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Node-weighted Steiner tree approximation in unit disk graphs

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Abstract Given a graph $G = (V, E)$ with node weight $w : V \rightarrow R^+$ and a subset $S \subseteq V$, find a minimum total weight tree interconnecting all nodes in S. This is the node-weighted Steiner tree problem which will be studied in this paper. In general, this problem is NP-hard and cannot be approximated by a polynomial time algorithm with performance ratio $a \ln n$ for any $0 < a < 1$ unless $NP \subseteq DTIME(n^{O(\log n)})$, where n is the number of nodes in s. In this paper, we are the first to show that even though for unit disk graphs, the problem is still NP-hard and it has a polynomial time constant approximation. We present a 2.5ρ-approximation where ρ is the best known performance ratio for polynomial time approximation of classical Steiner minimum tree problem in graphs. As a corollary, we obtain that there is a polynomial time $(9.875 + \varepsilon)$-approximation algorithm for minimum weight connected dominating set in unit disk graphs, and also there is a polynomial time $(4.875 + \varepsilon)$-approximation algorithm for minimum weight connected vertex cover in unit disk graphs.

Keywords Node-weighted Steiner tree · Approximation algorithm · Unit disk graphs

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1 Introduction

Given a graph $G = (V, E)$ with weight function w on E and a subset S, *Steiner tree problem* (STP) is to find a minimum subgraph of G interconnecting all nodes in S. We call the set S as *terminal set*. For any Steiner tree T for S and node $u \in V(T)$, we call u as a *terminal node* if $u \in S$, otherwise, we call it as a *Steiner node*. The Steiner tree problem, which is of great interest nowadays, is a classical problem in networks. The problem is known to be NP-hard in most metrics (Garey and Johnson 1978). Lots of effort have been devoted to study the approximation algorithms for this problem (Berman and Ramaiyer 1994; Hougardy and Prömel 1998; Kou et al. 1981; Robins and Zelikovski 2000; Zelikovsky 1993) and have successfully achieved constant ratios.

Node-Weighted Steiner Tree problem (NWST) is a variation of the classical STP. Given a graph $G = (V, E)$ with node weight $w : V \to R^+$ and a subset S of V, the node-weighted Steiner tree problem is to find a Steiner tree interconnecting nodes in the set S such that its total weight is minimum. Since all of nodes in terminal set will be contained in any Steiner tree constructed, for convenience, we usually set $w(u) = 0$ for all nodes $u \in S$.

In this paper, we study NWST problem in a special type of graphs called unit disk graph, which has a wide application in networks. A *unit disk graph* is associated with a set of unit disks in the Euclidean plane. Each node is the center of a unit disk. An edge exists between two nodes u and v if and only if $|uv| \le 1$, where $|uv|$ is the Euclidean distance between u and v. As far as we know, there is no constant approximation algorithms for NWST problem in this special type of graphs right now. In our paper, we propose a constant approximation algorithm for NWST problem with approximation ratio of 2.5ρ, where $\rho = 1 + \frac{\ln 3}{2} \approx 1.55$ is the best known approximation for the classical Steiner tree problem (Robins and Zelikovski 2000). Also, as an application, we obtain a $(9.875 + \varepsilon)$-approximation algorithm for the Minimum Weighted Connected Dominating Set (MWCDS) problem, the target of which is to construct a minimum weight connected subgraph dominating every node in set S in a node-weighted graph. It improves the best previous approximation ratio $10 + \varepsilon$ (Huang et al. 2009). Also for the Minimum Weighted Connected Vertex Cover problem (MWCVC) in unit disk graphs, we obtain a $(4.875 + \varepsilon)$-approximation algorithm.

The rest of this paper is organized as follows. In Sect. 2, we introduce the related work for NWST problem. In Sect. 3, we first present the main idea of our algorithms. Then, we give some useful definitions and denotations first before we introduce the algorithm and prove their approximation ratio. As corollaries, we obtain better approximations for MWCDS and MWCVC problem in unit disk graphs in Sect. 3.3. Finally, we conclude our results.

2 Related work

As we know, quite a lot work has been done for STP problem starting from early 80s. Most of the early work (Aneja 1980; Segev 1987; Beasley 1984; Shore et al.

1982) focus on solving the problem and its variations using linear programming. For instance, Aneja (1980) used a specialized integer programming (set covering) formulation to represent the STP and used the row generation scheme to solve the exponential increase of the number of constraints in the formulation related to the size of the problem. Segev (1987) proposed an integer programming solution for an extension of the standard Steiner Tree problem using Lagrangian relaxation and subgradient optimization. Though these work presented solutions for Steiner tree problem from the perspective of linear programming, most of them did not provide solid theoretical proof for the approximation ratios of their algorithms.

As a variation of classical STP problem, NWST problem is also a NP-hard problem. And in 1991, Berman (see references in Klein and Ravi 1995) proved that cannot be approximated within a factor of $o(\ln k)$ by giving an approximation-preserving reduction from Set Cover (Feige 1998) to NWST problem, where k is the size of the terminal set. This conclusion is achieved in general graph. Later, Klein and Ravi (1995) presented the first asymptotically optimal solution of approximation ratio $2\ln k$, by constructing the Steiner tree with spiders, a tree with at most one node of degree greater than two. Later, this ratio is improved to be $1.35\ln k$ by Guha and Khuller (1999) by introducing a new concept called branch-spider. This is the best known ratio up till now. One thing needs to be noticed here is that all these research work was done in general graph.

As a special case, when the weights of all nodes are same, the NWST problem is equivalent to Steiner Tree with Minimum Number of Steiner Nodes (STP-MSP) problem. Given a set of n terminals S in the Euclidean plane and a positive constant c, the STP-MSP problem is to find a Steiner tree for S with minimum number of Steiner nodes such that each edge in the tree has a length no more than c. The best approximation algorithm known up till now is a 3-approximation algorithm by Chen et al. (2001).

Our focus in this paper is the NWST problem in unit disk graphs. Although known as NP-hard problem, no approximation algorithms known as far as we know up till now. We construct the first constant approximation algorithm implementable in polynomial time in this paper.

3 Node-weighted Steiner tree problem (NWST)

In this section, we study the NWST problem in unit disk graphs and propose an approximation algorithm for solving this problem. The main idea is based on the classical Steiner Tree problem. Firstly, we construct an edge-weighted graph G'' with the same node-set and edge-set as the original graph G. Then, we define the weight of every edge in G'' as the half of the sum of its endpoints' weights in G. Finally, we use ρ-approximation algorithm to obtain a Steiner tree of G''. From the Steiner tree of G'', we can get a Steiner tree of G with approximation ratio $2.5\rho \approx 3.875$. As corollaries, we obtain polynomial time algorithms for MWCDS and MWCVC in unit disk graphs with approximation ratio $9.875 + \varepsilon$ and $4.875 + \varepsilon$ separately.

3.1 Preliminaries

In this section, we give some useful definitions and denotations. For a node-weighted (or edge-weighted) graph G with weight function w and a subgraph H of G, denote $w(H)$ be the weight sum of all nodes (or edges) in H. For any two nodes u and v of G, denote $\mathrm{dist}_G(u, v)$ as the weight of the shortest path between u and v, which is calculated as $\min \sum_{v_k \in p} w(v_k)$ among all the possible paths between u and v. Here v_k represents those internal nodes on every possible path p.

Given an edge-weighted graph G and a node subset S, we denote the ρ-approximation algorithm for Minimum Steiner Tree as **SMT**(**G**, **S**) and the algorithm for finding Minimum Spanning Tree as **MST(G)**.

3.2 2.5ρ-approximation algorithm

The idea of this algorithm is to convert the node-weighted Steriner tree problem to the classical Steiner tree problem. Firstly, we construct an edge-weighted graph G'' from G as follows by initializing G'' with the same node-set and edge-set as G and an edge weight function w'. For every edge $e = (u, v)$ in G'', let the edge weight $w'(u, v) = \frac{1}{2}(w(u) + w(v))$. The second step of this algorithm is to compute a Steiner tree T of G'' on S through the ρ-approximation algorithm. Final, view T as the node-weighted Steiner tree of G on S and output it. The pseudo-code of this algorithm is presented in Algorithm 1.

Algorithm 1 NWST ($G = (V, E, w, S)$)

1: Initialize an edge-weighted graph $G' = (V', E', w', S')$ by setting $V' = V$, $S' = S$ and $E' = E$
2: **for** each edge (v_i, v_j) in graph G' **do**
3: Assign the weight of this edge $w'(v_i, v_j) = (w(v_i) + w(v_j))/2$.
4: **end for**
5: $T = \mathbf{SMT}(G', S)$
6: Output T

To better illustrate this algorithm, we give an example. Given a node-weighted graph G and a terminal set S as in Fig. 1a, firstly, we transform it into graph G' (Fig. 1b) according to the weight assignment. The numbers besides each vertex in G are the weights associated with them and we set the weight of all vertices in the terminal set as 0. Secondly, we calculate the SMT for graph G', which is the subgraph in Fig. 1c. Figure 1d presents the NWST for original graph G.

Since G is a unit disk graph, for any terminal set S, there always exists a Steiner minimum tree with maximum degree no more than 5. The following lemma proves the relationship between the weight of the optimal SMT in graph G'' and the weight of the optimal NWST we want.

Lemma 1 *Denote T_{OPT_SMT} as the optimal Steiner Minimum Tree for G'' on the set S and T_{OPT_NWST} as the optimal Node-weighted Steiner Tree of G on the same*

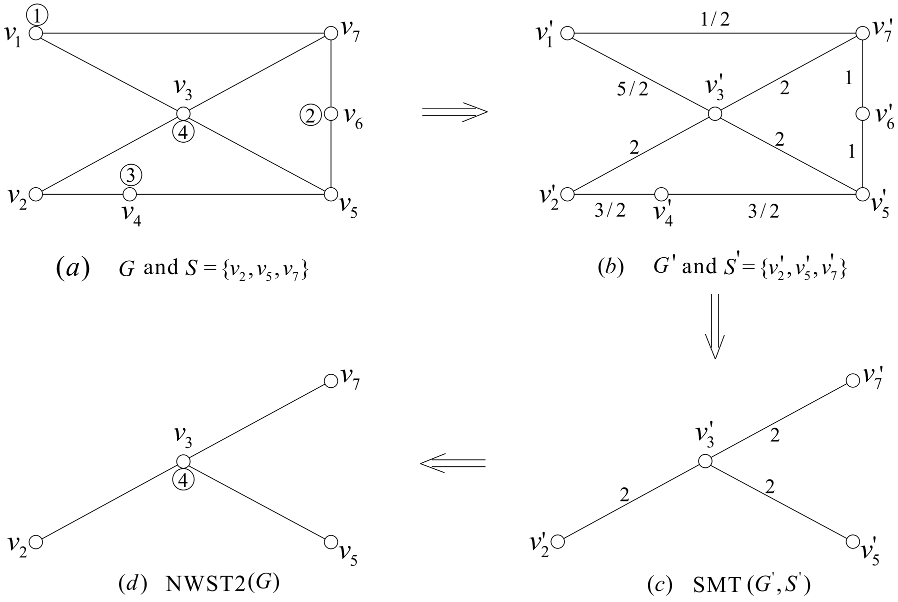

Fig. 1 An example of 2.5ρ algorithm

terminal set S, respectively. Then $w'(T_{OPT\_SMT}) \leq 2.5w(T_{OPT\_NWST})$ when G is a unit disk graph.

Proof Consider T_{OPT_NWST} as a Steiner tree on S of G''. For convenience, denote T as T_{OPT_NWST}, V as $V(T_{OPT\_NWST})$, E as $E(T_{OPT\_NWST})$ and $d_T(u)$ as the degree of node u in tree T, we have

$$\begin{aligned} w'(T_{OPT\_SMT}) &\leq w'(T_{OPT\_NWST}) \\ &= \sum_{e=uv\in E}\left(\frac{1}{2}(w(u)+w(v))\right) \\ &= \sum_{u\in V}\frac{d_T(u)}{2}w(u) \\ &\leq \frac{5}{2}\sum_{u\in V}(w(u)) = 2.5w(T). \end{aligned}$$

Hence, the lemma holds. □

If we consider the Steiner tree T of G'' as a subgraph of G, we can get the following lemma:

Lemma 2 *For any Steiner tree T of G'' on terminal set S, if we view it as a subgraph G, then T is also a Steiner tree of G on set S and $w(T) \leq w'(T)$.*

Proof Since for any Steiner tree, the degree of each Steiner node is not less than 2 and all weight of nodes in terminal set is 0, the lemma holds. □

Theorem 1 *Algorithm NWST is a* 2.5ρ*-approximation for node-weighted Steiner tree problem in unit disk graph.*

Proof Let T be the output of NWST. Denote T_{OPT_SMT} and T_{OPT_NWST} as in Lemma 1. By Lemmas 1 and 2, we have

$$w(T) \le w'(T) \le \rho w'(T_{OPT\_SMT}) \le 2.5\rho w(T_{OPT\_NWST}). \qquad \square$$

3.3 MWCDS & MWCVC

As we know, the problem of MWCDS is to construct the Connected Dominating Set(CDS) in a node-weighted graph with the minimum total weight. Normally, researchers start with calculating Dominating Set (DS) for the graph first and then interconnecting them. Obviously, the node-weighted Steriner tree can be used in the MWCDS problem to interconnect all nodes of the DS to get better approximation algorithm. Therefore, we can obtain the following corollary.

Corollary 1 *A* $(9.875 + \varepsilon)$*-approximation for MWCDS in unit disk graph exists by interconnecting all nodes of the DS using algorithm NWST.*

Proof For any node-weighted graph G and a given Dominating Set DS, denote T_{OPT_CDS} and T_{OPT} be the optimal CDS of the G and the optimal Steiner tree of G on the given DS respectively. Since the induced graph $G[DS \cup T_{OPT\_CDS}]$ is connected, this graph contains a Steiner tree of G on DS. Thus, $w(T_{OPT} \setminus DS) \le w(T_{OPT\_CDS})$.

By Huang et al. (2009), we can get a dominating set C of G with $w(C) \le (6 + \varepsilon)w(T_{OPT\_CDS})$. Then, using algorithm NWST2 for C, we can obtain a Steiner tree T with $w(T \setminus C) \le 2.5\rho w(T_{OPT} \setminus C)$. Clearly, $V(T)$ is a connected dominating set of G and

$$\begin{aligned} w(V(T)) &= w(C) + w(V(T) \setminus C) \\ &\le (6 + \varepsilon)w(V(T_{OPT\_CDS})) + 2.5\rho w(V(T_{OPT}) \setminus C) \\ &\le (9.875 + \varepsilon)w(V(T_{OPT\_CDS})). \end{aligned} \qquad \square$$

Besides, using NWST algorithm we proposed, we could achieve better approximation performance for the MWCVC problem as well. The MWCVC problem concerns about finding a Connected Vertex Cover (CVC) for a node-weighted graph, which has the minimum total weight. The connectivity of the Vertex Cover (VC) could be easily obtained by constructing node-weighted Steiner tree connecting vertices in the VC set. Therefore, applying our NWST algorithm into the MWCVC problem, we have

Corollary 2 *The MWCVC problem in unit disk graph could be approximated using a* $(4.875 + \varepsilon)$*-approximation algorithm by interconnecting all nodes in the set of VC using algorithm NWST.*

Proof For any node-weighted graph G and a given VC, denote V_{OPT_CVC} and V_{OPT} be optimal CVC of G and the optimal Steiner tree of G on VC respectively. Obviously, the induced graph of the $G[VC \cup V_{OPT\_CVC}]$ is connected and contains a Steiner tree of G on VC. Thus, $w(V_{OPT} \setminus VC) \le w(V_{OPT\_CVC})$.

As node-weighted vertex cover in unit disk graph admits a PTAS (Bar-Yehuda and Even 1985; Halperin 2000; Nieberg and Hurink 2005), we can get a node-weighted vertex cover set C of G with a weight of $w(C) \le (1+\varepsilon)w(V_{OPT\_CVC})$. Then, using algorithm NWST to interconnect C, we can obtain a Steiner tree T. Clearly, $V(T)$ is a connected vertex cover of G and

$$\begin{aligned} w(V(T)) &= w(C) + w(V(T) \setminus C) \\ &\le (1+\varepsilon)w(V_{OPT\_CVC}) + 2.5\rho w(V_{OPT} \setminus C) \\ &\le (4.875+\varepsilon)w(V_{OPT\_CVC}). \end{aligned}$$ □

4 Conclusion

In this paper, we propose a constant approximation algorithms for NWST problem in unit disk graph. The approximation ratios for this algorithm is proven to be 2.5ρ, respectively. As corollaries, we improve the approximation of MWCDS in unit disk graphs from $10+\varepsilon$ to $9.875+\varepsilon$ and also the approximation ratio of MWCVC to $4.875+\varepsilon$ in unit disk graphs.

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