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Embeddings of circulant networks

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Abstract In this paper we solve the edge isoperimetric problem for circulant networks and consider the problem of embedding circulant networks into various graphs such as arbitrary trees, cycles, certain multicyclic graphs and ladders to yield the minimum wirelength.

Keywords Circulant networks · Multicyclic graphs · Embedding · Congestion · Wirelength

1 Introduction

Graph embedding has been known as a powerful tool for implementation of parallel algorithms or simulation of different interconnection networks. A parallel algorithm can be modeled by a task interaction graph, where nodes and edges represent tasks and direct communications between tasks, respectively. Thus, the problem of efficiently executing a parallel algorithm A on a parallel computer M can be often reduced to the problem of mapping the graph G, representing A, on the graph H, representing M, so that the communication overhead is minimized. This is called graph embedding (Opatrny and Sotteau 2000), which is defined more precisely as follows:

Let $G(V,E)$ and $H(V,E)$ be finite graphs with n vertices. An embedding f of G into H is defined (Bezrukov et al. 1998) as follows:

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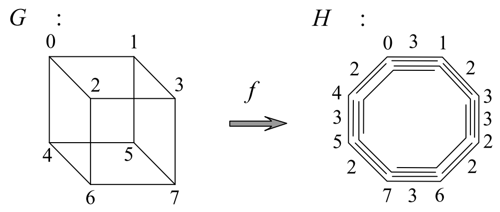
Fig. 1 Wiring diagram of a hypercube G into a cycle H with $WL_f(G, H) = 20$. The edge congestions are marked on the edges of H



1. f is a bijective map from $V(G) \rightarrow V(H)$
2. f is a one-to-one map from $E(G)$ to $\{P_f(f(u), f(v)) : P_f(f(u), f(v))$ is a path in H between $f(u)$ and $f(v)$ for $(u, v) \in E(G)\}$.

Here G is called the guest graph and H, the host graph. The *edge congestion* of an embedding f of G into H is the maximum number of edges of the graph G that are embedded on any single edge of H. Let $EC_f(G, H(e))$ denote the number of edges (u, v) of G such that e is in the path $P_f(f(u), f(v))$ between $f(u)$ and $f(v)$ in H. In other words,

$$EC_f(G, H(e)) = |\{(u, v) \in E(G) : e \in P_f(f(u), f(v))\}|$$

where $P_f(f(u), f(v))$ denotes the path between $f(u)$ and $f(v)$ in H with respect to f. For convenience of notation we write $EC_f(e)$ instead of $EC_f(G, H(e))$ in the sequel.

If we think of G as representing the wiring diagram of an electronic circuit, with the vertices representing components and the edges representing wires connecting them, then the edge congestion $EC(G, H)$ is the minimum, over all embeddings $f : V(G) \rightarrow V(H)$, of the maximum number of wires that cross any edge of H (Bezrukov et al. 2000a).

The Wirelength Problem The *wirelength* (Manuel et al. 2009) of an embedding f of G into H is given by

$$WL_f(G, H) = \sum_{e \in E(H)} EC_f(e) = \sum_{(u,v) \in E(G)} d_H(f(u), f(v))$$

where $d_H(f(u), f(v))$ denotes the length of the shortest path $P_f(f(u), f(v))$ in H. See Fig. 1. Then, the *minimum wirelength* of G into H is defined as

$$WL(G, H) = \min WL_f(G, H)$$

where the minimum is taken over all embeddings f of G into H. The *wirelength problem* (Bezrukov et al. 1998, 2000a; Chavez and Trapp 1998; Manuel et al. 2009; Opatrny and Sotteau 2000; Rajasingh et al. 2004) of a graph G into H is to find an embedding of G into H that induces the minimum wirelength $WL(G, H)$.

The wirelength of a graph embedding arises from VLSI designs, data structures and data representations, networks for parallel computer systems, biological models that deal with cloning and visual stimuli, parallel architecture, structural engineering and so on (Lai and Williams 1999; Xu 2001).

Grid embedding plays an important role in computer architecture. VLSI Layout Problem (Bhatt and Leighton 1984), Crossing Number Problem (Djidjev and Vrto 2003), Edge Embedding Problem (Garey and Johnson 1979) are all a part of grid embedding. Embedding problems have been considered for binary trees into paths (Lai and Williams 1999), complete binary trees into hypercubes (Bezrukov 2001), tori and grids into twisted cubes (Lai and Tsai 2010), meshes into locally twisted cubes (Han et al. 2010), paths into twisted cubes (Fan et al. 2007), cycles into twisted cubes (Fan et al. 2008), meshes into faulty crossed cubes (Yang et al. 2010), star graph into path (Yang 2009), snarks into torus (Vodopivec 2008), generalized ladders into hypercubes (Caha and Koubek 2001), grids into grids (Rottger and Schroeder 2001), binary trees into grids (Opatrny and Sotteau 2000), hypercubes into cycles (Chavez and Trapp 1998), generalized wheels into arbitrary trees (Rajasingh et al. 2004), and hypercubes into grids (Manuel et al. 2009). Even though there are numerous results and discussions on the wirelength problem, most of them deal with only approximate results and the estimation of lower bounds (Bezrukov et al. 1998; Chavez and Trapp 1998). All the embeddings discussed in this paper produce exact wirelengths.

Another interesting *NP*-complete problem (Garey and Johnson 1979) is the edge isoperimetric problem (Harper 2004) which will be used to solve the wirelength problem. We consider the following version of the edge isoperimetric problem of a graph $G(V,E)$.

Find a subset of vertices of a given graph, such that the number of edges in the subgraph induced by this subset is maximal among all induced subgraphs with the same number of vertices. Mathematically, for a given m, if $I_G(m) = \max_{A\subseteq V, |A|=m} |I_G(A)|$ where $I_G(A) = \{(u,v) \in E : u, v \in A\}$, then the problem is to find $A \subseteq V$ such that $|A| = m$ and $I_G(m) = |I_G(A)|$. We call such a set A optimal. Further for regular graphs a subset of vertices A is optimal, then its complement $V \setminus A$ is also an optimal set (Bezrukov et al. 2000b; Bezrukov and Elsässer 2003). In the literature, this problem is also defined as the maximum subgraph problem.

2 Background

Lemma 1 (Congestion lemma) (Manuel et al. 2009) *Let G be an r-regular graph and f be an embedding of G into H. Let S be an edge cut of H such that the removal of edges of S leaves H into 2 components H_1 and H_2 and let $G_1 = f^{-1}(H_1)$ and $G_2 = f^{-1}(H_2)$. Also S satisfies the following conditions*:

(i) *For every edge $(a,b) \in G_i$, $i = 1, 2$, $P_f(f(a), f(b))$ has no edges in S.*
(ii) *For every edge (a,b) in G with $a \in G_1$ and $b \in G_2$, $P_f(f(a), f(b))$ has exactly one edge in S.*
(iii) *G_1 is a maximum subgraph on k vertices where $k = |V(G_1)|$.*
Then $EC_f(S)$ is minimum and $EC_f(S) = rk - 2|E(G_1)|$.

Lemma 2 (Partition lemma) (Manuel et al. 2009) *Let $f : G \to H$ be an embedding. Let $\{S_1, S_2, \ldots, S_p\}$ be a partition of $E(H)$ such that each S_i is an edge cut of H.*

Then

$$WL_f(G,H)=\sum_{i=1}^{p}EC_f(S_i).$$

Lemma 3 (Generalized Partition lemma) *Let $f:G\to H$ be an embedding. For $1\le i\le k$, suppose $S_i=\{S_1^i,S_2^i,\dots,S_{p_i}^i\}$ partitions $E(H)\setminus F_i$ for mutually disjoint F_i's such that S_j^i, $1\le j\le p_i$, $1\le i\le k$ and $S=\bigcup_{i=1}^{k}F_i$ are all edge cuts of H. Then*

$$WL_f(G,H)=\frac{1}{k}\left[\sum_{i=1}^{k}\sum_{j=1}^{p_i}EC_f(S_j^i)+EC_f(S)\right].$$

Proof For $1\le i\le k$, we have

$$WL_f(G,H)=\sum_{j=1}^{p_i}EC_f(S_j^i)+EC_f(F_i).$$

By summing up for all i, we get

$$\begin{aligned}kWL_f(G,H)&=\sum_{j=1}^{p_1}EC_f(S_j^1)+\sum_{j=1}^{p_2}EC_f(S_j^2)\\&\quad+\cdots+\sum_{j=1}^{p_k}EC_f(S_j^k)+EC_f(F_1)+EC_f(F_2)+\cdots+EC_f(F_k).\end{aligned}$$

Therefore

$$WL_f(G,H)=\frac{1}{k}\left[\sum_{i=1}^{k}\sum_{j=1}^{p_i}EC_f(S_j^i)+EC_f(S)\right].$$

□

Remark 1 When $k=1$, we obtain the Partition lemma.

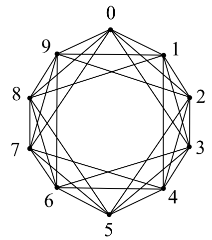
3 Edge isoperimetric problem for circulant networks

In 2004, L.H. Harper (2004) quotes "In analyzing Harary graph he wished to solve its edge isoperimetric problem". In this section we solve the maximum subgraph problem for circulant networks which is a particular class of Harary graphs.

Definition 1 (Xu 2001) A circulant undirected graph, denoted by $G(n;\pm S)$ where $S\subseteq\{1,2,\dots,\lfloor n/2\rfloor\}$, $n\ge 3$ is defined as a graph consisting of the vertex set $V=\{0,1,\dots,n-1\}$ and the edge set $E=\{(i,j):|j-i|\equiv s(\operatorname{mod} n),\ s\in S\}$.

The circulant graph shown in Fig. 2 is $G(10;\pm\{1,2,3\})$. It is clear that $G(n;\pm 1)$ is the undirected cycle C_n and $G(n;\pm\{1,2,\dots,\lfloor n/2\rfloor\})$ is the complete graph K_n. Further $G(n;\pm\{1,2,\dots,j\})$, $1\le j<\lfloor n/2\rfloor$, $n\ge 3$ is a $2j$-regular graph.

Fig. 2 Circulant graph $G(10; \pm\{1, 2, 3\})$



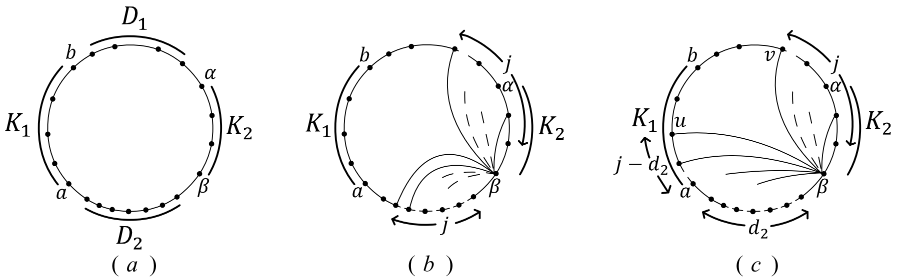


Fig. 3 (**a**) K_1, K_2, D_1 and D_2 are disjoint segments on C (**b**) $d_2 \geq j$, (**c**) $u \in K_1$, $v \in D_1$

Lemma 4 *Let C denote the cycle $G(n; \pm 1)$ in $G(n; \pm\{1, 2, \ldots, j\})$, $1 \leq j < \lfloor n/2 \rfloor$, $n \geq 3$. Let K_1 and K_2 be disjoint segments induced by k_1 and k_2 consecutive vertices on C respectively such that $k_1 + k_2 \leq \lfloor n/2 \rfloor$. Then $G[K_1 \cup K_2]$, the subgraph induced by $K_1 \cup K_2$, contains a vertex of degree at most j.*

Proof Let the complement of $K_1 \cup K_2$ in C consist of two disjoint segments D_1 and D_2 induced by d_1 and d_2 consecutive vertices on C respectively. Then $d_1 + d_2 \geq \lceil n/2 \rceil$. Without loss of generality, let $k_2 \leq \frac{1}{2}\lfloor n/2 \rfloor$ and $d_2 \geq \frac{1}{2}\lceil n/2 \rceil$. This implies $k_2 \leq d_2$. Suppose $k_1 = 1$ and $K_1 = \{u\}$. Then

$$\deg_{G[K_1 \cup K_2]} u = \begin{cases} 2j - (d_1 + d_2) & d_1, d_2 \leq j \\ 0 & d_1, d_2 \geq j \\ j - d_1 & d_2 \geq j, d_1 \leq j \\ j - d_2 & d_1 \geq j, d_2 \leq j \end{cases}$$

But $2j - (d_1 + d_2) \leq 2j - \lceil n/2 \rceil \leq 2j - j = j$. Therefore $\deg_{G[K_1 \cup K_2]} u \leq j$. The same argument holds when $k_2 = 1$. We now assume that $k_1 \geq 2$ and $k_2 \geq 2$. Let a, b be the end vertices of K_1 and α, β be the end vertices of K_2 taken in the clockwise sense. See Fig. 3(a). We claim that $\deg_{G[K_1 \cup K_2]} \beta \leq j$.

Case 1 ($d_2 \geq j$): In this case $\deg_{G[K_1 \cup K_2]} \beta \leq j$. See Fig. 3(b).

Case 2 ($d_2 < j$): Let u be a vertex on C such that $d_C(\beta, u) = j$ measured in the clockwise direction and let the shortest path of length j on C with origin β and passing through α have its other end at v.

Subcase 2.1 ($v \in K_1$): In this case $u \in K_1$. Therefore $\deg_{G[K_1 \cup K_2]} \beta \le 2j - (d_1 + d_2) \le j$.

Subcase 2.2 ($v \in D_1$): If u lies on K_1 then $\deg_{G[K_1 \cup K_2]} \beta \le (k_2 - 1) + j - d_2$. But $(k_2 - 1) + j - d_2 = (k_2 - d_2) + j - 1 \le (d_2 - d_2) + j - 1 < j$. Therefore $\deg_{G[K_1 \cup K_2]} \beta < j$. See Fig. 3(c). If u lies on D_1 then $\deg_{G[K_1 \cup K_2]} \beta \le (k_2 - 1) + k_1$. In this case $k_1 < j - d_2$. Therefore $\deg_{G[K_1 \cup K_2]} \beta < j$.

Subcase 2.3 ($v \in K_2$): Since $k_2 \le d_2$ and $d_2 < j$, we have $k_2 < j$. Hence this case does not occur. □

Lemma 5 *Let C denote the cycle $G(n; \pm 1)$ in $G(n; \pm\{1, 2, \dots, j\})$, $1 \le j < \lfloor n/2 \rfloor$, $n \ge 3$. Let K be a segment on C induced by k consecutive vertices on C where $k \le \lfloor n/2 \rfloor$. If K_1 and K_2 are disjoint segments induced by k_1 and k_2 consecutive vertices on C respectively such that $k_1 + k_2 = k$ then $|E(G[K_1 \cup K_2])| \le |E(G[K])|$.*

Proof The proof is by induction on k. Suppose $k = 2$. Then $|E(G[K])| = 1$. If u and v are nonadjacent vertices on C such that $d_C(u, v) \le j$ then $|E(G[\{u, v\}])| = 1$ and if $d_C(u, v) > j$, then $|E(G[\{u, v\}])| = 0$. Thus $|E(G[\{u, v\}])| \le |E(G[K])|$. Assume the result to be true for $k - 1$ consecutive vertices. Consider k consecutive vertices on C, $k \le \lfloor n/2 \rfloor$. If $k \le j + 1$ then $G[K]$ is the complete graph on k vertices and hence it contains the maximum number of edges. Suppose $k > j + 1$. Let K_1 and K_2 be disjoint segments induced by k_1 and k_2 consecutive vertices on C respectively such that $k_1 + k_2 = k$. By Lemma 4, $G[K_1 \cup K_2]$ contains an end vertex v in K_1 or K_2 of degree at most j. Without loss of generality, let v be an end vertex of K_1 with $\deg_{G[K_1 \cup K_2]} v \le j$. Delete the vertex v from $K_1 \cup K_2$ to obtain $K_1' \cup K_2$ with $k - 1$ vertices. By induction hypothesis $|E(G[K_1' \cup K_2])| \le |E(G[K'])|$ where K' is induced by $k - 1$ consecutive vertices on C. Thus, $|E(G[K_1 \cup K_2])| = |E(G[K_1' \cup \{v\} \cup K_2])| \le |E(G[K_1' \cup K_2])| + j \le |E(G[K'])| + j = |E(G[K])|$. □

Lemma 6 *A set of k consecutive vertices of $G(n; \pm 1)$ induces a maximum subgraph of $G(n; \pm S)$ on k vertices, $k \le \lfloor n/2 \rfloor$, $S = \{1, 2, \dots, j\}$, $1 \le j < \lfloor n/2 \rfloor$, $n \ge 3$.*

Proof Let the cycle $G(n; \pm 1)$ be denoted by C and let A be a set of k consecutive vertices on C. Let B be a set of k non-consecutive vertices on C. Then $B = \bigcup_{i=1}^{p} X_i$ where $p \ge 2$, X_i's are mutually disjoint and each X_i is a set of consecutive vertices on C such that $\sum_{i=1}^{p} |X_i| = k$. We claim that $|E(G[B])| \le |E(G[A])|$. We prove this claim by induction on p. When $p = 2$, by Lemma 5, we get $|E(G[B])| \le |E(G[A])|$. Assume that the claim is true for $p - 1$. Then $|E(G[\bigcup_{i=1}^{p-1} X_i])| \le |E(G[X])|$ where X is induced by $k - |X_p|$ consecutive vertices on C. Now, $|E(G[\bigcup_{i=1}^{p} X_i])| = |E(G[\bigcup_{i=1}^{p-1} X_i \cup X_p])| \le |E(G[X \cup X_p])| \le |E(G[A])|$. See Fig. 4. □

The following result shows that Lemma 6 is true even if the restriction on the upper bound of k is relaxed.

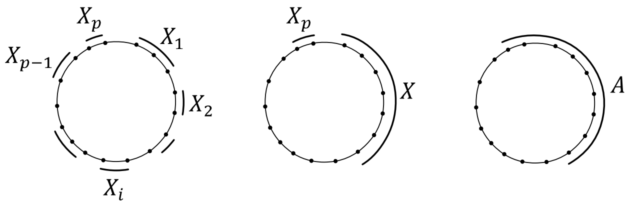


Fig. 4 $|X| = k - |X_p|$ and $|A| = k$

Theorem 1 *A set of k consecutive vertices of* $G(n; \pm 1)$, $1 \le k \le n$ *induces a maximum subgraph of* $G(n; \pm S)$, *where* $S = \{1, 2, \dots, j\}$, $1 \le j < \lfloor n/2 \rfloor$, $n \ge 3$.

Proof By Lemma 6, a set of $k \le \lfloor n/2 \rfloor$ consecutive vertices of $G(n; \pm 1)$ in $G(n; \pm\{1, 2, \dots, j\})$ induces a maximum subgraph of $G(n; \pm\{1, 2, \dots, j\})$. Since $G(n; \pm\{1, 2, \dots, j\})$, $1 \le j < \lfloor n/2 \rfloor$ is a regular graph, the remaining $n - k$ vertices also induce a maximum subgraph of G. $\square$

Theorem 2 *The number of edges in a maximum subgraph on k vertices of* $G(n; \pm S)$, $S = \{1, 2, \dots, j\}$, $1 \le j < \lfloor n/2 \rfloor$, $1 \le k \le n$, $n \ge 3$ *is given by*

$$\xi = \begin{cases} k(k-1)/2 & k \le j+1 \\ kj - j(j+1)/2 & j+1 < k \le n-j \\ \frac{1}{2}\{(n-k)^2 + (4j+1)k - (2j+1)n\} & n-j < k \le n. \end{cases}$$

Proof Let $K = \{v_1, v_2, \dots, v_k\}$ be a set of k consecutive vertices of $G(n; \pm 1)$ in $G(n; \pm S)$. By Theorem 1, $G[K]$ is a maximum subgraph of $G(n; \pm S)$ on k vertices.

Case 1 ($k \le n - j$): The number of edges induced by K in $G(n; \pm i)$, $1 \le i \le j$ is given by

$$\xi_i = \begin{cases} k - i & k > i \\ 0 & \text{otherwise.} \end{cases}$$

See Fig. 5. Therefore,

$$|E(G[K])| = \sum_{i=1}^{j} \xi_i = \sum_{i=1}^{\min\{j, k-1\}} (k - i).$$

This implies that

$$\xi = \begin{cases} k(k-1)/2 & k \le j+1 \\ kj - j(j+1)/2 & k > j+1. \end{cases}$$

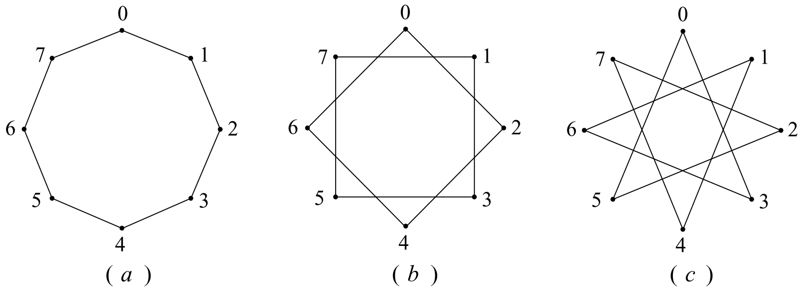


Fig. 5 Circulant graph (**a**) $G(8;\pm 1)$, (**b**) $G(8;\pm 2)$, (**c**) $G(8;\pm 3)$

Case 2 ($k > n - j$):

$$\begin{aligned}|E(G[K])| &= |E(G)| - \{2j(n-k)\text{-number of edges induced by } (n-k) \\ &\qquad \text{consecutive vertices of } G\} \\ &= nj - 2j(n-k) + (n-k)(n-k-1)/2 \\ &= \frac{1}{2}\{(n-k)^2 + (4j+1)k - (2j+1)n\}.\end{aligned}$$

□

4 Wirelength of circulant networks into arbitrary trees

A tree is a connected graph having no cycles. Any two vertices of a tree are joined by a unique path. The most common type of tree is the binary tree. It is so named because each node can have at most two descendents. A binary tree is said to be a complete binary tree if each internal node has exactly two descendents. These descendents are described as left and right children of the parent node. The binary search tree property (Cormen et al. 2001) of a binary tree states that all labels in the left subtree of any vertex x are all less than x, and all labels in the right subtree of x are greater than x. The consecutive label property is motivated by the binary search tree property (Rajasingh et al. 2004).

Definition 2 Let T be an ordered rooted tree with vertex labels $1, 2, \ldots, n$. A subtree T' of the tree T is consecutively labelled if the labels of T' are consecutive numbers $y+1, y+2, \ldots, y+k$ where k denotes the number of vertices of T'.

Definition 3 Let T be an ordered rooted tree with vertex labels $1, 2, \ldots, n$. A labeling of T satisfies the consecutive label property if for every vertex v of T, the subtrees $T_1, T_2, \ldots, T_m$ rooted at v are consecutively labeled.

Remark 2 Preorder, Inorder and Postorder labeling (Quadras 2005) of the vertices of trees induce consecutive label property in trees.

Embedding Algorithm A

Input: A circulant network $G(n; \pm\{1, 2, \dots, j\})$, $1 \le j < \lfloor n/2 \rfloor$ and an arbitrary rooted tree T on n vertices.

Algorithm: Label the consecutive vertices of $G(n; \pm 1)$ in $G(n; \pm\{1, 2, \dots, j\})$ as $0, 1, \dots, n-1$ in the clockwise sense. Label the vertices of the tree as $0, 1, \dots, n-1$ using inorder labeling.

Output: An embedding f of $G(n; \pm\{1, 2, \dots, j\})$ into T given by $f(x) = x$ with minimum wirelength.

Proof of correctness: Let e be an edge of T. Then $T - e$ yields a component T_1 which is consecutively labeled (Rajasingh et al. 2004). By Theorem 1, the subgraph of $G(n; \pm\{1, 2, \dots, j\})$ induced by $\{f^{-1}(v) : v \in T_1\}$ is maximum. By Congestion lemma, the congestion on e is minimum. This is true for every edge of T. Partition lemma implies that the wirelength is minimum.

The proof of the following result is an easy consequence of Lemma 2 and of the discussion in the proof of Theorem 2.

Theorem 3 *The wirelength of* $G(n; \pm\{1, 2, \dots, j\})$, $1 \le j < \lfloor n/2 \rfloor$ *into* T *is given by*

$$WL(G(n; \pm\{1, 2, \dots, j\}), T) = 2 \sum_{e \in E(T)} \left\{ jk(e) - \sum_{i=1}^{\min\{j, k(e)-1\}} \{k(e) - i\} \right\}$$

where $k(e)$ *is the number of vertices in the component* T_1 *of* $T - e$ *with* $k(e) \le \lfloor n/2 \rfloor$.

As $G(n; \pm\{1, 2, \dots, \lfloor n/2 \rfloor\}) \simeq K_n$, we have the following result.

Theorem 4 *The wirelength of the complete graph* K_n *into* T *is given by*

$$WL(K_n, T) = \sum_{e \in E(T)} k(e)\{n - k(e)\}$$

where $k(e)$ *is the number of vertices in the component* T_1 *of* $T - e$ *with* $k(e) \le \lfloor n/2 \rfloor$.

5 Wirelength of circulant networks into cycle related graphs

In this section we consider the embedding of circulant network into cycles and certain multicyclic graphs. Here C_n denotes a cycle on n vertices.

5.1 Even and odd cycles

Embedding Algorithm B

Input: A circulant network $G(n; \pm\{1, 2, \dots, j\})$, $1 \le j < \lfloor n/2 \rfloor$ and C_n.

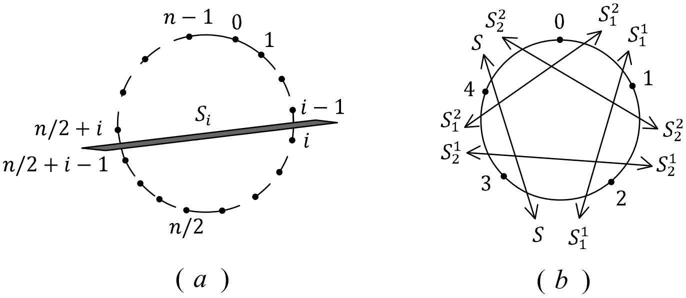


Fig. 6 (**a**) S_i contains two diametrically opposite edges of C_n, n even (**b**) edge cuts of C_5

Algorithm: Label the consecutive vertices of $G(n; \pm 1)$ in $G(n; \pm\{1, 2, \ldots, j\})$ and C_n as $0, 1, \ldots, n-1$ in the clockwise sense.

Output: An embedding f of $G(n; \pm\{1, 2, \ldots, j\})$ into C_n given by $f(x) = x$ with minimum wirelength.

Proof of correctness:

Case 1 (n **even**): Let the edge set E of C_n be partitioned into $\{S_1, S_2, \ldots, S_{n/2}\}$ where each S_i contains two diametrically opposite edges of C_n. In other words, $S_i = \{(i-1, i), (n/2+i-1, n/2+i)\}$, $1 \le i \le n/2$ where the labels are taken *modulo n*. See Fig. 6(a). For each i, $E(C_n) \setminus S_i$ has two components H_{i1} and H_{i2}. Let $G_{i1} = f^{-1}(H_{i1})$ and $G_{i2} = f^{-1}(H_{i2})$. Then each G_{ij}, $j = 1, 2$ is on $n/2$ consecutive vertices of $G(n; \pm 1)$. By Theorem 1, these vertices induce a maximum subgraph of $G(n; \pm\{1, 2, \ldots, j\})$, $1 \le j < \lfloor n/2 \rfloor$. Thus each S_i satisfies conditions (i), (ii) and (iii) of the Congestion lemma. Therefore $EC_f(S_i)$ is minimum. Partition Lemma implies that the wirelength is minimum.

Case 2 (n **odd**): For $1 \le i \le 2$, let $S_i = \{S_1^i, S_2^i, \ldots, S_{(n-1)/2}^i\}$ where $S_j^1 = \{(j-1, j), (\frac{n-3}{2} + j, \frac{n-1}{2} + j)\}$ and $S_j^2 = \{(j-1, j), (\frac{n-1}{2} + j, \frac{n+1}{2} + j)\}$, $1 \le j \le (n-1)/2$, the labels taken *modulo n*. Let $F_1 = \{(n-1, 0)\}$ and $F_2 = \{(\frac{n-1}{2}, \frac{n+1}{2})\}$. Then S_i partitions $E(C_n) \setminus F_i$, $i = 1, 2$. The sets F_1, F_2 are mutually disjoint and $S = F_1 \cup F_2$ is an edge cut of C_n. See Fig. 6(b). For each j, $E(C_n) \setminus S_j^i$ has two components H_{j1}^i and H_{j2}^i induced by consecutive vertices on C_n with $|H_{j1}^i| = \lfloor n/2 \rfloor$ and $|H_{j2}^i| = \lceil n/2 \rceil$. Let $G_{j1}^i = f^{-1}(H_{j1}^i)$ and $G_{j2}^i = f^{-1}(H_{j2}^i)$. Then G_{j1}^i is on $\lfloor n/2 \rfloor$ consecutive vertices of $G(n; \pm 1)$. By Theorem 1, these vertices induce a maximum subgraph of $G(n; \pm\{1, 2, \ldots, j\})$, $1 \le j < \lfloor n/2 \rfloor$. Thus each S_j^i satisfies conditions (i), (ii) and (iii) of the Congestion lemma. Therefore $EC_f(S_j^i)$ is minimum. Similarly $EC_f(S)$ is minimum. The Generalized Partition lemma (when $k = 2$) implies that the wirelength is minimum.

Theorem 5 $WL(G(n; \pm\{1, 2, \ldots, j\}); 1 \le j < \lfloor n/2 \rfloor, C_n) = \frac{nj(j+1)}{2}$.

Proof We have already proved that the embedding f defined in Embedding Algorithm B induces minimum wirelength of $G(n; \pm\{1, 2, \ldots, j\})$ onto C_n.

Case 1 (n **even**): Following the notation used in Case 1 of the algorithm, we have by Lemma 1, $EC_f(S_1) = 2j\lfloor n/2 \rfloor - 2\sum_{i=1}^{j}(\lfloor n/2 \rfloor - i) = j(j+1)$. But $C_n \setminus S_i$ is isomorphic to $C_n \setminus S_j$ for $i \neq j$. Therefore, $WL(G(n; \pm\{1, 2, \ldots, j\}), C_n) = \frac{n}{2}EC_f(S_1) = \frac{nj(j+1)}{2}$.

Case 2 (n **odd**): Following the notation used in Case 2 of the algorithm, we have by Lemma 1, $EC_f(S) = j(j+1)$. But $C_n \setminus S$ is isomorphic to $C_n \setminus S_j^i$ for $1 \leq i \leq 2$, $1 \leq j \leq (n-1)/2$. Therefore, $WL(G(n; \pm\{1, 2, \ldots, j\}), C_n) = \frac{1}{2}\{\sum_{i=1}^{2}\sum_{j=1}^{(n-1)/2} EC_f(S_j^i) + EC_f(S)\} = \frac{1}{2}\{(n-1)EC_f(S) + EC_f(S)\} = \frac{nj(j+1)}{2}$. □

Theorem 6 $WL(K_n, C_n) = \frac{n}{2}\lfloor n/2 \rfloor \lceil n/2 \rceil$.

5.2 Unicyclic graphs

A connected unicyclic graph arises from a tree by adding an extra edge. In other words, a graph which contains exactly one cycle $C : v_1 v_2 \ldots v_m v_1$ is said to be a unicyclic graph, denoted by UC_m. Let $T_1, T_2, \ldots, T_m$ be trees with roots at $v_1, v_2, \ldots, v_m$ respectively. Then UC_m contains n vertices where $n = |V(T_1)| + |V(T_2)| + \cdots + |V(T_m)|$.

Embedding Algorithm C

Input: A circulant network $G(n; \pm\{1, 2, \ldots, j\})$, $1 \leq j < \lfloor n/2 \rfloor$ and UC_m.

Algorithm: Label the consecutive vertices of $G(n; \pm 1)$ in $G(n; \pm\{1, 2, \ldots, j\})$ as $0, 1, \ldots, n-1$ in the clockwise sense. Label the vertices of the unicyclic graph as follows: Label the vertex v_1 as 0 and the vertices v_{i+1}, $1 \leq i \leq m-1$ as $\sum_{k=1}^{i} |V(T_k)|$. Label the vertices of $V(T_1) \setminus v_1$ from 1 to $|V(T_1)| - 1$ and the vertices of $V(T_{i+1}) \setminus v_{i+1}$, $1 \leq i \leq m-1$ from $\sum_{k=1}^{i} |V(T_k)| + 1$ to $\sum_{k=1}^{i} |V(T_k)| + |V(T_{i+1})| - 1$ using inorder labeling.

Output: An embedding f of $G(n; \pm\{1, 2, \ldots, j\})$ into UC_m given by $f(x) = x$ with minimum wirelength.

Proof of correctness: We partition the edges of each tree as in Sect. 4 and the edges of the cycle as in Sect. 5.1. A straightforward computation yields minimum wirelength.

5.3 Multicyclic graphs

A multicyclic graph is obtained from a tree by replacing at least two of the edges by two parallel edges and subdividing the parallel edges to obtain paths. The multicyclic graph can also be viewed as a particular class of series-parallel graphs. Series-parallel graphs are an important class of recursively defined graphs that can be characterized in many ways. The oldest and the most popular characterization due to Duffin (1965) provides a Kuratowski-like condition which states that a graph G is series-parallel if and only if it contains no subgraph homeomorphic to K_4, the complete graph on four vertices.

A series–parallel graph is usually defined recursively by using parallel and series compositions. This classical definition justifies another name of these graphs, namely, 2-terminal series–parallel graphs, since we assume that every such graph has two nodes distinguished as poles and denoted by S (for South) and N (for North). A series–parallel graph G with poles S and N is defined as either

(i) an edge (S, N)

or can be constructed as in (ii) or (iii):

(ii) G is a parallel composition of at least two series-parallel graphs $G_1, G_2, \dots, G_k (k \geq 2)$, denoted by $G = G_1 \parallel G_2 \parallel \dots \parallel G_k$. This operation identifies the South Poles S_i of the component graphs into the South Pole S of G, and similarly the North Poles N_i become N of G.

(iii) G is a series composition of at least two series-parallel graphs $G_1, G_2, \dots, G_l (l \geq 2)$, denoted by $G = G_1 \circ G_2 \circ \cdots \circ G_l$. This operation identifies N_i and S_{i+1} for $i = 1, 2, \dots, l-1$, and assigns S_1 to S and N_l to N.

Embedding Algorithm D

Input: A circulant graph $G(l(n-1)+1; \pm\{1, 2, \dots, j\})$, $1 \leq j < \lfloor \frac{l(n-1)+1}{2} \rfloor$ and a series-parallel graph $H = G_1 \circ G_2 \circ \cdots \circ G_l$ where $G_i \simeq C_n$, $1 \leq i \leq l$, n even and the south pole and the north pole of each G_i are diametrically opposite vertices of C_n.

Algorithm: Label the consecutive vertices of $G(l(n-1)+1; \pm 1)$ in $G(l(n-1)+1; \pm\{1, 2, \dots, j\})$ as $0, 1, \dots, l(n-1)$ in the clockwise sense. Label the vertices of the series-parallel graph as follows: Without loss of generality, we name north pole N as S_{l+1}. Label south poles S_i as $\frac{(i-1)n}{2}$, $1 \leq i \leq l+1$. For each G_i, there are two edge disjoint paths between the south pole and the north pole. For $1 \leq i \leq l$, the internal vertices of a path from S_i to S_{i+1} are labeled consecutively from $\frac{(i-1)n}{2} + 1$ to $\frac{in}{2} - 1$ and the internal vertices of the other path from S_i to S_{i+1} are labeled consecutively from $\frac{(2l-i+1)n}{2} - l + i - 1$ to $\frac{(2l-i)n}{2} - l + i + 1$.

Output: An embedding f of $G(l(n-1)+1; \pm\{1, 2, \dots, j\})$, $1 \leq j < \lfloor \frac{l(n-1)+1}{2} \rfloor$ into $H = G_1 \circ G_2 \circ \cdots \circ G_l$ where $G_i \simeq C_n$, $1 \leq i \leq l$, n even, given by $f(x) = x$ with minimum wirelength.

Proof of correctness: As in the case of embedding circulant graph into an even cycle, let edge set E of the series-parallel graph H be partitioned into $\{S_m^i : 1 \leq i \leq l, 1 \leq m \leq n/2\}$ where each S_m^i contains two diametrically opposite edges of G_i, $1 \leq i \leq l$, in $G_1 \circ G_2 \circ \cdots \circ G_l$. See Fig. 7. For $1 \leq i \leq l$, $1 \leq m \leq n/2$, $E(H) \setminus S_m^i$ has two components H_{m1}^i and H_{m2}^i induced by consecutive vertices on H. Let $G_{m1}^i = f^{-1}(H_{m1}^i)$ and $G_{m2}^i = f^{-1}(H_{m2}^i)$. Then G_{m1}^i is induced by consecutive vertices of $G(l(n-1)+1; \pm\{1, 2, \dots, j\})$. Again by Lemmas 1 and 2, f induces minimum wirelength.

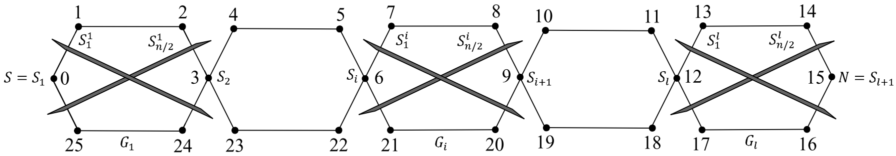


Fig. 7 S_1^i and $S_{n/2}^i$ contain two diametrically opposite edges of $G_i \simeq C_6$, $l = 5$, $n = 6$

Theorem 7 *The wirelength of* $G(l(n-1)+1; \pm\{1,2,\dots,j\})$, $1 \le j < \lfloor \frac{l(n-1)+1}{2} \rfloor$ *into* $H = G_1 \circ G_2 \circ \cdots \circ G_l$ *where* $G_i \simeq C_n$, $1 \le i \le l$, *n even, is given by*

$$WL(G,H) = \begin{cases} 2n\sum_{i=1}^{l/2}\{jx - \sum_{p=1}^{\min\{j,x-1\}}(x-p)\} & l \text{ even} \\ 2n\sum_{i=1}^{(l-1)/2}\{jx - \sum_{p=1}^{\min\{j,x-1\}}(x-p)\} & \\ \quad + n\{jy - \sum_{p=1}^{\min\{j,y-1\}}(y-p)\} & l \text{ odd} \end{cases}$$

where $x = [2(i-1)(n-1)+n]/2$ *and* $y = [(l-1)(n-1)+n]/2$.

Proof Clearly, $H \setminus S_m^i$ is isomorphic to $H \setminus S_m^{(l-i+1)}$ for $1 \le i \le \lfloor l/2 \rfloor$, $1 \le m \le n/2$. Therefore, $EC_f(S_m^i) = EC_f(S_m^{(l-i+1)})$.

Case 1 (l **even**):

$$\begin{aligned} &WL(G,H) \\ &= \sum_{i=1}^{l}\sum_{m=1}^{n/2} EC_f(S_m^i) \\ &= 2\sum_{i=1}^{l/2}[EC_f(S_1^i) + EC_f(S_2^i) + \cdots + EC_f(S_{(n/2)}^i)] \\ &= 2\sum_{i=1}^{l/2}\frac{n}{2}\left\{2jx - 2\sum_{p=1}^{\min\{j,x-1\}}(x-p)\right\} \quad \text{where } x = [2(i-1)(n-1)+n]/2 \\ &= 2n\sum_{i=1}^{l/2}\left\{jx - \sum_{p=1}^{\min\{j,x-1\}}(x-p)\right\}. \end{aligned}$$

Case 2 (l **odd**):

$$\begin{aligned} &WL(G,H) \\ &= \sum_{i=1}^{l}\sum_{m=1}^{n/2} EC_f(S_m^i) \end{aligned}$$

$$= 2\sum_{i=1}^{(l-1)/2}\sum_{m=1}^{n/2} EC_f(S_m^i) + \sum_{m=1}^{n/2} EC_f(S_m^{(\frac{l+1}{2})})$$

$$= 2\sum_{i=1}^{(l-1)/2} \frac{n}{2}\left\{2jx - 2\sum_{p=1}^{\min\{j,x-1\}}(x-p)\right\} + \frac{n}{2}\left\{2jy - 2\sum_{p=1}^{\min\{j,y-1\}}(y-p)\right\}$$

$$= 2n\sum_{i=1}^{(l-1)/2}\left\{jx - \sum_{p=1}^{\min\{j,x-1\}}(x-p)\right\} + n\left\{jy - \sum_{p=1}^{\min\{j,y-1\}}(y-p)\right\}$$

where $x = [2(i-1)(n-1)+n]/2$ and $y = [(l-1)(n-1)+n]/2$. □

Remark 3 The techniques adopted in this Section allow us to compute the exact wirelength of circulant networks into all classes of multicyclic graphs.

6 Wirelength of circulant networks into ladders

In this section we consider the embedding of circulant network $G(2n; \pm\{1, 2, \dots, j\})$, $1 \le j < n$ into the ladder $P_2 \times P_n$.

Embedding Algorithm E

Input: A circulant network $G(2n; \pm\{1, 2, \dots, j\})$, $1 \le j < n$ and the ladder $P_2 \times P_n$.

Algorithm: Label the consecutive vertices of $G(2n; \pm 1)$ in $G(2n; \pm\{1, 2, \dots, j\})$ as $0, 1, \dots, 2n-1$ in the clockwise sense. Label the vertices of the ladder as follows: The first row is labeled 0 to $n-1$ from left to right and the second row is labeled n to $2n-1$ from right to left.

Output: An embedding f of $G(2n; \pm\{1, 2, \dots, j\})$ into $P_2 \times P_n$ given by $f(x) = x$ with minimum wirelength.

Proof of correctness: Let X be a horizontal edge cut of the ladder such that X disconnects the ladder into two components R_1 and R_2 where $V(R_1) = \{0, 1, \dots, n-1\}$ and $V(R_2) = \{n, n+1, \dots, 2n-1\}$. Let Y_i be a vertical edge cut of the ladder such that Y_i disconnects the ladder into two components C_i and C_i' where $V(C_i) = \{0, 1, \dots, i-1\} \cup \{2n-i, 2n-i+1, \dots, 2n-1\}$ and $V(C_i') = V(P_2 \times P_n) \setminus V(C_i)$. See Fig. 8. Let G_1 and G_2 be the inverse images of R_1 and R_2 under this labeling. The edge cut X satisfies the conditions (i) and (ii) of the Congestion lemma. Since G_1 is on n consecutive vertices of $G(2n; \pm 1)$ in $G(2n; \pm\{1, 2, \dots, j\})$, by Theorem 1, $|E(G_1)|$ is maximum satisfying the condition (iii) of the Congestion lemma. Thus by the Congestion lemma, $EC_f(X)$ is minimum. Again let G_i and G_i' be the inverse images of C_i and C_i' under this labeling. The edge cut Y_i satisfies the conditions (i) and (ii) of the Congestion lemma. Also G_i is induced by $2i$ consecutive vertices of $G(2n; \pm 1)$ in $G(2n; \pm\{1, 2, \dots, j\})$. Thus $EC_f(Y_i)$ is minimum for $i = 1, 2, \dots, n-1$. Hence by Partition lemma the wirelength is minimum.

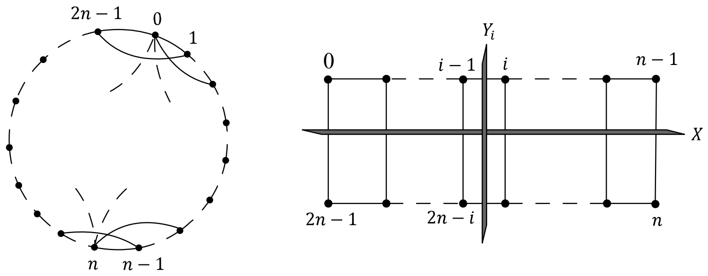


Fig. 8 X is a horizontal edge cut and Y_i is a vertical edge cut

Theorem 8 *The wirelength of* $G(2n; \pm\{1,2,\ldots,j\})$, $1 \le j < n$ *into* $P_2 \times P_n$ *is given by*

$$WL(G(2n; \pm\{1,2,\ldots,j\}), P_2 \times P_n) = \begin{cases} j(j+1) + j(n^2-1) - 4\sum_{m=1}^{(n-1)/2}\sum_{i=1}^{\min\{j,2m-1\}}(2m-i) & n \text{ odd} \\ jn(n+2) - 4\sum_{m=1}^{n/2}\sum_{i=1}^{\min\{j,2m-1\}}(2m-i) & n \text{ even.} \end{cases}$$

Proof We have

$$EC_f(X) = 2jn - 2\sum_{i=1}^{j}(n-i) = j(j+1).$$

Case 1 (n **odd**): Clearly, $EC_f(Y_i) = EC_f(Y_{n-i})$ for $1 \le i \le (n-1)/2$.

Therefore,

$$\begin{aligned}\sum_{i=1}^{n-1} EC_f(Y_i) &= 2\sum_{i=1}^{(n-1)/2} EC_f(Y_i) \\ &= 2\left\{4j - 2\sum_{i=1}^{\min\{j,1\}}(2-i) + 8j - 2\sum_{i=1}^{\min\{j,3\}}(4-i)\right. \\ &\quad \left. + \cdots + 2j(n-1) - 2\sum_{i=1}^{\min\{j,n-2\}}(n-1-i)\right\} \\ &= j(n^2-1) - 4\sum_{m=1}^{(n-1)/2}\sum_{i=1}^{\min\{j,2m-1\}}(2m-i).\end{aligned}$$

Thus,

$$\begin{aligned}&WL(G(2n; \pm\{1,2,\ldots,j\}), 1 \le j < n, P_2 \times P_n) \\ &\quad = j(j+1) + j(n^2-1) - 4\sum_{m=1}^{(n-1)/2}\sum_{i=1}^{\min\{j,2m-1\}}(2m-i).\end{aligned}$$

Case 2 (n **even**): Clearly, $EC_f(Y_i)=EC_f(Y_{n-i})$ for $1\le i\le n/2-1$.
Therefore,

$$\begin{aligned}
\sum_{i=1}^{n-1} EC_f(Y_i) &= \sum_{i=1}^{n/2-1} EC_f(Y_i) + EC_f(Y_{n/2}) + \sum_{i=n/2+1}^{n-1} EC_f(Y_i)\\
&= 2\Bigg\{4j-2\sum_{i=1}^{\min\{j,1\}}(2-i)+8j-2\sum_{i=1}^{\min\{j,3\}}(4-i)\\
&\quad+\cdots+2j(n-2)-2\sum_{i=1}^{\min\{j,n-3\}}(n-2-i)\Bigg\}\\
&\quad+2jn-2\sum_{i=1}^{j}(n-i)\\
&= 2\Bigg\{4j-2\sum_{i=1}^{\min\{j,1\}}(2-i)+8j-2\sum_{i=1}^{\min\{j,3\}}(4-i)\\
&\quad+\cdots+2jn-2\sum_{i=1}^{\min\{j,n-1\}}(n-i)\Bigg\}\\
&\quad-2jn+2\sum_{i=1}^{j}(n-i)\\
&= jn(n+2)-4\sum_{m=1}^{n/2}\sum_{i=1}^{\min\{j,2m-1\}}(2m-i)-j(j+1).
\end{aligned}$$

Thus,

$$\begin{aligned}
&WL(G(2n;\pm\{1,2,\ldots,j\}),\ 1\le j<n,\ P_2\times P_n)\\
&\quad= jn(n+2)-4\sum_{m=1}^{n/2}\sum_{i=1}^{\min\{j,2m-1\}}(2m-i).
\end{aligned}$$

□

7 Conclusion

We obtain the exact wirelength of circulant networks into arbitrary trees, certain multicyclic graphs and ladders. All the embeddings constructed in this paper are simple, elegant and produce exact wirelengths. It would be an interesting line of research to solve the following problem.

Open Problem To find the exact wirelength of circulant networks into grids.

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