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A PTAS for the minimum weighted dominating set problem with smooth weights on unit disk graphs

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Abstract In the minimum weighted dominating set problem (MWDS), we are given a unit disk graph with non-negative weight on each vertex. The MWDS seeks a subset of the vertices of the graph with minimum total weight such that each vertex of the graph is either in the subset or adjacent to some nodes in the subset. A weight function is called smooth, if the ratio of the weights of any two adjacent nodes is upper bounded by a constant. MWDS is known to be NP-hard. In this paper, we give the first polynomial time approximation scheme (PTAS) for MWDS with smooth weights on unit disk graphs, which achieves a $(1+\epsilon)$-approximation for MWDS, for any $\epsilon > 0$.

Keywords Dominating set · Maximal independent set · Polynomial time approximation scheme · Unit disk graphs

1 Introduction

Wireless ad hoc sensor networks is a recently emerged advanced technology with a lot of applications in many fields, such as surveillance of battlefield, search and rescue, disaster detection, and etc. Unlike wired networks, no physical infrastructure exists in wireless ad hoc sensor networks. It is usually beneficial to choose a virtual backbone formed by nodes in a connected dominating set for efficient routing, broadcasting and connectivity management in wireless ad hoc networks (Bharghavan and Das 1997).

A wireless ad hoc network is usually modelled as a unit disk graph (UDG), in which the sensor nodes are assumed to lie on the Euclidean plane, and there is an

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edge between two nodes iff their Euclidean distance is no more than one. Given a UDG $G = (V, E)$, a *dominating set* (DS) of G is a subset D of V such that each vertex of G is either in D or adjacent to some node in D. The minimum dominating set (MDS) problem seeks to find a dominating set of a graph G with the smallest cardinality.

Due to its wide applications in wireless ad hoc sensor networks and many other areas, MDS has been studied extensively in recent years. For MDS in general graphs, it was proved in Guha and Khuller (1998) that for any $0 < \rho < 1$, there is no polynomial time $\rho \ln n$-approximation unless NP $\subseteq$ DTIME($n^{O(\ln n)}$) (n is the number of vertices); also, a greedy ($\ln \Delta + 3$)-approximation was given, where Δ is the maximum degree of the graph. When restricted to UDGs, the MDS problem is still NP-hard (Clark et al. 1990). A distributed constant-approximation for MDS in UDGs was given in Wan et al. (2002). Polynomial time approximation schemes (PTASs) were given in Ambühl and Erlebach (2006), Gfeller and Vicari (2007) and Nieberg and Hurink (2006).

In practice, it is natural to assume that the vertices of the graph have some positive weights. In the context of wireless ad-hoc networks, these weights usually reflect residual energy or capabilities of a node for a specific task. So the minimum weighted dominating set (MWDS) problem is considered by several authors. Ambühl and Erlebach (2006) is the first to design a constant factor approximation algorithm for MWDS on UDGs. Later, Huang et al. (2008) and Dai and Yu (2009) improved the approximation ratio significantly. However, it is still an open question whether MWDS has a PTAS on UDGs.

In this paper, we will give the first PTAS for MWDS with smooth weights (which will be specified later, see Sect. 2) on UDGs. It should be noted that a constant factor approximation algorithm is implied in Wang et al. (2005) previously under the same conditions. The assumption of the smoothness of the weights is reasonable in practice, since in many applications such as wireless ad hoc networks, the weights of neighboring nodes do not vary significantly (Wang et al. 2005).

The rest of the paper is organized as follows. In Sect. 2, we give some preliminaries needed in the paper. In Sect. 3, we present the PTAS algorithm. The proof of the correctness of the algorithm is given in Sect. 4.

2 Preliminaries

In this section, we introduce some notions and notations that are needed in the rest of the paper.

Throughout, we denote the 1-hop closed neighborhood of a node v by

$$N(v) = \{u \in V | u \text{ is adjacent with } v\} \cup \{v\}.$$

For a subset $S \subset V$, $N(S) = \bigcup_{v \in S} N(v)$. And the r-hop closed neighborhood of v is defined recursively as $N^r(v) = N(N^{r-1}(v))$. In particular, $N^1(v) = N(v)$ and $N^0(v) = \{v\}$.

Let $S \subset V$ be a subset of vertices in G. In the following, we use $G[S]$ to denote the subgraph induced by S. In case of a weighted graph, we define the weight of a subset S by $w(S) := \sum_{v \in S} w(v)$.

Two vertices of a graph are called *independent* if they are not adjacent to one another. A subset $I \subset V$ is called *independent* if all vertices are not connected. A *maximal independent* (MIS) set is an independent set that cannot be extended by the addition of any other vertex from the graph without violating the independence property. We will denote by $MIS(S)$ for the MIS of the graph $G[S]$ induced by a subset $S \subset V$. It is easy to verify that an MIS of G is also a dominating set of G. In a UDG, we have $|MIS(N^r(v))| \leq (2r+1)^2$. Since if we draw a disk with radius centered at v and draw disks with radius $1/2$ centered at each nodes of the maximal independent set, then all small disks are pairwise disjoint and contained in the bigger disk. It follows that $|MIS(N^r(v))| \leq \frac{\pi(r+1/2)^2}{\pi(1/2)^2} = (2r+1)^2$. This fact will be used frequently in the sequel.

Furthermore, we give the following definitions.

Definition 2.1 (Minimum weighted dominating set problem, MWDS) Given a weighted graph $G = (V, E)$ with each vertex v having a non-negative weight $w(v)$. Find a dominating set D of G such that the total weight $\sum_{v \in D} w(v)$ is minimized.

Definition 2.2 Given a weighted graph $G = (V, E)$ with each vertex v having a non-negative weight $w(v)$. The wight function $w : V \to R^+$ is called smooth if there exists a constant $C \geq 1$ such that $\max_{(uv) \in E} \frac{w(u)}{w(v)} \leq C$.

A polynomial-time approximation scheme (PTAS) is an algorithm which, in addition to an input instance, requires a parameter $\epsilon > 0$, which then returns a solution with a relative error of at most $1 + \epsilon$ with respect to an optimal solution. The running time of such algorithms is allowed to depend on ϵ but should be polynomial in $n := |V|$ for fixed $\epsilon > 0$. For example, a PTAS for the MDS problem returns a dominating set of cardinality at most $(1+\epsilon)$ times the cardinality of a minimum cardinality dominating set.

3 The algorithm

Our idea follows that of Nieberg and Hurink (2006), in which they proposed a PTAS to compute an MDS in a UDG. To describe the main idea, we need the following

Definition 3.1 A collection of subsets $\{S_1, \ldots, S_k\}$ is defined to be *a 2-separated partition of* G if the distance $\mathrm{dist}(S_i, S_j) > 2$ holds for any $i \neq j$.

Denote by $D_{opt}(S)$ the set of nodes in $V(G)$ with the minimum cardinality that dominates S. Note that $D_{opt}(S)$ is computed w.r.t. entire underlying graph G and is *not* restricted to lie in S, however, it is always true that $D_{opt}(S) \subseteq N(S)$. Hence, for a 2-separated partition $\{S_1, \ldots, S_k\}$ of G, $D_{opt}(S_1), D_{opt}(S_2), \ldots, D_{opt}(S_k)$ are disjoint. As $D_{opt}(V) \cap N(S_i)$ is a dominating set of S_i and $D_{opt}(S_i)$ is the set with minimum cardinality that dominates S_i, we have $|D_{opt}(S_i)| \leq |D_{opt}(V) \cap N(S_i)|$ and hence,

$$\sum_{i=1}^{k} |D_{opt}(S_i)| \leq \sum_{i=1}^{k} |D_{opt}(V) \cap N(S_i)| = |D_{opt}(V)|.$$

Note that $\bigcup_{i=1}^{k} D_{opt}(S_i)$ is not necessarily a dominating set of G. To get a dominating set of G, we enlarge the S_i's to T_i's such that

(1) $S_i \subseteq T_i$;
(2) $|T_i| \leq (1+\epsilon)|S_i|$;
(3) $\bigcup_{i=1}^{k} T_i$ is a dominating set of $V(G)$.

Then $\bigcup_{i=1}^{k} T_i$ is a $(1+\epsilon)$-approximation of $V(G)$. To compute $\{S_1, \ldots, S_k\}$ and $\{T_1, \ldots, T_k\}$, pick an arbitrary node v_1, find the minimum non-negative integer r_1 satisfying

$$|D_{opt}((N^{r_1+2}(v_1))) \leq (1+\epsilon)|D_{opt}((N^{r_1}(v_1)))|. \tag{1}$$

Set $S_1 = N^{r_1}(v_1)$ and $T_1 = N^{r_1+2}(v_1)$, repeat this procedure in the remaining graph $G[V \setminus T_1]$, until no vertex left. Denote each node selected in the algorithm by v_i which is referred to as *core nodes*. Denote $S_i = N^{r_i}(v_i)$ and $T_i = N^{r_i+2}(v_i)$.

The key point of the above method lies in the fact that given an $\epsilon > 0$, there always exists a uniform constant $r(\epsilon) = O(\frac{1}{\epsilon} \ln \frac{1}{\epsilon})$ for all core node v_i which is dependent only on ϵ, such that $|D_{opt}(T_i)| \leq (1+\epsilon)|D_{opt}(S_i)|$ and $r_i \leq r(\epsilon)$. Thus, $|D_{opt}(T_i)|$ can be computed exactly locally by enumeration with time complexity $n^{O(r(\epsilon)^2)}$ (which is a polynomial in n whenever ϵ is given), since $|D_{opt}(T_i)| \leq |MIS(T_i)| = O(r(\epsilon)^2)$.

In the rest of this section, we give a PTAS for MWDS with smooth weights in unit disk graphs. The main line of the algorithms follows that of Nieberg and Hurink (2006). However, the algorithm in Nieberg and Hurink (2006) cannot be directly applied for the MWDS problem, this is because there may not exist a uniform constant $r(\epsilon)$ due to the fact that the ratio of the maximum weight to the minimum weight of two nodes in the graph G may have no upper bounds (even if the weight function is smooth). To over come this difficulty, the key improvement of our algorithm over (Nieberg and Hurink 2006) is that we select the core nodes carefully at each iteration (in Nieberg and Hurink (2006), the core nodes are chosen with some arbitrariness), i.e., at each time, we selected the core node with the largest weight among all nodes in the remaining graphs. With this significant modification, we are able to show that we can get a PTAS for MWDS.

In the following algorithm, $D_{opt}(N^r(u))$ denotes the dominating set of $N^r(u)$ (as before, $D_{opt}(N^r(u))$ is computed w.r.t. the whole underlying graph) with minimum total weights, where u is a core node, and $w(.)$ is used to denote total weights of the corresponding dominating set. We emphasize that the notation $N^r(u)$ in the algorithm is referred to those nodes within r-hop away from u in $G[U]$ (not in G), where U is used to record nodes waiting to be dealt with, and W is used to record the core nodes obtained.

Algorithm 1

Input: A unit disk graph $G=(V,E)$ with weight function $w:V\to R^+$, and parameter $\epsilon>0$.

1: $U\leftarrow V(G)$; $W\leftarrow\emptyset$.
2: **while** $U\neq\emptyset$ **do**
3: Choose a node $u\in U$ with the maximum weight; find the smallest non-negative integer r_u satisfying $w(D_{opt}(N^{r_u+2}(u)))\leq(1+\epsilon)w(D_{opt}(N^{r_u}(u)))$. Set $U\leftarrow U\setminus N^{r_u+2}(u)$, $W\leftarrow W\cup\{u\}$.
/* $w(D_{opt}(N^{r_u}(u)))$ is computed by exhausted search*/
4: **end while**
5: Output $D=\bigcup_{u\in W}D_{opt}(N^{r_u+2}(u))$.

4 Proof of correctness

In this section, we show the correctness of Algorithm 1. In line 3, a dominating $D_{opt}(N^{r_u}(u))$ with minimum total weight is computed for each cluster $N^{r_u}(u)$. This was done by an exhausted search. The following lemma guarantees that this can always be done in polynomial time.

Lemma 4.1 *Let G be a unit disk graph with smooth weights. Then, for any real number $\epsilon>0$, there exists a constant $r(\epsilon)$ which depends only on ϵ, but is independent of the topology of G, such that $w(D_{opt}(N^{r+2}(v)))\leq(1+\epsilon)w(D_{opt}(N^{r}(v)))$ for each $v\in W$ whenever $r\leq r(\epsilon)$, where the set of core nodes W is chosen as in Algorithm 1.*

Proof We prove the lemma by contradiction. If not, then there exists an $\epsilon_0>0$ and a node $v_0\in W$ such that

$$w(D_{opt}(N^{r+2}(v_0)))>(1+\epsilon_0)w(D_{opt}(N^{r}(v_0)))\quad\text{for } r=0,1,2,\ldots.$$

According to the rule of selecting core nodes in Algorithm 1, v_0 is the nodes with the maximum weight in the remaining graph induced by U. Note that the core node v_0 dominates v_0 and all its neighbors. Thus, we have

$$w(D_{opt}(N^{0}(v_0)))=\min\{w(v)|v\in N(v_0)\}\geq\frac{w(v_0)}{C}.$$

The inequality follows from the smoothness assumption of the weight function. Also, it is clear that $w_{opt}(N^1(v_0))\geq w_{opt}(N^0(v_0))\geq\frac{w(v_0)}{C}$.

If r is even, then we have

$$\begin{aligned}&w(D_{opt}(N^{r+2}(v_0)))\\&\quad>(1+\epsilon_0)w(D_{opt}(N^{r}(v_0)))>\cdots>(1+\epsilon_0)^{1+r/2}w(D_{opt}(N^{0}(v_0))).\end{aligned}$$

If r is odd, we have

$$w(D_{opt}(N^{r+2}(v_0))) > (1+\epsilon_0)w(D_{opt}(N^r(v_0))) > \cdots > (1+\epsilon_0)^{(1+r)/2}w(D_{opt}(N^1(v_0))).$$

Note that $MIS(N^{r+2}(v_0))$ is a dominating set of $N^{r+2}(v_0)$ and $D_{opt}(N^{r+2}(v_0))$ is the dominating set of $N^{r+2}(v_0)$ with minimum total weights. Therefore,

$$w(D_{opt}(N^{r+2}(v_0))) \le w(MIS(N^{r+2}(v_0))) \le w(v_0)|MIS(N^{r+2}(v_0))| \le w(v_0)(2r+1)^2.$$

If r is even, we obtain

$$(1+\epsilon_0)^{1+r/2}\frac{w(v_0)}{C} \le (1+\epsilon_0)^{1+r/2}w(D_{opt}(N^0(v_0))) \le w(v_0)(2r+1)^2.$$

It follows that

$$(1+\epsilon_0)^{1+r/2} \le C(2r+1)^2.$$

Similarly, if r is odd, we obtain $(1+\epsilon_0)^{(1+r)/2} \le C(2r+1)^2$. In either case, we obtain an inequality with its left hand side being exponential in r while its right hand side being a polynomial in r. Thus, if r is sufficient large, both inequalities cannot be true; a contradiction. □

Lemma 4.2 *The constant r_u in Lemma 4.1 is upper bounded by $r(\epsilon) = O(\frac{1}{\epsilon}\ln\frac{1}{\epsilon})$.*

Proof Consider the inequality $(1+\epsilon)^x \le 4Cx^2$, where $\epsilon > 0$ is fixed. Let $x_0 = 2\frac{1}{\epsilon}\ln\frac{1}{\epsilon}$. We show there exists an ϵ_0, such that $(1+\epsilon)^{x_0} \le 4Cx_0^2 \le C(2x_0+1)^2$ whenever $0 < \epsilon < \epsilon_0$.

Taking logarithm on both sides and using the fact $\ln(1+\epsilon) \le \epsilon$ for any $\epsilon \ge 0$, we obtain that $x_0\epsilon \le \ln(4C) + 2\ln x_0$ implies $(1+\epsilon)^{x_0} \le 4C{x_0}^2$. The former is equivalent to

$$2\ln\frac{1}{\epsilon} \le \ln(4C) + 2\ln 2 + 2\ln\frac{1}{\epsilon} + 2\ln\ln\frac{1}{\epsilon}.$$

Clearly, there exists an ϵ_0 such that the above inequality holds whenever $0 < \epsilon < \epsilon_0$. This shows that when ϵ is sufficiently small, $(1+\epsilon)^{x_0} \le C(2x_0+1)^2$ holds and, hence $r(\epsilon) \le x_0 = 2\frac{1}{\epsilon}\ln\frac{1}{\epsilon}$. □

Lemma 4.3 *The time complexity of Algorithm 1 is $n^{O(C^{1/\epsilon\ln(1/\epsilon)}1/\epsilon^2\ln^2(1/\epsilon))}$, where n is the order of the graph.*

Proof The time complexity of Algorithm 1 is dominated by enumerating the local optimal solution in $N^{r_u+2}(u)$ with the largest radius r_u. Denote by w_{min} the minimum weight among all weights for nodes in $N^{r_u+2}(u)$. Then we have

$$w_{min}|D_{opt}(N^{r_u+2}(u))| \le w(D_{opt}(N^{r_u+2}(u))) \le w(u)|MIS(N^{r_u+2}(u))| \le w(u)(2r_u+1)^2.$$

Furthermore, by the smoothness of the weight function, we have $\frac{w(u)}{w_{min}} \le C^{r_u}$. It follows that $|D_{opt}(N^{r_u+2}(u))| \le C^{r_u}(2r_u + 1)^2$. Therefore, if we enumerate all possible cases to compute $D_{opt}(N^{r_u+2}(u))$, the time complexity is at most $n^{O(C^{1/\epsilon \ln(1/\epsilon)} 1/\epsilon^2 \ln^2(1/\epsilon))}$. □

Lemma 4.4 *$D = \bigcup_{u \in W} D_{opt}(N^{r+2}(u))$ obtained in Algorithm 1 is a dominating set for graph G.*

Proof By Algorithm 1, $V(G) = \bigcup_{u \in W} N^{r+2}(u)$. Thus, for each $v \in V$, there exists some $N^{r+2}(u)$ such that $v \in N^{r+2}(u)$. Clearly, v is dominated by $D_{opt}(N^{r+2}(u))$. □

Theorem 4.5 *Algorithm 1 is a PTAS for the MWDS with smooth weights on unit disk graphs.*

Proof Let $S_i = N^{r_i}(v_i)$, $T_i = N^{r_i+2}(v_i)$ $(i = 1, 2, \ldots, k)$ be the sets of nodes constructed in Algorithm 1. Clearly $\{S_1, S_2, \ldots, S_k\}$ is a 2-separated collection of V. Let $D_{opt}(V)$ be the optimal solution to MWDS. Then $D_{opt}(V) \cap N(S_i)$ dominates S_i. By the definition, $D_{opt}(S_i)$ is the set of nodes with minimum total weights that dominates S_i. Thus, we obtain $w(D_{opt}(S_i)) \le w(D_{opt}(V) \cap N(S_i))$. By Lemma 4.1, $w(D_{opt}(T_i)) \le (1+\epsilon)w(D_{opt}(S_i))$. It follows that

$$w\left(\bigcup_{i=1}^{k} D_{opt}(T_i)\right) \le \sum_{i=1}^{k} w(D_{opt}(T_i)) \le (1+\epsilon)\sum_{i=1}^{k} w(D_{opt}(S_i)) \le (1+\epsilon)w(D_{opt}(V)).$$

Combined with Lemmas 4.3 and 4.4, we get the conclusion that $\bigcup_{i=1}^{k} D_{opt}(T_i)$ is a $(1+\epsilon)$-approximation of $D_{opt}(V)$. This completes the proof. □

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