Distance-*d* independent set problems for bipartite and chordal graphs

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Abstract The paper studies a generalization of the INDEPENDENT SET problem (IS for short). A distance-*d* independent set for an integer $d \geq 2$ in an unweighted graph $G = (V, E)$ is a subset $S \subseteq V$ of vertices such that for any pair of vertices $u, v \in S$, the distance between u and v is at least d in G. Given an unweighted graph G and a positive integer k, the DISTANCE-*d* INDEPENDENT SET problem (D*d*IS for short) is to decide whether G contains a distance-*d* independent set S such that $|S| \geq k$. D2IS is identical to the original IS. Thus D2IS is $\mathcal{NP}$-complete even for planar graphs, but it is in $\mathcal{P}$ for bipartite graphs and chordal graphs. In this paper we investigate the computational complexity of D*d*IS, its maximization version MaxD*d*IS, and its parameterized version ParaD*d*IS(k), where the parameter is the size of the distance-*d* independent set: (1) We first prove that for any $\varepsilon > 0$ and any fixed integer $d \geq 3$, it is $\mathcal{NP}$-hard to approximate MaxD*d*IS to within a factor of $n^{1/2-\varepsilon}$ for bipartite graphs of *n* vertices, and for any fixed integer $d \geq 3$, ParaD*d*IS(k) is $\mathcal{W}[1]$-hard for bipartite graphs. Then, (2) we prove that for every fixed integer $d \geq 3$, D*d*IS remains $\mathcal{NP}$-complete even for planar bipartite graphs of maximum degree three. Furthermore, (3) we show that if the input graph is restricted to chordal graphs, then D*d*IS can be solved in polynomial time for any even $d \geq 2$, whereas D*d*IS is $\mathcal{NP}$-complete for any odd $d \geq 3$. Also, we show the hardness of approximation of MaxD*d*IS and the $\mathcal{W}[1]$-hardness of ParaD*d*IS(k) on chordal graphs for any odd $d \geq 3$.

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1 Introduction

One of the most important and most investigated computational problems in theoretical computer science and combinatorial optimization is the INDEPENDENT SET problem (IS for short) because of its many applications in scheduling, computer vision, pattern recognition, coding theory, map labeling, computational biology, and some other fields. The input of IS is an unweighted graph $G = (V, E)$ and a positive integer $k \leq |V|$. An *independent set* of G is a subset $S \subseteq V$ of vertices such that, for all $u, v \in S$, the edge $\{u, v\}$ is not in E. IS asks whether G contains an independent set S having $|S| \geq k$. IS is among the first problems ever to be shown to be $\mathcal{NP}$-complete, and has been used as a starting point for proving the $\mathcal{NP}$-completeness of other problems (Garey and Johnson 1979). Moreover, it is well known that IS remains $\mathcal{NP}$-complete even for substantially restricted graph classes such as cubic planar graphs (Garey et al. 1976), triangle-free graphs (Poljak 1974), and graphs with large girth (Murphy 1992).

In this paper, we consider a generalization of IS, named the DISTANCE- d INDEPENDENT SET problem (DdIS for short). A distance-d independent set for an integer $d \geq 2$ in an unweighted graph $G = (V, E)$ is a subset $S \subseteq V$ of vertices such that for any pair of vertices $u, v \in S$, the distance between u and v is at least d in G. For a fixed constant $d \geq 2$, DdIS considered in this paper is formulated as the following class of problems (Agnarsson et al. 2004):

DISTANCE-d INDEPENDENT SET (DdIS)
    Input: An unweighted graph $G = (V, E)$ and a positive integer $k \leq |V|$.
    Question: Does G contain a distance-d independent set of size k or more?

The maximization version of DdIS can be also defined:

MAXIMUM DISTANCE-d INDEPENDENT SET (MaxDdIS)
    Input: An unweighted graph $G = (V, E)$.
    Output: A distance-d independent set of the maximum size.

The problem parameterized by the solution size k is as follows:

PARAMETERIZED DISTANCE-d INDEPENDENT SET (ParaDdIS(k))
    Input: An unweighted graph $G = (V, E)$.
    Parameter A positive integer $k \leq |V|$.
    Question: Does G contain a distance-d independent set of size k or more?

It is important to note that D2IS is identical to the original IS, and DdIS is equivalent to IS on the $(d-1)$th power graph G^{d-1} of the input graph G as pointed out in (Agnarsson et al. 2004).

Even when $d = 2$, DdIS (i.e., D2IS) is $\mathcal{NP}$-complete, and thus it would be easy to show that DdIS is $\mathcal{NP}$-complete in general. Fortunately, however, it is known that if the input graph is restricted to, for example, bipartite graphs (Harary 1969), chordal graphs (Gavril 1972), circular-arc graphs (Gavril 1974), comparability graphs (Golumbic 1977), and many other classes (Minty 1980; Lozin and Milanič 2008; Brandstädt and Giakoumakis 2012), then D2IS admits polynomial-time algorithms. Furthermore, Agnarsson et al. (2004) show the following tractability of DdIS by using the closure property under taking power (Flotow 1995, 1996; Raychaudhuri 1987):

Fact 1 [Agnarsson et al. 2004] *Let n denote the number of vertices in the input graph G. Then, for every integer $d \geq 2$, DdIS is solvable in $O(n)$ time for interval graphs, in $O(n(\log \log n + \log d))$ time for trapezoid graphs, and in $O(n)$ time for circular-arc graphs.*

This tractability suggests that if we restrict the set of instances to, for example, subclasses of bipartite graphs and chordal graphs, then DdIS for a fixed $d \geq 3$ might be also solvable efficiently. On the other hand, however, we have a "negative" fact that if G is planar/bipartite, then the $(d-1)$th power graph G^{d-1} is not necessarily planar/bipartite. From those points of view, this paper investigates DdIS, namely, our work focuses on the computational complexity of DdIS and/or the inapproximability of MaxDdIS on (subclasses of) bipartite graphs and chordal graphs.

Our main results are summarized in the following list:

(i) For every fixed integer $d \geq 3$, DdIS is $\mathcal{NP}$-complete even for bipartite graphs.
(ii) For any $\varepsilon > 0$ and fixed integer $d \geq 3$, it is $\mathcal{NP}$-hard to approximate MaxDdIS to within a factor of $n^{1/2-\varepsilon}$ for bipartite graphs of n vertices.
(iii) For every fixed integer $d \geq 3$, ParaDdIS(k) is $\mathcal{W}[1]$-hard for bipartite graphs.
(iv) For every fixed integer $d \geq 3$, DdIS remains $\mathcal{NP}$-complete even for planar bipartite graphs of maximum degree three.
(v) For every fixed even integer $d \geq 2$, DdIS is in $\mathcal{P}$ for chordal graphs.
(vi) For every fixed odd integer $d \geq 3$, DdIS is $\mathcal{NP}$-complete for chordal graphs.
(vii) For any $\varepsilon > 0$ and fixed odd integer $d \geq 3$, it is $\mathcal{NP}$-hard to approximate MaxDdIS to within a factor of $n^{1/2-\varepsilon}$ for chordal graphs of n vertices.
(viii) For every fixed odd integer $d \geq 3$, ParaDdIS(k) is $\mathcal{W}[1]$-hard for chordal graphs.

One can see that the complexity of DdIS depends on the parity of d if the set of input graphs is restricted to chordal graphs.

The organization of the paper is as follows: Section. 2 is devoted to our notation and terminology. In Sect. 3 we prove the $\mathcal{NP}$-hardness, the hardness of approximation, and the $\mathcal{W}[1]$-hardness of the problem for bipartite graphs. In Sect. 4, we provide tractable and intractable cases for chordal graphs.

2 Preliminaries

Let $G = (V, E)$ be an unweighted graph, where V and E denote the set of vertices and the set of edges, respectively. $V(G)$ and $E(G)$ also denote the vertex set and the edge set of G, respectively. We denote an edge with endpoints u and v by $\{u, v\}$. For a pair of vertices u and v, the length of a shortest path from u to v, i.e., the distance between u and v is denoted by $dist_G(u, v)$, and the diameter G is defined as $diam(G) = \max_{u,v \in V} dist_G(u, v)$.

A graph G_S is a subgraph of a graph G if $V(G_S) \subseteq V(G)$ and $E(G_S) \subseteq E(G)$. For a subset of vertices $U \subseteq V$, let $G[U]$ be the subgraph induced by U. For a subgraph $G_S = (V_S, E_S)$ of G, if $E_S = V_S \times V_S$, then G_S (or $G[V_S]$) and V_S are called a *clique* and a *clique set*, respectively.

For a positive integer $d \geq 1$ and a graph G, the *d*th power of G, denoted by $G^d = (V(G), E^d)$, is the graph formed from $V(G)$, where all pairs of vertices $u, v \in G$ such that $dist_G(u, v) \leq d$ are connected by an edge $\{u, v\}$. Note that $E(G) \subseteq E^d$, i.e., the original edges in $E(G)$ are retained.

A path of length ℓ, denoted by P_ℓ, from a vertex v_0 to a vertex v_ℓ is represented as a sequence of vertices such that $P_\ell = \langle v_0, v_1, \ldots, v_\ell \rangle$. A cycle of length ℓ, denoted by C_ℓ, is similarly written as $C_\ell = \langle v_0, v_1, \ldots, v_{\ell-1}, v_0 \rangle$. A *chord* of a path (cycle) is an edge between two vertices of the path (cycle) that is not an edge of the path (cycle).

A graph $G = (V, E)$ is *bipartite* if there is a partition of V into two disjoint independent sets V_1 and V_2 such that $V_1 \cup V_2 = V$. A *planar bipartite* graph is a bipartite graph that can be drawn in the plane without edge crossings. A graph G is *chordal* if each cycle in G of length at least four has at least one chord. A graph $G = (V, E)$ is *split* if there is a partition of V into a clique set V_1 and an independent set V_2 such that $V_1 \cap V_2 = \emptyset$ and $V_1 \cup V_2 = V$. Note that the split graphs are a subclass of the chordal graphs. A graph is *star* if it is a rooted tree of height one. See, e.g., (Brandstädt et al. 1999), for the definitions of interval, trapezoid, circular-arc, and comparability graphs, and inclusion relations among the graph classes.

For the maximization problems, an algorithm ALG is called a σ-approximation algorithm and the approximation ratio of ALG is σ if $OPT(G)/ALG(G) \leq \sigma$ holds for every input G, where $ALG(G)$ and $OPT(G)$ are the number of vertices of obtained subsets by ALG and the number of vertices of an optimal solution, respectively.

A *parameterized problem* is a pair (Q, k) where $Q \subseteq \Sigma^*$ is a decision problem over some alphabet Σ, and $k : \Sigma^* \to \mathbf{N}$ is a *parameterization* of the problem, assigning a *parameter* to each instance of Q. An algorithm is *fixed-parameter tractable* or *fpt* if it has a running time at most $f(k) \cdot n^c$ for some computable function f and a constant c, where n is the input length and k is the parameter assigned to the input. Given two parameterized problems (Q_1, k_1) and (Q_2, k_2) over the alphabet Σ, an *fpt-reduction* from (Q_1, k_1) to (Q_2, k_2) is a function $g : \Sigma^* \to \Sigma^*$, computable by an fpt-algorithm, such that $I \in Q_1$ if and only if $g(I) \in Q_2$ and $k_2(g(I)) \leq f(k_1(I))$ for some computable function f, for every $I \in \Sigma^*$.

3 Bipartite graphs

In this section we consider the class of bipartite graphs and its subclasses. As mentioned in Sect. 1, D2IS is solvable in polynomial time by using a polynomial time algorithm which finds the maximum matching in a given bipartite graph (Harary 1969). Unfortunately, however, we can show the $\mathcal{NP}$-hardness of DdIS, the hardness of approximation of MaxDdIS, and the $\mathcal{W}[1]$-hardness of ParaDdIS(k) on bipartite graphs when $d \geq 3$.

Theorem 1 *For every fixed integer $d \geq 3$, DdIS is $\mathcal{NP}$-complete even for bipartite graphs.*

Proof We first show the $\mathcal{NP}$-completeness of D3IS and then one of the general DdIS for $d \geq 4$ in order to make the basic ideas of this proof clear. It is obvious that DdIS is in $\mathcal{NP}$ for every $d \geq 3$. To show that D3IS is $\mathcal{NP}$-hard, we reduce the $\mathcal{NP}$-hard problem D2IS on any general graphs to D3IS on bipartite graphs. That is, given a graph $G_2 = (V_2, E_2)$ of D2IS with n vertices, $V_2 = \{v_1, v_2, \ldots, v_n\}$, and m edges, $E_2 = \{e_1, e_2, \ldots, e_m\}$, we construct a new bipartite graph G_3 in the following way. The constructed graph G_3 consists of (i) n vertices, u_1 through u_n, each u_i of which is corresponding to $v_i \in V_2$, (ii) m vertices, w_1 through w_m, each w_i of which is corresponding to $e_i \in E_2$, and (iii) two special vertices α and β. (iv) The vertex α is connected to each vertex in $\{\beta\} \cup \{w_1, \ldots, w_m\}$, i.e., the induced graph $G[\{\alpha, \beta\} \cup \{w_1, \ldots, w_m\}]$ is star. (v) If $e_i = \{v_j, v_k\} \in E_2$, then we add two edges $\{w_i, u_j\}$ and $\{w_i, u_k\}$. Since there is a partition of V_3 into two disjoint independent sets $\{\beta, w_1, \ldots, w_m\}$ and $\{\alpha, u_1, \ldots, u_n\}$, the reduced graph G_3 must be bipartite. See Fig. 1. For example, if the instance G_2 is the left graph, then the reduced graph G_3 is illustrated in the right graph. It is clear that this reduction can be done in polynomial time.

For the above construction of G_3, we show that G_3 has a distance-3 independent set S_3 such that $|S_3| \geq k + 1$ if and only if G_2 has a distance-2 independent set S_2 such that $|S_2| \geq k$.
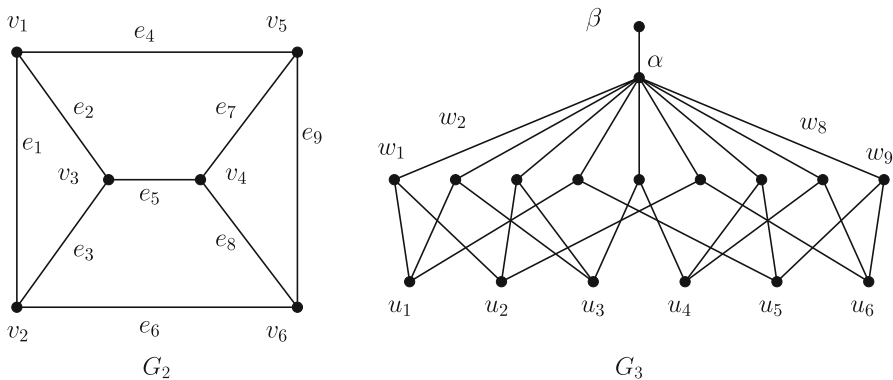

Fig. 1 (*Left*) graph G_2 of D2IS and (*Right*) reduced graph G_3 of D3IS from G_2.

(If part) Suppose that the graph G_2 of D2IS has the distance-2 independent set $S_2 = \{v_{1*}, v_{2*}, \ldots v_{k*}\}$ in G_2, where $\{1^*, 2^*, \ldots, k^*\} \subseteq \{1, 2, \ldots, n\}$. Then, we select a subset of vertices $S_3 = \{u_{1*}, u_{2*}, \ldots, u_{k*}\} \cup \{\beta\}$ of size $k + 1$. Note that the distance $dist_{G_3}(\beta, u_i)$ for every i is at least three. Since the distance $dist_{G_2}(v_{i*}, v_{j*})$ for any pair of vertices $v_{i*}, v_{j*} \in S_2$ ($i^* \neq j^*$) is at least two, the shortest path from u_{i*} to u_{j*} contains at least two vertices in $\{w_1, w_2, \ldots, w_m\}$. This means that the distance $dist_{G_3}(u_{i*}, u_{j*})$ for any $i^* \neq j^*$ is at least four. Thus, the selected vertex set S_3 of size $k + 1$ is a distance-3 independent set in G_3.

(Only-if part) Conversely, suppose that the constructed graph G_3 has the distance-3 independent set S_3 such that $|S_3| \geq k + 1$. First, take a look at the induced subgraph $G[\{\alpha, \beta\} \cup \{w_1, \ldots, w_m\}]$. Since its diameter $diam_{G_3}(G[\{\alpha, \beta\} \cup \{w_1, \ldots, w_m\}])$ is two, $|S_3 \cap V(G[\{\alpha, \beta\} \cup \{w_1, \ldots, w_m\}])| \leq 1$ holds, i.e., $|S_3 \cap \{u_1, u_2, \ldots, u_n\}| \geq k$ must be satisfied. Let $\{u_{1*}, u_{2*}, \ldots, u_{k*}\}$ be a subset of k vertices in $S_3 \cap \{u_1, u_2, \ldots, u_n\}$. Then, the pairwise distance of vertices in $\{v_{1*}, v_{2*}, \ldots, v_{k*}\}$ of G_2 corresponding to $\{u_{1*}, u_{2*}, \ldots, u_{k*}\}$ in G_3 is surely at least 2, i.e., G_2 has a distance-2 independent set S_2 such that $|S_2| \geq k$. This completes the proof of the $\mathcal{NP}$-hardness of D3IS.

To prove the $\mathcal{NP}$-hardness of DdIS for $d \geq 4$, we add the following two small modifications to the constructed graph G_3 in the above reduction, and construct a new bipartite graph G_d. Let $L = (d - 3) - \lceil \frac{d-1}{4} \rceil$ and let $\overline{L} = \lceil \frac{d-1}{4} \rceil$. Note that $L + \overline{L} = d - 3$. (1) The top vertex β in Fig. 1 is replace with a simple path of length L say, $\langle \beta, \beta_1, \ldots, \beta_L \rangle$, and (2) every bottom vertex u_j is replaced with a simple path of length $\overline{L}$, say, $\langle u_j, u_{j,1}, \ldots, u_{j,\overline{L}} \rangle$ for $1 \leq j \leq n$. Then, we can again show that G_d has a distance-d independent set S_d such that $|S_d| \geq k + 1$ if and only if G_2 has a distance-2 independent set S_2 such that $|S_2| \geq k$.

(If part for $d \geq 4$) If G_2 of D2IS has a distance-2 independent set $S_2 = \{v_{1*}, v_{2*}, \ldots v_{k*}\}$ in G_2 as before, then G_d has a subset of vertices $S_d = \{u_{1*,\overline{L}}, u_{2*,\overline{L}}, \ldots, u_{k*,\overline{L}}\} \cup \{\beta_L\}$ of size $k + 1$, which must be a distance-d independent set since $dist_{G_d}(\beta_L, u_{i*,\overline{L}}) = L + \overline{L} + 3 = (d - 3) + 3 = d$ and $dist_{G_d}(u_{i*,\overline{L}}, u_{j*,\overline{L}}) = 4(\overline{L} + 1) = 4\lceil \frac{d-1}{4} \rceil + 4 \geq d$ for any $i^* \neq j^*$.

(Only-if part for $d \geq 4$) Conversely, suppose that the constructed graph G_d has the distance-d independent set S_d such that $|S_d| \geq k + 1$. Similarly to the case of $d = 3$, since $diam_{G_d}(G[\{\alpha, \beta, \beta_1, \ldots, \beta_L\} \cup \{w_1, \ldots, w_m\}]) \leq d$, which means that $|S_d \cap (\{\alpha, \beta, \beta_1, \ldots, \beta_L\} \cup \{w_1, \ldots, w_m\})| \leq 1$ holds, $|S_d \cap \{u_1, u_{1,1}, \ldots, u_{1,\overline{L}}, u_2, u_{2,1}, \ldots, u_{2,\overline{L}}, \ldots, u_n, \ldots, u_{n,\overline{L}}\}| \geq k$ must be satisfied. Now we can assume that (at least) those k vertices in S_d are in the set of bottom vertices $\{u_{1,\overline{L}}, u_{2,\overline{L}}, \ldots, u_{n,\overline{L}}\}$, because $|S_d \cup \{u_{j,\overline{L}}\} \setminus \{u_{j,L'}\}| \geq |S_d|$ even if $u_{j,L'} \in S_d$ for $L' < L$. Let $\{u_{1*,\overline{L}}, u_{2*,\overline{L}}, \ldots, u_{k*,\overline{L}}\}$ be a subset of k vertices in $S_d \cap \{u_{1,\overline{L}}, \ldots, u_{n,\overline{L}}\}$. Then, the pairwise distance of vertices in $\{v_{1*}, v_{2*}, \ldots, v_{k*}\}$ of G_2 corresponding to $\{u_{1*,\overline{L}}, \ldots, u_{k*,\overline{L}}\}$ in G_d is surely at least two, i.e., G_2 has a distance-d independent set S_2 such that $|S_2| \geq k$. This completes the proof of the theorem. □

Next, we consider the maximization version MaxDdIS of DdIS, which asks for a distance-d independent set of the maximum size in an input graph G. Since MaxD2IS is equivalent to MAXIMUM INDEPENDENT SET, it cannot be approximated within a factor of $n^{1-\varepsilon}$ (Zuckerman 2007). In the following, we will show that the above reduction

can preserve the approximation-gap and thus gives us the following inapproximability of MaxDdIS for $d \geq 3$.

Corollary 1 *For any $\varepsilon > 0$ and a fixed integer $d \geq 3$, it is $\mathcal{NP}$-hard to approximate* MaxDdIS *to within a factor of $n^{1/2-\varepsilon}$ for bipartite graphs of n vertices.*

Proof Let $OPT(G_2)$ denote the number of vertices of an optimal solution for an n-vertex input graph G_2 of MaxD2IS. Let $OPT'(G_d)$ denote the number of vertices of an optimal solution for a v-vertex input bipartite graph G_d of MaxDdIS for a fixed $d \geq 3$. Let $g(n)$ be a parameter function of the instance G_2 of D2IS. Note that the reduction described in the proof of Theorem 1 is the following gap-preserving reduction: (1) If $OPT(G_2) \geq g(n)$, then $OPT'(G_d) \geq g(n) + 1$, and (2) if $OPT(G_2) < \frac{g(n)}{n^{1-\varepsilon}}$ for a positive constant ε, then $OPT'(G_d) < \frac{g(n)}{n^{1-\varepsilon}} + 1$.

The constructed graph G_d has at most $n \times \frac{n}{4}$ vertices labeled "u", $m \leq \frac{n^2}{2}$ vertices labeled "w", at most n vertices labeled "β", and one vertex α, i.e., $|V(G_d)| = v = O(n^2)$. Hence the approximation-gap is $n^{1-\varepsilon} = \Theta(v^{1/2-\varepsilon})$ for any $\varepsilon > 0$. By renaming v to n, we obtain the $n^{1/2-\varepsilon}$-inapproximability of MaxDdIS on bipartite graphs of n vertices.   □

Also, the reduction in the proof of Theorem 1 shows the following *fixed-parameter intractability* of ParaDdIS(k):

Corollary 2 *For every fixed integer $d \geq 3$,* ParaDdIS(k) *is $\mathcal{W}$[1]-hard for bipartite graphs.*

Proof It is known (Downey and Fellows 1995) that ParaD2IS(k) on general graphs is $\mathcal{W}$[1]-hard. Let (G_2, k) and (G_d, k') be the instances of ParaD2IS(k) and ParaDdIS(k') on bipartite graphs, respectively. Then, the reduction in the proof of Theorem 1 is the fpt-reduction such that (i) $k' \leq k + 1$, and (ii) (G_2, k) is a *yes*-instance of ParaD2IS(k) if and only if (G_d, k') is a *yes*-instance of ParaDdIS(k') on bipartite graphs.   □

Even if the input graph is restricted to planar bipartite graphs of maximum degree three, DdIS remains intractable for $d \geq 3$. Note that a planar bipartite graph is of course bipartite, and therefore D2IS on planar bipartite graphs is tractable.

Theorem 2 *For every fixed integer $d \geq 3$,* DdIS *is $\mathcal{NP}$-complete even for planar bipartite graphs of maximum degree three.*

Proof We first show the $\mathcal{NP}$-completeness of D3IS and then one of the general DdIS for $d \geq 4$. Obviously, DdIS is in $\mathcal{NP}$ for every $d \geq 3$. To show that D3IS is $\mathcal{NP}$-complete, we reduce the $\mathcal{NP}$-complete problem D2IS on any cubic planar graph $G_2 = (V_2, E_2)$ to D3IS on a new planar bipartite graph $G_3 = (V_3, E_3)$ of maximum degree three.

Let $V_2 = \{v_1, v_2, \ldots, v_n\}$ and $E_2 = \{e_1, e_2, \ldots, e_m\}$ be vertex and edge sets of the planar graph G_2. We construct the planar bipartite graph G_3 which consists of (i) n vertices, u_1 through u_n, which are associated with n vertices in V_2, v_1 through v_n, respectively, and (ii) m subgraphs, $SG_{3,1}$ through $SG_{3,m}$, which are associated with m edges in E_2, e_1 through e_m, respectively. For every i ($1 \leq i \leq m$), the ith
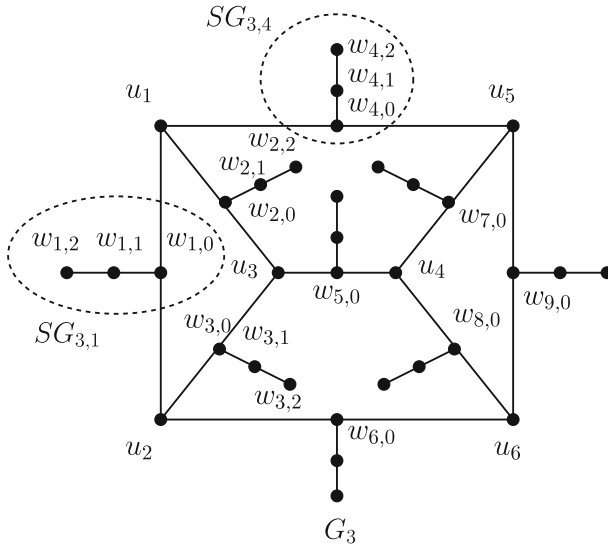
Fig. 2 An illustration of the construction when $d = 3$.

subgraph $SG_{3,i}$ contains three vertices, $w_{i,0}$, $w_{i,1}$, and $w_{i,2}$ and two edges, $\{w_{i,0}, w_{i,1}\}$ and $\{w_{i,1}, w_{i,2}\}$ such that $SG_{3,i}$ forms a path P_2 of length 2. (iii) If $e_i = \{v_j, v_k\} \in E_2$, then we introduce two edges $\{w_{i,0}, u_j\}$ and $\{w_{i,0}, u_k\}$. Note that every simple path $SG_{3,i}$ of length two becomes a single vertex by applying the edge-contraction twice, and also every path $\langle u_j, w_{i,0}, u_k \rangle$ becomes back an edge $\{u_j, u_k\}$ by applying one edge-contraction for $1 \le i \le m$ and $1 \le j, k \le n$. Namely, the constructed graph G_3 is a minor of the planar graph G_2 and thus it must be planar. The maximum degree is clearly three. The construction can be accomplished in polynomial time. For example, if the cubic planar graph G_2 is the left graph in Fig. 1, then the reduced graph G_3 is illustrated in Fig. 2.

For the above construction of G_3, we will show that G_3 has a distance-3 independent set S_3 such that $|S_3| \ge k + m$ if and only if G_2 has a distance-2 independent set S_2 such that $|S_2| \ge k$.

(If part) Suppose that the graph G_2 of D2IS has a distance-2 independent set $S_2 = \{v_{1^*}, v_{2^*}, \ldots v_{k^*}\}$ in G_2, where $\{1^*, 2^*, \ldots, k^*\} \subseteq \{1, 2, \ldots, n\}$. Then, we select two subsets of vertices $S_3' = \{u_{1^*}, u_{2^*}, \ldots, u_{k^*}\}$ and $S_3'' = \{w_{1,2}, w_{2,2}, w_{3,2}, \ldots, w_{m,2}\}$ such that $|S_3'| = k$ and $|S_3''| = m$. One can verify that $S_3 = S_3' \cup S_3''$ is a distance-3 independent set in G_3 since the pairwise distance in S_3' is at least four, the pairwise distance in S_3'' is at least six, and the distance between $w_{i,2}$ in S_3'' and every vertex in S_3' is at least three for each i.

(Only-if part) Conversely, suppose that the graph G_3 has the distance-3 independent set S_3 such that $|S_3| \ge k + m$. First, from each subgraph $SG_{3,i}$ which is the path of length 2, we can select at most one vertex as the distance-3 independent set since its diameter is two. Thus, the maximum size of the distance-3 independent set in $V(SG_{3,1}) \cup V(SG_{3,2}) \cup \ldots \cup V(SG_{3,m})$ is at most m, which means that $|S_3 \cap \{u_1, u_2, \ldots, u_n\}| \ge k$ holds. Let $\{u_{1^*}, u_{2^*}, \ldots, u_{k^*}\}$ be a subset of k vertices
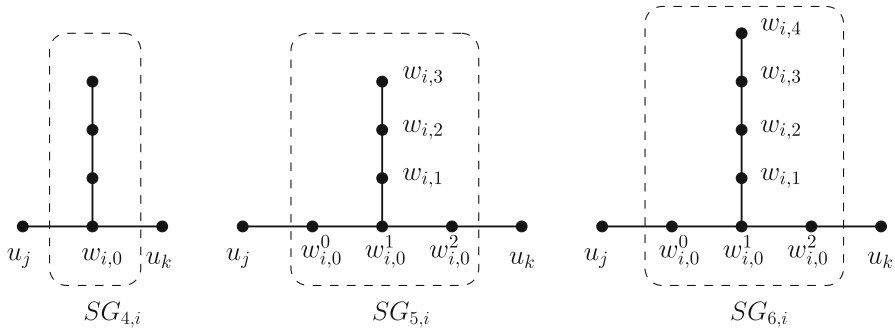
Fig. 3 (*Left*) subgraphs $SG_{4,i}$ for $d = 4$, (*Center*) $SG_{5,i}$ for $d = 5$, and (*Right*) $SG_{6,i}$ for $d = 6$.

in $S_3 \cap \{u_1, u_2, \ldots, u_n\}$. Then, the pairwise distance in the corresponding subset of vertices $\{v_{1*}, v_{2*}, \ldots, v_{k*}\}$ of G_2 is surely at least two, i.e., G_2 has a distance-2 independent set S_2 such that $|S_2| \geq k$. This completes the proof of the $\mathcal{NP}$-hardness of D3IS.

In the following, we give a brief sketch of the ideas to prove the $\mathcal{NP}$-hardness of DdIS for $d \geq 4$. In the case of D4IS, all we have to do is replace the 2-length path $SG_{3,i}$ corresponding to the edge e_i with a 3-length path $SG_{4,i} = (\{w_{i,0}, w_{i,1}, w_{i,2}, w_{i,3}\}, \{(w_{i,0}, w_{i,1}), (w_{i,1}, w_{i,2}), (w_{i,2}, w_{i,3})\})$ for each i. See the left graph in Fig. 3. In the case of D5IS, $SG_{3,i}$ is replaced with $SG_{5,i} = (V(SG_{5,i}), E(SG_{5,i}))$:

$$V(SG_{5,i}) = \{w_{i,0}^0, w_{i,0}^1, w_{i,0}^2, w_{i,1}, w_{i,2}, w_{i,3}\}$$
$$E(SG_{5,i}) = \{\{w_{i,0}^0, w_{i,0}^1\}, \{w_{i,0}^1, w_{i,0}^2\}, \{w_{i,0}^1, w_{i,1}\}, \{w_{i,1}, w_{i,2}\}, \{w_{i,2}, w_{i,3}\}\}.$$

Then, u_j (u_k) corresponding to the vertex v_j (v_k) is connected to $w_{i,0}^0$ ($w_{i,0}^2$) if $e_i = \{v_j, v_k\} \in E_2$ (see the center graph in Fig. 3). For $d = 6$, we connect one vertex $w_{i,4}$ to the top vertex $w_{i,3}$ of $SG_{5,i}$ (see the right graph in Fig. 3). Similarly, for a general $d \geq 7$, such a $\perp$-shape subgraph consists of one horizontal path of length $2\lceil \frac{d}{4} \rceil - 2$ and one vertical path of $d - \lceil \frac{d}{4} \rceil$. Since the diameter of $SG_{d,i}$ is less than d, we can select at most one vertex as the distance-d independent set from each subgraph $SG_{d,i}$ as before. Also, if $\{v_i, v_j\} \in E_2$, then $dist_{G_d}(u_i, u_j) < d$; on the other hand if $dist_{G_2}(v_i, v_j) \geq 2$, then $dist_{G_d}(u_i, u_j) = 2 \times 2\lceil \frac{d}{4} \rceil \geq d$. □

4 Chordal graphs

In this section we restrict the instances to chordal graphs. In 1972, Gavril showed that D2IS admits an efficient algorithm for chordal graphs:

Lemma 1 [Gavril 1972] D2IS *is in* $\mathcal{P}$ *for chordal graphs.*

Recall that if the dth power graph G^d is interval trapezoid, or circular-arc, resp.), then the $(d+1)$th power G^{d+1} is also interval (Raychaudhuri 1987) [trapezoid (Flotow

1995), or circular-arc (Flotow 1996), resp.] for any integer $d \geq 1$. The class of chordal graphs *does not* satisfy the closure property under the graph power operation, i.e., the square G^2 of a chordal graph G is not necessarily chordal, but it *does* satisfy the closure property under the graph *odd power* operation:

Lemma 2 [Agnarsson et al. 2000; Balakrishnan and Paulraja 1983] *Let $d_o \geq 1$ be an odd integer. If G is a chordal graph, then G^{d_o} is also chordal.*

Together with Lemma 1, this yields:

Theorem 3 *For every fixed even integer $d_e \geq 2$, $\mathsf{D}d_e\mathsf{IS}$ is in $\mathcal{P}$ for chordal graphs.*

Proof Given a chordal graph G, we first construct the odd power graph G^{d_e-1} from G in polynomial time, which must be chordal by Lemma 2. Then, by using a polynomial-time algorithm for $\mathsf{D2IS}$ in Lemma 1, we can obtain a solution of $\mathsf{D}d_e\mathsf{IS}$ in polynomial time. $\qquad\square$

For an odd d_o, $\mathsf{D}d_o\mathsf{IS}$ is hard:

Theorem 4 *For every fixed odd $d_o \geq 3$, $\mathsf{D}d_o\mathsf{IS}$ is $\mathcal{NP}$-complete for chordal graphs.*

Proof Obviously, $\mathsf{D}d_o\mathsf{IS}$ on chordal graphs is in $\mathcal{NP}$ for every odd $d_o \geq 3$. To show that $\mathsf{D}d_o\mathsf{IS}$ on chordal graphs is $\mathcal{NP}$-complete, we reduce $\mathsf{D2IS}$ on any graph $G_2 = (V_2, E_2)$ to $\mathsf{D}d_o\mathsf{IS}$ on a new chordal graph $G_{d_o} = (V_{d_o}, E_{d_o})$.

Given the graph $G_2 = (V_2, E_2)$ of $\mathsf{D2IS}$ with n vertices, $V_2 = \{v_1, v_2, \ldots, v_n\}$, and m edges, $E_2 = \{e_1, e_2, \ldots, e_m\}$, we construct the following chordal graph G_{d_o}: (i) We prepare n paths of length $(d_o - 3)/2$, $SG_{d_o,1} = \langle u_{1,1}, u_{1,2}, \ldots, u_{1,(d_o-1)/2}\rangle$ through $SG_{d_o,n} = \langle u_{n,1}, u_{n,2}, \ldots, u_{n,(d_o-1)/2}\rangle$, each $SG_{d_o,i}$ of which is corresponding to $v_i \in V_2$, and (ii) m vertices, w_1 through w_m, each w_i of which is corresponding to $e_i \in E_2$. (iii) All the vertices w_1 through w_m are connected such that $G[\{w_1, \ldots, w_m\}]$ forms a clique of m vertices. (iv) If $e_i = \{v_j, v_k\} \in E_2$, then we connect w_i to two vertices $u_{j,1}$ and $u_{k,1}$.

Figure 4 illustrates the reduced graph G_7 from G_2 which is illustrated in Fig. 1 when $d = 7$. The constructed graph G_{d_o} is chordal since all C_4's in the clique graph $G[\{w_1, \ldots, w_m\}]$ have chords and also $G[\{w_1, \ldots, w_m\} \cup \{v_{i,0}\}]$ contains only cycles C_3's for every i. G_{d_o} can be constructed in polynomial time from G_2.

We show that the reduction satisfies that if G_{d_o} has a distance-d_o independent set S_{d_o} such that $|S_{d_o}| \geq k$ if and only in G_2 has a distance-2 independent set S_2 such that $|S_2| \geq k$. In the remaining of this proof, the crucial observations are: (1) The distance between any vertex v in $V_{d_o} \setminus \{u_{1,(d_o-1)/2}, u_{2,(d_o-1)/2}, \ldots, u_{n,(d_o-1)/2}\}$ and another vertex u in $V_{d_o} \setminus \{v\}$ is at most $d_o - 1$. On the other hand, (2) the pairwise distance of any two vertices in $\{u_{1,(d_o-1)/2}, u_{2,(d_o-1)/2}, \ldots, u_{n,(d_o-1)/2}\}$ is at most d_o. The two observations (1) and (2) imply that the distance-d_o independent set S_{d_o} in G_{d_o} must be a subset of outside vertices $\{u_{1,(d_o-1)/2}, u_{2,(d_o-1)/2}, \ldots, u_{n,(d_o-1)/2}\}$. (3) If v_j and v_k are two endpoints of single edge e_i in G_2, then there must be a path

$$\langle u_{j,(d_o-1)/2}, u_{j,(d_o-3)/2}, \ldots, u_{j,1}, w_i, u_{k,1}, u_{k,2}, \ldots, u_{k,(d_o-1)/2}\rangle$$

by the above reduction rules. Thus, the distance between u_{j,d_o} and u_{k,d_o} in G_{d_o} is $(d_o - 1)/2 \times 2 = d_o - 1$.
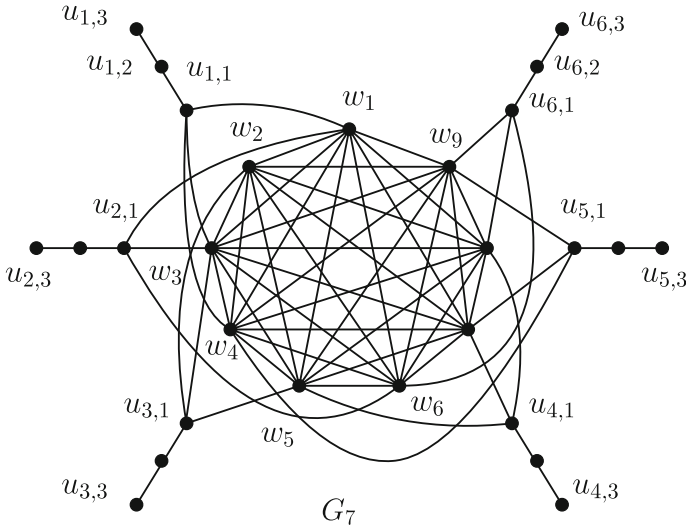
Fig. 4 An illustration of the construction when $d = 7$.

(If part) Now suppose that the graph G_2 of D2IS has a distance-2 independent set $S_2 = \{v_{1*}, v_{2*}, \ldots v_{k*}\}$ in G_2, where $\{1*, 2*, \ldots, k*\} \subseteq \{1, 2, \ldots, n\}$. Then, we select a subset $S_{d_o} = \{u_{1*,(d_o-1)/2}, u_{2*,(d_o-1)/2}, \ldots, u_{k*,(d_o-1)/2}\}$ of size k. It is easy to verify that the pairwise distance in S_{d_o} is exactly d_o.

(Only-if part) Conversely, suppose that the reduced graph G_{d_o} has the distance-d_o independent set $S_{d_o} = \{u_{1*,(d_o-1)/2}, u_{2*,(d_o-1)/2}, \ldots, u_{k*,(d_o-1)/2}\}$ of size k. Then, the pairwise distance in the corresponding subset of vertices $\{v_{1*}, v_{2*}, \ldots, v_{k*}\}$ of G_2 is surely at least two, i.e., G_2 has a distance-2 independent set S_2 such that $|S_2| \geq k$.
□

Corollary 3 D3IS *is* $\mathcal{NP}$-*complete for split graphs.*

Proof When $d = 3$ in the proof of Theorem 4, the constructed graph G_3 is a split graph since there is a partition of $V(G_3)$ into a clique set $\{w_1, w_2, \ldots, w_m\}$ and an independent set $\{u_{1,1}, u_{2,1}, \ldots, u_{n,1}\}$.
□

Similarly to the previous section, it can be shown that the reduction in the proof of Theorem 4 can preserve the approximation-gap, and also it is an fpt-reduction:

Corollary 4 *For any* $\varepsilon > 0$ *and fixed odd integer* $d_o \geq 3$, *it is* $\mathcal{NP}$-*hard to approximate* MaxDd_oIS *to within a factor of* $n^{1/2-\varepsilon}$ *for chordal graphs.*

Proof The proof is very similar to the proof of Corollary 1. Now, let $OPT'(G_{d_o})$ denote the number of vertices of an optimal solution for a v-vertex input chordal graph G_{d_o} of MaxDd_oIS for a fixed $d_o \geq 3$. Then, we can show that (1) if $OPT(G_2) \geq g(n)$, then $OPT'(G_{d_o}) \geq g(n)$, and (2) if $OPT(G_2) < \frac{g(n)}{n^{1-\varepsilon}}$ for a positive constant ε, then $OPT'(G_{d_o}) < \frac{g(n)}{n^{1-\varepsilon}}$. Hence the corollary follows from $v = O(n^2)$.
□

Corollary 5 *For every fixed odd integer* $d_o \geq 3$, ParaDd_oIS(k) *is* $\mathcal{W}[1]$-*hard for chordal graphs.*

Proof Let (G_2, k) and (G_{d_o}, k') be the inputs of $\mathsf{ParaD2IS}(k)$ and $\mathsf{ParaD}d_o\mathsf{IS}(k')$ on chordal graphs, respectively. Then, the reduction in the proof of Theorem 4 satisfies the condition $k' \leq k$. □

5 Concluding remarks

In the conference version (Eto et al. 2012) of this paper we claimed that the reduced graph G_d in the proof of Theorem 1 is *chordal bipartite* and thus $\mathsf{D}d\mathsf{IS}$ on chordal bipartite graphs is $\mathcal{NP}$-hard. However, G_d is not chordal bipartite since it includes an induced cycle of length six or more (for example, actually G_3 in Fig. 1 contains an induced cycle $\langle u_1, w_1, u_2, w_3, u_3, w_2, u_1 \rangle$ of length six). Therefore, the computational complexity of $\mathsf{D}d\mathsf{IS}$ on chordal bipartite graphs is still open.

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