_1} + \alpha_{i_2}$ yields $\alpha_t \geq \gamma$ hence $t \in T^*$, a contradiction to $t \in T \backslash T^*$.

To show (vi), let to the contrary $\alpha_i > 0$ for some $i \in I_2$. This yields $\gamma_2 > 0$ (by (28)), hence there is $i_2 \in I_2$ such that $\alpha_{i_2} = \gamma_2 = max\{a_i : i \in I_2\} > 0$. Since $I_1^* \neq \emptyset$, there is also $i_1 \in I_1^*$ such that $\alpha_{i_1} = \gamma_1 > 0$ and $\alpha_{i_1} + \gamma_2 = \gamma$ by (30). The latter is equivalent to $\gamma_1 + \alpha_{i_2} = \gamma$, thus yielding $i_2 \in I_2^*$ by (30), a contradiction to $I_2^* = \emptyset$.

Since $\alpha_i \leq 0$ for all $i \in I_2$, $\gamma_2 = 0$. Then, assuming to the contrary that $\alpha_t \geq \alpha_{i_1}$ yields $\alpha_t \geq \alpha_{i_1} + \gamma_2 = \gamma$ hence $t \in T^*$, a contradiction to $t \in T \backslash T^*$.

To show (vii) for $J_2^* \not\subset J_1^*$, let to the contrary $\alpha_t \leq \gamma_1$ and notice that $I_1^* \neq \emptyset$ and $J_2^* \not\subset J_1^*$ yield $I_2^* \neq \emptyset$ and, hence, $\gamma_2 > 0$ (by (28) and (i) shown above); then, $\alpha_t \leq \gamma_1 < \gamma_1 + \gamma_2 = \gamma$ yields $t \in T \backslash T^*$ by (30), a contradiction to $t \in T^*$. For $J_2^* \subset J_1^*$, observe $I_1^* \neq \emptyset$ yields $I_2^* = \emptyset$, therefore $\alpha_i \leq 0$ for all $i \in I_2$ (by (vi) shown above) hence $\gamma_2 = 0$; then, (30) yields $\gamma = \alpha_t = \gamma_1$. □

In several of the following proofs, the result is established by assuming $x' \in P^{(\alpha,\beta)}$ and then deriving a vertex $\tilde{x} \in P_I$ such that $\alpha\tilde{x} < \alpha x' = \beta$, thus showing a contradiction to the hypothesis that $\alpha x \geq \beta$ is valid for P_I. To avoid repeating the same argument, we show only the derivation of vertex $\tilde{x}$ from x' and then establish that $\alpha\tilde{x} - \alpha x' < 0$.

We list two intermediate lemmas, whose proofs appear in the Appendix. The first lemma applies always to J_1^* and also to J_2^* if $J_2^* \not\subset J_1^*$. The second lemma applies only to J_2^* if $J_2^* \subset J_1^*$.

Lemma 19 *Assume $l_k = 0$ for all $k \in K$. For any vertex $x' \in P^{(\alpha,\beta)}$ and $c \in C$ such that $J_c^* \not\subset J_{C\setminus\{c\}}^*$,*

(i) *if $x'_i = d_k$, $i \in I_c^*$, any value $d_{k'} < d_k$ appears $u_{k'}$ times in $J_c^* \cup T$;*
(ii) *if $x'_t = d_k$, $t \in T^*$, any value $d_{k'} < d_k$ appears $u_{k'}$ times in J_c^*.*

Example 3 (*cont.*) In Table 2, vertex x' violates Lemma 19i: although $x'_4 = 2$, $\{4\} = T^*$, value $1 < 2$ appears fewer than $u_1 = 2$ times in J_1^*. To show that x' cannot belong to $P^{(\alpha,\beta)}$, notice that $l_2 = 0 < 2 = o(x'; 1, 2) = o(x'; 2, 2)$ and, in addition, $o(x'; 1, 1) < 2 = o(x'; 2, 1)$. Then, the vertex $\tilde{x} = x'(4 \leftrightarrow 7)$ belongs to P_I, while $\alpha\tilde{x} - \alpha x' = (\alpha_4 - \alpha_7) \cdot (1 - 2) = (2 + 1) \cdot (-1) < 0$. Thus, $x' \in P^{(\alpha,\beta)}$ would yield $\alpha\tilde{x} < \beta$, a contradiction to $\alpha x \geq \beta$ being valid for P_I. In fact, $\alpha_4 > \alpha_7$ holds because $\alpha_4 > 0$ (by Lemma 18i) and $\alpha_7 \leq 0$ (by Lemma 18vi since $J_2^* \subset J_1^*$ yields $I_2^* = \emptyset$).

Lemma 20 *Assume $l_k = 0$ for all $k \in K$. For any vertex $x' \in P^{(\alpha,\beta)}$ and $J_2^* \subset J_1^*$, if $x'_i = d_k$, $i \in I'_2$, any value $d_{k'} > d_k$ appears $u_{k'}$ times in $J'_2 \setminus T^*$.*

Example 3 (*cont.*) In Table 2, vertex $\bar{x}$ violates Lemma 20: although $J_2^* \subset J_1^*$, $I'_2 = \{7, 8\}$ and $\bar{x}_8 = 1$, value $2 > 1$ appears fewer than $u_2 = 3$ times in $J'_2 \setminus T^* = \{5, 7, 8\}$. Since $l_1 = 0$ and $o(\bar{x}; 2, 2) < 3$, the vertex $\check{x} = \bar{x}(8; 1 \to 2)$ belongs to P_I, while $\alpha\check{x} - \alpha\bar{x} = \alpha_8 \cdot (2 - 1) = (-1) \cdot 1 < 0$; in fact, $\alpha_8 < 0$ by Lemma 18vi and the definition of I'_2. Alternatively, the vertex $\mathring{x} = \bar{x}(6 \leftrightarrow 8)$ (not shown in Table 2) also yields $\alpha\mathring{x} - \alpha\bar{x} < 0$ since $\alpha_6 = 0$; i.e., $6 \in I_2 \setminus I'_2$.

Let us outline of the proof of Theorem 16. This theorem holds if and only if, at any vertex of $P^{(\alpha,\beta)}$, all values d_k ($k \in K$) appear $p(|J_1^*|, k)$ times in J_1^*. Hence, it suffices to show a contradiction if a vertex $x' \in P^{(\alpha,\beta)}$ does not have this property; i.e., if $\sum_{j \in J_1^*} x'_j > \sum_{k \in K} p(|J_1^*|, k) \cdot d_k$. Then, at vertex x', some value $d_{k'}$ appears fewer than $p(|J_1^*|, k')$ times in J_1^*, whereas some value $d_k > d_{k'}$ appears more than $p(|J_1^*|, k)$ times (i.e., at least once) in J_1^*. Then, if $d_{\tilde{k}}$ is the maximum value appearing in J_1^*, it must be $d_{\tilde{k}} > d_{k'}$. Using Lemma 19 for J_1^*, we prove that $d_{\tilde{k}}$ appears in I_1^* and $d_{k'}$ appears in $T \setminus T^*$. Then, we distinguish two cases, defined by whether $J_2^* \not\subset J_1^*$.

For $J_2^* \not\subset J_1^*$, notice that Lemma 19 applies to both J_1^* and J_2^*. Then, if $d_{\tilde{k}}$ appears also in I_2^*, we show the contradiction directly by obtaining a vertex $\tilde{x} \in P_I$ such that $\alpha\tilde{x} < \alpha x'$. For $d_{\tilde{k}}$ not appearing in I_2^*, we establish a contradiction to Lemma 19i for J_2^* by showing that some value $d_{\hat{k}} > d_{\tilde{k}}$ appears in I_2^* and that the value $d_{\tilde{k}}$ (smaller than $d_{\hat{k}}$) appears fewer than $u_{\tilde{k}}$ times in $J_2^* \cup T$. This is done by a combinatorial argument relying on $|J_1^*| \leq |J_2^*|$, Lemma 19i for J_1^* ('any value $d_k < d_{\tilde{k}}$ appear u_k times in $J_1^* \cup T$') and the fact that $d_{\tilde{k}}$ appears in I_1^* but not in I_2^*.

For $J_2^* \subset J_1^*$, we show the contradiction directly or through Lemma 20 by showing that $d_{\tilde{k}}$ appears in I'_2 and some value $d_{\hat{k}} > d_{\tilde{k}}$ appears fewer than $u_{\hat{k}}$ times in $J'_2 \setminus T^*$.

Example 3 (*cont.*) In Table 2, vertex $\hat{x}$ belongs not to $P^l(J_1^*)$. Then, value 0 appears fewer than $p(|J_1^*|, 0) = p(3, 0) = 2$ times in J_1^*, while value 1 appears more than $p(|J_1^*|, 1) = 1$ times in J_1^*. Observe that $\hat{x}_1 = 1$ ($1 \in I_1^*$) and $\hat{x}_5 = 0$ ($5 \in T\backslash T^*$), thus Lemma 18vi yields $\alpha_5 = 1 < \alpha_1 = 2$ (since $J_2^* \subset J_1^*$ yields $I_2^* = \emptyset$). Moreover, $\hat{x}_6 = 1$ ($6 \in I_2\backslash I_2'$ and $\alpha_6 = 0$). Then, $\dot{x} = \hat{x}(5 \leftrightarrow \{1, 6\})$ is a vertex of P_I while $\alpha\dot{x} - \alpha\hat{x} = (\alpha_5 - \alpha_1 - \alpha_6) \cdot (1 - 0) = (-1) \cdot 1 < 0$, a contradiction to $\alpha x \geq \beta$ being valid for P_I. That is, vertex $\hat{x}$ not belonging to $P^l(J_1^*)$ implies that $\hat{x}$ cannot belong to $P^{(\alpha,\beta)}$.

Proof of Theorem 16 Assume to the contrary that there is a vertex $x' \in P^{(\alpha,\beta)}$ such that $\sum_{j\in J_1^*} x_j' > \sum_{k\in K} p(|J_1^*|, k) \cdot d_k$. Then, there is a value $d_{k'}$ that appears fewer than $p(|J_1^*|, k')$ times in J_1^*. As discussed above, if $d_{\tilde{k}} = \max\{x_j' : j \in J_1^*\}$, then it must be $d_{\tilde{k}} > d_{k'}$. The value $d_{\tilde{k}}$ cannot appear in T^*, since that would contradict Lemma 19ii for J_1^* because the value $d_{k'} < d_{\tilde{k}}$ appears fewer than $p(|J_1^*|, k') \leq u_{k'}$ times in J_1^*. Thus, let $x_{i_i}' = d_{\tilde{k}}$ for some $i_1 \in I_1^*$. Also by Lemma 19i, value $d_{k'} < d_{\tilde{k}}$ appears $u_{k'}$ times in $J_1^* \cup T$. Then, since $d_{k'}$ appears fewer than $p(|J_1^*|, k') \leq u_{k'}$ times in J_1^*, $d_{k'}$ must appear in $T\backslash T^*$; i.e., $x_t' = d_{k'}$ for some $t \in T\backslash T^*$.

Case 16.1 $J_2^* \not\subset J_1^*$

Since $i_1 \in I_1^* \neq \emptyset$, $J_2^* \not\subset J_1^*$ yields $I_2^* \neq \emptyset$.

If $d_{\tilde{k}}$ appears also in I_2^*, let $i_2 \in I_2^*$ be an index where this occurs, i.e., $x_{i_2}' = d_{\tilde{k}}$. Then, for the vertex $\tilde{x} = x'(t \leftrightarrow \{i_1, i_2\}) \in P_I$, it holds that $\alpha\tilde{x} - \alpha x' = (\alpha_t - \alpha_{i_1} - \alpha_{i_2}) \cdot (d_{\tilde{k}} - d_{k'}) < 0$, because $d_{\tilde{k}} > d_{k'}$ and $\alpha_{t_1} < \alpha_{i_1} + \alpha_{i_2}$ by Lemma 18v (since $i_1 \in I_1^*$, $i_2 \in I_2^*$ and $t \in T\backslash T^*$).

Otherwise, $x_i' \neq d_{\tilde{k}}$ for all $i \in I_2^*$. Then, $J_2^* \not\subset J_1^*$ together with the definition of J_1^* imply $|J_1^*| \leq |J_2^*|$ (recall that, if $J_2^* \not\subset J_1^*$, J_1^* is the set having the smallest cardinality). Since $T\backslash T^*$ is a subset of neither J_1^* nor J_2^*, it follows that $|J_1^* \cup (T\backslash T^*)| \leq |J_2^* \cup (T\backslash T^*)|$; i.e., $|J_1^* \cup T| \leq |J_2^* \cup T|$. Since, by Lemma 19i for J_1^*, all values smaller than $d_{\tilde{k}}$ appear the maximum number of times in $J_1^* \cup T$ (and $d_{\tilde{k}}$ appears in I_1^*), and at least as many values appear in $J_2^* \cup T$ but $d_{\tilde{k}}$ appears not in I_2^*, some value $d_{\hat{k}} > d_{\tilde{k}}$ appears in I_2^*. That is, $x_{i_2}' = d_{\hat{k}}$, $i_2 \in I_2^*$. Also, $d_{\tilde{k}}$ appearing in I_1^* but not in I_2^* and $|J_1^* \cup T| \leq |J_2^* \cup T|$ together imply that the value $d_{\tilde{k}} < d_{\hat{k}}$ appears fewer than $u_{\tilde{k}}$ times in $J_2^* \cup T$, a contradiction to Lemma 19i for J_2^*.

Case 16.2 $J_2^* \subset J_1^*$

It is clear that $J_2^* \subset J_1^*$ yields $I_2^* = \emptyset$. In that case, $i_1 \in I_1^*$ implies $\alpha_t < \alpha_{i_1}$ (by Lemma 18vi).

If $o(x'; 2, \tilde{k}) < u_{\tilde{k}}$, then $\tilde{x} = x'(t \leftrightarrow i_1)$ is a vertex of P_I satisfying $\alpha\tilde{x} - \alpha x' = (\alpha_t - \alpha_{i_1}) \cdot (d_{\tilde{k}} - d_{k'}) < 0$, since $d_{\tilde{k}} > d_{k'}$ and $\alpha_{t_1} < \alpha_{i_1}$.

Otherwise $o(x'; 2, \tilde{k}) = u_{\tilde{k}}$. This, together with $d_{\tilde{k}}$ appearing in I_1, imply that $d_{\tilde{k}}$ also appears in I_2. Thus, let $x_{i_2}' = d_{\tilde{k}}$, $i_2 \in I_2$, where $\alpha_{i_2} \leq 0$ by Lemma 18vi. For $\alpha_{i_2} = 0$, derive $\tilde{x} = x'(t \leftrightarrow \{i_1, i_2\}) \in P_I$ and notice that, as above, $\alpha\tilde{x} - \alpha x' = (\alpha_t - \alpha_{i_1}) \cdot (d_{\tilde{k}} - d_{k'}) < 0$.

For $\alpha_{i_2} < 0$, $i_2 \in I_2'$ by the definition of I_2'. We show that there is a value $d_{\hat{k}} > d_{\tilde{k}}$ appearing fewer than $u_{\hat{k}}$ times in $J_2'\backslash T^*$ thus contradicting Lemma 20. Observe that $I_2^* = \emptyset$ implies $J_2^* = T^*$. Then, the hypothesis $|J_1^*| + |J_2'\backslash T^*| \leq \sum_{k\in K} u_k - 1$, using the fact that $T \backslash T^*$ is not a subset of J_1^* but $T \backslash T^* \subset J_2' \backslash T^*$, yields $|J_1^* \cup (T \backslash T^*)| + |J_2' \backslash T| \leq \sum_{k \in K} u_k - 1$; i.e., $|J_1^* \cup T| + |I_2'| \leq \sum_{k \in K} u_k - 1$. That is, some value $d_{\hat{k}}$ appears fewer than $u_{\hat{k}}$ times in $J_1^* \cup T \cup I_2' \supset J_2'$ (where $d_{\hat{k}} \neq d_{\tilde{k}}$ since $o(x'; 2, \tilde{k}) = u_{\tilde{k}}$) thus appearing fewer than $u_{\hat{k}}$ times in J_2' (and hence in $J_2' \backslash T^*$). Since all values smaller than $d_{\tilde{k}}$ appear the maximum number of times in $J_1^* \cup T$ (by Lemma 19i), it must be $d_{\hat{k}} > d_{\tilde{k}}$. The proof is now complete. □

The proof of Theorem 17 is quite similar, hence appearing in the Appendix.

Notice that Theorem 15 holds also for the case of a single cardinality constraint. That is, if $l_k = 0$ for all $k \in K$, inequalities (19) and (20) describe the polytope defined as the convex hull of vectors satisfying a single cardinality constraint. For the same case, it has been claimed (Hooker 2012, Sect. 4.13.1)without a proof, that the polytope is completely described by inequalities (19) and (20), irrespectively of the values of $l_k, k \in K$. However, if some of the l_ks are strictly positive, the polytope associated with a single cardinality constraint has further facets not induced by (19) and (20) and, therefore, the result claimed in (Hooker 2012, Sect. 4.13.1) does not hold. An example suffices to show that.

Example 4 Consider the constraint $cardinality(x, \{1, 2, 3, 4\}; [0, 1, 0], [2, 2, 2])$ and let P_I denote the polytope defined by the convex hull of integer vectors satisfying that constraint. Notice that $l_1 = 1$ while $l_0 = l_2 = 0$. By Theorem 2, dim $P_I = 4$, thus any inequality satisfied at equality by 4 affinely independent points of P_I is facet-defining. The inequality

$$x_1 + x_2 - x_3 - x_4 \leq 3$$

is satisfied at equality by the vertices $\check{x} = (2, 1, 0, 0)$, $\bar{x} = (1, 2, 0, 0)$, $\check{x} = (2, 2, 1, 0)$, and $\hat{x} = (2, 2, 0, 1)$. The matrix $D = [\, \check{x}^T \; \bar{x}^T \; \check{x}^T \; \hat{x}^T \,]$, after deducting from the second row its first row multiplied by 0.5 (and replacing the second row by the result), becomes upper triangular hence non-singular. It follows that these four vertices are linearly, and thus affinely, independent.

A second counter-example of two cardinality constraints is

$$cardinality(x, \{1, 2, 3\}; [0, 1, 0], [2, 2, 2]),$$
$$cardinality(x, \{3, 4, 5\}; [0, 1, 0], [2, 2, 2]).$$

In this case, Theorem 2 yields dim $P_I = 5$, while the inequality $x_1 + x_2 - x_3 \geq 1$ becomes facet-defining. Overall, the polyhedral study of the cardinality constraint (and system) requires further investigation for arbitrary values of $l_k, k \in K$.

5 Concluding remarks

In this paper, we advance the polyhedral study of the cardinality system, thus offering also a polyhedral approach to systems of restricted representatives. For the case of two cardinality constraints, apart from providing necessary and sufficient conditions for known families of valid inequalities to be facet-defining, we show a condition under which these families completely describe the associated polytope P_I. This condition, together with the polytime separability of these inequalities (Hooker 2012, Sect. 7.10.1), establishes polynomial solvability for the problem of optimizing a linear function over P_I (Grötschel et al. 1981).

Motivated by the relationship between the alldifferent system and graph coloring (Magos and Mourtos 2011), we may also consider the following representation for the cardinality system (1) Define the *constraint graph* $G^{\#}(V_{G^{\#}}, E_{G^{\#}})$ as $V_{G^{\#}} = J$ and $E_{G^{\#}} = \{(i, j) : i, j \in J_c$ for some $c \in C\}$; i.e., $G^{\#}$ has one node per variable and two nodes are connected if and only if the associated variables appear together in some cardinality constraint. Then, any $c \in C$ can be associated with a *complete subgraph* in $G^{\#}$ in the sense that J_c is a set of pairwise connected nodes. Then, if D is regarded as a set of colors, we may define the *cardinality coloring* of $G^{\#}$ as a coloring in which color d is received by at least l_d and at most u_d nodes in any maximal clique of $G^{\#}$. Thus, the solutions to the cardinality system are exactly the cardinality colorings of $G^{\#}$. Whether this relationship can be insightful for the facietal structure of the cardinality system or for graph coloring remains to be investigated.

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6 Appendix

Proof of Lemma 19 (i) $x'_{i_1} = d_k, i \in I_1^*$

Assume to the contrary that some value $d_{k'} < d_k$ appears fewer than $u_{k'}$ times in $J_1^* \cup T$. If $o(x'; 1, k') < u_{k'}$ and given $l_k = 0$, derive the vertex $\tilde{x} = x'(i_1; d_k \rightarrow d_{k'}) \in P_I$; $\alpha\tilde{x} - \alpha x' = \alpha_{i_1} \cdot (d_{k'} - d_k) < 0$ since $i_1 \in I_1^*$ yields $\alpha_{i_1} > 0$ (by Lemma 18i) and $d_{k'} < d_k$.

Otherwise, $o(x'; 1, k') = u_{k'}$. Then, value $d_{k'}$ appearing fewer than $u_{k'}$ times in $J_1^* \cup T$ implies that $d_{k'}$ appears in $I_1 \backslash I_1^*$; i.e., $x'_{i_2} = d_{k'}$ for some $i_2 \in I_1 \backslash I_1^*$. Then, for the vertex $\tilde{x} = x'(i_1 \leftrightarrow i_2) \in P_I$, $\alpha\tilde{x} - \alpha x' = (\alpha_{i_1} - \alpha_{i_2}) \cdot (d_{k'} - d_k) < 0$ since $\alpha_{i_1} > \alpha_{i_2}$ (by Lemma 18ii) and $d_{k'} < d_k$.

(ii) $x'_{t_1} = d_k, t_1 \in T^*$

Assume to the contrary that some value $d_{k'} < d_k$ appears fewer than $u_{k'}$ times in J_1^*.

If $x'_{t_2} = d_{k'}$ for some $t_2 \in T \backslash T^*$, then for the vertex $\tilde{x} = x'(t_1 \leftrightarrow t_2) \in P_I$, $\alpha\tilde{x} - \alpha x' = (\alpha_{t_1} - \alpha_{t_2}) \cdot (d_{k'} - d_k) < 0$ since $\alpha_{t_1} > \alpha_{t_2}$ (by Lemma 18iii) and $d_{k'} < d_k$.

Otherwise, $x'_t \neq d_{k'}$ for all $t \in T \backslash T^*$. Then, four cases are defined by whether $o(x'; 1, k')$ or $o(x'; 2, k')$ is less than $u_{k'}$ (recall that $l_k = 0$).

Case 19.1 ($o(x'; 1, k') < u_{k'}, o(x'; 2, k') < u_{k'}$):] For $\tilde{x} = x'(t_1; d_k \rightarrow d_{k'})$, $\alpha\tilde{x} - \alpha x' = \alpha_{t_1} \cdot (d_{k'} - d_k) < 0$ since $t_1 \in T^*$ yields $\alpha_{t_1} > 0$ (by Lemma 18i).

Case 19.2 $(o(x'; 1, k') < u_{k'}, o(x'; 2, k') = u_{k'})$: Since $d_{k'}$ appears more times in J_2 than in J_1, it appears in I_2; i.e., $x'_{i_2} = d_{k'}$ for some $i_2 \in I_2$. For $\tilde{x} = x'(t_1 \leftrightarrow i_2), \alpha\tilde{x} - \alpha x' = (\alpha_{t_1} - \alpha_{i_2}) \cdot (d_{k'} - d_k) < 0$ since $\alpha_{t_1} > \alpha_{i_2}$. The latter follows from Lemma 18vii for $I_2^* \neq \emptyset$ (since $\alpha_{t_1} > \gamma_2 \geq \alpha_{i_2}$) and from Lemma 18i and 18vi for $I_2^* = \emptyset$ (since $\alpha_{t_1} > 0 \geq \alpha_{i_2}$).

Case 19.3 $(o(x'; 1, k') = u_{k'}, o(x'; 2, k') < u_{k'})$: Since value $d_{k'}$ appears u_k times in J_1 but not in $T \backslash T^*$ and appears fewer than u_k times in $J_1^* = I_1^* \cup T^*$, it must appear in $I_1 \backslash I_1^*$; i.e., $x'_{i_1} = d_{k'}$ for some $i_1 \in I_1 \backslash I_1^*$. For $\tilde{x} = x'(t_1 \leftrightarrow i_1), \alpha\tilde{x} - \alpha x' = (\alpha_{t_1} - \alpha_{i_1}) \cdot (d_{k'} - d_k) < 0$ since $\alpha_{t_1} > \alpha_{i_1}$. The latter holds because the opposite, i.e., $\alpha_{t_1} \leq \alpha_{i_1}$, implies $\gamma \leq \alpha_{i_1}$ hence $i_1 \in I_1^*$, a contradiction to $i_1 \in I_1 \backslash I_1^*$.

Case 19.4 $(o(x'; 1, k') = u_{k'}, o(x'; 2, k') = u_{k'})$: As in Case 19.3, $o(x'; 1, k') = u_{k'}$ implies that $x'_{i_1} = d_{k'}$ for some $i_1 \in I_1 \backslash I_1^*$. This, together with $o(x'; 1, k') = u_{k'} = o(x'; 2, k')$ imply that $x'_{i_2} = d_{k'}$ for some $i_2 \in I_2$. For $\tilde{x} = x'(t \leftrightarrow \{i_1, i_2\}), \alpha\tilde{x} - \alpha x' = (\alpha_{t_1} - \alpha_{i_1} - \alpha_{i_2}) \cdot (d_{k'} - d_k) < 0$, since $\alpha_{t_1} - \alpha_{i_1} - \alpha_{i_2} > 0$. If $i_2 \in I_2 \backslash I_2^*$, the latter holds by Lemma 18iv. If $i_2 \in I_2^*$, it holds because the opposite, i.e., $\alpha_{t_1} \leq \alpha_{i_1} + \alpha_{i_2}$, implies $i_1 \in I_1^*$, a contradiction to $i_1 \in I_1 \backslash I_1^*$.

□

Proof of Lemma 20 Let $x'_{i_2} = d_k$ for some $i_2 \in I'_2$. $J_2^* \subset J_1^*$ implies $I_1^* \neq \emptyset$ and $I_2^* = \emptyset$, hence $i_2 \in I'_2$ implies $\alpha_{i_2} < 0$ (by Lemma 18vi and the definition of I'_2). Assume to the contrary that some value $d_{k'} > d_k$ appears fewer than $u_{k'}$ times in $J'_2 \backslash T_2^*$. Recall that $l_k = 0$.

If $o(x'; 2, k') < u_{k'}$, the vertex $\tilde{x} = x'(i_2; d_k \rightarrow d_{k'}) \in P_I$ since $l_k = 0$. Then, $\alpha\tilde{x} - \alpha x' = \alpha_{i_2} \cdot (d_{k'} - d_k) < 0$ since $\alpha_{i_2} < 0$ and $d_{k'} > d_k$.

Otherwise, $o(x'; 2, k') = u_{k'}$. Then, value $d_{k'}$ appears $u_{k'}$ times in J_2 but fewer than $u_{k'}$ times in $J'_2 \backslash T^*$, hence it appears in $I_2 \backslash I'_2 = \{i \in I_2 : \alpha_i = 0\}$ or in T^*.

Case 20.1 $(x'_{i_3} = d_{k'}, i_3 \in I_2 \backslash I'_2)$: Recall that $x'_{i_2} = d_k$, $i_2 \in I'_2$. For $\tilde{x} = x'(i_2 \leftrightarrow i_3), \alpha\tilde{x} - \alpha x' = (\alpha_{i_2} - \alpha_{i_3}) \cdot (d_{k'} - d_k) < 0$ since $\alpha_{i_2} < 0 = \alpha_{i_3}$ and $d_{k'} > d_k$.

Case 20.2 $(x'_t = d_{k'}, t \in T^*)$: If value d_k appears also in I_1, let $i_1 \in I_1$ be an index where this occurs; i.e., $x'_{i_1} = d_k$. Also, $\alpha_{t_1} = \gamma_1$ by Lemma 18vii (because $I_1^* \neq \emptyset = I_2^*$) and $\alpha_{i_1} \leq \gamma_1$ by (28), thus $\alpha_{i_1} \leq \alpha_{t_1}$. Then, for $\tilde{x} = x'(t \leftrightarrow \{i_1, i_2\}), \alpha\tilde{x} - \alpha x' = (\alpha_{i_1} + \alpha_{i_2} - \alpha_{t_1}) \cdot (d_{k'} - d_k) < 0$ since $\alpha_{i_1} \leq \alpha_{t_1}, \alpha_{i_2} < 0$ and $d_{k'} > d_k$.

Otherwise, value d_k appears not in I_1 but appears in I_2 (because $x'_{i_2} = d_k$, $i_2 \in I'_2$) hence $o(x'; 1, k) < u_k$. Then, the vertex $\tilde{x} = x'(i_2 \leftrightarrow t) \in P_I$ yields $\alpha\tilde{x} - \alpha x' = (\alpha_{i_2} - \alpha_t) \cdot (d_{k'} - d_k) < 0$ since $\alpha_{i_2} < 0$ and $\alpha_t > 0$ by Lemma 18i (because $t \in T^*$).

□

Proof of Theorem 17 Observe that $J_1^* \supset J_2^*$ implies $I_1^* \neq \emptyset$ and $I_2^* = \emptyset$, hence $\alpha_i \leq 0$ for all $i \in I_2$ by Lemma 18vi. The result holds if and only if all values $d_k, k \in K$ appear $q(|J'_2 \backslash T^*|, k)$ times in $J'_2 \backslash T^*$, at all vertices of $P^{(\alpha,\beta)}$.

Assume to the contrary that there is a vertex $x' \in P^{(\alpha,\beta)}$ with $\sum_{j \in J_2' \backslash T^*} x_j' < \sum_{k \in K} q(|J_2' \backslash T^*|, k) \cdot d_k$. Then, at vertex x', some value $d_{\tilde{k}}$ appears fewer than $q(|J_2' \backslash T^*|, \tilde{k})$ times in $J_2' \backslash T^*$, whereas some other value $d_k < d_{\tilde{k}}$ appears more than $q(|J_2' \backslash T^*|, k)$ times in $J_2' \backslash T^*$. Then, if $d_{k'} = \min\{x_j' : j \in J_2' \backslash T^*\}$ it must be $d_{k'} < d_{\tilde{k}}$. Since $q(|J_2' \backslash T^*|, \tilde{k}) \leq u_{\tilde{k}}$, value $d_{\tilde{k}}$ appears fewer than $u_{\tilde{k}}$ times in $J_2' \backslash T^*$. Thus, if $d_{k'}$ appears in I_2' (i.e., if $x_{i_2}' = d_{k'}$ for some $i_2 \in I_2'$), Lemma 20 is contradicted.

Hence let $x_{t_2}' = d_{k'}$, $t_2 \in T \backslash T^*$. The hypothesis $|J_1^*| + |J_2' \backslash T^*| \geq \sum_{k \in K} u_k$ yields that each value d_k occurs u_k times in $J_1^* \cup (J_2' \backslash T^*)$. Therefore, value $d_{\tilde{k}}$, since appearing fewer than $u_{\tilde{k}}$ times in $J_2' \backslash T^*$ must appear in J_1^*; i.e., $x_{t_1}' = d_{\tilde{k}}$ for some $t_1 \in T^*$ or $x_{i_1}' = d_{\tilde{k}}$ for some $i_1 \in I_1^*$. In the former case, for $\tilde{x} = x'(t_1 \leftrightarrow t_2)$, $\alpha\tilde{x} - \alpha x' = (\alpha_{t_2} - \alpha_{t_1}) \cdot (d_{\tilde{k}} - d_{k'}) < 0$ since $\alpha_{t_1} > \alpha_{t_2}$ (by Lemma 18iii) and $d_{\tilde{k}} > d_{k'}$. In the latter case, $\alpha_{t_2} < \alpha_{i_1}$ (by Lemma 18vi) and the contradiction is established by using the vertex $\tilde{x} = x'(t_2 \leftrightarrow \{i_1, i_2\})$ (if $x_{i_2}' = d_{\tilde{k}}$, for some $i_2 \in I_2$) or the vertex $\tilde{x} = x'(t_2 \leftrightarrow i_1)$ (if $d_{\tilde{k}}$ appears not in I_2). □

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