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On domination number of Cartesian product of directed paths

Juan Liu · Xindong Zhang · Jixiang Meng

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Abstract Let $\gamma(G)$ denote the domination number of a digraph G and let $P_m \Box P_n$ denote the Cartesian product of P_m and P_n, the directed paths of length m and n. In this paper, we give a lower and upper bound for $\gamma(P_m \Box P_n)$. Furthermore, we obtain a necessary and sufficient condition for $P_m \Box P_n$ to have efficient dominating set, and determine the exact values: $\gamma(P_2 \Box P_n) = n$, $\gamma(P_3 \Box P_n) = n + \lceil \frac{n}{4} \rceil$, $\gamma(P_4 \Box P_n) = n + \lceil \frac{2n}{3} \rceil$, $\gamma(P_5 \Box P_n) = 2n + 1$ and $\gamma(P_6 \Box P_n) = 2n + \lceil \frac{n+2}{3} \rceil$.

Keywords Cartesian product · Directed path · Domination number

1 Introduction

Throughout this article, a digraph $G = (V(G), E(G))$ always means a finite directed graph without loops and multiple arcs, where $V = V(G)$ is the vertex set and $E = E(G)$ is the arc set. Given two vertices u and v in G, we say u dominates v if $u = v$ or $uv \in E$. For a vertex $v \in V$, $N_G^+(v)$ and $N_G^-(v)$ denote the set of out-neighbors and in-neighbors of v, $d_G^+(v) = |N_G^+(v)|$ and $d_G^-(v) = |N_G^-(v)|$ denote the out-degree and

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J. Liu · J. Meng (✉)
College of Mathematics and System Sciences, Xinjiang University, Urumqi, Xinjiang 830046, P.R. China
e-mail: mengjx@sina.cn

J. Liu
e-mail: liujuan1999@126.com

J. Liu · X. Zhang
College of Mathematics Sciences, Xinjiang Normal University, Urumqi, Xinjiang 830054, P.R. China

in-degree of v in G, respectively. Let $N_G^+[v] = N_G^+(v) \cup \{v\}$. A vertex v dominates all vertices in $N_G^+[v]$. A set $S \subseteq V$ is a *dominating set* of G if S dominates $V(G)$. The *domination number* of G, denoted by $\gamma(G)$, is the minimum cardinality of a dominating set of G. A dominating set S is called a $\gamma(G)$-set of G if $|S| = \gamma(G)$. Note that each dominating set of G contains all vertices with in-degree 0 in G. A set $S \subseteq V$ is efficient dominating set if each vertex of G is dominated by exactly one vertex in S. Note that any efficient dominating set in a digraph must be of size $\gamma(G)$. Let P_n denote a directed path with n vertices.

Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two digraphs which have disjoint vertex sets $V_1 = \{x_1, x_2, \dots, x_{n_1}\}$ and $V_2 = \{y_1, y_2, \dots, y_{n_2}\}$ and disjoint arc sets E_1 and E_2, respectively. The *Cartesian product* $G = G_1 \Box G_2$ has vertex set $V = V_1 \times V_2$, and $(x_i, y_j)(x_{i'}, y_{j'}) \in E(G_1 \Box G_2)$ if and only if either $x_i x_{i'} \in E_1$ and $y_j = y_{j'}$, or $x_i = x_{i'}$ and $y_j y_{j'} \in E_2$. The subdigraph $G_1^{y_i}$ of $G_1 \Box G_2$ has vertex set $V_1^{y_i} = \{(x_j, y_i) :$ for any $x_j \in V_1$, fixed $y_i \in V_2\} \cong V_1$, and arc set $E_1^{y_i} = \{(x_j, y_i)(x_{j'}, y_i) : x_j x_{j'} \in E_1\} \cong E_1$. It is clear that $G_1^{y_i} \cong G_1$. Similarly, the subdigraph $G_2^{x_i}$ of $G_1 \Box G_2$ has vertex set $V_2^{x_i} = \{(x_i, y_j) :$ for any $y_j \in V_2$, fixed $x_i \in V_1\} \cong V_2$, and arc set $E_2^{x_i} = \{(x_i, y_j)(x_i, y_{j'}) : y_j y_{j'} \in E_2\} \cong E_2$. It is clear that $G_2^{x_i} \cong G_2$. In 1983, Jacobson and Kinch (1983) first discussed the domination number of the Cartesian product of two undirected graphs, and they established the domination numbers of $P_m \Box P_n (m \le 4)$. Beyond $k = 4$ the problem becomes much more difficult. Hare (1986) developed an algorithm to compute the domination number of $P_m \Box P_n$ and using the output of an implementation of her algorithm she found simple formulas for $\gamma(P_m \Box P_n)$ when $1 \le m \le 10$ agreeing with her data. Chang and Clark (1993) proved that Hare's formulas for $m = 5$ and 6 and $n \ge 1$. Note that Hare's algorithm does not produce a dominating set for $P_m \Box P_n$ as it computers $\gamma(P_m \Box P_n)$. In 1994, Chang and Clark (1994) presented dominating set for $5 \le m \le 10$. Other related works for Cartesian product of undirected paths (see, for example El-Zahar and Pareek 1991; Faudree and Schelp 1990; Gravie and Mollard 1997; Hartnell and Rall 2004). However, to date no research has been done for Cartesian product of two directed paths, and the methods and findings existing for undirected paths could not be applied to Cartesian product of two directed paths with slight modifications. Thus, in this paper, we study the Cartesian product $P_m \Box P_n$ of P_m and P_n, the directed paths of length m and n. By the definition, we know that $P_m \Box P_n \cong P_n \Box P_m$. We give a lower and upper bound for $\gamma(P_m \Box P_n)$. Furthermore, we obtain a necessary and sufficient condition for $P_m \Box P_n$ to have efficient dominating set, and determine the exact values: $\gamma(P_2 \Box P_n) = n$, $\gamma(P_3 \Box P_n) = n + \lceil \frac{n}{4} \rceil$, $\gamma(P_4 \Box P_n) = n + \lceil \frac{2n}{3} \rceil$, $\gamma(P_5 \Box P_n) = 2n + 1$ and $\gamma(P_6 \Box P_n) = 2n + \lceil \frac{n+2}{3} \rceil$. And we present dominating sets for $2 \le m \le 6$.

Terminologies not given here are referred to Chartrand and Lesniak (2005).

2 Main results

We emphasize that the vertices of a directed path P_n are always denoted by $\{0, 1, \dots, n-1\}$ throughout this paper. This notation turned out to be convenient to formulate the proof of the following results. Note that if $m = 1$, then $P_m \Box P_n = P_n$, and if $n = 1$, then $P_m \Box P_n = P_m$. We know that $\gamma(P_n) = \lceil \frac{n}{2} \rceil$ for the directed path P_n. Thus, we discuss the cases that $m, n \ge 2$ in this paper.

Lemma 2.1 *Let $m \geq 2$. Then there exists a minimum dominating set S of $P_m \Box P_n$ such that $1 \leq |P_m^i \cap S| \leq m-1$ for every $i \in V(P_n)$.*

Proof Let S be a minimum dominating set of $P_m \Box P_n$. Suppose that $|P_m^{i_t} \cap S| = m$ holds for k subdigraphs $P_m^{i_t}$, $1 \leq k \leq n-1$, $1 \leq t \leq k$. We now construct a dominating set S' with $|S'| = |S|$ such that only $k-1$ subdigraphs $P_m^{i_t}$ have m vertices in common with S'.

If $|P_m^0 \cap S| = m$, then $|P_m^1 \cap S| = 0$ and $S' = (S \setminus \{(1,0)\}) \cup \{(1,1)\}$ has required properties. We find that $|P_m^{n-1} \cap S| \neq m$ since $(j, n-1)$ exactly dominates itself and another vertex $(j+1, n-1)$ for $0 \leq j < m-1$. Assume now that $|P_m^i \cap S| = m$ for $i \in V(P_n) \setminus \{0, n-1\}$. We have $|P_m^{i+1} \cap S| = 0$ and $S' = (S \setminus \{(1,i)\}) \cup \{(1, i+1)\}$ has required properties. Thus, $|P_m^i \cap S| \leq m-1$ for every $i \in V(P_n)$.

Let S be a minimum dominating set of $P_m \Box P_n$ such that $|P_m^i \cap S| \leq m-1$ for every $i \in V(P_n)$. We now prove that $|P_m^i \cap S| \geq 1$. Note that $(0,0) \in (P_m^0 \cap S)$ and thus $|P_m^0 \cap S| \geq 1$. Suppose that there exists a vertex $i \in V(P_n) \setminus \{0\}$ such that $|P_m^i \cap S| = 0$. $(0,i)$ is exactly dominated by the vertex $(0, i-1)$, so $(0, i-1) \in S$ and, each vertex (j, i) $(1 \leq j \leq m-1)$ could be dominated by $(j-1, i)$ and $(j, i-1)$. Since $(j-1, i) \notin S$, we have $(j, i-1) \in S$ for all $0 \leq j \leq m-1$. Thus $|P_m^{i-1} \cap S| = m$. A contradiction. Therefore, there exists a minimum dominating set S of $P_m \Box P_n$ such that $1 \leq |P_m^i \cap S| \leq m-1$ for every $i \in V(P_n)$. □

Theorem 2.2 *Let $n \geq 2$. Then $\gamma(P_2 \Box P_n) = n$.*

Proof By Lemma 2.1, we have $\gamma(P_2 \Box P_n) \geq n$. Let $S_0 = \{(0,i) : \forall i \in V(P_n)\} \subseteq V(P_2 \Box P_n)$, then S_0 is a dominating set of $P_2 \Box P_n$. Thus, $\gamma(P_2 \Box P_n) = n$. □

We now consider the Cartesian product $P_3 \Box P_n$ and define a set S_1 (see Fig. 1) as follows: S_1 consists of vertices $(0,i)$ and $(1,i)$, $i \equiv 0 \pmod 4$; $(2,i)$, $i \equiv 1 \pmod 4$; $(0,i)$, $i \equiv 2 \pmod 4$; $(1,i)$, $i \equiv 3 \pmod 4$. Note that $|S_1| = n + \lceil \frac{n}{4} \rceil$.

Theorem 2.3 *Let $n \geq 2$. Then $\gamma(P_3 \Box P_n) = n + \lceil \frac{n}{4} \rceil$.*

Proof Since the set S_1 defined above is a dominating set of $P_3 \Box P_n$, we have $\gamma(P_3 \Box P_n) \leq n + \lceil \frac{n}{4} \rceil$. We now prove that $\gamma(P_3 \Box P_n) \geq n + \lceil \frac{n}{4} \rceil$.

By Lemma 2.1, let S be a minimum dominating set of $P_3 \Box P_n$ such that $1 \leq |P_3^i \cap S| \leq 2$ for every $i \in V(P_n)$. Assume now that there are t vertices $\{i_1, i_2, \ldots, i_t\}$ such that $|P_3^{i_l} \cap S| = 2$ $(1 \leq l \leq t)$ and then $|S| = n + t$. Note that $|P_3^0 \cap S| = 2$. If $t < \lceil \frac{n}{4} \rceil$, then there exist four consecutive vertices $r, r+1, r+2, r+3 \in V(P_n)$ such that

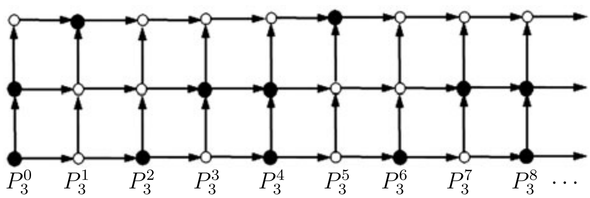
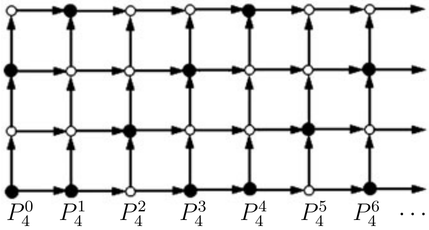


Fig. 1 The set S_1

Fig. 2 The set S_2



$|P_3^r \cap S| = |P_3^{r+1} \cap S| = |P_3^{r+2} \cap S| = |P_3^{r+3} \cap S| = 1$, it is impossible. Therefore, $t \geq \lceil \frac{n}{4} \rceil$. Thus, $\gamma(P_3 \Box P_n) = n + \lceil \frac{n}{4} \rceil$. □

Next, we consider the Cartesian product $P_4 \Box P_n$ and define a set S_2 (see Fig. 2) as follows: S_2 consists of vertices $(0, i)$ and $(2, i)$, $i \equiv 0 \pmod 3$; $(0, i)$ and $(3, i)$, $i \equiv 1 \pmod 3$; $(1, i)$, $i \equiv 2 \pmod 3$. Note that $|S_2| = n + \lceil \frac{2n}{3} \rceil$.

Lemma 2.4 *Let $n \geq 2$. Then there exists a minimum dominating set S of $P_4 \Box P_n$ such that $1 \leq |P_4^i \cap S| \leq 2$ for every $i \in V(P_n)$.*

Proof If $n = 2$ or 3, then S_0 and S_1 in proof of Theorems 2.2 and 2.3 have the properties that $1 \leq |P_4^i \cap S_0| \leq 2$ for every $i \in V(P_2)$ and $1 \leq |P_4^i \cap S_1| \leq 2$ for every $i \in V(P_3)$. Now we consider that $n \geq 4$. Let S be a minimum dominating set of $P_4 \Box P_n$ such that $1 \leq |P_4^i \cap S| \leq 3$ for every $i \in V(P_n)$ by Lemma 2.1. Suppose that $|P_4^{i_t} \cap S| = 3$ holds for k subdigraphs $P_4^{i_t}$, $1 \leq k \leq n-1$, $1 \leq t \leq k$. We now construct a dominating set S' with $|S'| = |S|$ such that only $k-1$ subdigraphs $P_4^{i_t}$ have 3 vertices in common with S'.

Let P_4^i be such a subdigraph with $|P_4^i \cap S| = 3$ and such that $|P_4^l \cap S| \leq 2$ for all $l \in \{i+1, \ldots, n-1\}$. Since $|P_4^{n-1} \cap S| \leq 2$, we have $i \leq n-2$. We consider four cases.

Case 1 $(1, i)$, $(2, i)$ and $(3, i)$ are in S.

Since S is a minimum dominating set of $P_4 \Box P_n$, it is easy to see that $(1, i+1)$, $(2, i+1)$ and $(3, i+1)$ must be not in S and $(0, i+1)$ is in S. Thus, $S' = (S \setminus \{(2, i)\}) \cup \{(2, i+1)\}$ has required properties.

Case 2 $(0, i)$, $(1, i)$ and $(2, i)$ are in S.

Note that $(0, i+1)$ and $(1, i+1)$ must be not in S in this case. If $i = n-2$, then there exists only one vertex of $\{(2, i+1), (3, i+1)\}$ belongs to S. Thus,

$$S' = (S \setminus \{(1, i)\}) \cup \{(1, i+1)\} \tag{1}$$

has required properties.

Assume now that $i \leq n-3$. If only one vertex of $\{(2, i+1), (3, i+1)\}$ belongs to S, then we have done as (1). If both $(2, i+1)$ and $(3, i+1)$ are in S, then $(1, i+2)$, $(2, i+2)$ and $(3, i+2)$ must be not in S and $(0, i+2)$ is in S. Thus, $S' = (S \setminus \{(1, i), (2, i+1)\}) \cup \{(1, i+1), (2, i+2)\}$ has required properties.

Case 3 $(0, i)$, $(1, i)$ and $(3, i)$ are in S.

If at least one vertex of $\{(2, i+1), (3, i+1)\}$ belongs to S, then we could discuss on $(S \setminus \{(3, i)\}) \cup \{(2, i)\}$ analogous to Case 2. If both $(2, i+1)$ and $(3, i+1)$ are not in S, then $(1, i+1)$ must be in S. If $i = n-2$, then $(0, i+1)$ is not in S. Thus,

$$S' = (S \setminus \{(1, i), (3, i)\}) \cup \{(2, i), (3, i+1)\} \tag{2}$$

has required properties.

Assume now that $i \le n-3$. If $(0, i+1)$ is not in S, then we have done as (2). If $(0, i+1)$ is in S, then we could discuss on $(S \setminus \{(1, i), (3, i)\}) \cup \{(2, i), (2, i+1)\}$ analogous to Case 2.

Case 4 $(0, i)$, $(2, i)$ and $(3, i)$ are in S.

Note that $(2, i+1)$ and $(3, i+1)$ must be not in S in this case. If $i = n-2$, then there exists only one vertex of $\{(0, i+1), (1, i+1)\}$ belongs to S. Thus,

$$S' = (S \setminus \{(3, i)\}) \cup \{(3, i+1)\} \tag{3}$$

has required properties.

Assume now that $i \le n-3$. If only one vertex of $\{(0, i+1), (1, i+1)\}$ belongs to S, then we have done as (3). If both $(0, i+1)$ and $(1, i+1)$ are in S, then we could discuss on $(S \setminus \{(3, i)\}) \cup \{(2, i+1)\}$ analogous to Case 2.

Therefore, there exists a minimum dominating set S of $P_m \Box P_n$ such that $1 \le |P_4^i \cap S| \le 2$ for every $i \in V(P_n)$. □

Theorem 2.5 *Let $n \ge 2$. Then $\gamma(P_4 \Box P_n) = n + \lceil \frac{2n}{3} \rceil$.*

Proof Since the set S_2 defined above is a dominating set of $P_4 \Box P_n$, we have $\gamma(P_4 \Box P_n) \le n + \lceil \frac{2n}{3} \rceil$. We now prove that $\gamma(P_4 \Box P_n) \ge n + \lceil \frac{2n}{3} \rceil$.

By Lemma 2.4, let S be a minimum dominating set of $P_4 \Box P_n$ such that $1 \le |P_4^i \cap S| \le 2$ for every $i \in V(P_n)$. Assume now that there are t vertices $\{i_1, i_2, \ldots, i_t\}$ such that $|P_4^{i_l} \cap S| = 2$ $(1 \le l \le t)$, and then $|S| = n + t$. Note that $|P_4^0 \cap S| = |P_4^1 \cap S| = 2$ and $(0, 0), (2, 0) \in P_4^0 \cap S$. It is easy to calculate that $\gamma(P_4 \Box P_2) = 4$, $\gamma(P_4 \Box P_3) = 5$, $\gamma(P_4 \Box P_4) = 7$ and $\gamma(P_4 \Box P_5) = 9$. Thus, $t = \lceil \frac{2n}{3} \rceil$ for $n = 2, 3, 4, 5$. Let $n \ge 6$, if $t = n-1$ or $t = n-2$, then $t \ge \lceil \frac{2n}{3} \rceil$. Assume now that $t < n-2$. Let p and q $(p < q)$ are two any vertices in $V(P_n)$ such that $|P_4^p \cap S| = |P_4^q \cap S| = 1$. We consider three cases.

Case 1 $q = p+1$.

Since S is a minimum dominating set of $P_4 \Box P_n$, this case is impossible.

This case illustrates that there does not exist two consecutive vertices r and $r+1$ in $V(P_n)$ such that $|P_4^r \cap S| = |P_4^{r+1} \cap S| = 1$.

Case 2 $q = p+2$.

We have $|P_4^p \cap S| = |P_4^{p+2} \cap S| = 1$ and $|P_4^{p+1} \cap S| = 2$. Thus, there exists only one the following situation: only four vertices $(2, p)$, $(0, p+1)$, $(3, p+1)$ and $(1, p+2)$ belong to S in $V(P_4^p \cup P_4^{p+1} \cup P_4^{p+2})$. We find that $(0, p-1)$ and $(1, p-1)$ must be in S. Thus $|P_4^{p-2} \cap S| = 2$ ($(3, p-2) \in S$ and one vertex of $\{(0, p-2), (1, p-2), (2, p-2)\}$ is in S) and $|P_4^{p+3} \cap S| = 2$ (see Fig. 3). Thus, there are six consecutive

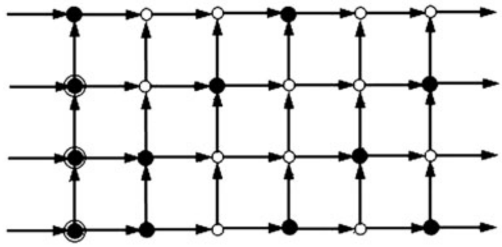


Fig. 3 An illustration for the proof of Theorem 2.5 (Case 2)

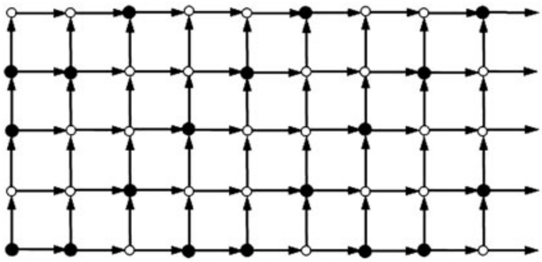


Fig. 4 The set S_3

vertices in $V(P_n)$ such that there are at most two subdigraphs with $|P_4^i \cap S| = 1$ and at least four subdigraphs with $|P_4^l \cap S| = 2$. Therefore, we have $t \geq \lceil \frac{2n}{3} \rceil$.

Case 3 $q \geq p+3$.

Clearly, $t \geq \lceil \frac{2n}{3} \rceil$ in this case.

From above we have $t \geq \lceil \frac{2n}{3} \rceil$ for all cases. Thus, $\gamma(P_4 \Box P_n) = |S| = n + t \geq n + \lceil \frac{2n}{3} \rceil$.

Therefore, $\gamma(P_4 \Box P_n) = n + \lceil \frac{2n}{3} \rceil$. □

Now we consider the Cartesian product $P_5 \Box P_n$ and define a set S_3 (see Fig. 4) as follows: S_3 consists of vertices $(0,0), (2,0), (3,0)$; $(0,i), (3,i)$, $i \equiv 1 \pmod 3$; $(1,i), (4,i)$, $i \equiv 2 \pmod 3$; $(0,i), (2,i)$, $i \equiv 0 \pmod 3$ for $i \geq 1$. Note that $|S_3| = 2n+1$.

Theorem 2.6 *Let $n \geq 2$. Then $\gamma(P_5 \Box P_n) = 2n+1$.*

Proof By Lemma 2.1, let S be a minimum dominating set of $P_5 \Box P_n$ such that $|P_5^i \cap S| \geq 1$ for $0 \leq i \leq n-1$. Note that $|P_5^0 \cap S| \geq 3$ and $|(P_5^0 \cup P_5^1) \cap S| \geq 5$. If $|P_5^i \cap S| = 1$ for $2 \leq i \leq n-1$, then $|P_5^{i-1} \cap S| \geq 3$. Thus $|S| \geq 2n+1$. We know that the set S_3 is a dominating set with $2n+1$ vertices. Hence, $\gamma(P_5 \Box P_n) = 2n+1$. □

Now we consider the Cartesian product $P_6 \Box P_n$.

Lemma 2.7 *Let $n \geq 6$. Then there exists a minimum dominating set S of $P_6 \Box P_n$ such that $2 \leq |P_6^i \cap S| \leq 3$ for every $i \in V(P_n)$.*

Proof Let $S^\star$ be a minimum dominating set of $P_6 \square P_n$ such that $1 \le |P_6^i \cap S^\star| \le 5$ for every $i \in V(P_n)$ by Lemma 2.1. Now we first prove that there exists a minimum dominating set $S^\star$ of $P_6 \square P_n$ such that $|P_6^i \cap S^\star| \ge 2$ for every $i \in V(P_n)$. Note that $(0,0) \in (P_6^0 \cap S^\star)$ and $|P_6^0 \cap S^\star| \ge 3$. Suppose that $|P_6^{i_t} \cap S^\star| = 1$ holds for k subdigraphs $P_6^{i_t}$, $1 \le k \le n-1, 1 \le t \le k$. We now construct a dominating set $S^{\star'}$ with $|S^{\star'}| = |S^\star|$ such that only $k-1$ subdigraphs $P_6^{i_t}$ have 1 vertices in common with $S^{\star'}$. Assume now that $|P_6^i \cap S^\star| = 1$ for a vertex $i \in (V(P_n) \setminus \{0\})$. If $(0,i) \in (P_6^i \cap S^\star)$, then $(2,i-1), (3,i-1), (4,i-1)$ and $(5,i-1)$ must be in $S^\star$, thus, $S^{\star'} = (S^\star \setminus \{(3,i-1)\}) \cup \{(3,i)\}$ has required properties. Similarly, we can construct required $S^{\star'}$ for $(j,i) \in (P_6^i \cap S^\star)$ when $j \in \{1,2,\dots,5\}$. By a similar argument, we can prove that there exists a minimum dominating set $S^\star$ of $P_6 \square P_n$ such that $|P_6^i \cap S^\star| \le 4$ for every $i \in V(P_n)$.

Next, we will prove that there exists a minimum dominating set S of $P_6 \square P_n$ such that $|P_6^i \cap S| \le 3$ for every $i \in V(P_n)$. Let S be a minimum dominating set of $P_6 \square P_n$ such that $2 \le |P_6^i \cap S| \le 4$ for every $i \in V(P_n)$ by above argument. Suppose that $|P_6^{i_t} \cap S| = 4$ holds for k subdigraphs $P_6^{i_t}$, $1 \le k \le n-1$, $1 \le t \le k$. We now construct a dominating set S' with $|S'| = |S|$ such that only $k-1$ subdigraphs $P_6^{i_t}$ have 4 vertices in common with S'. Let P_6^i be such a subdigraph with $|P_6^i \cap S| = 4$ and such that $|P_6^l \cap S| \le 3$ for all $l \in \{i+1,\dots,n-1\}$. Since $|P_6^{n-1} \cap S| \le 3$, we have $i \le n-2$. It is clear that there exists an integer $r \in \{i+1,\dots,n-1\}$ such that $|P_6^r \cap S| = 2$ and $|P_6^h \cap S| = 3$ for $i < h < r$ (h is an integer) since S is a minimum dominating set. We consider two cases.

Case 1 $r = i+1$.

If $|P_6^i \cap S| = 4$ and there exist three consecutive integers $q, q+1$ and $q+2$, such that $\{(q,i),(q+1,i),(q+2,i)\} \in (P_6^i \cap S)(0 \le q \le 3)$, then $S' = (S \setminus \{(q+1,i)\}) \cup \{(q+1,i+1)\}$ has required properties. If $\{(4,i),(5,i)\} \in (P_6^i \cap S)$, then $S' = (S \setminus \{(5,i)\}) \cup \{(5,i+1)\}$ has required properties. Now we consider the others cases.

Subcase 1.1 $(5,i) \notin (P_6^i \cap S)$ and $(4,i) \in (P_6^i \cap S)$.

In this case $(2,i)$ must be not in S, otherwise, there will exist three consecutive vertices in $P_6^i \cap S$. Hence, there are at least one vertex of $\{(1,i+1),(2,i+1)\}$ in S. If $(1,i+1) \in S$, then $S' = (S \setminus \{(1,i),(3,i)\}) \cup \{(2,i),(3,i+1)\}$ has required properties. Similarly, if $(2,i+1) \in S$, then $S' = (S \setminus \{(1,i),(3,i)\}) \cup \{(2,i),(1,i+1)\}$ has required properties.

Subcase 1.2 $(5,i) \in (P_6^i \cap S)$ and $(4,i) \notin (P_6^i \cap S)$.

If $(1,i) \notin (P_6^i \cap S)$, then there are at least one vertex of $\{(3,i+1),(4,i+1)\}$ in S. If $(3,i+1) \in S$, then we consider the set $(S \setminus \{(3,i)\}) \cup \{(4,i)\}$ as the above argument; If $(4,i+1) \in S$, then we consider the set $(S \setminus \{(5,i)\}) \cup \{(4,i)\}$ as the above argument.

If $(2,i) \notin (P_6^i \cap S)$, then there are at least one vertex of $\{(1,i+1),(2,i+1)\}$ in S and at least one vertex of $\{(3,i+1),(4,i+1)\}$ in S. If $(1,i+1) \in S$, then we consider the set $(S \setminus \{(1,i)\}) \cup \{(2,i)\}$ as the above argument; If $(4,i+1) \in S$, then we consider the set $(S \setminus \{(5,i)\}) \cup \{(4,i)\}$ as the above argument. Now we consider that the vertices $(2,i+1)$ and $(3,i+1)$ are in S.

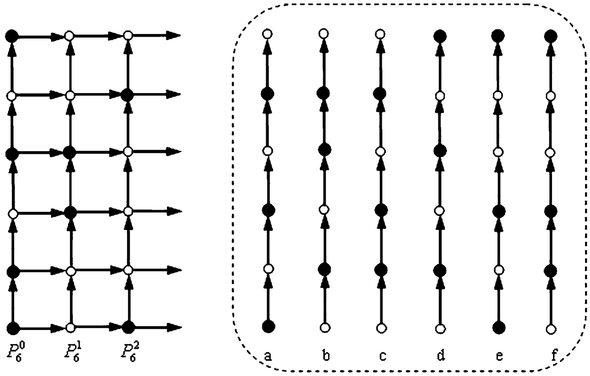


Fig. 5 An illustration for the proof of Lemma 2.7 (Subcase 1.2.2)

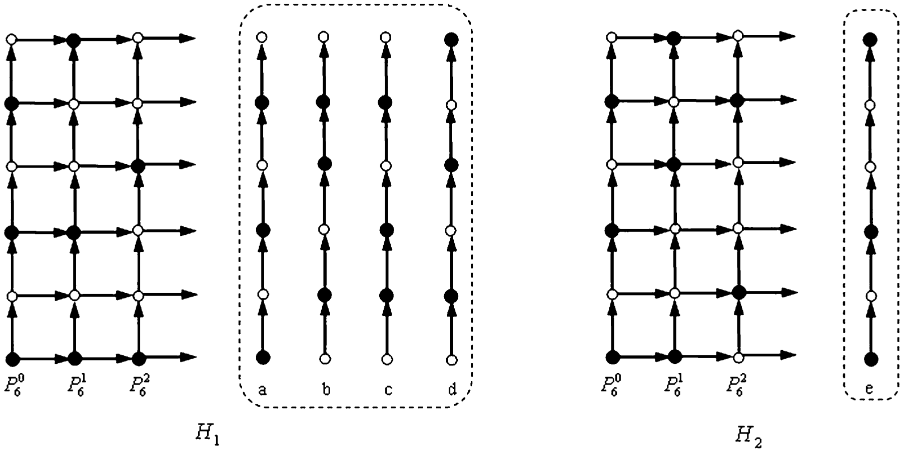


Fig. 6 An illustration for the proof of Lemma 2.7 (Subcase 1.2.2)

Subcase 1.2.1 $1 \leq i \leq n-2$.

If $(5, i-1) \in (P_6^{i-1} \cap S)$, then $S' = (S \setminus \{(5, i)\}) \cup \{(5, i+1)\}$ has required properties.

If $(4, i-1) \in (P_6^{i-1} \cap S)$, then $S' = (S \setminus \{(1, i), (3, i)\}) \cup \{(2, i), (1, i+1)\}$ has required properties.

If $(3, i-1) \in (P_6^{i-1} \cap S)$, then $S' = (S \setminus \{(3, i), (5, i)\}) \cup \{(4, i), (5, i+1)\}$ has required properties.

If $(2, i-1) \in (P_6^{i-1} \cap S)$, then $S' = (S \setminus \{(1, i)\}) \cup \{(1, i+1)\}$ has required properties.

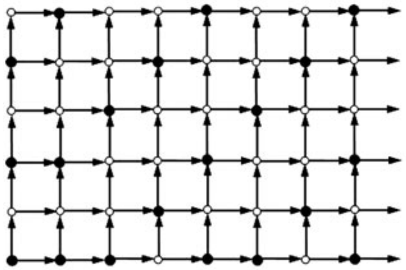
If $(0, i-1) \in (P_6^{i-1} \cap S)$, then $S' = (S \setminus \{(0, i)\}) \cup \{(0, i+1)\}$ has required properties.

The case $(1, i-1) \in (P_6^{i-1} \cap S)$ does not need discuss since $|P_6^{i-1} \cap S| \geq 2$.

Subcase 1.2.2 $i = 0$.

Since $n \geq 6$, the vertices of $P_6^0 \cap S$, $P_6^1 \cap S$ and $P_6^2 \cap S$ must be as Fig. 5. The vertices in $P_6^3 \cap S$ may occur six cases (see Fig. 5 $a, \ldots, f$ in the closed dashed curve). If we take one of the cases a, b, c and d, then we could obtain a required set

Fig. 7 The set S_4



as H_1 in Fig. 6 by adjusting the vertices in $P_6^0 \cap S$, $P_6^1 \cap S$ and $P_6^2 \cap S$. If we take the case e, then we could obtain a required set as H_2 in Fig. 6. If we take the case f, then the vertex $(0, 4)$ must be in S, thus, the case $(S \setminus \{(1, 3)\}) \cup \{(0, 3)\}$ is the same as the case e.

Case 2 $r > i + 1$.

In this case, we can obtain a set S such that $|P_6^q \cap S| = 3$ $(q = i, \dots, r - 2)$ and $|P_6^{r-1} \cap S| = 4$ by adjustment. Then the case is the same as Case 1.

From above all, we prove that there exists a minimum dominating set S of $P_6 \Box P_n$ such that $2 \le |P_6^i \cap S| \le 3$ for every $i \in V(P_n)$ with $n \ge 6$. $\square$

Now we define a set S_4 (see Fig. 7) as follows: S_4 consists of vertices $(0, 0), (2, 0), (4, 0)$; $(0, i), (2, i), (5, i)$, $i \equiv 1 \pmod 3$; $(0, i), (3, i)$, $i \equiv 2 \pmod 3$; $(1, i), (4, i)$, $i \equiv 0 \pmod 3$ for $i \ge 1$. Note that $|S_4| = 2n + \lceil \frac{n+2}{3} \rceil$.

Theorem 2.8 *Let $n \ge 6$. Then $\gamma(P_6 \Box P_n) = 2n + \lceil \frac{n+2}{3} \rceil$.*

Proof Note that the set S_4 is a dominating set of $P_6 \Box P_n$, thus, $\gamma(P_6 \Box P_n) \le 2n + \lceil \frac{n+2}{3} \rceil$. Now we prove that $\gamma(P_6 \Box P_n) \ge 2n + \lceil \frac{n+2}{3} \rceil$. By Lemma 2.7, we have $\gamma(P_6 \Box P_n) \ge 2n$. Let S be a minimum dominating set of $P_6 \Box P_n$ such that $2 \le |P_6^i \cap S| \le 3$ for every $i \in V(P_n)$ by Lemma 2.7. First, we give the following two claims.

Claim A There do not exist four consecutive integers $r, r + 1, r + 2$ and $r + 3$ such that $|P_6^r \cap S| = |P_6^{r+1} \cap S| = |P_6^{r+2} \cap S| = |P_6^{r+3} \cap S| = 2$.

Proof Suppose that $|P_6^r \cap S| = |P_6^{r+1} \cap S| = |P_6^{r+2} \cap S| = |P_6^{r+3} \cap S| = 2$. In fact, there are at least 12 vertices in $(P_6^{r+1} \cup P_6^{r+2} \cup P_6^{r+3})$ need be dominated by $(P_6^{r+1} \cup P_6^{r+2}) \cap S$. If there are more than 12 vertices in $(P_6^{r+1} \cup P_6^{r+2} \cup P_6^{r+3})$ need be dominated, a contradiction, since $|(P_6^{r+1} \cup P_6^{r+2}) \cap S| = 4$ and each vertex dominates at most three vertices including itself. If there are exact 12 vertices in $(P_6^{r+1} \cup P_6^{r+2} \cup P_6^{r+3})$ need be dominated, then each vertex in $(P_6^{r+1} \cup P_6^{r+2}) \cap S$ must exactly dominate three vertices including itself. Thus, $(5, r + 1)$ and $(5, r + 2)$

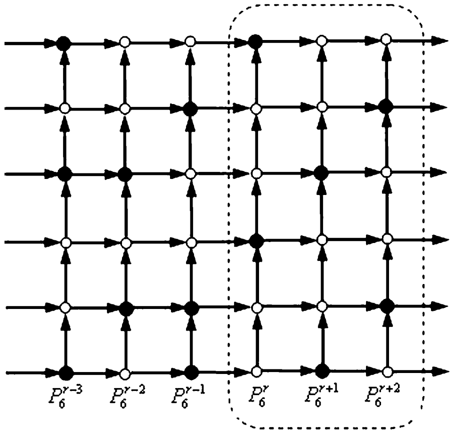


Fig. 8 An illustration for the proof of Theorem 2.8 (Claim B)

are not in S, $(4, r+2)$ and $(1, r+2)$ must be in S. In this case $|P_6^{r+3} \cap S| \geq 3$, a contradiction. $\square$

Claim B If there exists three consecutive integers $r, r+1$ and $r+2$ such that $|P_6^r \cap S| = |P_6^{r+1} \cap S| = |P_6^{r+2} \cap S| = 2$, then they must be as the set of Fig. 8 in the closed dashed curve and $r > 4$.

Let $|P_6^r \cap S| = |P_6^{r+1} \cap S| = |P_6^{r+2} \cap S| = 2$, then $|P_6^{r+3} \cap S| \geq 3$ and the vertices in $P_6^{r-1} \cap S$ must be as Fig. 8. If $|P_6^{r-2} \cap S| \geq 3$, then $|S| \geq |S_4|$. If $|P_6^{r-2} \cap S| = 2$, then it must be as Fig. 8. We could make $|P_6^{r-3} \cap S| = 3$ by adjusting the vertices in $P_6^{r-3} \cap S$ and $P_6^{r-4} \cap S$ (see Fig. 8). Thus, there are six consecutive vertices in $V(P_n)$ such that there are at least two subdigraphs with $|P_6^i \cap S| = 3$ and at most four subdigraphs with $|P_6^l \cap S| = 2$, clearly, $|S| \geq |S_4|$. From above all, we have $\gamma(P_6 \Box P_n) \geq 2n + \lceil \frac{n+2}{3} \rceil$. $\square$

Next, we will give a lower and upper bound for dominating set of Cartesian product $P_m \Box P_n$ of two directed paths. We define three sets S_5 (see Fig. 9), S_5', S_5'' as follows: S_5 consists of vertices $(0, i), (3, i), (6, i), \ldots, i \equiv 0 \pmod 3$; $(1, i), (4, i), (7, i), \ldots, i \equiv 1 \pmod 3$; $(2, i), (5, i), (8, i), \ldots, i \equiv 2 \pmod 3$; S_5' consists of vertices $(2, 0), (5, 0), (8, 0), \ldots$; S_5'' consists of vertices $(0, 2), (0, 5), (0, 8), \ldots$.

Theorem 2.9 *The Cartesian product digraph $P_m \Box P_n$ $(2 \leq m \leq n)$ has an efficient dominating set if and only if $m = 2$ and $n = 2$.*

Proof If $m = 2$ and $n = 2$, then $P_m \Box P_n$ has an efficient dominating set. Let $n \neq 2$. Suppose that S is an efficient dominating set of $P_m \Box P_n$, then $(0, 0), (0, 2) \in S$, in order to dominate the vertex $(1, 1)$, one vertex of $\{(0, 1), (1, 1), (1, 0)\}$ must be in S. If $(0, 1)$ or $(1, 0)$ is in S, then it is dominated twice by S, if $(1, 1)$ is in S, then the vertex $(1, 2)$ is dominated twice by S. A contradiction. $\square$

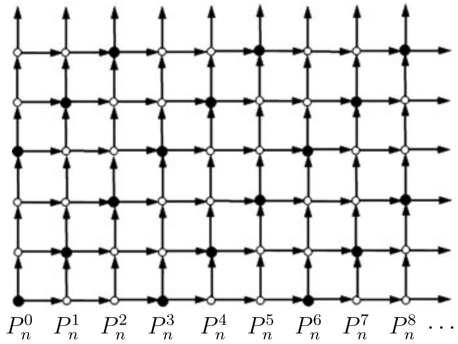


Fig. 9 The set S_5

Theorem 2.10 *Let $m, n \geq 3$. Then*

$$\left\lceil \frac{mn}{3} \right\rceil + 1 \leq \gamma(P_m \Box P_n)$$

$$\leq \begin{cases} 3k_1k_2 + k_1 + k_2 & \textit{if } m = 3k_1 \textit{ and } n = 3k_2; \\ k_1(n+1) + k_2 & \textit{if } m = 3k_1 \textit{ and } n \neq 3k_2; \\ k_2(m+1) + k_1 & \textit{if } m \neq 3k_1 \textit{ and } n = 3k_2; \\ 3k_1k_2 + 2k_1 + 2k_2 + 1 & \textit{if } m = 3k_1 + 1 \textit{ and } n = 3k_2 + 1; \\ 3k_1k_2 + 3k_1 + 2k_2 + 1 & \textit{if } m = 3k_1 + 1 \textit{ and } n = 3k_2 + 2; \\ 3k_1k_2 + 2k_1 + 3k_2 + 1 & \textit{if } m = 3k_1 + 2 \textit{ and } n = 3k_2 + 1; \\ 3k_1k_2 + 3k_1 + 3k_2 + 2 & \textit{if } m = 3k_1 + 2 \textit{ and } n = 3k_2 + 2. \end{cases}$$

Proof Since $P_m \Box P_n$ is a digraph with mn vertices and maximum out-degree is 2, every vertex in $P_m \Box P_n$ dominates at most three vertices including itself, thus, $\gamma(P_m \Box P_n) \geq \lceil \frac{mn}{3} \rceil$. If $3 \mid m$ or $3 \mid n$, then $P_m \Box P_n$ has no efficient dominating set by Theorem 2.9, and thus, $\lceil \frac{mn}{3} \rceil + 1 \leq \gamma(P_m \Box P_n)$. Now assume that $3 \nmid m$ and $3 \nmid n$. By analogous, in order to dominate vertices $(m-1, n-1)$, $(m-1, 1)$ and $(1, n-1)$, there are at least three times repeated domination. Thus, $\lceil \frac{mn}{3} \rceil$ vertices dominate at most $3\lceil \frac{mn}{3} \rceil - 3$ $(< mn)$ vertices. Hence $\gamma(P_m \Box P_n) \geq \lceil \frac{mn}{3} \rceil + 1$.

In the following, we will discuss the upper bound. Let $\lfloor \frac{m}{3} \rfloor = k_1$, $\lfloor \frac{n}{3} \rfloor = k_2$, we have

$$|S_5| = \begin{cases} 3k_1k_2 & \text{if } m = 3k_1 \text{ and } n = 3k_2; \\ nk_1 & \text{if } m = 3k_1 \text{ and } n \neq 3k_2; \\ mk_2 & \text{if } m \neq 3k_1 \text{ and } n = 3k_2; \\ 3k_1k_2 + k_1 + k_2 + 1 & \text{if } m = 3k_1 + 1 \text{ and } n = 3k_2 + 1; \\ 3k_1k_2 + 2k_1 + k_2 + 1 & \text{if } m = 3k_1 + 1 \text{ and } n = 3k_2 + 2; \\ 3k_1k_2 + k_1 + 2k_2 + 1 & \text{if } m = 3k_1 + 2 \text{ and } n = 3k_2 + 1; \\ 3k_1k_2 + 2k_1 + 2k_2 + 2 & \text{if } m = 3k_1 + 2 \text{ and } n = 3k_2 + 2, \end{cases}$$

$|S_5'| = \lfloor \frac{m}{3} \rfloor = k_1$ and $|S_5''| = \lfloor \frac{n}{3} \rfloor = k_2$.

It is clear that $S_5 \cup S_5' \cup S_5''$ is a dominating set of $P_m \Box P_n$. Thus, the upper bound can be obtain as theorem. □

The lower bound is sharp for $P_3 \Box P_3$, $P_3 \Box P_4$ and $P_4 \Box P_4$; the upper bound can be obtained when $m = 5$ and $n \geq 3$, or $m = 4$ and $n = 3k + 2$, or $m = 6$ and $n = 3k + 2$ where k is an integer.

3 Conclusions

This paper determined the exact value of $P_m \Box P_n$ ($m \leq 6$), and then gave a lower and upper bound for the domination number of $P_m \Box P_n$. It is our hope that the methods used here will eventually lead to a determination of $\gamma(P_m \Box P_n)$ for all m and n. Furthermore, we hope that discuss the bounds for the domination number of general digraphs.

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