J Comb Optim (2010) 20: 63–75
DOI 10.1007/s10878-008-9194-5

Periodic complementary binary sequences and Combinatorial Optimization algorithms

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Published online: 26 November 2008

Abstract We establish a new formalism for problems pertaining to the periodic autocorrelation function of finite sequences, which is suitable for Combinatorial Optimization methods. This allows one to bring to bear powerful Combinatorial Optimization methods in a wide array of problems that can be formulated via the periodic autocorrelation function. Using this new formalism we solve all remaining open problems regarding periodic complementary binary sequences, in the context of the Bömer and Antweiler diagram and thus complete the program that they started in 1990.

Keywords Periodic complementary binary sequences · Periodic Autocorrelation Function · Combinatorial Optimization · Algorithms

1 Introduction

In this work we establish a new formalism that allows one to use the powerful methods available in Combinatorial Optimization (Floudas and Pardalos 2001; Pardalos and Rodgers 1990), in the study of sequences with Periodic Autocorrelation Function zero.

This research is partially supported by NSF, AirForce and NSERC grants.

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We begin with the definition of the Periodic Autocorrelation Function, PAF, from Koukouvinos (1996):

Definition 1 Let $A = \{a_1, a_2, \dots, a_n\}$ be a sequence of length n, where $a_1, a_2, \dots, a_n$ are real numbers. The *periodic autocorrelation function*, $P_A(s)$, of A is defined as:

$$P_A(s) = \sum_{i=1}^{n} a_i a_{i+s}, \quad s = 0, 1, \dots, n-1$$

where we consider $i + s$ modulo n (when $i + s > n$).

The following lemma records a symmetry property of the elements of the PAF vector, the proof is easy and is left to the reader, also see Koukouvinos (1996).

Lemma 1

$$P_A(s) = P_A(n-s), \quad s = 1, 2, \dots, n-1.$$

The importance of Lemma 1 lies in the fact that for a sequence of length n one needs to consider only the first $[\frac{n}{2}]$ elements of the PAF vector, where $[x]$ denotes the integer part of x.

Notation In the rest of this paper we denote $m = [\frac{n}{2}]$.

2 PAF and quadratic forms

In this section we view the elements of the PAF vector as *Quadratic Forms* and we associate certain matrices with them. These matrices are used to formulate problems about sequences with PAF zero, as binary feasibility problems.

Definition 2 Let $a = [a_1, a_2, \dots, a_n]^T$ be a column $n \times 1$ vector, where $a_1, a_2, \dots, a_n \in \{-1, +1\}$ and consider the elements of the PAF vector $P_A(1), \dots, P_A(m)$. Define the following m symmetric matrices (which are independent of the sequence a)

$$M_i = (m_{jk}),$$
$$\text{s.t.} \quad \begin{cases} m_{jk} = m_{kj} = \frac{1}{2}, & \text{when } a_j a_k \in P_A(i),\ j, k \in \{1, \dots, n\}, \\ 0, & \text{otherwise,} \end{cases} \qquad i = 1, \dots, m.$$

The following lemma can be proved by straightforward computation.

Lemma 2 *The matrices M_i can be used to write the PAF equations in a matrix form*:

- *For n odd*:

$$a^T M_i a = P_A(i), \quad i = 1, \dots, m.$$

- *For n even*:

$$a^T M_i a = P_A(i), \quad i = 1, \ldots, m-1 \quad \textit{and} \quad a^T M_m a = \frac{1}{2} P_A(m).$$

Example 1 Let $n = 8$, $a = [a_1, \ldots, a_8]$. Then we have that $m = 4$ and

$$a^T M_i a = P_A(i), \quad i = 1, 2, 3 \quad \text{and} \quad a^T M_4 a = \frac{1}{2} P_A(4),$$

where

$$M_1 = \begin{bmatrix} 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 \end{bmatrix},$$

$$M_2 = \begin{bmatrix} 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 \end{bmatrix},$$

$$M_3 = \begin{bmatrix} 0 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 \end{bmatrix},$$

$$M_4 = \begin{bmatrix} 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 \end{bmatrix}.$$

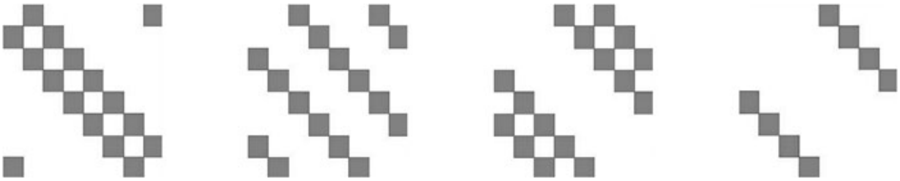

Fig. 1 Graphical representations of the four symmetric matrices M_1, M_2, M_3, M_4

Figure 1 shows graphical representations of the matrices M_1, M_2, M_3, M_4, where elements with value 0 are colored white and elements with value $1/2$ elements are colored grey.

Problem 1 Now suppose that we are looking for two $\{-1, +1\}$ sequences A and B of lengths n, such that

$$P_A(i) + P_B(i) = 0, \quad i = 1, \ldots, m.$$

Using Lemma 2 we can reformulate this problem as follows:

Problem 2 Find two binary sequences a, b (viewed as $n \times 1$ column vectors) such that

$$a^T M_i a + b^T M_i b = 0, \quad i = 1, \ldots, m,$$

where $a = [a_1, \ldots, a_n]$ and $b = [b_1, \ldots, b_n]$ and $a_i, b_i \in \{-1, +1\}$.

We are interested in ± 1 values of the $2n$ variables a_i, b_i such that

$$P_A(i) + P_B(i) = 0, \quad i = 1, \ldots, m$$

or equivalently

$$a^T M_i a + b^T M_i b = 0, \quad i = 1, \ldots, m.$$

Example 2 (Continued) Let $n = 8$ and consider the sequences:

$$a := [1, 1, 1, 1, 1, 1, -1, -1];$$

$$b := [1, 1, -1, 1, -1, 1, -1, -1].$$

Then we have that $P_A(i) + P_B(i) = 0, i = 1, 2, 3, 4$. One can easily verify that we also have:

$$a^T M_i a + b^T M_i b = 0, \quad i = 1, 2, 3, 4.$$

3 A Combinatorial Optimization formalism for PCS

We now use Lemma 2 to provide a Combinatorial Optimization formalism for the general problem of finding Periodic Complementary binary Sequences, denoted as PCS(n, p).

Definition 3 Let p be an integer such that $p \geq 2$ and n be a positive integer. A collection $A_1, \ldots, A_p$ of p sequences of length n each, with elements from $\{-1, +1\}$ is called a $\mathrm{PCS}(n, p)$, if

$$P_{A_1}(i) + \cdots + P_{A_p}(i) = 0, \quad i = 1, \ldots, m.$$

The study of $\mathrm{PCS}(n, p)$ was discussed systematically by Bömer and Antweiler (1990). They gave necessary existence conditions and produced a diagram summarizing their results for the existence and nonexistence of PCS for all values $2 \leq p \leq 12$ and $2 \leq n \leq 50$. They also describe synthesis methods for PCS, using the concepts of mates, interleaving, matrices with orthogonal columns, perfect arrays and periodic products.

Via Lemma 2, the search for $\mathrm{PCS}(n, p)$ can be expressed as a Combinatorial Optimization problem:

Find a set of binary sequences $a_1, a_2, \ldots, a_p$ (viewed as $n \times 1$ column vectors) such that

$$a_1^T M_i a_1 + \cdots + a_p^T M_i a_p = 0, \quad \text{for } i = 1, \ldots, m.$$

The $n \times n$ matrices $M_1, \ldots, M_m$ that appear in the Combinatorial Optimization formalism of the $\mathrm{PCS}(n, p)$ problem, satisfy a simple additive property, given in the following lemma.

Lemma 3

$$M_1 + \cdots + M_m = \frac{1}{2} M$$

where M is an $n \times n$ matrix with elements $m_{ij} = 1 - \delta_{ij}$ where δ_{ij} denotes the usual Kronecker's delta.

Proof We use the notations of Definition 2. The terms $P_A(1), \ldots, P_A(m)$ are made up from $\frac{n(n-1)}{2}$ different monomials of the form $a_j a_k$ (with $j \neq k$). The n monomials of the form $a_j a_k$ (with $j = k$) i.e. the squares, do not appear in the terms $P_A(1), \ldots, P_A(m)$. □

Example 3 (Continued) Let $n = 8$ and consider the sequences of Example 2, that form a $\mathrm{PCS}(8, 2)$. The relevant matrices M_1, M_2, M_3, M_4, given in Example 1, satisfy the additive property:

$$M_1 + M_2 + M_3 + M_4 = \frac{1}{2} \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \end{bmatrix}.$$

4 Applications of PCS and related sequences

We give some references to works describing applications of PCS. We do not aim to provide a comprehensive, or by all means complete, treatment of the subject, as this is not the purpose of the present paper. We are merely interesting in giving a flavor of the many different application areas involved.

As noted in Bömer and Antweiler (1990), PCS are used to construct sequences with desirable properties for radar applications, as described in Weathers and Holiday (1983). Again as noted in Bömer and Antweiler (1990), PCS intervene in coded aperture imaging (Fenimore and Cannon 1978) and higher-dimensional signal processing applications such as time-frequency-coding (Golomb and Taylor 1982) or spatial correlation (Hersheya and Yarlagadda 1983).

The book (Stinson 2004) contains a chapter on applications of combinatorial designs in general, in authentication codes, threshold schemes and group testing algorithms.

The book (Golomb and Gong 2005) is a rich source of information on applications of sequences with low autocorrelation function properties in general, in signal design for communications, radar, and cryptography applications.

Some categories of PCS, e.g. PCS$(n,2)$, are used as first rows of circulant matrices to construct Hadamard matrices from two circulant submatrices, see Kotsireas and Koukouvinos (2007) and references therein. These Hadamard matrices are then used for Coding Theory purposes, i.e. to construct binary linear codes with desirable properties.

5 Optimization algorithm

The formulation described in the previous section enabled us to use the algorithms for the Unconstrained Binary Quadratic Optimization Problem from Pardalos et al. (2008) with slight modifications. Firstly, the substitution of a_i with $2x_i - 1$, where $x_i \in \{0, 1\}$, transformed the problem to 0–1 domain:

$$\begin{aligned}
&a^T M_i a = 0 \quad \Longrightarrow \quad x^T Q_i x + n \cdot \mathbf{1}_n = 0, \quad i = 1, \dots, m, \text{ and } \mathbf{1}_n \text{ is a vector of 1s,}\\
&\quad Q_i = (q_{jk}), \qquad M_i = (m_{kj}),\\
&\quad q_{jj} = 4, \quad j = 1, \dots, n,\\
&\quad q_{jk} = -2, \quad \text{if } m_{jk} = \frac{1}{2},\\
&\quad q_{jk} = 0, \quad \text{if } m_{jk} = 0.
\end{aligned}$$

Let $d_i(x) = x^T Q_i x + n \cdot \mathbf{1}_n$ denote an error corresponding to the ith equation (positive, negative or zero). The sum of the absolute errors for each equation is used as the cost function to minimize:

$$f(x) = \sum_{i=1}^{m} \mathtt{abs}(x^T Q_i x + n \cdot \mathbf{1}_n) = \sum_{i=1}^{m} \mathtt{abs}(d_i(x)).$$

Search methods used in Pardalos et al. (2008) are based on the 1-flip neighborhood. A solution b belongs to 1-flip neighborhood of binary vector a if $b_j = 1 - a_j$ for some index j and $b_k = a_k$ for $k \neq j$. In order to accelerate 1-flip move evaluations, the special data structure is maintained throughout the search process. Let $gains(x)$ denote a vector in which the j-th element represents a gain in the cost function resulting from applying 1-flip operator to the variable x_j:

$$gains_j(x) = \sum_{i=1}^{m} \left(\mathtt{abs}\left[d_i(x) + q_{jj}(1 - 2x_j) + 2 \sum_{i=1, i \neq j}^{n} q_{ij}(1 - 2x_j)x_i \right] - \mathtt{abs}[d_i(x)] \right).$$

The negative elements of the vector $gains(x)$ indicate the variables that can be inverted in order to improve the current solution x.

The tabu search (Glover and Laguna 1993) was implemented based on 1-flip neighbourhood (see Pardalos et al. 2008 for implementation details). At each iteration, the tabu algorithm explores 1-flip neighbourhood of the current solution. The best solution from the neighbourhood is set to be a current solution and the procedure is repeated. Prohibition rules are used to escape from locally optimal solutions. If the variable x_j was inverted during the move, then it is prohibited to invert x_j for a fixed number of iterations, referred to as a tabu tenure. A random binary sequence was used to initialize the tabu search. The algorithm is restarted after every 10000 iterations with a new random initial solution. The algorithm is stopped whenever the solution x is obtained, such that $f(x) = 0$. The tabu tenure parameter of the algorithm was set to 21 iterations.

6 Results

In this section we give a number of new results, namely solutions to all open problems on p. 319 of the second edition of the Handbook of Combinatorial Designs (Colbourn and Dinitz 2007).

We use the standard notation, $+$ stands for $+1$ and $-$ stands for -1.

6.1 PCS(50, 2)

A pair of PCS(50, 2) has first been found in Kotsireas and Koukouvinos (2008), using refined pruning methods.

However, finding PCS(50, 2) remains a tough algorithmic challenge for conventional methods and so we tested our new formalism with it. We found many PCS(50, 2) and below we give two such pairs.

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6.2 Consequences of our new results

The new PCS that we found in this paper, imply a certain number of consequences, stated in the next theorem:

Theorem 1

- *There is a* $\mathrm{PCS}(50, 2)$, *see* Kotsireas and Koukouvinos (2008), *and hence*, $\mathrm{PCS}(50, 2k)$ *exist for all* $k \geq 1$.
- *There is a* $\mathrm{PCS}(40, 3)$, *see Appendix*, *and hence*, $\mathrm{PCS}(40, 3k)$ *exist for all* $k \geq 1$.
- *There is a* $\mathrm{PCS}(44, 3)$, *see Appendix*, *and hence*, $\mathrm{PCS}(44, 3k)$ *exist for all* $k \geq 1$.
- *There is a* $\mathrm{PCS}(48, 3)$, *see Appendix*, *and hence*, $\mathrm{PCS}(48, 3k)$ *exist for all* $k \geq 1$.
- *There is a* $\mathrm{PCS}(40, 5)$, *see Appendix*, *and hence*, $\mathrm{PCS}(40, 5k)$ *exist for all* $k \geq 1$.
- *There is a* $\mathrm{PCS}(44, 5)$, *see Appendix*, *and hence*, $\mathrm{PCS}(44, 5k)$ *exist for all* $k \geq 1$.
- *There is a* $\mathrm{PCS}(48, 5)$, *see Appendix*, *and hence*, $\mathrm{PCS}(48, 5k)$ *exist for all* $k \geq 1$.

Proof It is well-known (see Bömer and Antweiler 1990) that the existence of $\mathrm{PCS}(n, p)$ implies the existence of $\mathrm{PCS}(n, pk)$ for all positive integers $k \geq 1$. □

The previous theorem has an important corollary, that (based on our results) settles six more open cases in the table of open cases listed on p. 319 of the second edition of the Handbook of Combinatorial Designs (Colbourn and Dinitz 2007).

Corollary 1

- *There exist* $\mathrm{PCS}(44, 6)$ *and* $\mathrm{PCS}(50, 6)$.
- *There exist* $\mathrm{PCS}(40, 9)$, $\mathrm{PCS}(44, 9)$, $\mathrm{PCS}(48, 9)$.
- *There exist* $\mathrm{PCS}(50, 10)$.

7 Conclusion

In view of our new results (as included in the appendix) and their consequences (as stated in the previous section), it is clear that we have solved all open cases for $\mathrm{PCS}(n, p)$ stated in Table 8.17 on p. 319 of the second edition of the Handbook of Combinatorial Designs (Colbourn and Dinitz 2007). Therefore, the program started by Bömer and M. Antweiler in 1990 can now be declared completed.

Appendix

We present the new $\mathrm{PCS}(n, p)$ that we found using our new technique. In conjunction with corollary 1, this settles all open cases in the updated Bömer and Antweiler diagram stated on p. 319 of Colbourn and Dinitz (2007).

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PCS(40,3)

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PCS(44,3)

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PCS(48,3)

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PCS(40,5)

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PCS(44,5)

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PCS(48,5)

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PCS(38,6)

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PCS(42,6)

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PCS(46,6)

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PCS(40,7)

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PCS(44,7)

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PCS(48,7)

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PCS(38,10)

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PCS(42,10)

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PCS(44,10)

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PCS(46,10)

```
[---+++++++--+----+-+-+++-+-++-+---+++--+++-+-+]
[+-+-+--+-+-+++-+++++-+-++-++--+-+-+-+-+-++--+-]
[+-+++++++---+----+---++-++--+-++-++--+++--+-++]
[--+--+++-----+-----+++-++--+-++-+--+---+++---+]
[--+++++-++-+++++--++++-++---+--+--+---++--++++]
[+--+--+++-++---++++---+++++++---+--+---+-++-++]
[-++++++-+-+++--++-++-+-++-++++---+--+-----+--+]
[------++++---------+++-----+++-+--+-+--+--++--]
[-++---++-+-+++-+------+--+-+----++++--+++++-++]
[-+---+--+-+-++-+-+-+---+---+--+++--+----++++--]
```

PCS(40,11)

```
[--+---+-+---+-++-++++-++++-+++-+++++-++-]
[++--++----+++--+-++++-+++----+++-+-++-++]
[---+----+--++--++-++-+++-++-+-++----+--+]
```

[-+++-+-++-+---+++++++---++++++----+++---]
[-++++-++--+-+--+--+-+-+--++--+-+--+-----]
[--++-+-++--+-++-+-+++++----+-+----++++++]
[++--+++---+++-++++++++-++-+-+--++-+---+-]
[++---+++++-+++-+----++-+++-+-+-++-+-+-+-]
[+--+---+++--+-------+---++-++-++-----+-+]
[++--+--++++++-+-++----+++-++-++---+--+++]
[++-----+--+++--+++-+-+++-+-+-+---+--+++-]

PCS(44,11)

[+-+-++--++++-++--+-+---+-++-+-++-+-++-+-+-++]
[++-++---+--+++------+-+++--+-+++-------++-++]
[++--------+++--+--++-+-+++-+-+-+++-+-+-++-+-]
[--+--+-+-----+-+----+-----+++-++++---+-+++--]
[-++++---+------+-++-++-+--+---++---++++-++--]
[++-++-+----+------+-+++---+++++-+++----+--+-]
[+-+---++-+++-+--++-++--++++-++++++-+--++-+++]
[---+--+-+-+--+++++--+--+++++++--+++++-++++++]
[+-+-+---++----+++-++--+++--++-++---++-++--++]
[-+++--+-+++++--++-----+---+---++-++-+-+++++-]
[-+-++-----+-+-+++-+--+---+-+---++-++-++--+++]

PCS(48,11)

[-++-+-++++++-----+-+-+++++--+++-+++--++++-+--+--]
[--+------+++-----+--++---+--+--+----+-++-++-+-+-]
[-----+---+++-----+++-++-+-++++++-+-++--+--++-++-]
[-+-+----+-++++++-++-++-+--++--+--+-+----+-+++++-]
[+++-++---+-+-++-+-+-+++-+++-++++-+-++--+++----++]
[-+--+-+++++---+++--+---+-+-+-+++-+++-+--+--++-+-]
[++++++-+-++++--+++------++--++---+++--+--+--+-+-]
[----++-----+--++-++++---+++++++--+++--++-++--+++]
[+-+++--+++-++-+--+-+----+++++---+-+--+------+---]
[--+-+-+----++--+---+-+---++-+-++++---+-+-+++++-+-]
[--+--++-+-++-+--+++++++-++---++-+++--++++-+++--+]

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