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Graphs with large paired-domination number

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Abstract In this paper, we continue the study of paired-domination in graphs introduced by Haynes and Slater (1998) Networks 32: 199–206. A paired-dominating set of a graph G with no isolated vertex is a dominating set of vertices whose induced subgraph has a perfect matching. The paired-domination number of G, denoted by $\gamma_{\rm pr}(G)$, is the minimum cardinality of a paired-dominating set of G. Let G be a connected graph of order n with minimum degree at least two. Haynes and Slater (1998) Networks 32: 199–206, showed that if $n \geq 6$, then $\gamma_{\rm pr}(G) \leq 2n/3$. In this paper, we show that there are exactly ten graphs that achieve equality in this bound. For $n \geq 14$, we show that $\gamma_{\rm pr}(G) \leq 2(n-1)/3$ and we characterize the (infinite family of) graphs that achieve equality in this bound.

Keywords Bounds · Paired-domination · Minimum degree two

1 Introduction

Domination and its variations in graphs are now well studied. The literature on this subject has been surveyed and detailed in the two books by Haynes et al. (1998a,b). In this paper we investigate bounds on the paired-domination of a graph with minimum degree at least two.

A *matching* in a graph G is a set of independent edges in G. A *perfect matching* M in G is a matching in G such that every vertex of G is incident to an edge of M. A *paired-dominating set*, abbreviated PDS, of a graph G is a set S of vertices of G such that every vertex is adjacent to some vertex in S and the subgraph $G[S]$ induced by S contains a perfect matching M (not necessarily induced). Two vertices joined by an edge of M are said to be *paired* and are also called *partners* in S. Every graph without isolated vertices has a PDS since the end-vertices

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of any maximal matching form such a set. The *paired-domination number* of G, denoted by $\gamma_{\mathrm{pr}}(G)$, is the minimum cardinality of a PDS. A PDS of cardinality $\gamma_{\mathrm{pr}}(G)$ we call a $\gamma_{\mathrm{pr}}(G)$-set. Paired-domination was introduced by Haynes and Slater (1998, 1995) as a model for assigning backups to guards for security purposes, and is studied, for example, in Chellali and Haynes (2004a,b, 2005), Favaron and Henning (2004), Fitzpatrick and Hartnell (1998), Haynes and Henning (2006), Henning (2006), Henning and Plummer (2005), Proffitt et al. (2001), and Qiao et al. (2003) and elsewhere.

For notation and graph theory terminology we in general follow Haynes et al. (1998a). Specifically, let $G = (V, E)$ be a graph with vertex set V of order n and edge set E. For a set $S \subseteq V$, the subgraph induced by S is denoted by $G[S]$. We denote the degree of a vertex v in G by $d_G(v)$, or simply by $d(v)$ if the graph G is clear from context. A vertex of degree k we call a *degree-k vertex*. The minimum degree (resp., maximum degree) among the vertices of G is denoted by $\delta(G)$ (resp., $\Delta(G)$). If $\delta(G) \geq 2$, then we define a vertex of G as *small* if it has degree 2 in G, and *large* if it has degree more than 2 in G. The *open neighborhood* of $v \in V$ is $N(v) = \{u \in V \mid uv \in E\}$ and the *closed neighborhood* of v is $N[v] = \{v\} \cup N(v)$.

A *star* is the tree $K_{1,n-1}$ of order $n \geq 2$. A *subdivided star* is a star where each edge is subdivided exactly once. A *cycle* on n vertices is denoted by C_n and a *path* on n vertices by P_n. For $m \geq 3$ and $n \geq 1$, we denote by $L_{m,n}$ the graph obtained by joining with an edge a vertex in C_m to an end-vertex of P_n. The graph $L_{m,n}$ is called a *key*.

A *daisy* with $k \geq 2$ *petals* is a connected graph that can be constructed from $k \geq 2$ disjoint cycles by identifying a set of k vertices, one from each cycle, into one vertex. In particular, if the k cycles have lengths $n_1, n_2, \ldots, n_k$, we denote the daisy by $D(n_1, n_2, \ldots, n_k)$. Further, if $n = n_1 = n_2 = \cdots = n_k$, then we write $D(n_1, n_2, \ldots, n_k)$ simply as $D_k(n)$.

2 Known results

The decision problem to determine the paired-domination number of a graph is known to be NP-complete (Haynes and Slater, 1998). Hence it is of interest to determine upper bounds on the paired-domination number of a graph. Haynes and Slater (1998) obtained the following upper bound on the paired-domination number of a connected graph in terms of the order of the graph.

Theorem 1 (Haynes and Slater, 1998). *If G is a connected graph of order $n \geq 3$, then $\gamma_{\mathrm{pr}}(G) \leq n - 1$ with equality if and only if G is C_3, C_5 or a subdivided star.*

If we restrict the minimum degree to be at least two and the order to be at least six, then the upper bound in Theorem 1 on the paired-domination number can be improved from one less than its order to two-thirds its order.

Theorem 2 (Haynes and Slater, 1998). *If G is a connected graph of order $n \geq 6$ with $\delta(G) \geq 2$, then $\gamma_{\mathrm{pr}}(G) \leq 2n/3$.*

Haynes and Slater (1998) remark that "the bound of Theorem 2 is sharp as can be seen with the cycle C_6. Although there is no known infinite family of graphs which achieves this upper bound, the family of graphs shown in Fig. 1 has $\gamma_{\mathrm{pr}}(G)$ approaching $2n/3$ for large n."

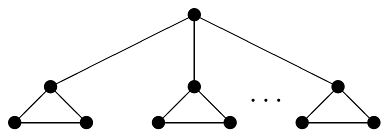

Fig. 1 $\gamma_{\rm pr}(G)$ approaching $2n/3$ for large n

Our aim in this paper is threefold: First to characterize the graphs that achieve equality in the bound of Theorem 2. Second to prove that if G is a connected graph of order $n \geq 10$ with $\delta(G) \geq 2$, then $\gamma_{\rm pr}(G) \leq 2(n-1)/3$. Third to characterize (the infinite family of) such graphs of order $n \geq 14$ that achieve equality in this bound.

3 The families $\mathcal{B}$, $\mathcal{C}$, $\mathcal{D}$, $\mathcal{F}$, $\mathcal{G}$ and $\mathcal{H}$

In this section, we define six families of graphs. Let $\mathcal{B} = \{B_1, B_2, \ldots, B_{12}\}$ be the family of twelve graphs shown in Fig. 2. If $G \in \mathcal{B}$, we call a degree-2 vertex of G with two large neighbors a *special vertex* of G.

Let

$$\mathcal{C} = \{C_3, C_4, C_5, C_6, C_7, C_9, C_{10}, C_{13}\}, \quad \text{and}$$

$$\mathcal{D} = \{D(3,5), D(5,5), D(5,6), D(5,9), D_3(5)\},$$

be a family of cycles and daisies, respectively. The family $\mathcal{D}$ of five daisies is shown in Fig. 3.

Let $\mathcal{F} = \{F_1, F_2, \ldots, F_6\}$ be the family of six graphs shown in Fig. 4.

Let $U_1, U_2, \ldots, U_{13}$ be the thirteen graphs shown in Fig. 5. We define a **unit** to be a graph that is isomorphic to the graph U_i for some i, $1 \leq i \leq 13$. The vertex named v in each unit

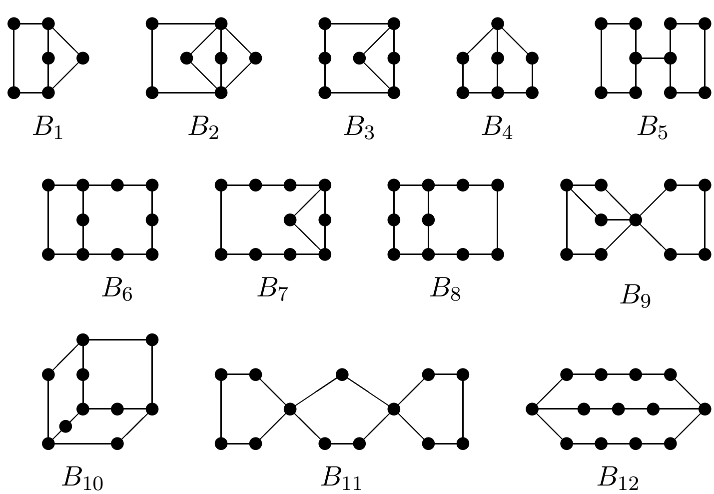

Fig. 2 The family $\mathcal{B}$ of twelve graphs

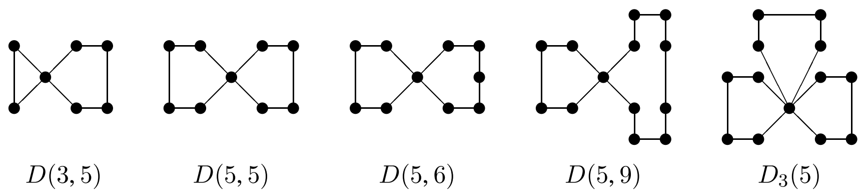

Fig. 3 The family $\mathcal{D}$ of five daisies

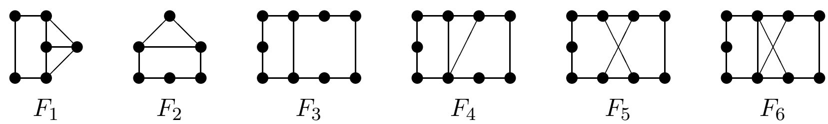

Fig. 4 The family $\mathcal{F}$ of six graphs

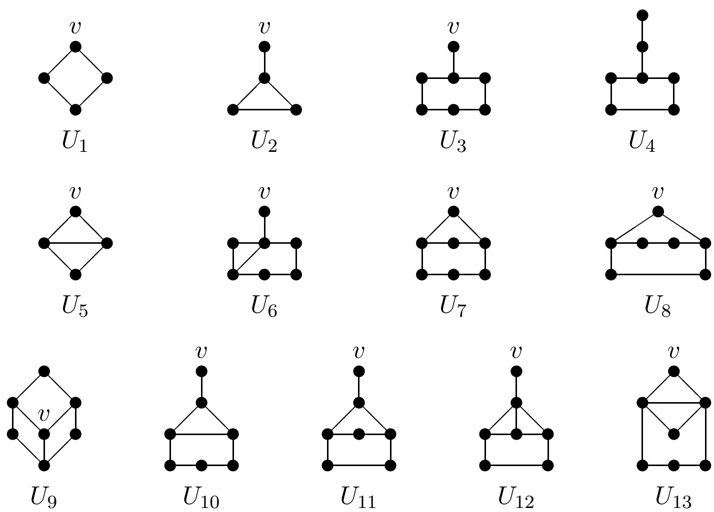

Fig. 5 The thirteen units

in Fig. 5 we call the *link vertex* of the unit. For $i = 1, 2, \ldots, 13$, we call a unit isomorphic to the graph U_i a *type-i unit*.

For $n = n_1 + n_2 + n_3 + n_4 \geq 2$, let $G = G(n_1, n_2, n_3, n_4)$ be the graph obtained from the disjoint union of n_1 units of type-1, n_2 units of type-2, n_3 units of type-3 and n_4 units of type-4, by identifying the n link vertices, one from each unit, into one new vertex which we call the *identified vertex* of G. Let $\mathcal{G}$ denote the family of all such graphs G. We call each graph in the family $\mathcal{G}$ a *good graph*. Observe that for $k \geq 2$, the graph $G(k, 0, 0, 0)$ is the daisy $D_k(4)$, while the family of graphs illustrated in Fig. 1 is the family $\{G(0, k, 0, 0) \mid k \geq 2\}$. The graph $G(2, 3, 1, 1)$ with seven units and with identified vertex v is shown in Fig. 6.

Let H be the graph obtained from the disjoint union of $n \geq 2$ units by identifying the n link vertices, one from each unit, into one new vertex. Let $\mathcal{H}$ denote the family of all such graphs H. Notice that the family $\mathcal{G}$ is a subfamily of the family $\mathcal{H}$.

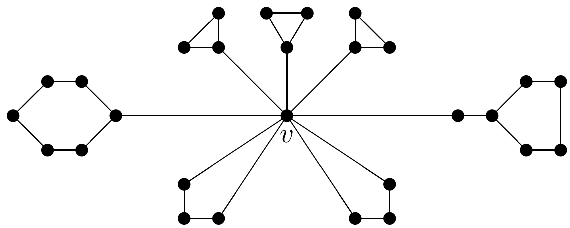

Fig. 6 A graph G in the family $\mathcal{G}$

4 Main results

We will refer to a graph G as a $\frac{2}{3}$**-minimal graph** if G is edge-minimal with respect to satisfying the following three conditions: (i) $\delta(G) \geq 2$, (ii) G is connected, and (iii) $\gamma_{\mathrm{pr}}(G) \geq 2(n-1)/3$, where n is the order of G. The following result, a proof of which is given in Section 5.2, characterizes $\frac{2}{3}$-minimal graphs

Theorem 3. *A graph G is a $\frac{2}{3}$-minimal graph if and only if $G \in \mathcal{B} \cup \mathcal{C} \cup \mathcal{D} \cup \mathcal{G}$.*

As a consequence of Theorem 3 we have the following result which gives the Haynes-Slater $2n/3$-bound of Theorem 2 and characterizes the graphs that achieve equality in this bound. A proof of Theorem 4 is given in Section 5.3.

Theorem 4. *If G is a connected graph of order $n \geq 6$ with $\delta(G) \geq 2$, then $\gamma_{\mathrm{pr}}(G) \leq 2n/3$. Furthermore, $\gamma_{\mathrm{pr}}(G) = 2n/3$ if and only if $G \in \{B_1, C_6, C_9, D(5,5)\} \cup \mathcal{F}$.*

Our main result provides a characterization of connected graphs with minimum degree at least two and order at least fourteen that have maximum possible paired-domination number. A proof of Theorem 5 is given in Section 5.4.

Theorem 5. *If G is a connected graph of order $n \geq 10$ with $\delta(G) \geq 2$, then*

$$\gamma_{\mathrm{pr}}(G) \leq \frac{2(n-1)}{3}.$$

Furthermore for $n \geq 14$, $\gamma_{\mathrm{pr}}(G) = 2(n-1)/3$ if and only if $G \in \mathcal{H}$.

5 Proof of main results

To prove our main results we introduce the concept of a near-paired-dominating set. Let $G = (V, E)$ be a graph and let $v \in V$. We define a *near-paired-dominating set*, abbreviated near-PDS, *of G relative to v* as a set $S \subseteq V$ such that $v \in S$, S dominates V, and $G[S - \{v\}]$ contains a perfect matching. The *near-paired-domination number of G relative to v*, denoted $\gamma_{\mathrm{npr}}(G; v)$, is the minimum cardinality of a near-PDS of G relative to v. A near-PDS of G relative to v of cardinality $\gamma_{\mathrm{npr}}(G; v)$ we call a $\gamma_{\mathrm{npr}}(G; v)$-set. (Note that it is possible that v itself may not be adjacent to any vertex of a $\gamma_{\mathrm{npr}}(G; v)$-set.)

5.1 Preliminary results

Before presenting a proof of our main results, we first establish some preliminary results. We omit the proofs of these preliminary results which can be readily checked by the reader.[1]

We begin with the following observation about graphs in the families $\mathcal{B} \cup \mathcal{C} \cup \mathcal{D} \cup \mathcal{G}$.

Observation 6. Let $G \in \mathcal{B} \cup \mathcal{C} \cup \mathcal{D} \cup \mathcal{G}$ have order n. Then, G is a connected graph with $\delta(G) = 2$, and $\gamma_{\mathrm{pr}}(G) = 4$ if $G = C_5$, $\gamma_{\mathrm{pr}}(G) = 2n/3$ if $G \in \{B_1, C_3, C_6, C_9, D(5,5)\}$ and $\gamma_{\mathrm{pr}}(G) = 2(n-1)/3$, otherwise.

Observation 7. Each graph in $\mathcal{B} \cup \mathcal{C} \cup \mathcal{D} \cup \mathcal{G}$ is a $\frac{2}{3}$-minimal graph.

Observation 8. Let $G \in \mathcal{B} \cup \mathcal{C} \cup \mathcal{D} \cup \mathcal{G}$ have order n, and let $v \in V(G)$. Then, (i) there is a $\gamma_{\mathrm{pr}}(G)$-set that contains v; (ii) if v is neither a special vertex of B_3 or B_6 nor the identified vertex of a good graph, then $\gamma_{\mathrm{npr}}(G; v) = \gamma_{\mathrm{pr}}(G) - 1$; (iii) if $G \in \{B_1, C_9, D(5,5)\}$ and if v is not a special vertex of B_1, then $\gamma_{\mathrm{pr}}(G - v) = \gamma_{\mathrm{pr}}(G) - 2$.

Observation 9. Let G be a connected graph with $\delta(G) \geq 2$ and let F be obtained from G by subdividing any edge four times. Then, $\gamma_{\mathrm{pr}}(F) \leq \gamma_{\mathrm{pr}}(G) + 2$.

Next we establish the value of $\gamma_{\mathrm{pr}}(C_n)$ for a cycle C_n and we characterize the $\frac{2}{3}$-minimal graphs that are cycles.

Observation 10. For $n \geq 3$, $\gamma_{\mathrm{pr}}(C_n) = 2\lceil \frac{n}{4} \rceil$.

Corollary 11. *A cycle G is a $\frac{2}{3}$-minimal graph if and only if $G \in \mathcal{C}$.*

The daisies with large paired-domination numbers are characterized in Observation 12.

Observation 12. If G is a daisy of order n, then $\gamma_{\mathrm{pr}}(G) \leq 2n/3$. Furthermore, $\gamma_{\mathrm{pr}}(G) = 2n/3$ if and only if $G = D(5,5)$, while $\gamma_{\mathrm{pr}}(G) = 2(n-1)/3$ if and only if $G = D_k(4)$ where $k \geq 2$ or $G \in \{D(3,5), D(5,6), D(5,9), D_3(5)\}$.

As observed earlier, for $k \geq 2$ the daisy $D_k(4) = G(k, 0, 0, 0) \in \mathcal{G}$. Hence we have the following characterization of $\frac{2}{3}$-minimal graphs that are daisies.

Corollary 13. *A daisy G is a $\frac{2}{3}$-minimal graph if and only if $G \in \mathcal{D} \cup \mathcal{G}$.*

The following lemma characterizes $\frac{2}{3}$-minimal graphs of small order.

Observation 14. If G is a $\frac{2}{3}$-minimal graph of order n, $3 \leq n \leq 7$, then $G \in \mathcal{B} \cup \mathcal{G}$.

[1] We provide proofs of some of these preliminary results in the appendix.

5.2 Proof of Theorem 3

The sufficiency follows from Observation 7. To prove the necessary, we proceed by induction on the order $n \geq 3$ of a $\frac{2}{3}$-minimal graph. By Observation 14, the result is true for $n \leq 7$. Let $n \geq 8$, and assume that the result is true for all $\frac{2}{3}$-minimal graphs G' of order n', where $3 \leq n' < n$. Let $G = (V, E)$ be a $\frac{2}{3}$-minimal graph of order n. Before proceeding further, we present two observations that will be useful in what follows. If e is an edge of G, then $\gamma_{\text{pr}}(G - e) \geq \gamma_{\text{pr}}(G)$. Hence, by the minimality of G, we have the following observation.

Observation 15. If $e \in E$, then either e is a bridge of G or $\delta(G - e) = 1$.

Since the paired-domination number of a graph cannot decrease if edges are removed, the next result is a consequence of the inductive hypothesis.

Observation 16. If G' is a connected subgraph of G of order $n' < n$ with $\delta(G') \geq 2$, then either $G' \in \mathcal{B} \cup \mathcal{C} \cup \mathcal{D} \cup \mathcal{G}$ or $\gamma_{\text{pr}}(G') < 2(n' - 1)/3$.

We now return to the proof of Theorem 3. Suppose $G = C_n$ (and still $n \geq 8$). Then, by Corollary 11, $G \in \mathcal{C}$. So we may assume that G is not a cycle. Let $\mathcal{L}$ be set of all large vertices of G, i.e., $\mathcal{L} = \{v \in V \mid d_G(v) \geq 3\}$. By assumption, $|\mathcal{L}| \geq 1$. If $|\mathcal{L}| = 1$, then G is a daisy, and so, by Corollary 13, $G \in \mathcal{D} \cup \mathcal{G}$. Hence we may assume $|\mathcal{L}| \geq 2$.

Let C be any component of $G - \mathcal{L}$; it is a path. If C has only one vertex, or has at least two vertices but the two ends of C are adjacent in G to different large vertices, then we say that C is a 2-*path*. Otherwise we say that C is a 2-*handle*.

Lemma 17. *If $\mathcal{L}$ is not an independent set, then $G = B_5$ or $G \in \mathcal{G}$.*

Proof: Suppose $e = uv$ is an edge, where $u, v \in \mathcal{L}$. By Observation 15, e must be a bridge of G. Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be the two components of $G - e$ where $u \in V_1$. For $i = 1, 2$, let $|V_i| = n_i$. Each G_i satisfies $\delta(G_i) \geq 2$ and is connected. Hence by Observation 16, for $i = 1, 2$, $G_i \in \mathcal{B} \cup \mathcal{C} \cup \mathcal{D} \cup \mathcal{G}$ or $\gamma_{\text{pr}}(G_i) < 2(n_i - 1)/3$. If $\gamma_{\text{pr}}(G_i) \leq 2(n_i - 1)/3$ for $i = 1, 2$, then $\gamma_{\text{pr}}(G) \leq \gamma_{\text{pr}}(G_1) + \gamma_{\text{pr}}(G_2) \leq 2(n - 2)/3$, a contradiction. Hence we may assume that $\gamma_{\text{pr}}(G_1) \geq 2n_1/3$.

Suppose first that $G_1 = C_5$. We show then that $G = B_5$. Let G_1 be the 5-cycle $u_1, u_2, \ldots, u_5, u_1$, where $u = u_1$. Suppose that $G_2 \notin \mathcal{B} \cup \mathcal{C} \cup \mathcal{D} \cup \mathcal{G}$. Then, $\gamma_{\text{pr}}(G_2) < 2(n_2 - 1)/3$, implying that $\gamma_{\text{pr}}(G_2) \leq 2(n_2 - 2)/3 = 2(n - 7)/3$ since $\gamma_{\text{pr}}(G_2)$ is an even integer. If $\gamma_{\text{pr}}(G_2) < 2(n - 7)/3$, then $\gamma_{\text{pr}}(G) \leq \gamma_{\text{pr}}(G_1) + \gamma_{\text{pr}}(G_2) < 2(n - 1)/3$, a contradiction. Hence, $\gamma_{\text{pr}}(G_2) = 2(n - 7)/3$. Let G^* be the graph of order $n^* = n - 3$ obtained from G by deleting the three vertices u_3, u_4, u_5 and adding the edge $u_2 v$. Since G is a $\frac{2}{3}$-minimal graph, it follows from construction that either G^* is a $\frac{2}{3}$-minimal graph or $\gamma_{\text{pr}}(G^*) < 2(n^* - 1)/3$. If G^* is a $\frac{2}{3}$-minimal graph, then, by the inductive hypothesis, $G^* \in \mathcal{B} \cup \mathcal{C} \cup \mathcal{D} \cup \mathcal{G}$, a contradiction since G^* contains a vertex, namely v, of degree 4 that belongs to a triangle. Hence, $\gamma_{\text{pr}}(G^*) < 2(n^* - 1)/3 = 2(n - 4)/3$. By construction of G^*, there exists a $\gamma_{\text{pr}}(G^*)$-set S^* that does not contain the vertex u_2 (if there is a $\gamma_{\text{pr}}(G^*)$-set that contains u_2, simply replace u_2 and its partner in this set with u_1 and v). But then $S^* \cup \{u_3, u_4\}$ is a PDS of G, and so $\gamma_{\text{pr}}(G) \leq |S^*| + 2 < 2(n - 4)/3 + 2 = 2(n - 1)/3$, a contradiction. Hence, $G_2 \in \mathcal{B} \cup \mathcal{C} \cup \mathcal{D} \cup \mathcal{G}$. If $G_2 \neq C_5$, then $\gamma_{\text{pr}}(G_2) \leq 2n_2/3 = 2(n - 5)/3$ and, by Observation 8(i), there exists a $\gamma_{\text{pr}}(G_2)$-set containing v. Such a $\gamma_{\text{pr}}(G_2)$-set can be extended to a

PDS of G by adding to it the pair $\{u_3, u_4\}$, and so $\gamma_{\rm pr}(G) \le 2(n-5)/3 + 2 = 2(n-2)/3$, a contradiction. Hence we must have that $G_2 = C_5$, and so $G = B_5$. Therefore we may assume that neither G_1 nor G_2 is a 5-cycle, for otherwise $G = B_5$. This implies that $\gamma_{\rm pr}(G_1) = 2n_1/3$.

By Observation 6, $G_1 \in \{B_1, C_3, C_6, C_9, D(5,5)\}$. By Observation 8(ii), $\gamma_{\rm npr}(G_1; u) = \gamma_{\rm pr}(G_1) - 1 \le 2n_1/3 - 1$. If $\gamma_{\rm pr}(G_2) < 2(n_2-1)/3$, then $\gamma_{\rm pr}(G) \le \gamma_{\rm pr}(G_1) + \gamma_{\rm pr}(G_2) < 2(n-1)/3$, a contradiction. Hence, $G_2 \in \mathcal{B} \cup \mathcal{C} \cup \mathcal{D} \cup \mathcal{G}$. Since G is a $\frac{2}{3}$-minimal graph, we cannot have that $G_2 \in \{B_3, B_6\}$ with v a special vertex of G_2. Hence if $G_2 \notin \mathcal{G}$ or if $G_2 \in \mathcal{G}$ and v is not the identified vertex of G_2, then, by Observation 8(ii), $\gamma_{\rm npr}(G_2; v) = \gamma_{\rm pr}(G_2) - 1 \le 2n_2/3 - 1$. But then $\gamma_{\rm pr}(G) \le \gamma_{\rm npr}(G_1, u) + \gamma_{\rm npr}(G_2, v) \le 2n/3 - 2$, a contradiction. Hence we must have that $G_2 \in \mathcal{G}$ with v the identified vertex of G_2. Suppose that $G_1 \in \{B_1, C_9, D(5,5)\}$. Since G is a $\frac{2}{3}$-minimal graph, we cannot have that $G_1 = B_1$ with u a special vertex of B_1. Hence, by Observation 8(iii), $\gamma_{\rm pr}(G_1 - u) = \gamma_{\rm pr}(G_1) - 2 = 2n_1/3 - 2$. By Observation 8(i), there exists a $\gamma_{\rm pr}(G_2)$-set containing v. Such a $\gamma_{\rm pr}(G_2)$-set can be extended to a PDS of G by adding to it the vertices in a $\gamma_{\rm pr}(G_1 - u)$-set, and so $\gamma_{\rm pr}(G) \le \gamma_{\rm pr}(G_1 - u) + \gamma_{\rm pr}(G_2) \le 2n_1/3 - 2 + 2n_2/3 = 2n/3 - 2$, a contradiction. Hence, $G_1 \notin \{B_1, C_9, D(5,5)\}$. Thus, $G_1 \in \{C_3, C_6\}$, and so $G \in \mathcal{G}$. □

By Lemma 17, we may assume that $\mathcal{L}$ *is an independent set*, for otherwise $G \in \mathcal{B} \cup \mathcal{G}$.

Lemma 18. *If G contains a path on six vertices with the two ends of the path not adjacent and with every internal vertex of the path a degree-2 vertex in G, then $G \in \{B_6, B_7, B_8, B_{12}\}$.*

Proof: Let u and v be two nonadjacent vertices that are joined by a path u, u_1, u_2, u_3, u_4, v every internal vertex of which has degree 2 in G. Let G' be the graph obtained from G by removing the vertices u_1, u_2, u_3, u_4, and adding the edge $e = uv$. Then, G' is a connected graph of order $n' = n - 4$ with $\delta(G') \ge 2$. By Observation 9, $\gamma_{\rm pr}(G) \le \gamma_{\rm pr}(G') + 2$. If $\gamma_{\rm pr}(G') < 2n'/3$, then $\gamma_{\rm pr}(G) < 2(n-1)/3$, a contradiction. Hence, $\gamma_{\rm pr}(G') \ge 2n'/3$.

On the one hand, suppose $G' - e$ is disconnected or $G' - e$ is connected and $\delta(G' - e) = 1$. Then since G is a $\frac{2}{3}$-minimal graph, it follows that G' is a $\frac{2}{3}$-minimal graph. The degree of each large vertex is unchanged in G and G', and so the graph G' has at least two large vertices. Hence applying the inductive hypothesis to G', $G' = B_1$ by Observation 6, whence $G \in \{B_6, B_7\}$.

On the other hand, suppose $G' - e$ is a connected subgraph of G and $\delta(G' - e) \ge 2$. Since $\gamma_{\rm pr}(G' - e) \ge \gamma_{\rm pr}(G') = 2n'/3$, Observations 6 and 16 imply that $G' - e \in \{B_1, C_3, C_5, C_6, C_9, D(5,5)\}$. If $G' - e = B_1$, then $n = 10$ and $\gamma_{\rm pr}(G) = 4 = 2(n-4)/3$, a contradiction. Since the set $\mathcal{L}$ is independent in G, $G' - e \ne C_3$. If $G' - e = C_5$, then $n = 9$ and $\gamma_{\rm pr}(G) = 4 = 2(n-3)/3$, a contradiction. Suppose $G' - e = C_6$. Then, $n = 10$. If u and v are at distance 3 apart in $G' - e$, then $\gamma_{\rm pr}(G) = 4 = 2(n-4)/3$, a contradiction. Hence, u and v are at distance 2 apart in $G' - e$, whence $G = B_8$. Suppose $G' - e = C_9$. Then, $n = 13$. If u and v are at distance 2 or 3 apart in $G' - e$, then $\gamma_{\rm pr}(G) = 6 = 2(n-4)/3$, a contradiction. Hence, u and v are at distance 4 apart in $G' - e$, whence $G = B_{12}$. If $G' - e = D(5,5)$, then $n = 13$ and $\gamma_{\rm pr}(G) = 6 = 2(n-4)/3$, a contradiction. □

By Lemma 18, we may assume that *there is no path on six vertices in G with the two ends of the path not adjacent and with every internal vertex of the path a degree-2 vertex in G*, for otherwise $G \in \mathcal{B}$. With this assumption, every 2-path in G contains at most three vertices, while every 2-handle of G contains at most five vertices.

Lemma 19. *If G contains a degree-3 vertex that is adjacent to the ends of a 2-handle, then $G \in \mathcal{G}$.*

Proof: Suppose that G contains a degree-3 vertex v that is adjacent to the ends of a 2-handle C. Let P be the 2-path that has an end adjacent with v, and let u be the other large vertex adjacent with an end of P. Let C contain $r-1$ vertices and P contain s vertices. By our earlier assumptions, $3 \le r \le 6$ and $1 \le s \le 3$.

Let $G_1 = G[V(C) \cup \{v\} \cup V(P)]$ and let $G_2 = G - V(G_1)$. Then, G_1 is a key $L_{r,s}$, while G_2 is a connected graph with $\delta(G_2) \ge 2$. For $i = 1, 2$, let G_i have order n_i, and so $n = n_1 + n_2$. It is a simple exercise to check that $\gamma_{\rm pr}(G_1) \le 2n_1/3$ with equality if and only if $G_1 \in \{L_{3,3}, L_{4,2}, L_{5,1}, L_{6,3}\}$. If $\gamma_{\rm pr}(G_2) < 2(n_2-1)/3$, then $\gamma_{\rm pr}(G) \le \gamma_{\rm pr}(G_1) + \gamma_{\rm pr}(G_2) < 2(n-1)/3$, a contradiction. Hence, by Observation 16, $G_2 \in \mathcal{B} \cup \mathcal{C} \cup \mathcal{D} \cup \mathcal{G}$.

If $G_2 = C_5$, then $\gamma_{\rm pr}(G) \le 2(n-1)/3$ with equality if and only if $G_1 \in \{L_{3,2}, L_{4,1}, L_{5,3}, L_{6,2}\}$, i.e., if and only if $G \in \{G(1,0,0,1), G(0,1,0,1), G(0,0,1,1), G(0,0,0,2)\}$. However G is a $\frac{2}{3}$-minimal graph, and so $\gamma_{\rm pr}(G) \ge 2(n-1)/3$, whence $\gamma_{\rm pr}(G) = 2(n-1)/3$ and $G \in \mathcal{G}$. Thus we may assume that $G_2 \ne C_5$, and so $\gamma_{\rm pr}(G_2) \le 2n_2/3$.

Suppose $\gamma_{\rm pr}(G_2) = 2n_2/3$. Then, by Observation 6, $G_2 \in \{B_1, C_3, C_6, C_9, D(5,5)\}$. If $G_2 = B_1$, then since $\mathcal{L}$ is an independent set, u is a vertex of degree-3 in B_1 and a simple check shows that $\gamma_{\rm pr}(G) \le 2(n-2)/3$, a contradiction. If $G_2 = C_3$, then $\gamma_{\rm pr}(G) \le 2(n-1)/3$ with equality if and only if $G \in \{G(0,2,0,0), G(0,1,0,1), G(0,1,1,0)\}$. If $G_2 = C_6$, then $\gamma_{\rm pr}(G) \le 2(n-1)/3$ with equality if and only if $G \in \{G(0,1,1,0), G(0,0,2,0), G(0,0,1,1)\}$. Hence if $G_2 \in \{C_3, C_6\}$, then $\gamma_{\rm pr}(G) = 2(n-1)/3$ and $G \in \mathcal{G}$. Since every 2-handle of G contains at most five vertices, $G_2 \ne C_9$. If $G_2 = D(5,5)$, then either u is the vertex of degree-3 in $D(5,5)$ or u is a degree-2 vertex with two small neighbors. In either case, a simple check shows that $\gamma_{\rm pr}(G) \le 2(n-2)/3$, a contradiction. Hence if $\gamma_{\rm pr}(G_2) = 2n_2/3$, then $G_2 \in \{C_3, C_6\}$ and $G \in \mathcal{G}$. Thus we may assume that $\gamma_{\rm pr}(G_2) < 2n_2/3$, i.e., $\gamma_{\rm pr}(G_2) \le 2(n_2-1)/3$. However, $G_2 \in \mathcal{B} \cup \mathcal{C} \cup \mathcal{D} \cup \mathcal{G}$, and so, by Observation 6, $\gamma_{\rm pr}(G_2) = 2(n_2-1)/3$.

If $\gamma_{\rm pr}(G_1) < 2n_1/3$, then $\gamma_{\rm pr}(G) \le \gamma_{\rm pr}(G_1) + \gamma_{\rm pr}(G_2) \le 2(n-2)/3$, a contradiction. Hence, $\gamma_{\rm pr}(G_1) = 2n_1/3$, whence, as observed earlier, $G_1 \in \{L_{3,3}, L_{4,2}, L_{5,1}, L_{6,3}\}$. Suppose $G_1 \ne L_{5,1}$. Then, by Observation 8(i), there exists a $\gamma_{\rm pr}(G_2)$-set that contains u. Any such $\gamma_{\rm pr}(G_2)$-set can be extended to a PDS of G by adding $2(n_1-3)/3$ vertices of G_1, implying that $\gamma_{\rm pr}(G) \le 2(n-3)/3$, a contradiction. Hence, $G_1 = L_{5,1}$. Let w be the degree-1 vertex of G_1. Then, $\gamma_{\rm npr}(G_1; w) = \gamma_{\rm pr}(G_1) - 1$. Since $\mathcal{L}$ is an independent set, u is not a special vertex of a graph in $\mathcal{B}$. If $G_2 \notin \mathcal{G}$ or if $G_2 \in \mathcal{G}$ and u is not the identified vertex of G_2, then $\gamma_{\rm npr}(G_2; u) = \gamma_{\rm pr}(G_2) - 1$, whence $\gamma_{\rm pr}(G) \le \gamma_{\rm npr}(G_1; w) + \gamma_{\rm npr}(G_2; u) = \gamma_{\rm pr}(G_1) + \gamma_{\rm pr}(G_2) - 2 < 2(n-1)/3$, a contradiction. Hence we must have that $G_2 \in \mathcal{G}$ and u is the identified vertex of G_2, implying that $G \in \mathcal{G}$. □

By Lemma 19, we may assume that *if G contains a 2-handle, then the large vertex adjacent to the ends of this 2-handle has degree at least 4.*

Lemma 20. *If G has a 2-handle containing three or four vertices, then $G \in \{B_9, B_{11}\}$ or $G \in \mathcal{G}$.*

Proof: Suppose that G contains a 2-handle C with $|C| \in \{3, 4\}$. Let v be the vertex of $\mathcal{L}$ adjacent to the two ends of C. By assumption, $d_G(v) \ge 4$. Let C be the path $v_1, v_2, \ldots, v_t$, and so $3 \le t \le 4$.

Let $G' = G - V(C)$. Then, G' is a connected subgraph of G of order $n' = n - t$ with $\delta(G') \geq 2$. By Observation 16, $G' \in \mathcal{B} \cup \mathcal{C} \cup \mathcal{D} \cup \mathcal{G}$ or $\gamma_{\rm pr}(G') < 2(n'-1)/3$. The degree of each large vertex other than v is unchanged in G and G', and so the graph G' has at least one large vertex. In particular, G' is not a cycle, and so by Observation 6, $\gamma_{\rm pr}(G') \leq 2n'/3$. Any $\gamma_{\rm pr}(G')$-set can be extended to a PDS of G by adding to it the two vertices v_2 and v_3, and so $\gamma_{\rm pr}(G) \leq \gamma_{\rm pr}(G') + 2$.

Suppose $t = 4$. If $\gamma_{\rm pr}(G') < 2n'/3 = 2(n-4)/3$, then $\gamma_{\rm pr}(G) < 2(n-1)/3$, a contradiction. Hence, $\gamma_{\rm pr}(G') \geq 2n'/3$, and so, by Observation 6, $G' \in \{B_1, D(5,5)\}$. If $G' = B_1$, then, since $\mathcal{L}$ is an independent set, the vertex v must be one of the two degree-3 vertices in B_1, whence $G = B_9$. If $G' = D(5,5)$, then since $\mathcal{L}$ is an independent set and $|\mathcal{L}| \geq 2$, the vertex v must be a degree-2 vertex of $D(5,5)$ with two small neighbors, whence $G = B_{11}$.

Suppose that $t = 3$. If $\gamma_{\rm pr}(G') < 2(n'-1)/3$, then $\gamma_{\rm pr}(G) < 2(n-1)/3$, a contradiction. Hence, $\gamma_{\rm pr}(G') \geq 2(n'-1)/3$, and so, Observation 6, $G' \in \mathcal{B} \cup \mathcal{C} \cup \mathcal{D} \cup \mathcal{G}$. Since $\mathcal{L}$ is an independent set, if $G' \in \mathcal{B}$, then v is not a special vertex of G'. Hence if $G' \notin \mathcal{G}$ or if $G' \in \mathcal{G}$ and v is not the identified vertex of G', then by Observation 8(ii), $\gamma_{\rm npr}(G'; v) = \gamma_{\rm pr}(G') - 1$. Any $\gamma_{\rm npr}(G'; v)$-set can be extended to a PDS of G by adding the vertex v_1 (with v and v_1 paired), and so $\gamma_{\rm pr}(G) \leq \gamma_{\rm npr}(G'; v) + |\{v_1\}| = \gamma_{\rm pr}(G') \leq 2n'/3 = 2(n-3)/3$, a contradiction. Hence we must have that $G' \in \mathcal{G}$ with v the identified vertex of G', implying that $G \in \mathcal{G}$. □

By Lemma 20, we may assume that *every* 2*-handle of G contains two vertices or five vertices.*

Lemma 21. *Every* 2*-path of G contains one vertex or two vertices.*

Proof: Assume that G has a 2-path $P\colon v_1, v_2, v_3$ containing three vertices. Let u and v be the large vertices adjacent to v_1 and v_3, respectively. Let $G' = G - V(P)$. Then, G' is a subgraph of G of order $n' = n - 3$ with $\delta(G') \geq 2$. Any $\gamma_{\rm pr}(G')$-set can be extended to a PDS of G by adding to it two adjacent vertices of P, and so $\gamma_{\rm pr}(G) \leq \gamma_{\rm pr}(G') + 2$. Hence if $\gamma_{\rm pr}(G') < 2(n'-1)/3$, then $\gamma_{\rm pr}(G) < 2(n-1)/3$, a contradiction. Consequently, $\gamma_{\rm pr}(G') \geq 2(n'-1)/3$.

Assume first that G' is disconnected. Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be the two components of G' where $u \in V_1$. For $i = 1, 2$, let $|V_i| = n_i$, and so $n = n_1 + n_2 + 3$. Each G_i satisfies $\delta(G_i) \geq 2$ and is connected. Our assumption that no large vertex of degree-3 is adjacent to the ends of a 2-handle implies that G_i is not a cycle. Hence by Observation 16, for $i = 1, 2$, $G_i \in \mathcal{B} \cup \mathcal{D} \cup \mathcal{G}$ or $\gamma_{\rm pr}(G_i) < 2(n_i - 1)/3$. If $\gamma_{\rm pr}(G_i) \leq 2(n_i-1)/3$ for $i = 1, 2$, then $\gamma_{\rm pr}(G) \leq \gamma_{\rm pr}(G_1) + \gamma_{\rm pr}(G_2) + 2 \leq 2(n_1 + n_2 - 2)/3 + 2 = 2(n-2)/3$, a contradiction. Hence we may assume that $\gamma_{\rm pr}(G_1) = 2n_1/3$. By our earlier assumption, no 2-handle of G contains five vertices. Hence, since G_1 is not a cycle, it follows from Observation 6 that $G = B_1$. Since $\mathcal{L}$ is an independent set in G, the vertex u is a degree-3 vertex in G_1. But then any $\gamma_{\rm pr}(G_2)$-set can be extended to a PDS of G by adding to it four additional vertices, namely v_1 and v_2 and the two vertices in G_1 at distance 2 from u. Hence, $\gamma_{\rm pr}(G) \leq \gamma_{\rm pr}(G_2) + 4 \leq 2n_2/3 + 4 = 2(n-9)/3 + 4 = 2(n-3)/3$, a contradiction. Therefore, G' is connected.

As observed earlier, $\gamma_{\rm pr}(G') \geq 2(n'-1)/3$. Hence, by Observation 16, $G' \in \mathcal{B} \cup \mathcal{C} \cup \mathcal{D} \cup \mathcal{G}$. By our earlier assumptions, we recall that $n \geq 8$ (and so, $n' \geq 5$), $\mathcal{L}$ is an independent set in G, every 2-handle in G, if any, contains two or five vertices, no 2-path in G contains four or more vertices, and the large vertex adjacent to the ends of a 2-handle has degree at least 4. In particular, we observe that the vertices u and v are not adjacent in G, and that if G' has a 2-path containing four or more vertices or a 2-handle that does not contain two or five vertices, then such a 2-path or 2-handle contains at least one of u and v. It follows that $G' \notin \{B_5, C_3, D(5,9), D_3(5)\}$ and if $G' \in \mathcal{G}$, then G' contains exactly two units. With these restrictions, it is a simple exercise to check that $\gamma_{\mathrm{pr}}(G) \le 2(n-2)/3$, a contradiction. □

Lemma 22. *There is no* 2*-handle in* G.

Proof: Assume that G contains a 2-handle C. Let v be the vertex in $\mathcal{L}$ adjacent to the two ends of C. By our earlier assumptions, $|C| = 2$ or $|C| = 5$ and $d_G(v) \ge 4$. Let C be the path $v_1, v_2, \dots, v_t$, where $t \in \{2, 5\}$. Since $|\mathcal{L}| \ge 2$, there is a 2-path P with one of its ends adjacent to v. Let u be the large vertex adjacent to the other end of P. By our earlier assumptions, $1 \le |P| \le 2$.

Suppose that $|P| = 1$. Let $V(P) = \{x\}$, and so $N(x) = \{u, v\}$. Let $G' = G - x$. Then, G' is a subgraph of G of order $n' = n - 1$ with $\delta(G') \ge 2$. Since v is adjacent to the ends of the 2-handle C, and $|C| \in \{2, 5\}$, there exists a $\gamma_{\mathrm{pr}}(G')$-set containing v. Such a $\gamma_{\mathrm{pr}}(G')$-set is a PDS of G, implying that $\gamma_{\mathrm{pr}}(G) \le \gamma_{\mathrm{pr}}(G')$. Let G_v be the component of G' containing v (possibly, $G_v = G'$). Since v is adjacent to the ends of the 2-handle C in G_v, and since $d_{G_v}(v) \ge 3$, the component G_v is not a cycle and $G_v \notin \{B_1, D(5,5)\}$. Hence, by Observation 16, $\gamma_{\mathrm{pr}}(G_v) \le 2(|V(G_v)| - 1)/3$. If G' is connected, then $G' = G_v$, and so $\gamma_{\mathrm{pr}}(G) \le 2(n-2)/2$, a contradiction. Therefore, G' is disconnected. Let G_u be the component of G' containing u. Since every large vertex adjacent to the ends of a 2-handle in G has degree at least 4, the component G_u is not a cycle. In particular, $G_u \ne C_5$, and so $\gamma_{\mathrm{pr}}(G_u) \le 2|V(G_u)|/3$ by Observation 16. Hence, $\gamma_{\mathrm{pr}}(G) \le \gamma_{\mathrm{pr}}(G_u) + \gamma_{\mathrm{pr}}(G_v) \le 2(|V(G_v)| - 1)/3 + 2|V(G_u)|/3 = 2(n-2)/3$, a contradiction. Hence, $|P| = 2$ and we may assume that every 2-path with one end adjacent to v contains two vertices.

Suppose that v is adjacent to the end of at least two 2-handles (each containing two or five vertices) in G. Let $F = G - V(C)$. Then, F is not a cycle, $d_F(v) \ge 3$ and v is adjacent to the ends of a 2-handle in F (containing two or five vertices). Thus by Observation 16, $\gamma_{\mathrm{pr}}(F) \le 2(|V(F)| - 1)/3 = 2(n - t - 1)/3$. Furthermore, there exists a $\gamma_{\mathrm{pr}}(F)$-set containing v. Hence if $t = 2$, then $\gamma_{\mathrm{pr}}(G) \le \gamma_{\mathrm{pr}}(F) \le 2(n-3)/3$, while if $t = 5$, then $\gamma_{\mathrm{pr}}(G) \le \gamma_{\mathrm{pr}}(F) + 2 \le 2(n-6)/3 + 2 = 2(n-3)/3$. Both cases produce a contradiction. Hence, C is the only 2-handle with its ends adjacent to v.

Since $d_G(v) \ge 4$, there are at least two 2-paths with one end adjacent to v. By assumption, every such 2-path contains two vertices. Let the 2-path P be given by x, y where x is the end of P adjacent to v (and y the end adjacent to u). Let $P_1, \dots, P_r$, $r \ge 1$, be the other 2-paths with one end adjacent to v. For $i = 1, \dots, r$, let x_i be the end of the 2-path P_i adjacent to v and let y_i be the other end of P_i. Let G' be obtained from $G - V(C) - \{v\}$ by adding the edges xx_i for $i = 1, \dots, r$. Then, G' is a connected graph of order $n' = n - |V(C)| - 1$ with $\delta(G') \ge 2$. Observe that every edge e incident with x in G' is a bridge of G' or satisfies $\delta(G' - e) = 1$, while every edge $f \in E(G')$ that is a bridge in G or satisfies $\delta(G - f) = 1$ is also a bridge of G' or satisfies $\delta(G' - f) = 1$.

Assume that $\gamma_{\mathrm{pr}}(G') \ge 2(n' - 1)/3$. It follows that since G is a $\frac{2}{3}$-minimal graph, so is G', whence, by the inductive hypothesis, $G' \in \mathcal{B} \cup \mathcal{C} \cup \mathcal{D} \cup \mathcal{G}$. Since G' is not a cycle, $G' \notin \mathcal{C}$. Since $\mathcal{L}$ is an independent set in G, the set of large vertices in G' is an independent set in G'. By construction of the graph G', if $d_G(v) = 4$, then $r = 1$ and y, x, x_1, y_1 is either a 2-path or a 2-handle in G' of cardinality 4, while if $d_G(v) \ge 5$, then $r \ge 2$ and the 2-paths $P_1, P_2, \dots, P_r$ in G are also 2-paths in G' with x the large vertex in G' adjacent to the ends x_i of each 2-path P_i, $1 \le i \le r$. Thus, G' has either a 2-path or a 2-handle (namely, y, x, x_1, y_1)

of cardinality 4 or a large vertex (namely, x) that is adjacent only to the ends of 2-paths with one such 2-path (namely, the 2-path consisting of the vertex y) of cardinality 1 and the remaining 2-paths (namely, the 2-paths $P_1, P_2, \ldots, P_r$) of cardinality 2. By our earlier assumptions, every 2-handle in G', except possibly for one 2-handle (namely, y, x, x_1, y_1 which occurs if $r = 1$ and if y and y_1 are adjacent to the common large vertex u), contains two or five vertices, while every large vertex, except possibly for the vertex u, adjacent to the ends of a 2-handle in G' has degree at least 4. The only graphs in $\mathcal{B} \cup \mathcal{D} \cup \mathcal{G}$ that satisfy our earlier assumptions and have either a 2-path or a 2-handle of cardinality 4 or a large vertex that is adjacent only to the ends of 2-paths, all but one of which have cardinality 2, are graphs in the subfamily $\{B_4, B_8, B_9, B_{12}, D(3,5), D(5,6), G(0,1,0,1), G(0,0,1,1)\}$. Thus, G' must be a graph in this subfamily. In all cases, it is a simple exercise to check that $\gamma_{\rm pr}(G) < 2(n-1)/3$, a contradiction. Hence, $\gamma_{\rm pr}(G') \leq 2(n'-2)/3$.

Let S' be a $\gamma_{\rm pr}(G')$-set. Assume that $x \notin S'$ or if $x \in S'$, then x is not paired with x_i in S' for some i, $1 \leq i \leq r$. Let $S = S' \cup \{v, v_1\}$ if $t = 2$ and let $S = S' \cup \{v, v_1, v_2, v_3\}$ if $t = 5$. Then, S is a PDS of G, and so $\gamma_{\rm pr}(G) \leq |S| \leq 2(n-2)/3$, a contradiction. Hence, renaming vertices if necessary, we may assume that $\{x, x_1\} \subset S'$ with x and x_1 paired in S'. Let u_1 be the degree-2 vertex adjacent with x_1 in G. Let $S = S' \cup \{v, u_1\}$ if $t = 2$ and let $S = S' \cup \{v, u_1, v_2, v_3\}$ if $t = 5$. Then, S is a PDS of G (with v paired with x, and with u_1 paired with x_1), and so $\gamma_{\rm pr}(G) \leq |S| \leq 2(n-2)/3$, a contradiction. □

By our assumptions to date, $\mathcal{L}$ is independent and every 2-path of G contains one vertex or two vertices. Further, G contains no 2-handle by Lemma 22. In particular, this implies that G contains no triangle.

Lemma 23. *If G contains a 4-cycle, then $G \in \{B_1, B_2\}$.*

Proof: Suppose that G contains a 4-cycle u, v, w, x, u. Since G has no 2-handle, we may assume that u and w are large vertices of G (and so v and x are small vertices). Let $G' = G - v$ have order $n' = n - 1$. Then, $\delta(G') \geq 2$ and G' is connected. Any $\gamma_{\rm pr}(G')$-set contains at least one of u and w and is therefore a PDS of G. Thus, $\gamma_{\rm pr}(G) \leq \gamma_{\rm pr}(G')$. Hence if $\gamma_{\rm pr}(G') < 2n'/3$, then $\gamma_{\rm pr}(G) < 2(n-1)/3$, a contradiction. Therefore, $\gamma_{\rm pr}(G') \geq 2n'/3$. By Observation 16, $G' \in \{B_1, C_3, C_5, C_6, C_9, D(5,5)\}$. By our earlier assumptions and results, $G' \notin \{C_3, C_6, C_9, D(5,5)\}$. If $G' = C_5$, then $G = B_1$, while if $G' = B_1$, then $G = B_2$. □

By Lemma 23, we may assume that G contains no 4-cycle. Hence the *smallest cycle in G has length* 5.

Lemma 24. *If G contains a large vertex that is adjacent only to the ends of 2-paths on one vertex, then $G = B_{10}$.*

Proof: Let v be a large vertex of G that is adjacent only to the ends of 2-paths on one vertex. By our assumption that G contains no 4-cycle, the vertex v is the only common neighbor of two vertices in $N(v)$. Let $G' = G - N[v]$ and let G' have order $n' = n - d_G(v) - 1 \leq n - 4$. Then, $\delta(G') \geq 2$.

We show first that G' has no C_5-component. Assume, to the contrary, that G' has a C_5-component C: a, b, c, d, e, a. By our earlier assumptions, this component contains two large vertices, say a and c, of G at distance 2 apart on the cycle. Let v_1 be the common neighbor of a and v, and let v_2 be the common neighbor of c and v. If $d(v) = 3$, let $G_1 = G' - V(C)$, while if $d(v) \geq 4$, let $G_1 = G - V(C) - \{v_1, v_2\}$. Let G_1 have order n_1. In both cases, G_1 is a connected graph with $\delta(G_1) \geq 2$.

Assume that $d(v) = 3$. Then, $n_1 = n - 9$. Let u be the vertex of G_1 that has a common neighbor with v in G. If $\gamma_{pr}(G') \leq 2(n' - 2)/3 = 2(n - 11)/3$, then $\gamma_{pr}(G) \leq \gamma_{pr}(G') + 6 \leq 2(n - 2)/3$, a contradiction. Hence, $\gamma_{pr}(G') \geq 2(n' - 1)/3$. By Observation 16, $G' \in \mathcal{B} \cup \mathcal{C} \cup \mathcal{D} \cup \mathcal{G}$. By our earlier assumptions, G_1 is not a cycle, and so $\gamma_{pr}(G_1) \leq 2n'/3 = 2(n - 9)/3$. By Observation 8(i), there is a $\gamma_{pr}(G_1)$-set that contains u. Such a $\gamma_{pr}(G_1)$-set can be extended to a PDS of G by adding to it the four vertices a, c, v_1, v_2, implying that $\gamma_{pr}(G) \leq \gamma_{pr}(G') + 4 \leq 2(n - 9)/3 + 4 = 2(n - 3)/3$, a contradiction. Hence, $d(v) \geq 4$, and so $n_1 = n - 7$. By our earlier assumptions, G_1 is not a cycle and $G_1 \notin \{B_1, D(5, 5)\}$. Hence, by Observations 6 and 16, $\gamma_{pr}(G_1) \leq 2(n' - 1)/3 = 2(n - 8)/3$. Any $\gamma_{pr}(G')$-set can be extended to a PDS of G by adding to it the four vertices a, c, v_1, v_2, implying that $\gamma_{pr}(G) \leq \gamma_{pr}(G') + 4 \leq 2(n - 8)/3 + 4 = 2(n - 2)/3$, a contradiction. Hence, G' has no C_5-component.

Applying the inductive hypothesis to each component of G', $\gamma_{pr}(G') \leq 2n'/3 \leq 2(n - 4)/3$. Any $\gamma_{pr}(G')$-set can be extended to a PDS of G by adding to it v and a neighbor of v. Hence, $2(n - 1)/3 \leq \gamma_{pr}(G) \leq \gamma_{pr}(G') + 2 \leq 2(n - 1)/3$. Consequently we must have equality throughout this inequality chain, implying that $d(v) = 3$ and that each component of G' is from the family $\{B_1, C_3, C_6, C_9, D(5, 5)\}$. Since the smallest cycle in G has length 5, no component of G' is B_1 or C_3. If some component of G' is a 9-cycle, then $G' = C_9$ with the three large vertices of G from $V(G')$ pairwise at distance 3 apart in G'. But then $n = 13$ and $\gamma_{pr}(G) = 6 = 2(n - 4)/3$, a contradiction. If some component of G' is $D(5, 5)$, then $G' = D(5, 5)$ with the three large vertices of G from $V(G')$ consisting of the central vertex of G' and the two vertices at distance 4 apart in G'. But then $n = 13$ and $\gamma_{pr}(G) = 6 = 2(n - 4)/3$, a contradiction. Hence every component of G' must be a 6-cycle. But then $G' = C_6$, implying that $G = B_{10}$. □

By Lemma 24, we may assume that *every large vertex is adjacent to the end of at least one* 2*-path on two vertices*.

Lemma 25. *Every large vertex of G is adjacent to the ends of at least two* 2*-paths that contain two vertices.*

Proof: Assume that G contains a large vertex v that is adjacent to the end of exactly one 2-path P on two vertices. By assumption, the smallest cycle in G has length 5. We consider two possibilities.

Case 1. The vertex v belongs to a 5-cycle $C: v, w, x, y, z, v$. Necessarily, $V(P) \subset V(C)$. We may assume that P is the 2-path w, x, and so y is a large vertex and z a small vertex. Let G' be obtained from G be deleting all neighbors of v not on C. Then, $\delta(G') \geq 2$ and G' has order $n' = n - d(v) + 2 \leq n - 1$. Notice that in G', the vertex v belongs to a 2-handle, namely z, v, w, x, on four vertices.

Suppose that G' contains a C_5-component $F: a, b, c, d, e, a$. By our earlier assumptions, this component contains two large vertices, say a and c, of G at distance 2 apart on the cycle F. Let v_1 be the common neighbor of a and v, and let v_2 be the common neighbor of c and v. Let $G_1 = G - V(F) - \{v_1, v_2\}$. Then, G_1 is a connected graph with $\delta(G_1) \geq 2$. Let G_1 have order $n_1 = n - 7$. Since G_1 is not a cycle and $G_1 \notin \{B_1, D(5, 5)\}$, it follows by Observations 6 and 16 that $\gamma_{pr}(G_1) \leq 2(n' - 1)/3 = 2(n - 8)/3$. Any $\gamma_{pr}(G')$-set can be extended to a PDS of G by adding to it the four vertices a, c, v_1, v_2, implying that $\gamma_{pr}(G) \leq \gamma_{pr}(G') + 4 \leq 2(n-8)/3 + 4 = 2(n-2)/3$, a contradiction. Hence, G' has no C_5-component.

Applying the inductive hypothesis to each component of G', $\gamma_{pr}(G') \leq 2n'/3$ with equality if and only if each component of G' belongs to the family $\{B_1, C_3, C_6, C_9, D(5,5)\}$. Let S' be a $\gamma_{pr}(G')$-set. If $v \notin S'$, then $\{w, x, y\} \subset S'$ with w and x paired in S'. Replacing x in S' with the vertex v, produces a new $\gamma_{pr}(G')$-set (with v and w paired). Hence we may assume $v \in S'$. But then S' is a PDS of G, implying that $\gamma_{pr}(G) \leq |S'| = \gamma_{pr}(G') \leq 2n'/3$. If $d(v) \geq 4$, then $\gamma_{pr}(G) \leq 2n'/3 \leq 2(n-2)/3$, a contradiction. Hence, $d(v) = 3$. If $\gamma_{pr}(G') < 2n'/3$, then $\gamma_{pr}(G') \leq 2(n'-1)/3 = 2(n-2)/3$, implying once again that $\gamma_{pr}(G) \leq 2(n-2)/3$, a contradiction. Hence, $\gamma_{pr}(G') = 2n'/3$. This implies that the component of G' containing the vertex v must be $D(5,5)$. However then G is obtained from $D(5,5)$ by adding a new vertex and joining it to two vertices at distance 4 apart in $D(5,5)$, whence $n = 10$ and $\gamma_{pr}(G) = 4 = 2(n-4)/3$, a contradiction.

Case 2. The vertex v belongs to no 5-cycle. Let G' be obtained from G by removing the vertices in $N[v]$ and both vertices on the 2-path that contain two vertices with one end adjacent to v.

Suppose that G' contains a C_5-component C. By our earlier assumptions, this component contains two large vertices of G at distance 2 apart on the cycle. On the one hand, suppose that $d(v) = 3$. Let $G_1 = G' - V(C)$. Then, G_1 is a connected graph with $\delta(G_1) \geq 2$ of order $n_1 = n - 10$. By our earlier assumptions, G_1 is not a cycle and $G_1 \neq D(5,5)$. If $G_1 = B_1$, then $n = 16$ and $\gamma_{pr}(G) \leq 8 = 2(n-4)/3$, a contradiction. Hence, $G_1 \neq B_1$. It follows from Observations 6 and 16 that $\gamma_{pr}(G_1) \leq 2(n_1 - 1)/3 = 2(n-11)/3$. Any $\gamma_{pr}(G_1)$-set can be extended to a PDS of G by adding to it six vertices, implying that $\gamma_{pr}(G) \leq \gamma_{pr}(G_1) + 6 \leq 2(n-11)/3 + 6 = 2(n-2)/3$, a contradiction. On the other hand, suppose that $d(v) \geq 4$. Let G^* be obtained from G by deleting $V(C)$ and the vertices on the two 2-paths that have an end adjacent with one of the two large vertices of C. Then, G^* is a connected graph with $\delta(G^*) \geq 2$ of order n^* where $n^* = n - 7$ or $n^* = n - 8$. Further, by our earlier assumptions, we know that the smallest cycle in G^* has length 5 and G^* contains at least two large vertices. In particular, G^* is not a cycle and $G^* \notin \{B_1, D(5,5)\}$. It follows from Observations 6 and 16 that $\gamma_{pr}(G^*) \leq 2(n^*-1)/3 \leq 2(n-8)/3$. Any $\gamma_{pr}(G^*)$-set can be extended to a PDS of G by adding to it four vertices, implying that $\gamma_{pr}(G) \leq \gamma_{pr}(G^*) + 4 \leq 2(n-8)/3 + 4 = 2(n-2)/3$, a contradiction. We deduce, therefore, that G' contains no C_5-component.

Applying the inductive hypothesis to each component of G', $\gamma_{pr}(G') \leq 2n'/3 \leq 2(n-5)/3$. Any $\gamma_{pr}(G')$-set can be extended to a PDS of G by adding to it the vertex v and its neighbor on the 2-path that contains two vertices. Hence, $\gamma_{pr}(G) \leq \gamma_{pr}(G') + 2 \leq 2(n-2)/3$, a contradiction. □

By Lemma 25, every large vertex of G is adjacent to the ends of at least two 2-paths that contain two vertices. Let $\mathcal{S}_1$ be the set of small vertices in G with both neighbors in $\mathcal{L}$ (and so, each vertex in $\mathcal{S}_1$ is a 2-path on one vertex). Let $\mathcal{L}_1$ be the set of large vertices in G that are dominated by $\mathcal{S}_1$. Let $\mathcal{S}_2$ be the set of small vertices in G that belong to 2-paths on two vertices and are dominated by $\mathcal{L}_1$ (and so, each vertex in $\mathcal{S}_2$ is adjacent to exactly one large vertex and this large vertex is adjacent to a vertex of $\mathcal{S}_1$). Let S_3 be the set of small vertices that are not dominated by $\mathcal{L}_1$. Let $\mathcal{L}_2$ be the set of large vertices that are dominated by $\mathcal{S}_3$.

For $i = 1, 2$, let $|\mathcal{L}_i| = \ell_i$ and for $j = 1, 2, 3$, let $|\mathcal{S}_j| = s_j$. Let $n_1 = \ell_1 + s_1 + s_2$ and let $n_2 = \ell_2 + s_3$. Then, $n = n_1 + n_2$. By Lemma 25, $s_2 \geq 2\ell_1$.

For sets A and B, let $[A, B]$ denote the set of edges between A and B. Then, $3\ell_1 \leq |[\mathcal{L}_1, \mathcal{S}_1 \cup \mathcal{S}_2]| = 2s_1 + s_2 = 2(n_1 - \ell_1) - s_2 \leq 2(n_1 - \ell_1) - 2\ell_1 = 2n_1 - 4\ell_1$, implying that $\ell_1 \leq 2n_1/7$. Moreover, $3\ell_2 \leq |[\mathcal{L}_2, \mathcal{S}_3]| = s_3 = n_1 - \ell_2$, implying that $\ell_2 \leq n_2/4$. We now construct a PDS S of G as follows. Initially, let S consist of the (independent) set $\mathcal{L}$ which dominates $V(G)$. We then consider each vertex $v \in \mathcal{L}$ in turn. If $N(v) \subset S$, then delete v from S; otherwise, add a neighbor of v to S. The PDS S of G constructed in this way is such that $|S| \leq 2|\mathcal{L}| = 2\ell_1 + 2\ell_2 \leq 4n_1/7 + n_2/2 \leq 4n/7 < 2(n-1)/3$ since $n > 7$. Consequently, $\gamma_{\rm pr}(G) \leq |S| < 2(n-1)/3$, a contradiction. This completes the proof of Theorem 3.

5.3 Proof of Theorem 4

Let G be a connected graph of order $n \geq 6$ with $\delta(G) \geq 2$. Since the paired-domination number of a graph cannot decrease if edges are removed, it follows from Theorem 3 and Observation 6 that $\gamma_{\rm pr}(G) \leq 2n/3$. Further, suppose $\gamma_{\rm pr}(G) = 2n/3$. Then by removing edges of G, if necessary, we produce a $\frac{2}{3}$-minimal graph G' satisfying $\gamma_{\rm pr}(G') = 2n/3$. By Theorem 3 and Observation 6, $G' \in \{B_1, C_6, C_9, D(5,5)\}$. In all cases it can be readily checked that $G = G'$ or $G' \in \mathcal{F}$ where $\mathcal{F}$ is the family of six graphs shown in Fig. 4.

5.4 Proof of Theorem 5

Let G be a connected graph of order $n \geq 10$ with $\delta(G) \geq 2$. Since the paired-domination number of a graph cannot decrease if edges are removed, it follows from Theorem 3 and Observation 6 that $\gamma_{\rm pr}(G) \leq 2(n-1)/3$. Further, suppose $n \geq 14$ and $\gamma_{\rm pr}(G) = 2(n-1)/3$. Then by removing edges of G, if necessary, we produce a $\frac{2}{3}$-minimal graph G' satisfying $\gamma_{\rm pr}(G') \geq 2(n-1)/3$. Since $n \geq 14$, $G' \in \mathcal{G}$ by Theorem 3 and Observation 6. It can now be readily checked that $G = G'$ or $G' \in \mathcal{H}$ where $\mathcal{H}$ is the family of graphs constructed in Section 3.

Appendix

In this Appendix, we provide proofs of some of the preliminary results in Section 5.1.

Recall Observation 9.

Observation 9. Let G be a connected graph with $\delta(G) \geq 2$ and let F be obtained from G by subdividing any edge four times. Then, $\gamma_{\rm pr}(F) \leq \gamma_{\rm pr}6(G) + 2$.

Proof: Let uv be the edge of G that is subdivided four times to obtain the graph F, and let u, u_1, u_2, u_3, u_4, v be the resulting path in F. Let S be a $\gamma_{\rm pr}(G)$-set. Then, S can be extended to a PDS of F as follows. If $\{u, v\} \subseteq S$ with u and v paired in $G[S]$, let $S' = S \cup \{u_1, u_4\}$ (with u paired with u_1 and v paired with u_4 in $F[S]$); if $\{u, v\} \subseteq S$ with u and v not paired in $G[S]$ or if $\{u, v\} \cap S = \emptyset$, let $S' = S \cup \{u_2, u_3\}$; if exactly one of u and v is in S, say $u \in S$, then let $S' = S \cup \{u_3, u_4\}$. Then, S' is a PDS of F, and so $\gamma_{\rm pr}(F) \leq |S'| = \gamma_{\rm pr}(G) + 2$. □

Recall Observation 10.

Observation 10. For $n \geq 3$, $\gamma_{\rm pr}(C_n) = 2\left\lceil \frac{n}{4} \right\rceil$.

Proof: We proceed by induction on the order n of a cycle C_n. The result is straightforward to verify for $n \in \{3, 4, 5, 6\}$. Assume then that $n \geq 7$ and that the result is true for all cycles on fewer than n vertices. Consider an n-cycle G given by $v_1, v_2, \ldots, v_n, v_1$. Let $G' = (G \setminus \{v_1, v_2, v_3, v_4\}) \cup \{v_5 v_n\}$. Then, G' is a cycle of order $n - 4 \geq 3$. By the inductive hypothesis, $\gamma_{\rm pr}(G') = 2\lceil (n-4)/4 \rceil = 2\lceil n/4 \rceil - 2$.

It remains for us to show that $\gamma_{\rm pr}(G) = \gamma_{\rm pr}(G') + 2$. Let D be a $\gamma_{\rm pr}(G)$-set. Then each component of $G[D]$ is a P_2 or a P_4. Suppose $G[D]$ contains no P_2-component. Then, since $n \geq 7$, we may assume (renaming vertices if necessary) that $\{v_1, v_2, v_3, v_4\} \subseteq D$ and $v_5 \notin D$. On the one hand, if $v_6 \in D$, then $\{v_6, v_7, v_8, v_9\} \subseteq D$, in which case $(D \setminus \{v_3, v_6, v_7\}) \cup \{v_5\}$ is a PDS of G. On the other hand, if $v_6 \notin D$, then $\{v_7, v_8, v_9, v_{10}\} \subseteq D$, $(D \setminus \{v_3, v_4, v_7, v_8\}) \cup \{v_5, v_6\}$ is a PDS of G. Both cases produce a PDS of cardinality $\gamma_{\rm pr}(G) - 2$, which is impossible. Hence, $G[D]$ contains a P_2-component. We may assume (renaming vertices if necessary) that $D \cap \{v_1, v_2, v_3, v_4\} = \{v_2, v_3\}$. Therefore, $D \setminus \{v_2, v_3\}$ is a PDS of G', and so $\gamma_{\rm pr}(G') \leq |D| - 2 = \gamma_{\rm pr}(G) - 2$. However, by Observation 9, $\gamma_{\rm pr}(G) \leq \gamma_{\rm pr}(G') + 2$. Consequently, $\gamma_{\rm pr}(G) = \gamma_{\rm pr}(G') + 2 = 2\lceil n/4 \rceil$. □

Recall Observation 12.

Observation 12. If G is a daisy of order n, then $\gamma_{\rm pr}(G) \leq 2n/3$. Furthermore, $\gamma_{\rm pr}(G) = 2n/3$ if and only if $G = D(5, 5)$, while $\gamma_{\rm pr}(G) = 2(n-1)/3$ if and only if $G = D_k(4)$ where $k \geq 2$ or $G \in \{D(3, 5), D(5, 6), D(5, 9), D_3(5)\}$.

A proof of Observation 12 follows from Observations 26 and 27.

Observation 26. If G is a daisy of order n with two petals, then $\gamma_{\rm pr}(G) \leq 2n/3$. Furthermore, $\gamma_{\rm pr}(G) = 2n/3$ if and only if $G = D(5, 5)$, while $\gamma_{\rm pr}(G) = 2(n-1)/3$ if and only if $G \in \{D(3, 5), D(4, 4), D(5, 6), D(5, 9)\}$.

Proof: If $n = 5$, then $G = D(3, 3)$, while if $n = 6$, then $G = D(3, 4)$. Hence if $n \in \{5, 6\}$, $\gamma_{\rm pr}(G) = 2 < 2(n-1)/3$. If $n = 7$, then $G = D(3, 5)$ or $G = D(4, 4)$. In both cases, $\gamma_{\rm pr}(G) = 2(n-1)/3$. Assume, then, that $n \geq 8$.

Let $G = D(n_1 + 1, n_2 + 1)$, and so $n = n_1 + n_2 + 1$. Let v denote the vertex of degree 4 in G and let F_1 and F_2 denote the two cycles passing through v, where $F_i \cong C_{n_i+1}$ for $i = 1, 2$. Let F_1 be the cycle $v, v_1, v_2, \ldots, v_{n_1}, v$ and let F_2 be the cycle $v, u_1, u_2, \ldots, u_{n_2}, v$. Let S_2 be a $\gamma_{\rm pr}(F_2)$-set that contains v. By Observation 10, $|S_2| = (n_2 + 4)/2$ if $n_2 \equiv 0 \pmod 4$ and $|S_2| \leq (n_2 + 3)/2$ otherwise. Let $S_1 = \{v_i \mid i \equiv 0 \text{ or } 3 \pmod 4\}$.

Suppose $n_1 \equiv 2 \pmod 4$ (and still $n \geq 8$). Then, $|S_1| = (n_1 - 2)/2$ and $\gamma_{\rm pr}(G) \leq |S_1 \cup S_2| \leq (n+1)/2 < 2(n-1)/3$. Hence we may assume $n_i \not\equiv 2 \pmod 4$ for $i = 1, 2$.

Suppose $n_1 \equiv 3 \pmod 4$ (and still $n \geq 8$). Let $D_1 = \{v_i \mid i \equiv 0 \text{ or } 1 \pmod 4\}$, and so $|D_1| = (n_1 - 1)/2$. Let $D_2 = \{u_i \mid i \equiv 2 \text{ or } 3 \pmod 4\}$, and so $|D_2| \leq (n_2 + 1)/2$. Thus, $\gamma_{\rm pr}(G) \leq |D_1 \cup D_2 \cup \{v\}| \leq (n+1)/2 < 2(n-1)/3$. Hence we may assume $n_i \not\equiv 3 \pmod 4$ for $i = 1, 2$, and so $n_i \geq 4$. Thus, $n \geq 9$.

Suppose $n_1 \equiv 1 \pmod 4$. Then, $n \geq 10$ and $|S_1| = (n_1 - 1)/2$. Thus, $\gamma_{\rm pr}(G) \leq |S_1 \cup S_2| \leq (n+2)/2 \leq 2(n-1)/3$ with strict inequality if $n \geq 11$. Hence, $\gamma_{\rm pr}(G) < 2(n-1)/3$ unless $G = D(5, 6)$, in which case $\gamma_{\rm pr}(G) = 2(n-1)/3$.

Suppose, finally, that $n_i \equiv 0 \pmod 4$ for $i = 1, 2$. Then, $|S_1| = n_1/2$ and $|S_2| = (n_2 + 4)/2$. Further, $n \equiv 1 \pmod 4$ and $n \geq 9$. If $n = 9$, then $G = D(5, 5)$ and $\gamma_{\rm pr}(G) = 2n/3$. If $n = 13$, then $G = D(5, 9)$ and $\gamma_{\rm pr}(G) = 2(n-1)/3$. Hence we may assume $n \geq 17$. Then, $\gamma_{\rm pr}(G) \leq |S_1 \cup S_2| \leq (n + 3)/2 < 2(n - 1)/3$. □

Observation 27. If G is a daisy of order n and $G \neq D(5, 5)$, then $\gamma_{\rm pr}(G) \leq 2(n-1)/3$ with equality if and only if $G = D_k(4)$ where $k \geq 2$ or $G \in \{D(3, 5), D(5, 6), D(5, 9), D_3(5)\}$.

Proof: We proceed by induction on the order $n \geq 5$ of a daisy G, where $G \neq D(5, 5)$. If $n = 5$, then $G = D(3, 3)$, while if $n = 6$, then $G = D(3, 4)$. This establishes the base cases. Assume, then, that $n \geq 7$ and that the result holds for all daisies of order less than n. Let G be a daisy of order n with $k \geq 2$ petals where $G \neq D(5, 5)$.

If $k = 2$, then the result follows from Observation 26. Hence we may assume $k \geq 3$. Let v denote the vertex of degree $2k$ in G, and let $F_1, F_2, \ldots, F_k$ denote the k cycles passing through v, where $F_i \cong C_{n_i+1}$ for $i = 1, 2, \ldots, k$. Thus, $n = 1 + \sum_{i=1}^{k} n_i$. Renaming if necessary, we may assume $n_1 = \min(n_1, n_2, \ldots, n_k)$. Let F_1 be the cycle $v, v_1, v_2, \ldots, v_{n_1}, v$.

Let $G' = D(n_2, \ldots, n_k)$. Then, G' is a daisy of order $n' = n - n_1$. If $G' = D(5, 5)$, then either $G \in \{D(3, 5, 5), D(4, 5, 5)\}$, in which case $\gamma_{\rm pr}(G) = 6 < 2(n-1)/3$, or $G = D_3(5)$, in which case $n = 13$ and $\gamma_{\rm pr}(G) = 2(n-1)/3$. Hence we may assume $G' \neq D(5, 5)$. Applying the inductive hypothesis to G', $\gamma_{\rm pr}(G') \leq 2(n'-1)/3$ with equality if and only if $G' = D_{k-1}(4)$ or $G' \in \{D(3, 5), D(5, 6), D(5, 9), D_3(5)\}$.

Let S' be a $\gamma_{\rm pr}(G')$-set. The restriction of S' to the vertices of at least one cycle in G' must be a PDS in that cycle. Hence we may choose S' to contain the vertex v. If $n_1 = 2$, then S' is a PDS of G, and so $\gamma_{\rm pr}(G) \leq |S'| < 2(n-1)/3$. Hence we may assume that $n_1 \geq 3$. We now extend S' to a PDS of G as follows. If $n_1 \equiv 1$ or $2 \pmod 4$, let $S_1 = \{v_i \mid i \equiv 0 \text{ or } 3 \pmod 4\}$. If $n_1 \equiv 0$ or $3 \pmod 4$, let $S_1 = \{v_i \mid i \equiv 2 \text{ or } 3 \pmod 4\}$. Then, $|S_1| \leq (n_1 + 1)/2$ and $\gamma_{\rm pr}(G) \leq |S' \cup S_1| \leq 2(n'-1)/3 + (n_1+1)/2 \leq 2(n-1)/3 + (3 - n_1)/6 \leq 2(n-1)/3$. Furthermore, if $\gamma_{\rm pr}(G) = 2(n-1)/3$, then we must have equality throughout this inequality chain. In particular, $\gamma_{\rm pr}(G') = 2(n'-1)/3$ and $n_1 = 3$. Thus, $G' = D_{k-1}(4)$ or, by our choice of n_1, $G' \in \{D(5, 6), D(5, 9), D_3(5)\}$. However if $G' = D(5, 6)$, then $n = 13$, $G = D(4, 5, 6)$ and $\gamma_{\rm pr}(G) = 6 < 2(n-1)/3$; if $G' = D(5, 9)$, then $n = 16$, $G = D(4, 5, 9)$ and $\gamma_{\rm pr}(G) = 8 < 2(n-1)/3$; if $G' = D_3(5)$, then $n = 16$, $G = D(4, 5, 5, 5)$ and $\gamma_{\rm pr}(G) = 8 < 2(n-1)/3$. Hence if $\gamma_{\rm pr}(G) = 2(n-1)/3$, then $G' = D_{k-1}(4)$, and so $G = D_k(4)$. □

Recall Observation 14.

Observation 14. If G is a $\frac{2}{3}$-minimal graph of order n, $3 \leq n \leq 7$, then $G \in \mathcal{B} \cup \mathcal{G}$.

Proof: Let $G = (V, E)$ and let $\Delta = \Delta(G)$. If $3 \leq n \leq 6$, then $G = C_n$ or $G = B_1$. Suppose $n = 7$. Then, $\gamma_{\rm pr}(G) \geq 4$, and so $\Delta \leq 4$. If $\Delta = 2$, then $G = C_7$. Suppose $\Delta = 3$. Let u be a vertex of degree 3. If some vertex in $V - N[u]$ is not adjacent to any vertex in $N(u)$, then $G = B_3$ or $G = G(1, 1, 0, 0)$ or $G = G(0, 2, 0, 0)$. If every vertex is within distance 2 from u, then $G = B_4$ or $G = G(1, 1, 0, 0)$. Suppose $\Delta = 4$. Let u be a vertex of degree 4 in G, and let $V - N[u] = \{v, w\}$. If v and w have a common neighbor, then $\gamma_{\rm pr}(G) = 2$, a contradiction. Hence, v and w have no common neighbor. If both v and w are adjacent to exactly one vertex in $N(u)$, then $G = D(3, 5)$. If exactly one of v and w is adjacent to only one vertex in $N(u)$, then $G = B_2$. If both v and w are adjacent to two vertices in $N(u)$, then $G = G(2, 0, 0, 0)$. □

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