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Critical edges/nodes for the minimum spanning tree problem: complexity and approximation

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Abstract In this paper, we study the complexity and the approximation of the k most vital edges (nodes) and min edge (node) blocker versions for the minimum spanning tree problem (MST). We show that the k most vital edges MST problem is *NP*-hard even for complete graphs with weights 0 or 1 and 3-approximable for graphs with weights 0 or 1. We also prove that the k most vital nodes MST problem is not approximable within a factor $n^{1-\epsilon}$, for any $\epsilon > 0$, unless $NP = ZPP$, even for complete graphs of order n with weights 0 or 1. Furthermore, we show that the min edge blocker MST problem is *NP*-hard even for complete graphs with weights 0 or 1 and that the min node blocker MST problem is *NP*-hard to approximate within a factor 1.36 even for graphs with weights 0 or 1.

Keywords Most vital edges/nodes · Min edge/node blocker · Minimum spanning tree · Complexity · Approximation

1 Introduction

For problems of security or reliability, it is important to assess the capacity of a system to resist to a destruction or a failure of a number of its entities. This amounts

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to identifying critical entities which can be determined with respect to a measure of performance or a cost associated to the system. Modeling the network as a weighted connected graph where entities are edges or nodes and costs are weights associated to edges, one way of identifying critical entities is to determine a subset of edges or nodes whose removal from the graph causes the largest cost increase. Another way is to find a subset of edges or nodes of minimum cardinality whose removal involves that the optimal cost in the residual network is larger than a given threshold. In the literature these problems are referred to respectively as the *k most vital edges/nodes* problem and *min edge/node blocker* problem. In this paper the k most vital edges/nodes and min edge/node blocker versions for the minimum spanning tree problem are investigated.

The problem of finding the k most vital edges of a graph has been studied for various problems including shortest path (Bar-Noy et al. 1995; Khachiyan et al. 2008; Nardelli et al. 2001), maximum flow (Wollmer 1964; Ratliff et al. 1975; Wood 1993), 1-median and 1-center (Bazgan et al. 2010). For the minimum spanning tree problem, Frederickson and Solis-Oba (1996) showed that k MOST VITAL EDGES MST is *NP*-hard and proposed an $O(\log k)$-approximation algorithm. For a fixed k, the problem is obviously polynomial. The case $k = 1$ has been largely studied in the literature (Hsu et al. 1991; Iwano and Katoh 1993; Suraweera et al. 1995). Several exact algorithms based on an explicit enumeration of possible solutions have been proposed (Liang 2001; Liang and Shen 1997; Shen 1999; Bazgan et al. 2011).

After introducing some preliminaries in Sect. 2, we show in Sect. 3 that k MOST VITAL EDGES MST is *NP*-hard even for complete graphs with weights 0 or 1 and 3-approximable for graphs with weights 0 or 1. We also prove, in Sect. 4, that k MOST VITAL NODES MST is not approximable within a factor $n^{1-\epsilon}$, for any $\epsilon > 0$, unless *NP* = *ZPP*, even for complete graphs of order n with weights 0 or 1. In Sect. 5, we establish that MIN EDGE BLOCKER MST is *NP*-hard even for complete graphs with weights 0 or 1. In Sect. 6, we show that MIN NODE BLOCKER MST is *NP*-hard to approximate within a factor 1.36 even for graphs with weights 0 or 1. Final remarks are provided in Sect. 7.

2 Basic concepts and preliminary results

Let $G = (V, E)$ be a weighted undirected connected graph where $|V| = n$, $|E| = m$ and $w(e) \geq 0$ is the integer weight of each edge $e \in E$. Denote by $G - R$ the graph obtained from G by removing the subset R of edges or nodes.

We consider in this paper the k most vital edges (nodes) and min edge (node) blocker versions of the minimum spanning tree problem. These problems are defined as follows:

k MOST VITAL EDGES (resp. NODE) MST

Input: A connected weighted graph $G = (V, E)$ where each edge $e \in E$ has an integer weight $w_e \geq 0$ and a positive integer k.

Output: A subset $S^* \subseteq E$ (resp. $S^* \subseteq V$), with $|S^*| = k$, such that the weight of a minimum spanning tree in $G - S^*$ is maximum.

For an instance of k MOST VITAL EDGES MST defined on a graph G, we consider that $k \leq \lambda(G) - 1$ where $\lambda(G)$ is the edge-connectivity of G. Otherwise, any selection of k edges including the edges of a minimum cardinality cut would lead to a solution with infinite value since we disconnect G.

For an instance of k MOST VITAL NODES MST defined on a graph G, we consider that $k \leq \kappa(G) - 1$, where $\kappa(G)$ is the node-connectivity of G. Otherwise, any selection of k nodes including the nodes of a minimum node separator would lead to a solution with infinite value since we disconnect G.

MIN EDGE (resp. NODE) BLOCKER MST

Input: A connected weighted graph $G = (V, E)$ where each edge $e \in E$ has an integer weight $w_e \geq 0$ and a positive integer U.
Output: A subset $S^* \subseteq E$ (resp. $S^* \subseteq V$) of minimum cardinality such that the weight of a minimum spanning tree in $G - S^*$ is greater than or equal to U.

An optimal solution S^* of an instance of MIN EDGE (resp. NODE) BLOCKER MST defined on a graph G is such that $|S^*| \leq \lambda(G)$ (resp. $|S^*| \leq \kappa(G)$) since, at worst, it is necessary to disconnect G so as to exceed the threshold U.

Given an optimization problem in NPO and an instance I of this problem, we use $|I|$ to denote the size of I, $opt(I)$ to denote the optimum value of I, and $val(I, S)$ to denote the value of a feasible solution S of instance I. The *performance ratio* of S (or *approximation factor*) is $r(I, S) = \max\{\frac{val(I,S)}{opt(I)}, \frac{opt(I)}{val(I,S)}\}$. The *error* of S, $\varepsilon(I, S)$, is defined by $\varepsilon(I, S) = r(I, S) - 1$.

For a function f, an algorithm is an $f(|I|)$-*approximation*, if for every instance I of the problem, it returns a solution S such that $r(I, S) \leq f(|I|)$.

The notion of a *gap*-reduction was introduced in Arora and Lund (1996). A maximization problem Π is called *gap-reducible* to a maximization problem Π' with parameters (c, ρ) and (c', ρ'), $\rho, \rho' \geq 1$, if there exists a polynomial time computable function f which maps any instance I of Π to an instance I' of Π', while satisfying the following properties.

- If $opt(I) \geq c$ then $opt(I') \geq c'$
- If $opt(I) < \frac{c}{\rho}$ then $opt(I') < \frac{c'}{\rho'}$.

The interest of a *gap*-reduction is that if Π is not approximable within a factor ρ then Π' is not approximable within a factor ρ'.

The notion of an E-reduction (*error-preserving* reduction) was introduced by Khanna et al. (1994). A problem Π is called E-*reducible* to a problem Π', if there exist polynomial time computable functions f, g and a constant β such that

- f maps an instance I of Π to an instance I' of Π' such that $opt(I)$ and $opt(I')$ are related by a polynomial factor, i.e. there exists a polynomial p such that $opt(I') \leq p(|I|)opt(I)$,
- g maps any solution S' of I' to one solution S of I such that $\varepsilon(I, S) \leq \beta\varepsilon(I', S')$.

An important property of an E-reduction is that it can be applied uniformly to all levels of approximability; that is, if Π is E-reducible to Π' and Π' belongs to $\mathcal{C}$ then Π belongs to $\mathcal{C}$ as well, where $\mathcal{C}$ is a class of optimization problems with any kind of approximation guarantee (see also Khanna et al. 1994).

A problem Π is called *E-equivalent* to a problem Π' if Π is *E-reducible* to Π' and Π' is *E-reducible* to Π.

3 *k* most vital edges MST

Frederickson and Solis-Oba (1996) show that k Most Vital Edges MST is *NP*-hard even for graphs with weights 0 or 1 and that the problem is $O(\log k)$-approximable for graphs with arbitrary weights. In this section, we strengthen the *NP*-hardness result of Frederickson and Solis-Oba by specifying a more restricted class of instances for which the problem remains *NP*-hard. Moreover, we establish a constant approximation result for graphs with weights 0 or 1.

First we show that we can decide in polynomial time if the optimum value is a fixed constant.

Proposition 1 *For any fixed value $c \geq 0$, it can be checked in polynomial time if the optimum value of k* Most Vital Edges MST *on graphs with weights* 0 *or* 1 *on edges is c.*

Proof Consider an instance I of k Most Vital Edges MST formed by a weighted graph $G = (V, E)$, with weights 0 or 1, and by a positive integer k. Denote by $G_0 = (V, E_0)$ the subgraph induced by the edges of weight 0. Let $E_1 = E \setminus E_0$ and $m_1 = |E_1|$.

We have that $opt(I) = 0$ if and only if G_0 is $(k + 1)$ edge-connected. Indeed, if $opt(I) = 0$ then G_0 must be $(k + 1)$ edge-connected otherwise $opt(I) > 0$. Conversely, if G_0 is $(k + 1)$ edge-connected, then removing any subset of k edges from G_0 induces a minimum spanning tree of weight 0. Consequently, it is polynomial to verify if $opt(I) = 0$ since it is polynomial to determine the edge-connectivity of a given graph. Once we checked iteratively that $opt(I) \neq \ell$, for $0 \leq \ell \leq c - 1$, we consider all the $\binom{m_1}{c}$ graphs $G_0 \cup R$, for any subset $R \subseteq E_1$ with $|R| = c$. We can decide in polynomial time if $opt(I) = c$ by verifying if $G_0 \cup R$ is $(k + 1)$ edge-connected. □

We show in the following that k Most Vital Edges MST is E-equivalent to Max Component defined as follows.

Max Component

Input: a connected graph and a positive integer k.
Output: a subset of k edges to be removed such that the number of connected components in the obtained graph is maximum.

Theorem 1 *k* Most Vital Edges MST *for graphs with weights* 0 *or* 1 *is E-equivalent to* Max Component.

Proof We first show that Max component is E-reducible to k Most Vital Edges MST. Given an instance I of Max component formed by a graph $G = (V, E)$ with n nodes, we construct an instance I' of k Most Vital Edges MST consisting of a complete graph $G' = (V, E')$ where each edge $(i, j) \in E'$ is assigned a weight 0 if $(i, j) \in E$ and 1 otherwise.

Let $S^* \subseteq E$ be a subset of k edges whose deletion from G generates a maximum number of connected components. By removing S^* from G', all the connected components of $G - S^*$ are linked in $G' - S^*$ by edges of weight 1. Thus, the weight of a minimum spanning tree in $G' - S^*$ is equal to the number of connected components in $G - S^*$ minus 1. Therefore, we have $opt(I') \geq opt(I) - 1$.

Let $S' \subseteq E'$ be a subset of k edges whose deletion from G' generates a minimum spanning tree in $G' - S'$ of weight v. If S' contains edges of weight 1 then by replacing these edges by edges of weight 0, either the weight of a minimum spanning tree in the modified graph remains unchanged or it increases. Thus, considering S defined from S' by replacing edges of weight 1 with edges from $E' \setminus S'$ of weight 0, se define a subset $S \subseteq E$ such that $G - S$ contains at least $v + 1$ connected components. Hence, $val(I, S) \geq val(I', S') + 1$. In particular, when S is an optimum solution, we have $opt(I') + 1 \leq val(I, S) \leq opt(I)$. It follows from the previous result that $opt(I) = opt(I') + 1$.

Therefore, we have $opt(I') \leq opt(I)$ and $\varepsilon(I, S) = \frac{opt(I)}{val(I,S)} - 1 \leq \frac{opt(I')+1}{val(I',S')+1} - 1 = \frac{opt(I')-val(I',S')}{val(I',S')+1} \leq \frac{opt(I')-val(I',S')}{val(I',S')} = \varepsilon(I', S')$.

We show now that k MOST VITAL EDGES MST is E-reducible to MAX COMPONENT. Consider an instance I of k MOST VITAL EDGES MST formed by a graph $G = (V, E)$ with edges of weight 0 or 1. From Proposition 1, we can consider that $opt(I) > 0$. We construct an instance I' of MAX COMPONENT consisting of the graph $G' = (V, E')$ obtained from G by considering only edges of weight 0.

Let S^* be a subset of k edges whose removal from G generates a minimum spanning tree T in $G - S^*$ of maximum weight. The weight of T being equal to the number of edges of T of weight 1, by deleting edges of $S^* \cap E'$ plus any $k - |S^* \cap E'|$ edges from E', the number of connected components in $G' - S^*$ is at least equal to the weight of T plus 1. Thus, we have $opt(I') \geq opt(I) + 1$.

Consider a subset S' of k edges whose deletion from G' partitions G' into $val(I', S')$ connected components. If $val(I', S') = 1$ then we can replace S' by another solution with value at least 2 obtained by selecting k edges including a minimum cut since from Proposition 1, G' is not $(k + 1)$ edge-connected. Thus, we can assume that $val(I', S') \geq 2$. By removing S' from G, all connected components of $G' - S'$ are linked in $G - S'$ by edges of weight 1. Thus, the weight of a minimum spanning tree in $G - S'$ is equal to $val(I', S') - 1$. Then, $val(I, S') \geq val(I', S') - 1$. In particular, when S' is an optimum solution in G', we have $val(I, S') = opt(I') - 1$ and thus $opt(I) \geq opt(I') - 1$. It follows from the previous result that $opt(I') = opt(I) + 1$.

Therefore, since $opt(I) > 0$, we have $opt(I') \leq 2opt(I)$ and $\varepsilon(I, S') = \frac{opt(I)}{val(I,S')} - 1 \leq \frac{opt(I')-1}{val(I',S')-1} - 1 = \frac{opt(I')-val(I',S')}{val(I',S')-1} = \frac{val(I',S')}{val(I',S')-1} \frac{opt(I')-val(I',S')}{val(I',S')} \leq 2\frac{opt(I')-val(I',S')}{val(I',S')} = 2\varepsilon(I', S')$. □

From Theorem 1, we obtain the two following results. First, we slightly strengthen the *NP*-hardness result of Frederickson and Solis-Oba (1996) by specifying a more restricted class of instances for which the problem remains *NP*-hard.

Corollary 1 *k* MOST VITAL EDGES MST *is NP-hard even for complete graphs with weights* 0 *or* 1.

Proof The E-reduction from MAX COMPONENT to k MOST VITAL EDGES MST constructs from any graph G a *complete* graph G' with weights 0 or 1. Since MAX COMPONENT is *NP*-hard (Frederickson and Solis-Oba 1996), the results follows. □

Second, we establish a constant approximation result for graphs with weights 0 or 1.

Corollary 2 *k* MOST VITAL EDGES MST *is* 3*-approximable for graphs with weights* 0 *or* 1.

Proof In the E-reduction from k MOST VITAL EDGES MST to MAX COMPONENT, we have shown that any solution S of I' is such that $\varepsilon(I,S) \leq 2\varepsilon(I',S)$. Thus, $r(I,S) - 1 \leq 2(r(I',S) - 1)$ and then $r(I,S) \leq 2r(I',S) - 1$. Since $r(I',S) = 2$ as established in Frederickson and Solis-Oba (1996), we have $r(I,S) \leq 3$. □

4 *k* most vital nodes MST

We study in this section the complexity of k MOST VITAL NODES MST. First we show that k MOST VITAL NODES MST is at least as hard as k MOST VITAL EDGES MST by establishing an E-reduction from the edge version to the node version. As far as we know, this is the first result in the literature that establishes a direct relationship between the k most vital edge version and the k most vital node version of a problem. Using the *NP*-hardness of the edge version even for graphs with weights 0 or 1 (Frederickson and Solis-Oba 1996), this reduction implies the *NP*-hardness of k MOST VITAL NODES MST on the same class of graphs. We strengthen this result by proving that k MOST VITAL NODES MST is not approximable within a factor $n^{1-\epsilon}$, for any $\epsilon > 0$, if $NP \neq ZPP$, even for complete graphs with weights 0 or 1.

Theorem 2 *k* MOST VITAL EDGES MST *is E-reducible to k* MOST VITAL NODES MST.

Proof Consider an instance I of k MOST VITAL EDGES MST formed by a weighted graph $G = (V, E)$ with $V = \{v_1, \dots, v_n\}$ and $|E| = m$. We construct an instance I' of k MOST VITAL NODES MST formed by a graph $G' = (V', E')$ as follows (see Fig. 1). We consider in G' the nodes of V and m nodes $r_1, \dots, r_m$. Let $R = \{r_1, \dots, r_m\}$. To each edge $e_\ell = (v_i, v_j) \in E$ of weight w_{ij}, $\ell = 1, \dots, m$ and $i < j$, we associate two edges in E' : (v_i, r_ℓ) of weight w_{ij} and (r_ℓ, v_j) of weight 0. Let $K_k^{v_i}$, for $i = 1, \dots, n$, be n complete graphs of size k with $X_{v_i} = \{v_i^1, \dots, v_i^k\}$ and weights 0 on their edges. We connect each node v_i, for $i = 1, \dots, n$, to the k nodes of $K_k^{v_i}$ and assign a weight 0 to these added edges. We also add, for each edge $(v_i, r_\ell) \in E'$ the edges (v_i^h, r_ℓ), for $h = 1, \dots, k$, with the same weight as the weight of the edge (v_i, r_ℓ).

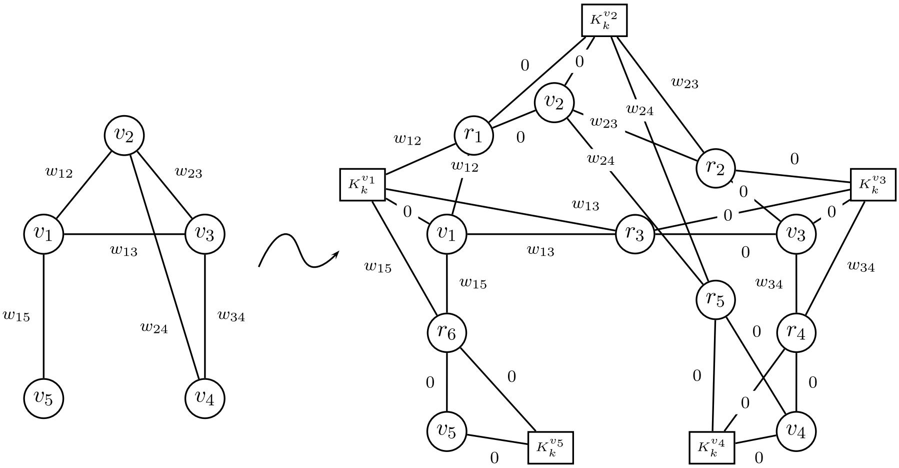

Fig. 1 Construction of an instance of k MOST VITAL NODES MST from an instance of k MOST VITAL EDGES MST

Suppose first that there exists a subset $S^* \subseteq E$, with $|S^*| = k$, such that a minimum spanning tree T in $G - S^*$ has a maximum weight. We set $N^* = \{r_\ell : e_\ell \in S^*\}$. By deleting N^* from G', we construct a spanning tree T' in $G' - N^*$ as follows: we take for each edge $e_\ell = (v_i, v_j) \in T$ with $i < j$, the edges (v_i, r_ℓ) and (r_ℓ, v_j) in T', for each edge $e_h = (v_i, v_j) \notin T$ with $i < j$, the edge (r_h, v_j) in T', and we add the paths $v_i, v_i^1, \ldots, v_i^k$, $i = 1, \ldots, n$. We prove, by contradiction, that T' is a minimum spanning tree in $G' - N^*$. Suppose that there exists a spanning tree T'' in $G' - N^*$ of weight strictly inferior to that of T'. Then, the spanning tree constituted by the edges $e_\ell = (v_i, v_j)$ such that $(v_i, r_\ell) \in T''$ has a smaller weight than T in $G - S^*$, contradicting the optimality of T. Thus, T' is a minimum spanning tree in $G' - N^*$. Therefore, we have $opt(I') \geq opt(I)$.

Consider now a subset N, with $|N| = k$, and a minimum spanning tree T' in $G' - N$. If N contains v_i or one node v_i^h, for a given i and h, then the weight of a MST in $G' - N$ is the same as in $G' - (N \setminus \{v_i\})$ or $G' - (N \setminus \{v_i^h\})$. When removing all nodes v_i, v_i^h from N we obtain a subset $N' \subseteq R$, $|N'| \leq k$. Since N' corresponds to edges in G, any subset $N'' \subseteq R$ containing N' such that $|N''| = k$ is such that the weight of a MST in $G' - N''$ is at least as large as the weight of a MST in $G' - N'$. Let $S = \{e_\ell : r_\ell \in N''\}$. Consider T the spanning tree in $G - S$ constituted by the edges $e_\ell = (v_i, v_j)$ such that the edge $(v_i, r_\ell) \in T'$. T is optimal, since otherwise, the existence of a spanning tree T'' of weight strictly inferior to that of T would imply that the corresponding spanning tree constructed from T'' in $G' - N''$, as explained above, has a weight strictly inferior to that of T'. Thus, T is a minimum spanning tree in $G - S$ of the same weight as T'. Hence, $val(I, S) = val(I', N'')$. In particular, when N'' is an optimal solution in G', we have $opt(I') = val(I, S) \leq opt(I)$. It follows from the previous result that $opt(I) = opt(I')$. Therefore, we have $\varepsilon(I, S) = \varepsilon(I', N'')$. $\square$

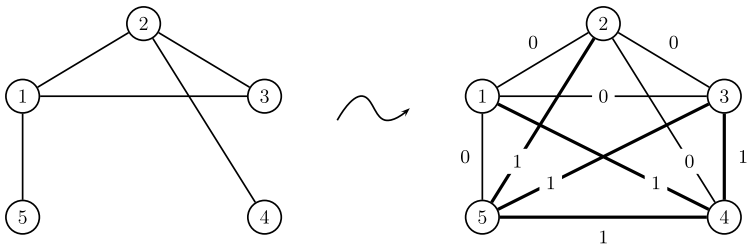

Fig. 2 Construction of an instance of k MOST VITAL NODES MST from an instance of MAX INDEPENDENT SET

Theorem 3 k MOST VITAL NODES MST *is not approximable within a factor* $n^{1-\epsilon}$, *for any* $\epsilon > 0$, *unless NP* = *ZPP*, *even for complete graphs of order n with weights* 0 *or* 1.

Proof We propose a *gap*-reduction from MAX INDEPENDENT SET to k MOST VITAL NODES MST.

Denote by $\alpha(G)$ the cardinality of maximum independent set of G. Let g be the non approximation gap of MAX INDEPENDENT SET. Thus, for a given integer ℓ, it is *NP*-hard to decide if $\alpha(G) = \ell$ or $\alpha(G) < \frac{\ell}{g}$.

Given an instance I of MAX INDEPENDENT SET formed by a graph $G = (V, E)$, we construct an instance I' of k MOST VITAL NODES MST constituted by a complete graph $G' = (V, E')$ where each edge $(i, j) \in E'$ is assigned a weight 0 if $(i, j) \in E$ and 1 otherwise (see Fig. 2). We set $k = n - \ell$. We show that:

1. $\alpha(G) = \ell \Rightarrow opt(I') \geq \ell - 1$
2. $\alpha(G) < \frac{\ell}{g} \Rightarrow opt(I') < \frac{\ell-1}{g}$.

1. Suppose first that there exists an independent set V^* in G of cardinality ℓ and let $N^* = V \backslash V^*$. By removing N^* from G', all nodes of $G' - N^*$ are connected by edges of weight 1 only. Thus, we obtain a minimum spanning tree in $G' - N^*$ of value $\ell - 1$. Therefore, $opt(I') \geq \ell - 1$.

2. Suppose now that $\alpha(G) < \frac{\ell}{g}$. Hence, there exists a maximum independent set V^* such that $|V^*| < \frac{\ell}{g}$. If the node set N^* of cardinality $n - \ell$ to be removed from G' is such that $N^* \cap V^* = \emptyset$ then let $V_1 = V \backslash (N^* \cup V^*)$. Each node of V_1 is at least connected to one node of V^* by an edge of weight 0, otherwise $V^* \cup \{v\}$ would be an independent set in G of larger cardinality. Thus, the weight of a minimum spanning tree in $G' - N^*$ cannot exceed $\frac{\ell}{g} - 1$. Since $g > 1$, we have $\frac{\ell}{g} - 1 < \frac{\ell-1}{g}$. Therefore if $\alpha(G) < \frac{\ell}{g}$ then $opt(I') < \frac{\ell-1}{g}$. If $N^* \cap V^* \neq \emptyset$ then a minimum spanning tree in $G' - N^*$ would have a weight strictly inferior to $\frac{\ell}{g} - 1$.

Since MAX INDEPENDENT SET is not approximable within a factor $n^{1-\epsilon}$, for any $\epsilon > 0$, unless $NP = ZPP$ (Håstad 1999), we deduce that k MOST VITAL NODES MST is also not $n^{1-\epsilon}$-approximable, for any $\epsilon > 0$, unless $NP = ZPP$. □

From Theorem 3 and Corollary 2, we can give the following result.

Corollary 3 *There is no E-reduction from k* MOST VITAL NODES MST *for graphs with weights* 0 *or* 1 *to k* MOST VITAL EDGES MST *for graphs with weights* 0 *or* 1.

5 Min edge blocker MST

We present in the following a relationship between k MOST VITAL EDGES MST and MIN EDGE BLOCKER MST.

Proposition 2 *k* MOST VITAL EDGES MST *and* MIN EDGE BLOCKER MST *are polynomial-time equivalent.*

Proof If an algorithm $\mathcal{A}_k$ solves k MOST VITAL EDGES MST defined on graph G for all $1 \leq k \leq \lambda(G) - 1$, then we can run $\mathcal{A}_k$ for $k = 1, \ldots, \lambda(G) - 1$ and choose the smallest k yielding optimum at least U. If no k exists then the optimum for MIN EDGE BLOCKER MST is $\lambda(G)$. Conversely, if an algorithm $\mathcal{B}_U$ solves MIN EDGE BLOCKER MST with any bound U, we can apply binary search to locate the largest U that requires the removal of at most k nodes. □

Theorem 4 MIN EDGE BLOCKER MST *is NP-hard even for complete graphs with weights* 0 *or* 1.

Proof Follows from Proposition 2 and Corollary 1. □

6 Min node blocker MST

The equivalent of Proposition 2 applied to nodes also holds (with a similar proof).

Proposition 3 *k* MOST VITAL NODES MST *and* MIN NODE BLOCKER MST *are polynomial-time equivalent.*

Theorem 5 MIN NODE BLOCKER MST *is NP-hard even for complete graphs with weights* 0 *or* 1.

Proof Follows from Proposition 3 and Theorem 3. □

This result could also be established by the following *gap*-reduction from MIN EDGE BLOCKER MST.

Theorem 6 MIN EDGE BLOCKER MST *is gap-reducible to* MIN NODE BLOCKER MST.

Proof Consider an instance I for MIN EDGE BLOCKER MST formed by a graph $G = (V, E)$, with $|V| = n$ and $|E| = m$, and a positive integer U. We construct an instance I' for MIN NODE BLOCKER MST, constituted by a graph $G' = (V', E')$ and a positive integer U, using the same construction as in Theorem 2, but we modify the size of the n complete graphs which we set to be $m + 1$. We show that

1. $opt(I) \leq c \Rightarrow opt(I') \leq c$
2. $opt(I) > c\rho \Rightarrow opt(I') > c\rho$.

1. Let $S^* \subseteq E$ be a subset of minimum cardinality such that a minimum spanning tree T in $G - S^*$ has a weight at least U. We set $N^* = \{r_\ell : e_\ell \in S^*\}$. By deleting N^* from G', we construct a minimum spanning tree T' in $G' - N^*$ of the same weight as that of T as explained in Theorem 2. Thus, the weight of T' is at least U. Therefore, $opt(I') \leq opt(I) \leq c$.

2. Suppose now that $opt(I) > c\rho$. When we remove all nodes of R from G', the weight of a minimum spanning tree is infinite. Hence, $opt(I') \leq m$. Let $N \subseteq V'$ be an optimal solution whose deletion generates a minimum spanning tree T' in $G' - N$ of weight at least U. If N contains v_i or one node v_i^h, for a given i and h, then N must contain all the $m + 1$ nodes v_i and X_{v_i}, since otherwise the weight of a minimum spanning in $G' - N$ is the same as in $G' - (N \setminus \{v_i\})$ or $G' - (N \setminus \{v_i^h\})$. Therefore, since $opt(I') \leq m$, we can consider that $N \subseteq R$. Let $S = \{e_\ell : r_\ell \in N\}$. We construct a minimum spanning tree T in $G - S$ as explained in Theorem 2. The weight of T being equal to the weight of T' is at least U. Hence, $opt(I) \leq val(I, S) = val(I', N) = opt(I')$ and thus $opt(I') > c\rho$. □

In the absence of known inapproximability results for MIN EDGE BLOCKER MST, we can only exploit the above *gap*-reduction to establish the *NP*-hardness of MIN NODE BLOCKER MST. Nevertheless, we can obtain the following stronger result.

Theorem 7 MIN NODE BLOCKER MST *is NP-hard to approximate within a factor* 1.36 *even for graphs with weights* 0 *or* 1.

Proof We propose a *gap*-reduction from MIN VERTEX COVER. Consider an instance I of MIN VERTEX COVER formed by a graph $G = (V, E)$ with $V = \{v_1, \ldots, v_n\}$. We construct from I, an instance I' of MIN NODE BLOCKER MST constituted by a graph $G' = (V', E')$ and a positive integer U as follows (see Fig. 3). G' is a copy of G to which we add a path $x_1, x_2, \ldots, x_n$ with $X = \{x_1, \ldots, x_n\}$ and we connect each node x_i to the nodes $x_i^1, \ldots, x_i^n$ of a complete graph K_n^i of size n. We also connect each node x_i^r to node x_{i+1} and each node x_i to node x_{i+1}^r for $i = 1, \ldots, n - 1$ and $r = 1, \ldots, n$. We connect each node v_i to nodes x_i and x_i^r, for $i = 1, \ldots, n$ and $r = 1, \ldots, n$. We associate a weight 1 to all edges of the path $(x_1, x_2), (x_2, x_3), \ldots, (x_{n-1}, x_n)$ and to edges (x_i^r, x_{i+1}) and (x_i, x_{i+1}^r) for $i = 1, \ldots, n - 1$ and $r = 1, \ldots, n$, and a weight 0 to all other edges in E'. We set $U = n - 1$.

We show that

1. $opt(I) \leq c \Rightarrow opt(I') \leq c$
2. $opt(I) > c\rho \Rightarrow opt(I') > c\rho$

which establishes that MIN NODE BLOCKER MST is *NP*-hard to approximate within a factor 1.36, since MIN VERTEX COVER is *NP*-hard to approximate within a factor 1.36 (Dinur and Safra 2005).

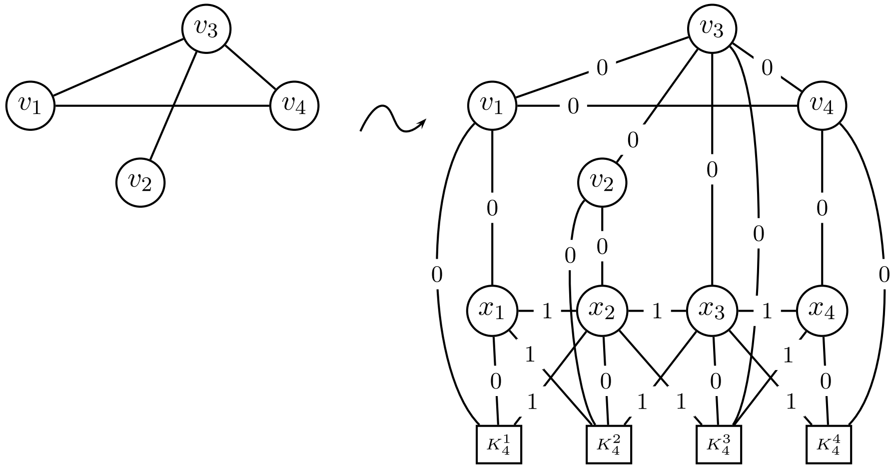

Fig. 3 Construction of an instance of MIN NODE BLOCKER MST from an instance of MIN VERTEX COVER

1. Let $V^* \subseteq V$ be a minimum vertex cover in G. By deleting the nodes of V^* from G', the nodes of $V \backslash V^*$ form an independent set in $G' - V^*$. Then, connecting any two nodes x_i, x_j in $G' - V^*$ requires to use a path of weight at least 1. Thus, a minimum spanning tree in $G' - V^*$, of weight $U = n - 1$, is obtained by connecting the nodes x_i through the path $x_1, x_2, \dots, x_n$ and each node $v_i \in V \backslash V^*$ and x_i^r to node x_i, for $i = 1, \dots, n$ and $r = 1, \dots, n$. Therefore, we get $opt(I') \leq opt(I) \leq c$.

2. Suppose now that $opt(I) > c\rho$. When we remove all nodes v_i, $i = 1, \dots, n$ from G', the weight of a minimum spanning tree in the resulting graph is U. Hence, $opt(I') \leq n$. Let $N \subseteq V'$ be an optimal solution. If N contains nodes x_i or x_i^ℓ for a given i and ℓ, then N must contain all the nodes x_i and x_i^r for $r = 1, \dots, n$, otherwise the weight of a minimum spanning tree in $G' - N$ is the same as in $G' - (N \backslash \{x_i\})$ or $G' - (N \backslash \{x_i^\ell\})$. Therefore, since $opt(I') \leq n$, we can consider in the following that N is included in V. We show in the following that N is a vertex cover in G. Suppose that there exists an edge $(v_i, v_j) \in E$ such that $v_i \notin N$ and $v_j \notin N$. By deleting N from G', the weight of a minimum spanning tree in $G' - N$ is at most equal to $n - 2$. Indeed, in such a minimum spanning tree the nodes x_i, v_i, v_j, x_j are not connected by the edges (v_i, x_i), (x_j, v_j) and the path on X from x_i to x_j but by the path (x_i, v_i), (v_i, v_j), (v_j, x_j) of weight 0, thus contradicting the fact that the weight of a minimum spanning tree in $G' - N$ must be at least $n - 1$. Thus, N is a vertex cover in G and $opt(I) \leq val(I, N) = val(I', N) = opt(I')$ and then $opt(I') > c\rho$. □

7 Conclusions

As a first result, we established or strengthened the *NP*-hardness of the four studied problems. Regarding approximation, negative results were obtained only for the node related versions and positive results were obtained only for k MOST VITAL EDGES MST. This situation, combined with our reductions from edge related versions to node related versions (see Theorems 2 and 6, and Corollary 3) clearly shows that node related versions are more difficult than edge related versions. An interesting perspective is to look for approximability results for k MOST VITAL NODES MST and MIN EDGE (NODE) BLOCKER MST and for inapproximability results for edge related versions.

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