J Comb Optim (2014) 27:451–461
DOI 10.1007/s10878-012-9531-6

On the total outer-connected domination in graphs

O. Favaron · H. Karami · S.M. Sheikholeslami

Published online: 10 July 2012

Abstract A set S of vertices of a graph G is a *total outer-connected dominating set* if every vertex in $V(G)$ is adjacent to some vertex in S and the subgraph induced by $V \setminus S$ is connected. The *total outer-connected domination number* $\gamma_{toc}(G)$ is the minimum size of such a set. We give some properties and bounds for γ_{toc} in general graphs and in trees. For graphs of order n, diameter 2 and minimum degree at least 3, we show that $\gamma_{toc}(G) \leq \frac{2n-2}{3}$ and we determine the extremal graphs.

Keywords Total outer-connected dominating set · Total outer-connected domination number · Diameter

1 Introduction

For domination problems, multiple edges and loops are irrelevant, so we forbid them. We use $V(G)$ and $E(G)$ (or simply V and E) for the vertex set and edge set of a graph G and denote $|V(G)| = n$, $|E(G)| = m$. For a vertex $v \in V(G)$, the *open neighborhood* $N(v)$ is the set $\{u \in V(G)\colon uv \in E(G)\}$ and the *closed neighborhood* $N[v]$ is the set $N(v) \cup \{v\}$. The *open neighborhood* $N(S)$ of a set $S \subseteq V$ is the set $\bigcup_{v \in S} N(v)$, and the *closed neighborhood* $N[S]$ of S is the set $N(S) \cup S$. For a vertex

O. Favaron
LRI, UMR 8623, University Paris Sud and CNRS, Orsay, 91405, France
e-mail: Odile.Favaron@lri.fr

H. Karami · S.M. Sheikholeslami (✉)
Department of Mathematics, Azarbaijan Shahid Madani University, Tabriz, Iran
e-mail: s.m.sheikholeslami@azaruniv.edu

S.M. Sheikholeslami
School of Mathematics, Institute for Research in fundamental sciences (IPM), P.O. Box 19395-5746, Tehran, Iran

$x \in V \setminus S$, we write $N_S(x)$ for $N(x) \cap S$. A vertex x of $V \setminus S$ is a private neighbor of a vertex u of S with respect to S, or a (u, S)-private neighbor, if $N_S(x) = \{u\}$. The degree of a vertex v and the minimum degree in G are respectively denoted by $d_G(v)$ and $\delta(G)$ ($d(v)$ and δ for short). We denote by P_n and C_n the path and the cycle on n vertices. We say that we attach a vertex u to a vertex v of a graph if we join u and v by an edge. The 2-corona G of a graph H is obtained by attaching a leaf of a path P_2 at each vertex of H (all the P_2's are distinct and disjoint). The set $V(H)$ is called the handle of G and each attached P_2 is called a pendant P_2.

For a graph G, a set $S \subseteq V(G)$ is a *dominating set* if $N[S] = V(G)$. A dominating set S is an *outer-connected dominating set* if the subgraph induced by $V \setminus S$, denoted $G[V \setminus S]$, is connected. The set $S \subseteq V(G)$ is a *total dominating set* if $N(S) = V(G)$. A total dominating set S is a *total outer-connected dominating set* (TOCD set for short) if the subgraph induced by $V \setminus S$ is connected. The minimum size of a dominating set, a total dominating set, an outer-connected dominating set, a total outer-connected dominating set are the *domination number* $\gamma(G)$, the *total domination number* $\gamma_t(G)$, the *outer-connected domination number* $\gamma_{oc}(G)$, and the *total outer-connected domination number* $\gamma_{toc}(G)$, respectively. A $\gamma_{toc}(G)$-set ($\gamma_t(G)$-set) is a total outer-connected dominating set of G of size $\gamma_{toc}(G)$ (a total dominating set of size $\gamma_t(G)$). The total outer-connected domination number was introduced by Cyman (2010) and has been studied particularly for trees (see Cyman and Raczek 2009; Hattingh and Joubert 2010).

We note the following results related to the total domination.

If a graph G has k components $G_1, G_2, \ldots, G_k$, then $\gamma_t(G) = \sum_{i=1}^{k} \gamma_t(G_i)$. If the connected graph G is not a star, then it admits a $\gamma_t(G)$-set containing no leaf of G.

Theorem A *Let H be a connected graph of order $n \geq 3$. Then*

1. (Cockayne et al. 1980) $\gamma_t(H) \leq \lfloor \frac{2n}{3} \rfloor$.
2. (Brigham et al. 2000) $\gamma_t(H) = \frac{2n}{3}$ *if and only if H is a 2-corona or a cycle C_3 or C_6.*

   $\gamma_t(H) = \frac{2n-1}{3}$ *if and only if H is a 2-corona minus a pendant edge or a cycle C_5.*

2 General properties of $\gamma_{toc}(G)$

We begin with some easy properties of γ_{toc} in graphs of minimum degree at least 1. First it is immediate to check that for paths P_n, cycles C_n, complete graphs K_n, complete bipartite graphs $K_{p,q}$ with $p, q \geq 2$, stars $K_{1,n-1}$, we have $\gamma_{toc}(P_n) = n - 1$ for $3 \leq n \leq 5$, $\gamma_{toc}(P_n) = n - 2$ for $n \geq 6$, $\gamma_{toc}(C_n) = n - 1$ for $n = 3$, $\gamma_{toc}(C_n) = n - 2$ for $n \geq 4$, $\gamma_{toc}(K_n) = \gamma_{toc}(K_{p,q}) = 2$, $\gamma_{toc}(K_{1,n-1}) = n - 1$.

From the definition of γ, γ_t, γ_{oc} and γ_{toc}, it is clear that for every graph G,

$$1 \leq \gamma(G) \leq \gamma_t(G) \leq \gamma_{toc}(G) \leq n \quad \text{and} \quad \gamma(G) \leq \gamma_{oc}(G) \leq \gamma_{toc}(G).$$

It is known that $\frac{\gamma_t(G)}{\gamma(G)} \leq 2$ for every graph. However the other ratios $\frac{\gamma_{oc}}{\gamma}$, $\frac{\gamma_{toc}}{\gamma_t}$, $\frac{\gamma_{toc}}{\gamma_{oc}}$ are not bounded. For the first two ratios, the star $K_{1,n-1}$ satisfies $\gamma = 1$, $\gamma_t = 2$, $\gamma_{oc} = \gamma_{toc} = n - 1$. For the third ratio, we consider for instance an arbitrarily large integer q and a graph G defined from a path $P_{3q} = x_1^1 x_2^1 \cdots x_q^1 x_1^2 x_2^2 \cdots x_q^2 x_1^3 x_2^3 \cdots x_q^3$ and three vertices y_1, y_2, y_3 by adding the edges $y_i x_j^i$ with $i \in \{1, 3\}$ and $1 \leq j \leq q$ or $i = 2$ and $1 \leq j \leq 4q$. For this graph, $\gamma = \gamma_{oc} = 3$, $\gamma_t = 6$ and $\gamma_{toc} = q + 5$.

Proposition 1 *Let D be a γ_{toc}-set of a connected graph G of order $n \geq 3$ and minimum degree $\delta = 1$. Then D contains all the support vertices of G. If moreover $\gamma_{toc}(G) \leq n - 2$, then D also contains all the leaves.*

Proof Let v be a support vertex and vu a pendant edge of G. If $v \notin D$, then $u \notin N[D]$ or u is an isolated vertex of D, a contradiction. If $u \notin D$, then u is an isolated vertex in $G[V \setminus D]$, a contradiction except if $\gamma_{toc}(G) = n - 1$. □

Proposition 2 *Let $G_1, G_2, \ldots, G_p$ be the components of a graph G. Then*

$$\gamma_{toc}(G) = \min_{1 \leq i \leq p} \left\{ \sum_{j \neq i} n(G_j) + \gamma_{toc}(G_i) \right\}.$$

Proof For every TOCD set D of G, $V \setminus D$ is entirely contained in one component G_i and thus D contains all the components different from G_i. □

Proposition 3 *Let G be a graph of order n and minimum degree $\delta \geq 1$. Then $\gamma_{toc}(G) = n$ if and only if each component of G is a K_2.*

Proof Suppose $\gamma_{toc}(G) = n$ and let v be a vertex of degree Δ. The set $V \setminus \{v\}$ is an OCD set of G but not a TOCD set since $\gamma_{toc}(G) = n$. Therefore at least one neighbor v' of v has degree 1 in G. If v has another neighbor in G then $V \setminus \{v'\}$ is a TOCD set of G, a contradiction to $\gamma_{toc}(G) = n$. Hence $\Delta = 1$ and G is the disjoint union of K_2's. Conversely, if each component of G is a K_2, then $\gamma_{toc}(G) = n$ by Proposition 2. □

We already saw that the connected graphs of order $n = 3$, namely K_3 and P_3, satisfy $\gamma_{toc}(G) = n - 1$. We consider now those of order at least 4.

Proposition 4 *Let G be a connected graph of order $n \geq 4$. Then $\gamma_{toc}(G) = n - 1$ if and only if every edge is pendant or is adjacent to a pendant edge.*

Proof Suppose $\gamma_{toc}(G) = n - 1$ and let uv be a non-pendant edge of G with $d(v) \geq d(u)$. The set $D = V(G) \setminus \{u, v\}$ is an OCD set of G, but not a TOCD set since $\gamma_{toc}(G) = n - 1$. Hence some vertex of $(N(u) \cup N(v)) \setminus \{u, v\}$ is isolated in $G - \{u, v\}$. If no vertex of $N(u) \setminus N[v]$ and $N(v) \setminus N[u]$ is isolated in $G - \{u, v\}$, then $N(u) \cap N(v)$ contains a vertex z isolated in $G - \{u, v\}$. Since $n \geq 4$ and $d(v) \geq d(u)$, the set $V(G) \setminus \{u, z\}$ is a TOCD set of G, in contradiction with $\gamma_{toc}(G) = n - 1$. Therefore u or v is the end-vertex of a pendant edge.

Conversely, suppose that every edge of G is pendant or adjacent to a pendant edge. Since $n \geq 4$, $\gamma_{toc}(G) < n$ by Proposition 3. If every vertex of G is a leaf or a support vertex, then $\gamma_{toc}(G) = n - 1$ by Proposition 1. Otherwise, let D be a $\gamma_{toc}(G)$-set, L the set of leaves of G and X the set of the vertices of G which are neither leaves nor support vertices. By the definition of X and G, the neighbors of the vertices of X are support vertices. Hence the set $X \cup L$ is independent. Since $V \setminus D \subseteq X \cup L$ by Proposition 1 and $G[V \setminus D]$ is connected, $|V \setminus D| = 1$ and $\gamma_{toc}(G) = n - 1$. □

The following corollary is a consequence of Propositions 2, 3 and 4.

Corollary 5 *Let G be a graph of order n and minimum degree $\delta \geq 1$. Then $\gamma_{toc}(G) = n - 1$ if and only if each component of G is K_2, P_3, C_3 or is such that every edge is pendant or adjacent to a pendant edge, and at least one component is not K_2.* □

Cyman (2007) gave lower bounds on the outer-connected domination number respectively in terms of n and m and in terms of n and Δ. In Theorems 8 and 13, we establish similar bounds on the total outer-connected domination number.

Definition 6 For $1 \leq i \leq 5$, let $\mathcal{A}_i$ be the family of graphs $G = (V, E)$ defined as follows. The graph G is connected and V is a disjoint union $D \cup X$. Moreover
$\mathcal{A}_1$: The components of $G[D]$ are paths P_2, $G[X]$ is a tree, and every vertex of X has exactly one neighbor in D.
$\mathcal{A}_2$: The components of $G[D]$ are paths P_2 except one which is a path P_3, $G[X]$ is a tree, and every vertex of X has exactly one neighbor in D.
$\mathcal{A}_3$: The components of $G[D]$ are paths P_2 except one of them which is a path P_4 or two of them which are paths P_3, $G[X]$ is a tree, and every vertex of X has exactly one neighbor in D.
$\mathcal{A}_4$: The components of $G[D]$ are paths P_2, $G[X]$ is a tree, and every vertex of X has exactly one neighbor in D except one of them which has two neighbors in D.
$\mathcal{A}_5$: The components of $G[D]$ are paths P_2, $G[X]$ is connected and unicyclic, and every vertex of X has exactly one neighbor in D.
Let $\mathcal{A} = \bigcup_{i=1}^{5} \mathcal{A}_i$.

Proposition 7 *If $G \in \mathcal{A}$, then $\gamma_{toc}(G) \leq \lceil \frac{2(2n-m-1)}{3} \rceil$.*

Proof In each graph G of all families $\mathcal{A}_i$, the set D is a TOCD set of G. Hence $\gamma_{toc}(G) \leq |D|$. Since $|D|$ is an integer, it is sufficient in order to prove the result to show that $|D|$ is equal to $\frac{4n-2m-2}{3}$, $\frac{4n-2m-1}{3}$, or $\frac{4n-2m}{3}$. In all of these graphs, $n = |D| + |X|$ and $m = m(D) + m(D, X) + m(X)$.
In $\mathcal{A}_1$, $m(D) = \frac{|D|}{2}$, $m(D, X) = |X|$ and $m(X) = |X| - 1$. Hence $|D| = \frac{4n-2m-2}{3}$.
In $\mathcal{A}_2$, $m(D) = \frac{|D|+1}{2}$, $m(D, X) = |X|$ and $m(X) = |X| - 1$. Hence $|D| = \frac{4n-2m-1}{3}$.
In $\mathcal{A}_3$, $m(D) = \frac{|D|}{2} + 1$, $m(D, X) = |X|$ and $m(X) = |X| - 1$. Hence $|D| = \frac{4n-2m}{3}$.
In $\mathcal{A}_4$, $m(D) = \frac{|D|}{2}$, $m(D, X) = |X| + 1$ and $m(X) = |X| - 1$. Hence $|D| = \frac{4n-2m}{3}$.
In $\mathcal{A}_5$, $m(D) = \frac{|D|}{2}$, $m(D, X) = |X|$ and $m(X) = |X|$. Hence $|D| = \frac{4n-2m}{3}$. □

Theorem 8 *Let G be a connected graph with $n \geq 2$ vertices and m edges. Then*

$$\gamma_{toc}(G) \geq \left\lceil \frac{2(2n-m-1)}{3} \right\rceil$$

with equality if and only if $G \in \mathcal{A}$.

Proof Let D be a $\gamma_{toc}(G)$-set and $X = V \setminus D$. Since D is a total dominating set and $G[X]$ is connected, we have

$$m(D) \geq \frac{|D|}{2}, \tag{1}$$

$$m(D, X) \geq |X| \quad \text{and} \tag{2}$$

$$m(X) \geq |X| - 1. \tag{3}$$

Hence $m(G) \geq \frac{|D|}{2} + 2|X| - 1$ with $|X| = n - |D|$ and thus $m \geq 2n - \frac{3|D|}{2} - 1$. Since $|D| = \gamma_{toc}(G)$ is an integer, we get $\gamma_{toc}(G) \geq \lceil \frac{2(2n-m-1)}{3} \rceil$.

Assume $\gamma_{toc}(G) = \lceil \frac{2(2n-m-1)}{3} \rceil$. If $\gamma_{toc}(G) = \frac{4n-2m-2}{3}$ then $m(G) = \frac{|D|}{2} + 2|X| - 1$ (hence $|D|$ is even) and equality holds in (1), (2) and (3). From equality in (1) and since $\delta(D) \geq 1$, all the components of $G(D)$ are paths P_2. From equality in (2) and since D is dominating, every vertex of X has exactly one neighbor in D. From equality in (3) and since $G[X]$ is connected, $G[X]$ is a tree. Therefore $G \in \mathcal{A}_1$.

If $\gamma_{toc}(G) = \frac{4n-2m-1}{3}$ then $m(G) = \frac{|D|+1}{2} + 2|X| - 1$ (hence $|D|$ is odd). The unique possibility is that $m(D) = \frac{|D|+1}{2}$ and equality holds in (2) and (3). From $m(D) = \frac{|D|+1}{2}$, all the components of $G[D]$ are paths P_2 except one which is a path P_3, from equality in (2) every vertex of X has exactly one neighbor in D and from equality in (3) $G[X]$ is a tree. Therefore $G \in \mathcal{A}_2$.

If $\gamma_{toc}(G) = \frac{4n-2m}{3}$ then $m(G) = \frac{|D|}{2} + 2|X|$ (hence $|D|$ is even) and we have the following three possibilities. Equality holds in (2) and (3) and $m(D) = \frac{|D|}{2} + 1$, in which case $G \in \mathcal{A}_3$. Equality holds in (1) and (3) and $m(D, X) = |X| + 1$, in which case $G \in \mathcal{A}_4$. Equality holds in (1) and (2) and $m(X) = |X|$, in which case $G \in \mathcal{A}_5$.

Therefore $\gamma_{toc}(G) = \lceil \frac{2(2n-m-1)}{3} \rceil$ implies $G \in \mathcal{A}$. Conversely $G \in \mathcal{A}$ implies $\gamma_{toc}(G) = \lceil \frac{2(2n-m-1)}{3} \rceil$ by Proposition 7, which completes the proof. □

When G is a tree, then $m = n - 1$ and the lower bound of Theorem 8 becomes $\lceil \frac{2n}{3} \rceil$. This bound was already obtained in Cyman (2010). Moreover two different constructions of the extremal trees by recursively applying several operations are given in Cyman (2010) and Hattingh and Joubert (2010). In Corollary 10 we give an explicit description of these trees.

Definition 9 For $1 \leq i \leq 4$, let $\mathcal{B}_i$ be the family of trees T defined as follows and let $\mathcal{B} = \bigcup_{i=1}^{4} \mathcal{B}_i$.

$\mathcal{B}_1$: T is the 2-corona of a tree.

$\mathcal{B}_2$: T is obtained from the 2-corona of a tree by replacing a pendant P_2 by a pendant P_3.

$\mathcal{B}_3$: T is obtained from the 2-corona of a tree by replacing a pendant P_2 by a pendant P_4 or two pendant P_2 by two disjoint pendant P_3.

$\mathcal{B}_4$: T is obtained from the 2-corona of a tree by attaching a second pendant P_2 at one vertex of the handle of the 2-corona.

Corollary 10 *Let T be a tree with $n \geq 2$ vertices. Then $\gamma_{toc}(T) \geq \lceil \frac{2n}{3} \rceil$ with equality if and only if $T \in \mathcal{B}$.*

Proof By Theorem 8, $\gamma_{toc}(T) \geq \lceil \frac{4n-2m-2}{3} \rceil = \lceil \frac{2n}{3} \rceil$ with equality if and only if T is a tree of $\mathcal{A}$. There is no tree in $\mathcal{A}_5$ and the graphs of $\cup_{i=1}^{4} \mathcal{A}_i$ are trees if and only if exactly one vertex of each component of D has a neighbor in X (for the components P_3 and P_4, this vertex can be a leaf or a support vertex). Therefore for $1 \leq i \leq 4$, $\mathcal{B}_i$ is the class of the trees of $\mathcal{A}_i$ and $\mathcal{B}$ is the class of the trees of $\mathcal{A}$. Note that $\gamma_{toc}(T) = \frac{2n}{3}$ if $T \in \mathcal{B}_1$, $\gamma_{toc}(T) = \frac{2n+1}{3}$ if $T \in \mathcal{B}_2$ and $\gamma_{toc}(T) = \frac{2n+2}{3}$ if $T \in \mathcal{B}_3 \cup \mathcal{B}_4$. □

By Theorem A and Corollary 10 we get the following corollary.

Corollary 11 *A tree T satisfies $\gamma_t(T) = \gamma_{toc}(T)$ if and only if it is the 2-corona of a tree.*

Our second lower bound on γ_{toc} is an easy lower bound on γ_t which remains sharp for γ_{toc}.

Definition 12 Let $k \geq 1$ be an integer and let $\mathcal{C}_k$ be the family of the graphs G defined as follows. The vertex set V is a disjoint union $D \cup X$ with $|D|$ even and $|X| = k|D|$. Each component of $G[D]$ is a P_2 and $G[X]$ is connected of maximum degree at most k. The neighborhoods in X of the vertices of D are all disjoint and of size k. Let $\mathcal{C} = \bigcup_{k=1}^{\infty} \mathcal{C}_k$.

Theorem 13 *Let G be a connected graph of order $n \geq 2$ and maximum degree Δ. Then*

$$\gamma_{toc}(G) \geq \gamma_t(G) \geq \frac{n}{\Delta}.$$

Equalities $\gamma_{toc}(G) = \gamma_t(G) = \frac{n}{\Delta}$ occur if and only if $G \in \mathcal{C}$ or G is a K_2.

Proof If $\Delta = 1$, then G is a K_2 and $\gamma_{toc}(G) = \gamma_t(G) = 2 = \frac{n}{\Delta}$. Assume now $\Delta \geq 2$. Let D be a $\gamma_{toc}(G)$-set and $X = V \setminus D$. Since D is a total dominating set, $d_D(v) \geq 1$, $d_X(v) \leq \Delta - 1$ for every $v \in D$, and $X \subseteq N(D)$. Hence $|X| \leq (\Delta - 1)|D|$ and $n \leq \Delta|D|$. Note that we did not use the fact that $G[V \setminus D]$ is connected. This implies also $\gamma_t(G) \geq n/\Delta$.

If $|D| = \frac{n}{\Delta}$, then all the inequalities above become equalities. Every vertex v of D has degree 1 in D, thus each component of $G[D]$ is a path P_2, and degree $\Delta - 1$ in X. Since $|X| = (\Delta - 1)|D|$, the neighborhoods in X of the vertices of D are disjoint and each vertex of X has one neighbor in D and degree at most $\Delta - 1$ in X. Finally, $G[X]$ is connected since D is a TOCD set of G. Hence G is the cycle C_4, the unique graph of $\mathcal{C}_1$, if $\Delta = 2$ and belongs to $\mathcal{C}_{\Delta-1}$ if $\Delta \geq 3$.

Conversely, let $G \in \mathcal{C}_k$ for some integer $k \geq 1$. Then $\Delta(G) = k + 1$, $n = |D|(k + 1) = |D|\Delta$ and D is a TOCD set of G of order $\frac{n}{\Delta}$. Hence $\gamma_{toc}(G) \leq \frac{n}{\Delta}$ and thus $\gamma_{toc}(G) = \frac{n}{\Delta}$ by the direct part of the theorem. □

3 Graphs of diameter 2

The value of $\gamma_{toc}(G)$ may be very large for graphs with large diameter since for the long paths and cycles, $\gamma_{toc}(G) = n - 2$. In this section we limit the diameter for the graph and we establish a sharp upper bound on $\gamma_{toc}(G)$ in term of the number of vertices for the graphs G for which diam$(G) \leq 2$ and $\delta(G) \geq 3$. In order to characterize the extremal graphs, we first describe and study an infinite family of graphs.

Definition 14 A graph G belongs to Family $\mathcal{F}$ if G is obtained by joining a vertex u to every vertex of the disjoint union of a cycles C_3 and b cycles C_6 with $b = 0$ and $a \geq 1$ or $b \geq 1$ and $a + 2b \geq 5$.

Proposition 15 *Let G be a graph of order n obtained by joining a vertex u to every vertex of the disjoint union of $a \geq 0$ cycles C_3 and $b \geq 0$ cycles C_6 with $a + b \geq 1$. Then $\gamma_{toc}(G) \leq \frac{2n-2}{3}$ with equality if and only if $G \in \mathcal{F}$.*

Proof **Case 1** If $a + b = 1$, then G is a wheel and $\gamma_{toc}(G) = 2$. If $G \in \mathcal{F}$, then $a = 1$ and $b = 0$, $G \simeq K_4$ and $\gamma_{toc}(G) = 2 = \frac{2n-2}{3}$. If $G \notin \mathcal{F}$, then $a = 0$ and $b = 1$, $n = 7$ and $\gamma_{toc}(G) = 2 < \frac{2n-2}{3}$.

Case 2 If $a + b \geq 2$, then $n \geq 7$ and u is a cut-vertex of G. Let $H_1, \ldots, H_{a+b}$ be the components of $G - u$. We note that $\gamma_t(H_i) = \frac{2n(H_i)}{3}$ for $1 \leq i \leq a + b$. Let D_1 be a minimum TOCD set of G containing u and D_2 a minimum TOCD set of G not containing u. Then $\gamma_{toc}(G) = \min\{|D_1|, |D_2|\}$. The set D_1 contains of all the components of $G - u$ but the largest one and $|D_1| = n - 6$ if $b \geq 1$, $|D_1| = n - 3$ if $b = 0$. The set D_2 consists of a minimum total dominating set of each component of $G - u$ and $|D_2| = \sum_{i=1}^{a+b} \frac{2n(H_i)}{3} = \frac{2n-2}{3}$. If $b = 0$, then $G \in \mathcal{F}$ and since $n \geq 7$ we have $\gamma_{toc}(G) = \min\{n - 3, \frac{2n-2}{3}\} = \frac{2n-2}{3}$. If $b \geq 1$ then $\gamma_{toc}(G) = \min\{n - 6, \frac{2n-2}{3}\} \leq \frac{2n-2}{3}$. Moreover, $\gamma_{toc}(G) = \frac{2n-2}{3}$ if and only if $n \geq 16$, i.e., since $n = 3a + 6b + 1$, if and only if $a + 2b \geq 5$, in other terms if and only if $G \in \mathcal{F}$. □

Proposition 16 *Let G be a graph of order $n = 7$, minimum degree $\delta = 3$ and such that $\gamma_{toc}(G) = 4$. Then $G \in \mathcal{F}$.*

Proof Let v be a vertex of degree 3, $N(v) = \{v_1, v_2, v_3\}$ and $X = V \setminus N[v] = \{x, y, z\}$. Since $\delta = 3$, each vertex of X has at least one neighbor in $N(v)$.

Case 1 $G[X]$ is connected. Then $G[X]$ is a path xyz or a cycle C_3.

Subcase 1.1 Only one vertex of $N(v)$, say v_1, has a neighbor in X. Then v_1 is adjacent to each of x, y, z. Since each vertex has degree at least 3, each of X and $N(v)$ induces a C_3. Therefore $G \in \mathcal{F}$ with $u = v_1$, $a = 2$ and $b = 0$.

Subcase 1.2 Exactly two vertices of $N(v)$, say v_1 and v_2, have neighbors in X. Then v_3 is adjacent to v_1 and v_2. Without loss of generality, suppose v_1 adjacent to y. If v_1 is also adjacent to x and z, then $\{v, v_1\}$ is a TOCD set of order two of G. If v_1 is not adjacent to z (the case v_1 is not adjacent to x is similar), then v_2 is adjacent to z and $G[X]$ is a C_3 since z has degree 3. Then $\{v_2, v_3, z\}$ is a TOCD set of order three of G. This contradicts $\gamma_{toc}(G) = 4$.

Subcase 1.3 Each vertex v_1, v_2, v_3 has a neighbor in X.

If there exists no perfect matching between $N(v)$ and X, there are, without loss of generality, two possibilities. Either $\{v_1x, v_1y, v_2z, v_3z\} \subseteq E(G)$ and $\{y, z, v_3\}$ is a

TOCD set of order 3 of G. Or $\{v_1x, v_1z, v_2y, v_3y\} \subseteq E(G)$ and $\{v, v_1, z\}$ is a TOCD set of order 3 of G.

Suppose there exists a perfect matching v_1x, v_2y, v_3z between $N(v)$ and X. If $G[X]$ is a cycle, $\{v, v_1, x\}$ is a TOCD set of order 3 of G. Suppose $xz \notin E(G)$. If v_3 is adjacent to one of x, y or v_2, then $\{x, y, v_2\}$ is a TOCD set of order 3 of G. Suppose v_3 is not adjacent to $\{x, y, v_2\}$ and similarly v_1 is not adjacent to $\{y, z, v_2\}$. Since $\delta = 3$, v_1v_3, v_2x and v_2z are in $E(G)$ and $\{v_1, v_2, x\}$ is a TOCD set of order 3 of G. This contradicts $\gamma_{toc}(G) = 4$.

Case 2 X is an independent set of G.

Since $\delta = 3$, all the edges exist between $N(v)$ and X and $\{v, v_1, x\}$ is a TOCD set of order three of G. This contradicts $\gamma_{toc}(G) = 4$.

Case 3 $G[X]$ consists of a path xy and an isolated vertex z.

Since $\delta = 3$, z is adjacent to v_1, v_2, v_3 and each of x, y has at least two neighbors in $N(v)$. Hence at least one of v_1, v_2, v_3, say v_1 without loss of generality, is adjacent to x, y and z. Then $\{v, v_1\}$ is a TOCD set of order two of G. This contradicts $\gamma_{toc}(G) = 4$ and completes the proof. □

Theorem 17 *Let G be a graph of order n*, diam $(G) \leq 2$ *and* $\delta(G) \geq 3$. *Then*

$$\gamma_{toc}(G) \leq \left\lfloor \frac{2n-2}{3} \right\rfloor.$$

Furthermore, $\gamma_{toc}(G) = \frac{2n-2}{3}$ *if and only if* $G \in \mathcal{F}$.

Proof Let $v \in V(G)$ be a vertex of minimum degree δ, $N(v) = \{v_1, \ldots, v_\delta\}$ and let $X = V - N[v]$. If $X = \emptyset$, then $V(G) = N[v]$ and since $d(v) = \delta$ we deduce that G is a complete graph. Thus

$$\gamma_{toc}(G) = 2 \leq \left\lfloor \frac{2n-2}{3} \right\rfloor \quad \text{and} \quad \gamma_{toc}(G) = \frac{2n-2}{3} \quad \text{if and only if } G \simeq K_4 \in \mathcal{F}.$$

Now assume $X \neq \emptyset$. It follows that diam $(G) = 2$ and every vertex of X has at least one neighbor in $N(v)$. Furthermore since $d(v) = \delta$, every vertex v_i with no neighbor in X is adjacent to every vertex of $N(v) \setminus \{v_i\}$ and every vertex isolated in $G[X]$ is adjacent to every vertex of $N(v)$. Without loss of generality, we may assume v_1 is adjacent to some vertex in X. Then $\{v_1\} \cup X$ is a TOCD set of G and hence $\gamma_{toc}(G) \leq n - \delta$. If $\delta \geq \frac{n+3}{3}$, then

$$\gamma_{toc}(G) \leq n - \frac{n+3}{3} = \frac{2n-3}{3} < \frac{2n-2}{3}.$$

Thus we assume

$$3 \leq \delta \leq \frac{n+2}{3}. \tag{4}$$

This implies $n \geq 7$. If the induced subgraph $G[X]$ is connected, then the total dominating set $N[v]$ is a TOCD set of G and

$$\gamma_{toc}(G) \leq \delta + 1 \leq \frac{n+5}{3} \leq \frac{2n-2}{3}.$$

If $\gamma_{toc}(G) = \frac{2n-2}{3}$ then $n = 7$, $\delta = 3$, $\gamma_{toc}(G) = 4$ and $G \in \mathcal{F}$ by Proposition 16.

Let now $G_1, G_2, \ldots, G_k$ be the components of $G[X]$ with $k \geq 2$. Then for $1 \leq i \leq k$,

$$\left|V(G_i)\right| \leq n - \delta - 2 \leq n - 5. \tag{5}$$

Case 1 Some v_i, say v_1, has a private neighbor in X with respect to $N(v)$. Without loss of generality, let G_1 be the component of $G[X]$ containing a private neighbor x of v_1 and such that $|V(G_i) \setminus \mathrm{pn}(v_1, N(v))|$ is minimum, where $\mathrm{pn}(v_1, N(v))$ is the set of the $N(v)$-private neighbors of v_1. Since x is at distance at most two from any other vertex, v_1 is adjacent to each vertex in $\bigcup_{i=2}^{k} V(G_i)$ and for $2 \leq i \leq \delta$, each vertex v_i has a neighbor in $V(G_1) \cup \{v_1\}$. Hence the set $\{v_1\} \cup V(G_1)$ is a total dominating set of G.

Subcase 1.1 If $V(G_1) \setminus \mathrm{pn}(v_1, N(v)) \neq \emptyset$, i.e., each component G_i contains a vertex which is not a $(v_1, N(v))$-private neighbor, then $\{v_1\} \cup V(G_1)$ is a TOCD set of G and $\gamma_{toc}(G) \leq |V(G_1)| + 1$. If $n \leq 9$, then by (5),

$$\gamma_{toc}(G) \leq n - 4 < \frac{2n-2}{3}.$$

Let now $n \geq 10$ The set $V(G) \setminus V(G_1)$ is another TOCD set of G. Therefore

$$\gamma_{toc}(G) \leq \min\left\{\left|V(G_1)\right| + 1, n - \left|V(G_1)\right|\right\} \leq \frac{n+1}{2} < \frac{2n-2}{3}.$$

Subcase 1.2 If $V(G_1) \setminus \mathrm{pn}(v_1, N(v)) = \emptyset$, i.e., all the vertices of G_1 are private neighbors of v_1, then v_1 dominates G. Therefore any γ_t-set of $G - v_1$ is a TOCD set of G. Since $\delta(G - v_1) \geq 2$, each component of $G - v_1$ has order at least 3 and by Theorem A.1,

$$\gamma_{toc}(G) \leq \gamma_t(G - v_1) \leq \frac{2n-2}{3}.$$

If $\gamma_{toc}(G) = \frac{2n-2}{3}$, then $\gamma_t(G - v_1) = \frac{2n(G-v_1)}{3}$. By Theorem A.2 and since $\delta(G - v_1) \geq 2$, each component of $G - v_1$ is a C_3 or a C_6 and G is obtained by joining the vertex v_1 to each vertex of a disjoint union of cycles C_3 and C_6. By Proposition 15, and since $\gamma_{toc}(G) = \frac{2n-2}{3}$, $G \in \mathcal{F}$.

Case 2 For each i, the vertex v_i does not have any private neighbor in X with respect to $N(v)$. Hence each vertex in X has at least two neighbors in $N(v)$. Let T_1 and T_2 be respectively the sets of the vertices of the K_1- and K_2-components of $G[X]$ and let $T_3 = X - (T_1 \cup T_2)$. Every vertex of T_1 is adjacent to v_i for $1 \leq i \leq \delta$. If $T_3 \neq \emptyset$, let D be a $\gamma_t(G[T_3])$-set. By Theorem A, $|D| \leq \frac{2|T_3|}{3}$. We distinguish three cases.

Subcase 2.1 $T_1 \cup T_2 \neq \emptyset$ and if $T_2 \neq \emptyset$ then $N(v) \subseteq N(t)$ for every vertex t in T_2. Let $S = \{v, v_1\}$ if $T_3 = \emptyset$ and $S = D \cup \{v, v_1\}$ if $T_3 \neq \emptyset$. In both cases, S is a total dominating set since v_1 dominates $T_1 \cup T_2$, and $G - S$ is connected since every vertex of $T_3 \setminus D$ has a neighbor in $N(v) \setminus \{v_1\}$. Hence $\gamma_{toc}(G) \leq |S|$ and

$$\gamma_{toc}(G) \leq \frac{2|T_3|}{3} + 2 \leq \frac{2(n-\delta-2)}{3} + 2 = \frac{2n-2\delta+2}{3} \leq \frac{2n-4}{3} < \frac{2n-2}{3}.$$

Subcase 2.2 $T_2 \neq \emptyset$ and there exists a vertex t of T_2 such that $|N(t) \cap N(v)| < \delta$. Then $\deg(t) = \delta$ and without loss of generality, $N(t) \cap N(v) = N(v) \setminus \{v_1\}$. Let t' be the neighbor of t in X. If some neighbor of t has a private neighbor with respect to $N(t)$, then we are done by exchanging v and t and applying Case 1. Hence we suppose that for $2 \leq i \leq \delta$, v_i has no private neighbor with respect to $N(t)$. Every vertex x in $X \setminus \{t, t'\}$ has at least one neighbor v_i in $\{v_2, v_3, \ldots, v_\delta\}$. Moreover, since x is not a $N(t)$-private neighbor of v_i and is not adjacent to t', it has actually at least two neighbors in $\{v_2, v_3, \ldots, v_\delta\}$ and thus at least one in $\{v_3, \ldots, v_\delta\}$. Let $S = \{v, v_1, v_2\}$ if $T_3 = \emptyset$ and $S = D \cup \{v, v_1, v_2\}$ if $T_3 \neq \emptyset$. Every vertex of $T_1 \cup T_2$ has at least one neighbor in $\{v_1, v_2\}$. Therefore S is a total dominating set of G. Moreover $V(G) \setminus S$ is connected in G since t is adjacent to $v_3, v_4, \ldots, v_\delta, t'$ and every vertex of $X \setminus \{t, t'\}$ has a neighbor in $\{v_3, \ldots, v_\delta\}$. Hence S is a TOCD set of G and

$$\gamma_{toc}(G) \leq \frac{2|T_3|}{3} + 3 \leq \frac{2(n - \delta - 3)}{3} + 3 = \frac{2n - 2\delta + 3}{3} \leq \frac{2n - 3}{3} < \frac{2n - 2}{3}.$$

Subcase 2.3 $T_1 \cup T_2 = \emptyset$ and thus every component G_i of $G[X]$ has at least three vertices. Let without loss of generality $\{v_1, v_2, \ldots, v_r\}$ be the set of vertices of $N(v)$ with at least one neighbor in X. For $i = 1, 2$, we choose a vertex u_i of G_i such that u_i is a support vertex of G_i if $\delta(G_i) = 1$, u_i is any vertex of G_i otherwise. Since each of u_1, u_2 has at least two neighbors in $N(v)$, there exist two non-adjacent edges between $N(v)$ and $\{u_1, u_2\}$, say without loss of generality v_1u_1 and v_2u_2. When $r \geq 3$, we associate to each vertex v_i with $3 \leq i \leq r$ one of its neighbors u_i in X. Let H be the subgraph of G defined by $V(H) = X \cup \{v_1, \ldots, v_r\}$ and $E(H) = E(G[X]) \cup \{u_1v_1, \ldots, u_rv_r\}$. The graph H has order $n(H) = n - \delta - 1 + r \leq n - 1$. Let $H_1, \ldots, H_k$ be the components of H. Each of them consists of a component of $G[X]$ plus possibly some pendant edges and thus has at least three vertices. Moreover, by the choice of u_1 and u_2, the components H_1 and H_2 are not cycles, 2-coronas, nor 2-coronas minus a pendant edge. By Theorem A, $\gamma_t(H_i) \leq \frac{2n(H_i)-2}{3}$ for $1 \leq i \leq 2$. Let Y be a $\gamma_t(H)$-set not intersecting the set of leaves $\{v_1, \ldots, v_r\}$ and thus containing $\{u_1, \ldots, u_r\}$. It follows from Theorem A that

$$\begin{aligned}
|Y| &= \sum_{i=1}^{k} \gamma_t(H_i) \\
&\leq \gamma_t(H_1) + \gamma_t(H_2) + \sum_{i \geq 3} \frac{2n(H_i)}{3} \\
&\leq \frac{2(n(H_1) + n(H_2)) - 4}{3} + \sum_{i \geq 3} \frac{2n(H_i)}{3} \\
&= \frac{2n(H) - 4}{3} \\
&\leq \frac{2n - 6}{3}.
\end{aligned}$$

The set $S = Y \cup \{v_1\}$ is a total dominating set of G since every vertex v_i with $r + 1 \leq i \leq \delta$, if any, is adjacent to v_1. The set $V(G) \setminus S$ is connected in G since every vertex of $X \setminus Y$ has at least one neighbor in $N(v) \setminus \{v_1\}$. Therefore S is a TOCD set of G and $\gamma_{toc}(G) \leq |Y| + 1 \leq \frac{2n-3}{3} < \frac{2n-2}{3}$.

The proof of the direct part of the theorem is complete. Conversely, Proposition 15 shows that if $G \in \mathcal{F}$, then $\gamma_{toc}(G) = \frac{2n-2}{3}$. □

Remark 1 The Dutch-windmill graph, $K_3^{(m)}$, is a graph which consists of m copies of K_3 with a vertex in common. Since $\operatorname{diam}(K_3^{(m)}) = 2$ and $\gamma_{toc}(K_3^{(m)}) = n - 2$, the graph $K_3^{(m)}$ shows that the condition $\delta(G) \geq 3$ in Theorem 17 cannot be replaced by $\delta(G) \geq 2$.

4 Complexity issues

We consider the following decision problem.

TOTAL OUTER-CONNECTED DOMINATING SET (TOCDS)
INSTANCE: A graph $G = (V, E)$ and a positive integer $k \leq |V|$.
QUESTION: Does G have a total outer-connected dominating set of cardinality at most k?

Cyman (2007) showed that the similar decision problem for outer-connected dominating set (OCDS) was NP-complete in bipartite graphs by making a reduction from the problem Exact cover by 3-sets. The reader can check in Cyman (2007) that the construction of her proof shows the same result if we replace outer-connected dominating sets by total outer-connected dominating sets. Hence we can state without any new proof:

Theorem 18 *TOCDS is NP-complete for bipartite graphs.*

Note that the same proof also shows the NP-completeness in bipartite graphs of the decision problem related to the doubly connected dominating number (the dominating set S and $V \setminus S$ are connected). This result was already established by Cyman et al. (2006) with another construction.

Acknowledgement This research was in part supported by a grant from IPM (No.90050043)

References

Brigham RC, Carrington JR, Vitray RP (2000) Connected graphs with maximal total domination number. J Comb Math Comb Comput 34:81–95

Cockayne EJ, Dawes RM, Hedetniemi ST (1980) Total domination in graphs. Networks 10:211–219

Cyman J (2007) The outer-connected domination number of a graph. Australas J Comb 38:35–46

Cyman J (2010) Total outer-connected domination numbers in trees. Discuss Math, Graph Theory 30:377–383

Cyman J, Lemańska M, Raczek J (2006) On the doubly connected domination number of a graph. Cent Eur J Math 4:34–45

Cyman J, Raczek J (2009) Total outer-connected domination numbers of trees. Discrete Appl Math 157:3198–3202

Hattingh JH, Joubert EJ (2010) A note on total outer-connected domination numbers of a tree. AKCE J Graphs Comb 7:223–227