J Comb Optim (2010) 20: 361–368
DOI 10.1007/s10878-009-9213-1

An approximation algorithm for the k-level capacitated facility location problem

Donglei Du · Xing Wang · Dachuan Xu

Published online: 27 January 2009

Abstract We consider the k-level capacitated facility location problem (k-CFLP), which is a natural variant of the classical facility location problem and has applications in supply chain management. We obtain the first (combinatorial) approximation algorithm with a performance factor of $k+2+\sqrt{k^2+2k+5}+\varepsilon$ ($\varepsilon>0$) for this problem.

Keywords Capacitated facility location problem · Approximation algorithm · Local search

1 Introduction

In the *k-level capacitated facility location problem* (k-CFLP), we are given a complete $(k+1)$-bipartite graph $G=(D\cup F_1\cup\cdots\cup F_k; E)$ whose node set is the union of $k+1$ disjoint sets D, $F_1,\ldots,F_k$ and the edge set E of all edges between these sets. The nodes in D are called demand points or clients and the nodes in $F=F_1\cup\cdots\cup F_k$ are facilities (of levels $1,\ldots,k$ respectively). We use p to denote a sequence of facilities $i_t\in F_t$, $t=1,\ldots,k$, and refer to p as a path of facilities. The set of all possible paths is denoted by $\mathcal{P}$. Each client $j\in D$ has a demand d_j that must be served by

D. Du
Faculty of Business Administration, University of New Brunswick, Fredericton, NB E3B 5A3, Canada
e-mail: ddu@unb.ca

X. Wang · D. Xu (✉)
Department of Applied Mathematics, Beijing University of Technology, 100 Pingleyuan, Chaoyang District, Beijing 100124, People's Republic of China
e-mail: xudc@bjut.edu.cn

X. Wang
e-mail: wangxing198491@emails.bjut.edu.cn

one or more paths along with open facilities. Each facility $i_t \in F_t$ $(t = 1, \dots, k)$ is specified by a cost f_{i_t}, which is incurred when facility i_t is open, and by a capacity u_{i_t}, which is the maximum demand that facility i_t can serve. We are given edge costs $c \in R_+^E$. The objective is to open some facilities $X_t \subseteq F_t$ on each level $t = 1, \dots, k$ such that the demands of all the clients are met by the paths along with the open facilities and the total cost of facility opening and client service is minimized. We assume that c is induced by a metric on the whole set of nodes $V = D \cup F_1 \cup \cdots \cup F_k$.

For convenience, in our paper, we consider unit demands, i.e., $d_j = 1$ for all j. The k-CFLP can be formulated as a mixed integer linear program. Let x_{jp} be the fraction of demand of client j served by path p. Let y_{i_t} be equal to 1 if facility i_t is open at level t, and 0 otherwise.

$$
\begin{aligned}
\min \quad & \sum_{p \in \mathcal{P}} \sum_{j \in D} c_{jp} x_{jp} + \sum_{t=1}^{k} \sum_{i_t \in F_t} f_{i_t} y_{i_t} \\
\text{s.t.} \quad & \sum_{p \in \mathcal{P}} x_{jp} = 1, \quad \forall j \in D, \\
& \sum_{j \in D} \sum_{p: i_t \in p} x_{jp} \le u_{i_t} y_{i_t}, \quad \forall i_t \in F_t,\ t = 1, \dots, k, \\
& x_{jp} \ge 0, \quad \forall j \in D,\ p \in \mathcal{P}, \\
& y_{i_t} \in \{0, 1\}, \quad \forall i_t \in F_t,\ t = 1, \dots, k.
\end{aligned}
$$

In the above formulation, the first constraint imposes that demand of each client must be served by one or more paths. The second constraint says that at most u_{i_t} amount of demand may be assigned to each facility $i_t \in F_t$ $(t = 1, \dots, k)$.

The k-CFLP is NP-hard and unifies several existing facility location problems which have been extensively studied from approximation algorithm points of view in the last few years.

- The (metric) *uncapacitated facility location problem* (UFLP): This is a special case of the 1-CFLP with all $u_i = \infty$. The first constant factor approximation algorithm is given by Shmoys et al. (1997). The currently best approximation factor is 1.50 due to Byrka and Aardal (2009). On the negative side, Guha and Kuller showed that the existence of an 1.463-approximation algorithm for the UFLP would imply P = NP (Guha and Kuller 1999). We refer to Jain et al. (2003), Jain and Vazirani (2001), Mahdian et al. (2006), Xu and Du (2006), Xu and Zhang (2008) and the references therein for further discussions on approximation algorithms for the UFLP and its variants.
- The (metric) *k-level uncapacitated facility location problem* (k-FLP): This is the special case of k-CFLP with all $u_i = \infty$. It is known that the k-FLP can be solved within a factor of 3 by a linear programming (LP) rounding algorithm (Aardal et al. 1999), and later improved to 1.77 for the 2-FLP by Zhang (2006). The best combinatorial algorithm with a performance factor of 3.27 for the k-FLP is due to Ageev et al. (2005).

- The (metric) 1-*level capacitated facility location problem* (1-CFLP): This is the special case of k-CFLP when $k = 1$. Since the natural LP relaxation for the 1-CFLP is known to have an unbounded integrality gap, most known approximation algorithms for the 1-CFLP are based on local search techniques. When the capacities are uniform, Korupolu et al. (1998) give an 8-approximation local search algorithm for the 1-CFLP, which is improved to 5.83 by Chudak and Williamson (1999). When the capacities are not uniform, Pal et al. (2001) present a local search algorithm with performance guarantee 8.53, which is improved to 7.88 by Mahdian and Pal (2003), and further to the currently best factor $3 + 2\sqrt{2} + \varepsilon$ ($\varepsilon > 0$) by Zhang et al. (2005) through the so-called *multiexchange local search* (MLS) algorithm, which will be used in the design of our algorithm shortly. When all facility opening costs are equal, Levi et al. (2004) present an LP rounding algorithm with a performance factor of 5 which is the first constant upper bound on the integrality gap of the natural LP relaxation in this special case.

The main focus of this work is on designing an approximation algorithm for the k-CFLP due to its NP-hardness. We propose a combinatorial algorithm with a performance factor of $k + 2 + \sqrt{k^2 + 2k + 5} + \varepsilon$ for any given constant ε, the first such a result for the k-CFLP.

2 The algorithm

In this section, we present an algorithm for the k-CFLP, which will call upon, as a subroutine, the MLS algorithm for the 1-CFLP developed by Zhang et al. (2005). The following two results from Zhang et al. (2005) will be useful in our analysis shortly.

Lemma 2.1 (Zhang et al. 2005) *The MLS algorithm can identify and implement any admissible in polynomial time for any given* $\varepsilon > 0$. *The algorithm terminates in polynomial time with approximation guarantee of* $3 + 2\sqrt{2} + \varepsilon$.

Throughout the paper, we denote F^{SOL} and C^{SOL} as the total open cost and connection cost for any solution SOL of the problem, respectively.

Lemma 2.2 (Zhang et al. 2005) *Let* FEA *be any feasible solution of the* 1-*CFLP*, ZCY *be the solution obtained by the MLS algorithm. Then, we have*

$$F^{\mathrm{ZCY}} \le 5F^{\mathrm{FEA}} + 4C^{\mathrm{FEA}},$$
$$C^{\mathrm{ZCY}} \le F^{\mathrm{FEA}} + C^{\mathrm{FEA}}.$$

For any instance M of the k-CFLP, we define an instance M_{k-1} of $(k-1)$-CFLP and an instance S of 1-CFLP, respectively, in the following way:

1. M_{k-1} is obtained from M by deleting all the facilities at level 1. Thus, in M_{k-1} the set of facilities on level r ($r = 1, \ldots, k-1$) is F_{r+1}, and the connection cost between client $j \in D$ and facility $i_2 \in F_2$ is

$$\min_{v \in F_1} \{c_{jv} + c_{vi_2}\}. \tag{1}$$

2. S is obtained from M by deleting all facilities at levels greater than 1, and doubling all the edges costs between D and F_1.

Let $\mathcal{P}$ be the set of all paths such that each of which passes each level exactly once, that is, $\mathcal{P} = \{(i_1, i_2, \ldots, i_k) : i_1 \in F_1, \ldots, i_k \in F_k\}$. For any path $p = (i_1, i_2, \ldots, i_k) \in \mathcal{P}$, denote $c(p) = \sum_{t=2}^{k} c_{i_{t-1}, i_t}$.

Now we are ready to present our algorithm as follows.

Algorithm 2.3

Input. An instance M of the k-CFLP.
Output. A solution ALG for the instance M.
Step 1. *Solving instance S.* Solve S by the MLS algorithm. In this solution, let $\alpha_{ji_1(j)}$ denote the amount of demand of client $j \in D$ served by facility $i_1(j) \in F_1(j)$, where $F_1(j) \subseteq F_1$ is the set of level 1 facilities serving client $j \in D$.
Step 2. *Solving instance M_{k-1}.* Recursively solve M_{k-1} to obtain a solution for M_{k-1}. In this solution, let $\beta_{jp_{k-1}(j)}$ denote the amount of demand of client $j \in D$ served by path $p_{k-1}(j) = (i_2(j) \in F_2(j), \ldots, i_k(j) \in F_k(j)) \in \mathcal{P}_{k-1}(j)$, where $F_t(j) \subseteq F_t$ $(t = 2, \ldots, k)$ is the set of level t facilities serving client $j \in D$, and $\mathcal{P}_{k-1}(j)$ is the set of paths to which each client $j \in D$ is assigned in the solution of M_{k-1}, that is, $\mathcal{P}_{k-1}(j) = \{(i_2(j), \ldots, i_k(j)) : i_2(j) \in F_2(j), \ldots, i_k(j) \in F_k(j)\}$.
Step 3. *Constructing a transportation problem for each client $j \in D$.* For each client $j \in D$, solve the following transportation problem resulting in a solution such that the amount of demand of client $j \in D$ served by path $p_{k-1}(j) \in \mathcal{P}_{k-1}(j)$ is $\gamma_{i_1(j)p_{k-1}(j)}$:

$F_1(j)$:	the set of supply centers;
$\mathcal{P}_{k-1}(j)$:	the set of receiving centers;
$\alpha_{ji_1(j)}$:	the supply for each facility $i_1(j) \in F_1(j)$;
$\beta_{jp_{k-1}(j)}$:	the demand for each path $p_{k-1}(j) \in \mathcal{P}_{k-1}(j)$;
$c_{ji_1(j)} + c(p_{k-1}(j))$:	the connection cost between facility $i_1(j) \in F_1(j)$ and path $p_{k-1}(j) \in \mathcal{P}_{k-1}(j)$.

Step 4. *Constructing a solution for instance M.* Construct a solution ALG for M by connecting client j to path $p_k(j) = (i_1(j), p_{k-1}(j))$, for each facility $i_1(j) \in F_1(j)$ and each path $p_{k-1}(j) \in \mathcal{P}_{k-1}(j)$.

3 Analysis

Theorem 3.1 *Let $k \geq 2$, for any solution* SOL *of M, the solution* ALG *retrieved by Algorithm 2.3 satisfies*

$$F^{\mathrm{ALG}} + C^{\mathrm{ALG}} \leq 6F^{\mathrm{SOL}} + 5(1 + 2(k-1))C^{\mathrm{SOL}}.$$

Proof Let ALG(S) and ALG(M_{k-1}) denote the solutions for S and M_{k-1} in Steps 1 and 2 respectively in Algorithm 2.3. From Step 3 of Algorithm 2.3, we have

$$\sum_{i_1(j)\in F_1(j)} \gamma_{i_1(j)p_{k-1}(j)} = \beta_{jp_{k-1}(j)},$$

$$\sum_{p_{k-1}(j)\in \mathcal{P}_{k-1}(j)} \gamma_{i_1(j)p_{k-1}(j)} = \alpha_{ji_1(j)}, \quad \forall j \in D.$$

It follows from the construction of ALG and the triangle inequality that

$$\begin{aligned}
&F^{\mathrm{ALG}} + C^{\mathrm{ALG}} \\
&\quad = F^{\mathrm{ALG}(S)} + \frac{1}{2}C^{\mathrm{ALG}(S)} + F^{\mathrm{ALG}(M_{k-1})} \\
&\qquad + \sum_{j\in D}\sum_{i_1(j)\in F_1(j)}\sum_{p_{k-1}(j)\in\mathcal{P}_{k-1}(j)} c_{i_1(j)p_{k-1}(j)}\gamma_{i_1(j)p_{k-1}(j)} \\
&\quad \le F^{\mathrm{ALG}(S)} + \frac{1}{2}C^{\mathrm{ALG}(S)} + F^{\mathrm{ALG}(M_{k-1})} \\
&\qquad + \sum_{j\in D}\sum_{i_1(j)\in F_1(j)}\sum_{p_{k-1}(j)\in\mathcal{P}_{k-1}(j)} (c_{ji_1(j)} + (c_{ji_2(j)} + c(p_{k-1}(j))))\gamma_{i_1(j)p_{k-1}(j)} \\
&\quad = F^{\mathrm{ALG}(S)} + \frac{1}{2}C^{\mathrm{ALG}(S)} + F^{\mathrm{ALG}(M_{k-1})} \\
&\qquad + \sum_{j\in D}\sum_{i_1(j)\in F_1(j)} c_{ji_1(j)} \sum_{p_{k-1}(j)\in\mathcal{P}_{k-1}(j)} \gamma_{i_1(j)p_{k-1}(j)} \\
&\qquad + \sum_{j\in D}\sum_{p_{k-1}(j)\in\mathcal{P}_{k-1}(j)} (c_{ji_2(j)} + c(p_{k-1}(j))) \sum_{i_1(j)\in F_1(j)} \gamma_{i_1(j)p_{k-1}(j)} \\
&\quad = F^{\mathrm{ALG}(S)} + \frac{1}{2}C^{\mathrm{ALG}(S)} + F^{\mathrm{ALG}(M_{k-1})} + \sum_{j\in D}\sum_{i_1(j)\in F_1(j)} c_{ji_1(j)}\alpha_{ji_1(j)} \\
&\qquad + \sum_{j\in D}\sum_{p_{k-1}(j)\in\mathcal{P}_{k-1}(j)} (c_{ji_2(j)} + c(p_{k-1}(j)))\beta_{jp_{k-1}(j)} \\
&\quad \le F^{\mathrm{ALG}(S)} + \frac{1}{2}C^{\mathrm{ALG}(S)} + F^{\mathrm{ALG}(M_{k-1})} + \frac{1}{2}C^{\mathrm{ALG}(S)} + C^{\mathrm{ALG}(M_{k-1})} \\
&\quad = F^{\mathrm{ALG}(S)} + C^{\mathrm{ALG}(S)} + F^{\mathrm{ALG}(M_{k-1})} + C^{\mathrm{ALG}(M_{k-1})}. \qquad (2)
\end{aligned}$$

The total cost of the solution SOL is broken down to

$$F^{\mathrm{SOL}} + C^{\mathrm{SOL}} = \sum_{t=1}^{k} F_t^{\mathrm{SOL}} + \sum_{t=1}^{k} C_t^{\mathrm{SOL}},$$

where F_t^{SOL} and C_t^{SOL} ($t = 1, \dots, k$) denote the total open cost of facilities on level t, and the total connection cost between the open facilities on level $t-1$ and the open facilities on level t (C_1^{SOL} stands for the total connection cost between the clients in D and the facilities on level 1).

Suppose that in SOL, X_t $(t = 1, 2, \ldots, k)$ are the open facilities, and $X_t(j) \subseteq X_t$ $(t = 1, \ldots, k)$ are those serving client $j \in D$. Then the amount of demand of client j served by path $p(j) \in \mathcal{P}(j) = \{(i_1(j),\ i_2(j), \ldots, i_k(j)) : i_t(j) \subseteq X_t(j),\ t = 1, 2, \ldots, k\}$ is $x_{jp(j)}$.

Observe that SOL induces two solutions: (i) a solution SOL(S) for S, in which the amount of demand of client j served by facility $i_1(j) \in X_1(j)$ is $\sum_{p(j)\in\mathcal{P}(j):i_1(j)\in p(j)} x_{jp(j)}$, incuring connection cost $\sum_{p(j)\in\mathcal{P}(j):i_1(j)\in p(j)} 2c_{ji_1(j)} \times x_{jp(j)}$, and also (ii) a solution SOL(M_{k-1}) for M_{k-1}, in which the amount of demand of client j served by path $(i_2(j), \ldots, i_k(j))$ is $\sum_{p(j):(i_2(j),\ldots,i_k(j))\in p(j)} x_{jp(j)}$, incurring connection cost at most $\sum_{p(j)\in\mathcal{P}(j):(i_2(j),\ldots,i_k(j))\in p(j)} c_{jp(j)} x_{jp(j)}$ by the construction of connection costs. Then, recalling that the connection costs between D and F_1 in instance S are doubled comparing to those in instance M, we have

$$F^{\mathrm{SOL}(S)} = F_1^{\mathrm{SOL}}, \qquad C^{\mathrm{SOL}(S)} = 2C_1^{\mathrm{SOL}}, \tag{3}$$

$$F^{\mathrm{SOL}(M_{k-1})} = F_2^{\mathrm{SOL}} + \cdots + F_k^{\mathrm{SOL}}, \tag{4}$$

$$C^{\mathrm{SOL}(M_{k-1})} \le C_1^{\mathrm{SOL}} + C_2^{\mathrm{SOL}} + \cdots + C_k^{\mathrm{SOL}}. \tag{5}$$

By Lemma 2.2 and (3), we have

$$F^{\mathrm{ALG}(S)} + C^{\mathrm{ALG}(S)} \le 6F^{\mathrm{SOL}(S)} + 5C^{\mathrm{SOL}(S)} = 6F_1^{\mathrm{SOL}} + 10C_1^{\mathrm{SOL}}. \tag{6}$$

We proceed by induction on k. Assume now that $k = 2$. In this case M_1 is an instance of 1-CFLP and by Lemma 2.2 and (4)–(5),

$$\begin{aligned} F^{\mathrm{ALG}(M_1)} + C^{\mathrm{ALG}(M_1)} &\le 6F^{\mathrm{SOL}(M_1)} + 5C^{\mathrm{SOL}(M_1)} \\ &\le 6F_2^{\mathrm{SOL}} + 5(C_1^{\mathrm{SOL}} + C_2^{\mathrm{SOL}}). \end{aligned} \tag{7}$$

By (2), (6), and (7), we have

$$F^{\mathrm{ALG}} + C^{\mathrm{ALG}} \le 6F^{\mathrm{SOL}} + 15C_1^{\mathrm{SOL}} + 5C_2^{\mathrm{SOL}} \le 6F^{\mathrm{SOL}} + 15C^{\mathrm{SOL}}, \tag{8}$$

which indicates that the theorem holds for 2-CFLP.

Suppose that the theorem were true for $(k-1)$-CFLP. Now we want to prove that the theorem is true for k-CFLP. Since M_{k-1} is an instance of $(k-1)$-CFLP, by the inductive hypothesis and (4)–(5), we have

$$\begin{aligned} F^{\mathrm{ALG}(M_{k-1})} + C^{\mathrm{ALG}(M_{k-1})} &\le 6F^{\mathrm{SOL}(M_{k-1})} + 5(1 + 2(k-2))C^{\mathrm{SOL}(M_{k-1})} \\ &\le 6\sum_{t=2}^{k} F_t^{\mathrm{SOL}} + 5(1 + 2(k-2))\sum_{t=1}^{k} C_t^{\mathrm{SOL}}. \end{aligned} \tag{9}$$

By (2), (3) and (9), we have

$$
\begin{aligned}
&F^{\mathrm{ALG}} + C^{\mathrm{ALG}} \\
&\quad \le 6F^{\mathrm{SOL}} + 5(1+2(k-1))C_1^{\mathrm{SOL}} + 5(1+2(k-2))\sum_{t=2}^{k} C_t^{\mathrm{SOL}} \\
&\quad \le 6F^{\mathrm{SOL}} + 5(1+2(k-1))C^{\mathrm{SOL}},
\end{aligned}
$$

which concludes the theorem. □

For any given instance of the k-CFLP, if we scale the facility costs f_{i_t} by a factor $\delta = (k-3) + \sqrt{k^2+2k+5}$ and apply Algorithm 2.3 to solve the scaled instance, by Lemma 2.2, we have

$$\delta F^{\mathrm{ZCY}} \le 5\delta F^{\mathrm{FEA}} + 4C^{\mathrm{FEA}}$$

and

$$C^{\mathrm{ZCY}} \le \delta F^{\mathrm{FEA}} + C^{\mathrm{FEA}}.$$

From the above two inequalities, we have

$$F^{\mathrm{ZCY}} + C^{\mathrm{ZCY}} \le (5+\delta)F^{\mathrm{FEA}} + \left(\frac{4}{\delta}+1\right)C^{\mathrm{FEA}}.$$

Similar to the proof of Theorem 3.1, we can obtain

$$F^{\mathrm{ALG}} + C^{\mathrm{ALG}} \le (5+\delta)F^{\mathrm{SOL}} + \left(\frac{4}{\delta}+1\right)(1+2(k-1))C^{\mathrm{SOL}}.$$

Together with Lemma 2.1, we get the corollary:

Corollary 3.2 *For any constant $\varepsilon > 0$, Algorithm 2.3 runs in polynomial time with approximation guarantee of $h(k) := k+2+\sqrt{k^2+2k+5}+\varepsilon$.*

One can compute that $h(1) \le 5.83$ and $h(2) \le 7.61$. Furthermore, it is easy to identify that $2k+3 < h(k) < 2k+4$ for any k.

4 Discussions

In this paper, we present a combinatorial algorithm with a performance factor of $k+2+\sqrt{k^2+2k+5}+\varepsilon$ for the k-CFLP, where ε is any given positive constant. One may observe from the proof of Theorem 3.1 that the coefficient for C_t^{SOL} ($t = 2, \ldots, k$) could be strictly smaller than that for C_1^{SOL}. This fact may be explored to further improve the overall approximation factor for the k-CFLP.

As a future research question, it is interesting to give a lower bound on the polynomial-time approximation factor for k-CFLP directly, instead of using the lower bound of UFLP as a special case.

Acknowledgements We would like to thank the anonymous referee for the insightful comments. The research of the first author is supported in part by the Natural Sciences and Engineering Research Council of Canada (NSERC) grant 283103. The third author's research is supported by NSF of China (Grant 60773185) and Program for Beijing Excellent Talents.

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