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Polyhedral combinatorics of the cardinality constrained quadratic knapsack problem and the quadratic selective travelling salesman problem

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Abstract This paper considers the Cardinality Constrained Quadratic Knapsack Problem (QKP) and the Quadratic Selective Travelling Salesman Problem (QSTSP). The QKP is a generalization of the Knapsack Problem and the QSTSP is a generalization of the Travelling Salesman Problem. Thus, both problems are NP hard. The QSTSP and the QKP can be solved using branch-and-cut methods. Good bounds can be obtained if strong constraints are used. Hence it is important to identify strong or even facet-defining constraints. This paper studies the polyhedral combinatorics of the QSTSP and the QKP, i.e. amongst others we identify facet-defining constraints for the QSTSP and the QKP, and provide mathematical proofs that they do indeed define facets.

Keywords Quadratic knapsack · Quadratic selective travelling salesman · Polyhedral analysis · Facets

1. Introduction

A well-known extension of the Travelling Salesman Problem (TSP) is the Selective (or Prize-collecting) TSP: In addition to the edge-costs, each node has an associated reward (referred to as the node-reward) and instead of visiting all nodes, only profitable nodes are visited. The Quadratic Selective TSP (QSTSP) has additional complications: (1) each *pair* of nodes

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have an associated reward (referred to as the edge-reward) which can be gained only if both nodes are visited; and (2) a constraint on the number of nodes selected is imposed, which we refer to as the cardinality constraint. The objective of an QSTSP is to maximize the total node-rewards and edge-rewards gained minus the edge-costs incurred subject to the satisfaction of the cardinality constraint.

Conceptually the QSTSP consists of two interacting problems, a cardinality-constrained min-cost circuit problem with respect to the edge-costs and a cardinality-constrained max-reward clique problem with respect to the edge-rewards.

The cardinality constrained circuit problem (CCCP) is considered in Bauer (1997) where polyhedral results are presented and in Bauer et al. (2002) where a branch and cut algorithm is discussed. The max-reward clique problem is a special case of the quadratic knapsack problem where the knapsack constraints have unit coefficients. We denote this problem the cardinality constrained quadratic knapsack problem (QKP). The quadratic knapsack problem (when coefficients are not necessarily unit) is considered in e.g. Johnson et al. (1993), Billionnet and Calmels (1996), and Caprara et al. (1999). For cases when edge-rewards are non-negative, the cardinality constraint will be met with equality. The problem will then be similar to the p-dispersion problem considered in Erkut (1990) wherein the objective is to maximize the minimum edge-reward. The p-dispersion problem is considered in Pisinger (1999) with an objective equivalent to the one considered here.

Various TSP-like problems are similar to QSTSP in the way that a subset of nodes has to be selected. E.g. the Prize-collecting TSP (Balas, 1989, 1995), the Selective TSP (Gendreau et al., 1998; Laporte and Martello, 1990), the Orienteering problem (Fischetti et al., 1998), and the Generalized TSP (Fischetti et al., 1995, 1997). Problems that consider the combination of a clique problem and a cycle problem has been studied in Gendreau et al. (1995) and Gouveia and Manuel Pires (2001). Gendreau et al. (1995) study a problem where instead of imposing the cardinality constraint, an upper bound on the sum of the edge-costs are imposed. Gouveia and Manuel Pires (2001) study a QSTSP-like problem with the additional requirement that some nodes must be in the cycle.

In this paper we study the polyhedral combinatorics of the QKP and the QSTSP. Our interest in studying the QSTSP is due to the fact that this problem arose as a subproblem from another combinatorial optimisation problem which deals with designing hierarchical ring (cycle) networks (see Stidsen and Thomadsen, 2005). Naturally, the faster we can solve the QSTSP, the better. The QKP, however, is an interesting problem in its own right, but we study the QKP mostly for its relevance in understanding the QSTSP. Both problems are NP hard, as QKP is a generalization of the Knapsack Problem and QSTSP is a generalization of the Travelling Salesman Problem.

A promising approach in solving these combinatorial optimisation problems is the branch-and-cut method. A significant factor in the success of the method is the use of strong constraints that at least partially describe the convex hull of the incidence vectors of all feasible solutions, in other words, the use of facet-defining cuts.

The contribution of this research is therefore the identification of some of the strong cuts, the mathematical proofs that these cuts are indeed facet-defining, and the various mathematical techniques used in proving these results.

We begin with, in Section 2, giving an integer programming model for the QSTSP and define the polyhedra of the QKP, CCCP, and the QSTSP. In Sections 3 and 4, we present our polyhedral results on the QKP and the QSTSP polytopes. Finally, in Section 5, we conclude our findings.

2. Integer programming model and the polyhedra

In this research, we consider the QSTSP defined on the undirected graph $G = (V, E)$ for G complete. This is not restrictive in terms of implementation, as we can always introduce an arbitrary high cost for edges that do not exist.

For convenience of notation, we use (U_1, U_2) to denote $\{(i, j) \in E \mid i \in U_1, j \in U_2\}$, for any $U_1, U_2 \subseteq V$; use $\delta(S)$ to denote $\{(i, j) \in E \mid i \in S, j \in V \backslash S\}$; and use $E(S)$ to denote $\{(i, j) \in E | i, j \in S\}$.

We use r_i to denote the reward for selecting node i, w_e the reward for using edge e, c_e the cost of using edge e, and b the maximum number of nodes allowed in the cycle. Let x_e be the decision variable with $x_e = 1$ if $e \in E$ is chosen in the cycle and 0 otherwise; y_i be the decision variable with $y_i = 1$ if $i \in V$ is on the cycle, 0 otherwise; and z_e be the decision variable with $z_e = 1$, for $(i, j) \in E$ if node i and j are both on the cycle, 0 otherwise. If $e = (i, j) \in E$, then z_{ij} is sometimes used in place of z_e. Given these, the QSTSP is formulated as follows.

$$\max \quad \sum_{i \in V} r_i \cdot y_i + \sum_{e \in E} w_e \cdot z_e - \sum_{e \in E} c_e \cdot x_e \tag{1}$$

$$\text{s.t.} \quad \sum_{e \in \delta(i)} x_e = 2y_i, \quad \forall i \in V \tag{2}$$

$$z_e \leq y_i, \quad \forall i \in V, e \in \delta(i) \tag{3}$$

$$z_{ij} \geq y_i + y_j - 1, \quad \forall (i, j) \in E, i < j \tag{4}$$

$$\sum_{e \in \delta(S)} x_e \geq 2(y_k + y_l - 1), \quad \forall \emptyset \subset S \subset V, k \in S, l \notin S \tag{5}$$

$$\sum_{i \in V} y_i \leq b, \quad \forall i \in V \tag{6}$$

$$x_e \in \{0, 1\}, \quad \forall e \in E \tag{7}$$

$$y_i \in \{0, 1\}, \quad \forall i \in V \tag{8}$$

$$z_e \in \{0, 1\}, \quad \forall e \in E. \tag{9}$$

Constraints (2) ensure that a node is selected if and only if the degree of the node is two. Constraints (3) and (4) establish the fact that $z_{ij} = 1$ if and only if $y_i = y_j = 1$. Constraints (5) and (6) are the subtour elimination constraints and the cardinality constraints respectively. Let $n = |V|$. The Quadratic Selective Travelling Salesman(QSTS) polytope is defined to be

$$P_{QS}^{nb} = \text{ conv } \{(x, y, z) \in \mathcal{R}^{2|E|+n} | (x, y, z) \text{ satisfies (2)–(9)}\}. \tag{10}$$

We identify two related polytopes. The cardinality constrained quadratic knapsack(QK) polytope, given by

$$P_{QK}^{nb} = \text{ conv } \{(y, z) \in \mathcal{R}^{|E|+n} | (y, z) \text{ satisfies (3), (4), (6), (8) and (9)}\}; \tag{11}$$

and the cardinality constrained circuit polytope, given by

$$P_{C}^{nb} = \text{ conv } \{(x, y) \in \mathcal{R}^{|E|+n} | (x, y) \text{ satisfies (2), (5)–(8)}\}. \tag{12}$$

Note that P_{QS}^{nb} is contained in the intersection of P_{QK}^{nb} and P_{C}^{nb}. Thus any valid inequality for either P_{QK}^{nb} or P_{C}^{nb} is valid for P_{QS}^{nb}. Note also that for the CCCP and QSTSP we consider in this paper, we assume that the empty cycle is considered as a feasible solution, whereas in Bauer (1997), it is not considered as a feasible solution.

The contribution of this paper is the study of the QK polytope (P_{QK}^{nb}), and the polytope of an integer programming model of the QSTSP without the y variables denoted by $\tilde{P}_{QS}^{nb}$. We show that $\tilde{P}_{QS}^{nb}$ and P_{QS}^{nb} are in fact describing the same set of feasible solutions for the QSTSP, and that any facet-defining inequality defined for $\tilde{P}_{QS}^{nb}$ is also facet-defining for P_{QS}^{nb}. Then, we present our results on P_{QK}^{nb} and $\tilde{P}_{QS}^{nb}$: we establish the dimensions of these polytopes, and for each of them, we develop a number of classes of facet-defining constraints.

3. Polyhedral results for the QK polytope

In this section, we present our polyhedral results on the dimension of P_{QK}^{nb} and that four classes of constraints are facet-defining for P_{QK}^{nb}: the non-negativity constraints on the z-variables, the generalizations of Constraints (3) and (4), and a modification of Constraints (6).

In what follows, we use incidence vectors $(y, z) \in \{0, 1\}^{|V|+|E|}$, for $y \in \{0, 1\}^{|V|}$ and $z \in \{0, 1\}^{|E|}$ to represent our solutions. Each element in y corresponds to a node $j \in V$ and each element in z corresponds to an edge $(i, j) \in E$. We also use $e_j \in \{0, 1\}^{|V|+|E|}$, for $j \in V$, to represent a vector with the value of the element corresponding to node j equals 1 and the values of all other elements of e_j equal 0; and use $e_{ij} \in \{0, 1\}^{|V|+|E|}$, for $(i, j) \in E$, to represent a vector with the value of the element corresponding to edge (i, j) equals 1 and the values of all other elements of e_{ij} equal 0.

Theorem 3.1. *Given any $G = (V, E)$, $2 \leq b \leq |V|$, the dimension of the QK polytope, P_{QK}^{nb}, is $|E| + |V|$, i.e., it is full dimensional.*

Proof: Consider the sequential insertion of the following feasible solutions:

1. $(y, z)^0 = 0$;
2. $(y, z)^1 = e_j$, for all $j \in V$; and
3. $(y, z)^2 = e_i + e_j + e_{ij}$, for all $(i, j) \in E$.

Clearly, these give us $|E| + |V| + 1$ affinely independent feasible solutions, and therefore the dimension of the QK polytope is $|E| + |V|$. □

Theorem 3.2. *Given any $G = (V, E)$, for $2 \leq b \leq |V|$, the bound constraints given as*

$$z_f \geq 0, \quad \forall f \in E, \quad (13)$$

are facet-defining for P_{QK}^{nb}.

Proof: We need to show that the dimension of $F = P_{QK}^{nb} \cap \{z_f = 0\}$ is $|E| + |V| - 1$. First of all, F defines a proper face as there is at least one feasible solution that satisfies $z_f = 0$ (consider any cycle that does not contain edge f) and at least one feasible solution that does not satisfy $z_f = 0$ (consider any cycle that does contain edge f). Therefore $\dim(F) \leq |E| + |V| - 1$. Now consider the following feasible solutions:

1. $(y, z)^0 = 0$;
2. $(y, z)^1 = e_j$, for all $j \in V$; and
3. $(y, z)^2 = e_i + e_j + e_{ij}$, for all $(i, j) \in E \backslash \{f\}$.

Clearly, these give us $|E| + |V|$ affinely independent feasible solutions, and therefore $\dim(F)$ is $|E| + |V| - 1$. □

Proposition 3.1. *Given any $G = (V, E)$, for $|V| \geq 2$, $1 \leq b \leq |V|$, the constraints given as:*

$$\sum_{e\in(i,S)} z_e \leq y_i + \sum_{e\in E(S)} z_e, \quad \forall i \in V, S \subseteq V\backslash\{i\}, |S| \geq 1, \qquad (14)$$

are valid for P^{nb}_{QK}.

(Note that (3) is a special case of (14)).

Proof: Let $\tilde{S} \subseteq S$ be the nodes in S that are selected (i.e. that are in the cycle), we have that $\sum_{e\in(i,S)} z_e = |\tilde{S}|$ and $\sum_{e\in E(S)} z_e = \frac{1}{2}|\tilde{S}|(|\tilde{S}| - 1)$, hence it is easy to verify that (14) is valid for P^{nb}_{QK}. □

Theorem 3.3. *Given any $G = (V, E)$, for $3 \leq b \leq |V|$, Constraints (14) are facet-defining for P^{nb}_{QK} if $|S| \geq 2$.*

Proof: Let $F = P^{nb}_{QK} \cap \{\sum_{e\in(i,S)} z_e = y_i + \sum_{e\in E(S)} z_e\}$. F defines a proper face for P^{nb}_{QK}: consider $(y, z) = 0$ which satisfy the constraint at equality and $(y, z) = e_j + e_k$, for $j, k \in S$ which does not. Thus, $\dim(F) \leq \dim(P^{nb}_{QK}) - 1$. Now, we show that $\dim(F) \geq \dim(P^{nb}_{QK}) - 1$ by finding exactly $\dim(P^{nb}_{QK}) = |V| + |E|$ affinely independent feasible solutions that satisfy the constraints at equality. We do so by sequentially introducing the following vectors, each representing a feasible solution.

1. $(y, z)^1 = \{0\}$;
2. $(y, z)^2 = \{(y, z)^2_j \mid \forall j \in V\backslash\{i\}\}$ where $(y, z)^2_j = e_j$, (we have $|V| - 1$ of these solutions);
3. $(y, z)^3 = \{(y, z)^3_{ij} \mid \forall j \in S\}$ where $(y, z)^3_{ij} = e_i + e_j + e_{ij}$, for all $j \in S$, (we have $|S|$ of these solutions);
4. $(y, z)^4 = \{(y, z)^4_{jk} \mid \forall j \in S, k \in \bar{S} \setminus \{i\}\}$ where $(y, z)^4_{jk} = e_j + e_k + e_{jk}$, (we have $|(S, \bar{S} \setminus \{i\})|$ of these solutions);
5. $(y, z)^5 = \{(y, z)^5_{jk} \mid \forall j, k \in \bar{S} \setminus \{i\}, j < k\}$ where $(y, z)^5_{jk} = e_j + e_k + e_{jk}$, (we have $|E(\bar{S} \setminus \{i\})|$ of these solutions);
6. $(y, z)^6 = \{(y, z)^6_{jk} \mid \forall j, k \in S, j < k\}$ where $(y, z)^6_{jk} = e_i + e_j + e_k + e_{ij} + e_{ik} + e_{jk}$, (we have $|E(S, S)|$ of these solutions); and
7. $(y, z)^7 = \{(y, z)^7_{jk} \mid j \in S, \forall k \in \bar{S} \setminus i\}$, where $(y, z)^7_{jk} = e_i + e_j + e_k + e_{ij} + e_{ik} + e_{jk}$, (we have $|\bar{S}| - 1$ of these solutions).

Case 1 is the 0-vector, and as each of Case 2 corresponds to a distinct node in $V\backslash\{i\}$, we have $|V|$ affinely independent feasible solutions in total so far. Since each of Cases 3–7 corresponds to a distinct edge in E, (Cases 3 and 7 cover the edge set $(i, V\backslash\{i\})$, Case 4 covers edge set $(S, \bar{S}\backslash\{i\})$, Case 5 covers edge set $(\bar{S}\backslash\{i\}, \bar{S}\backslash\{i\})$, and Case 6 covers edge set

(S, S)), so the solutions are affinely independent to each other and to those in Cases 1 and 2. Hence, we have $|V| + |E|$ affinely independent feasible solutions in total. $\square$

Proposition 3.2. *Given any $G = (V, E)$, for $|V| \geq 2$, $1 \leq b \leq |V|$, the constraints, given by:*

$$\sum_{e \in E(S)} z_e + 1 \geq \sum_{i \in S} y_i, \quad \forall S \subseteq V, |S| \geq 2, \quad (15)$$

are valid for the QK polytope.

Note that (4) is a special case of (15). The proof is similar to that of Proposition 3.1.

Theorem 3.4. *Given any $G = (V, E)$, $3 \leq b \leq |V|$, Constraints (15), are facet-defining for the QK polytope if $|S| \geq 2$.*

Proof: Let $F = P_{QK}^{nb} \cap \{\sum_{e \in E(S)} z_e + 1 = \sum_{i \in S} y_i\}$. F defines a proper face for P_{QK}^{nb}: consider $(y, z) = 0$ which does not satisfy the constraint at equality and $(y, z) = e_i$ which does. Again, we find exactly $\dim(P_{QK}^{nb}) = |V| + |E|$ affinely independent feasible solutions that satisfy the constraints at equality. We do so by taking the following steps.

1. $(y, z)^1 = \{(y, z)_i^1 \mid \forall i \in S\}$, where $(y, z)_i^1 = e_i$ (we have $|S|$ of these solutions);
2. $(y, z)^2 = \{(y, z)_{ij}^2 \mid \forall i \in S, j \in V\}$, where $(y, z)_{ij}^2 = e_i + e_j + e_{ij}$, (we have $|(S, S)| + |(S, \bar{S})|$ of these solutions);
3. $(y, z)^3 = \{(y, z)_k^3 \mid \forall k \in \bar{S}\}$, where $(y, z)_k^3 = e_i + e_j + e_k + e_{ij} + e_{ik} + e_{jk}$, for a fixed $i \in S$, and a fixed $j \in S \setminus \{i\}$, (we have $|\bar{S}|$ of these solutions); and
4. $(y, z)^4 = \{(y, z)_{jk}^4 \mid \forall j, k \in \bar{S}\}$, where $(y, z)_{jk}^4 = e_i + e_j + e_k + e_{ij} + e_{ik} + e_{jk}$, for a fixed $i \in S$, (there are $|(\bar{S}, \bar{S})|$ of these solutions).

It is obvious that the $|S| + |(S, S)| + |(S, \bar{S})|$ feasible solutions introduced in Step 1 and Step 2 are affinely independent as these edge sets are disjoint sets. We now show that the solutions introduced in Step 3 are affinely independent to all the previously introduced solutions. We do so by contradiction. We assume that, w.l.o.g., the first solution introduced in Step 3 is $(y, z)_l^3$, for any $l \in \bar{S}$, and that $(y, z)_l^3 = \sum_i \lambda_i (y, z)_i^1 + \sum_{ij} \mu_{ij} (y, z)_{ij}^2$, for some $\lambda \in \mathbb{R}^{|S|}$, $\mu \in \mathbb{R}^{|(S,S)|+|(S,\bar{S})|}$, $(\lambda, \mu) \neq 0$. Now, to obtain the elements in $(y, z)_l^3$ corresponding to the z variables, we need to set $\mu_{ij} = \mu_{il} = \mu_{jl} = 1$, and $\mu_f = 0$ for all $f \in E \setminus \{(i, j), (i, l), (j, l)\}$. Observe further that in $(y, z)^1$, as $l \in \bar{S}$, the value of the element corresponding to node l is always 0. So, the value of the element in $(y, z)^3$ that corresponds to node l should be 2 instead of 1. Hence there is a contradiction. Clearly as the nodes in $\bar{S}$ are all distinct, we conclude that the incidence vectors in $(y, z)^3$ are all affinely independent. Last of all, the solutions introduced in Step 4, i.e. $(y, z)^4$ are obviously affinely independent to all the previously introduced solutions as the edge sets in Steps 2, 3, and 4 are disjoint sets. Thus the theorem is proved. $\square$

Definition 3.1. Given any $G = (V, E)$, for $|V| \geq 2$, $1 \leq b \leq |V|$, the constraints, given as:

$$\sum_{e \in \delta(i)} z_e \leq (b - 1) y_i, \quad \forall i \in V, \quad (16)$$

are valid for P_{QK}^{nb}.

Multiplying (6) by y_i, we get Constraints (16). (Note that $y_i y_i = y_i$ and $y_i y_j = z_{ij}$).

Theorem 3.5. *Given any $G = (V, E)$, $3 \leq b \leq |V| - 1$, Constraints (16) are facet-defining for P^{nb}_{QK}.*

Proof: Let $F = P^{nb}_{QK} \cap \{\sum_{e \in \delta(i)} z_e = (b-1)y_i\}$. F defines a proper face for P^{nb}_{QK}: consider $(y, z) = e_i$ which does not satisfy the constraint at equality since b is at least 3, and $(y, z) = e_i + \sum_{j \in S} e_j$, for any $S \subseteq V \backslash \{i\}$, $|S| = b - 1$, which does satisfy the constraint at equality. Now consider the following feasible solutions which do not select node i:

1. $(y, z)^0 = 0$;
2. $(y, z)^1 = e_k$, for all $k \in V \backslash \{i\}$, (we have $|V| - 1$ of these solutions); and
3. $(y, z)^2 = e_k + e_l + e_{kl}$, for all $\{k, l\} \subseteq V \backslash \{i\}$, (we have $|E| - (|V| - 1)$ of these solutions).

Clearly, these $|E| + 1$ points are affinely independent, and satisfy (16) at equality.

Now, we are left with finding the remaining $|V| - 1$ affinely independent feasible solutions. We do so by inspecting the set of all feasible solutions that selects exactly b nodes, including node i. We define such a set of solutions to be $\Omega^{V,b} = \{I_1, \ldots, I_m \mid I_l = \{i\} \cup U_l,\ |I_l| = b,\ \forall l = 1, \ldots, m\}$, where $U_l \subset V \backslash \{i\}$. (Note that m is finite). We denote each solution I_l by $\{i, j^l_1, \ldots, j^l_{b-1}\}$ for $l = 1, \ldots, m$.

Our inductive hypothesis is that there are precisely $|V| - 1$ affinely independent feasible solutions among the m solutions in $\Omega^{V,b}$. Our proof takes the following steps: Step 1 concerns the initial case for $|V| = 4$ and $b = 3$; Step 2 concerns induction on $|V|$ while holding b constant; and Step 3 concerns induction on both b and $|V|$.

Step 1. There are precisely 3 affinely independent feasible solutions for the case when $|V| = 4$ and $b = 3$. W.l.o.g., suppose the four node are indexed $\{1, 2, 3, 4\}$ and that $i = 3$. We have the following three basis solutions: (1) $e_1 + e_2 + e_3 + e_{12} + e_{13} + e_{23}$; (2) $e_1 + e_3 + e_4 + e_{13} + e_{14} + e_{34}$; and (3) $e_2 + e_3 + e_4 + e_{23} + e_{34} + e_{24}$. (Note that the only remaining feasible solution that selects three nodes does not include node 3.)

Step 2. We assume that our inductive hypothesis is true for $|V| = 4, \ldots, s$ and $b = t$, for $t \leq s - 1$. We now show that it is true for $|V| = s + 1$ and $b = t$. Consider the QKP defined on $\tilde{G} = (\tilde{V}, \tilde{E})$, for $\tilde{V} = V \cup \{q\}$, $\tilde{E} = (q, V) \cup E(V)$. We show that $\Omega^{\tilde{V},t}$ contains exactly s affinely independent feasible solutions. By our inductive hypothesis, there exists $\Omega^{V,t}$ that contains $s - 1$ affinely independent feasible solutions, and w.l.o.g., let these $s - 1$ solutions be $I_1, \ldots, I_{s-1}$. As b was held constant at t, these $s - 1$ points are also feasible for $\tilde{G}$ and satisfy (16) at equality. Now consider a new solution $I_s = \{i, j^1_1, \ldots, j^1_{t-2}, q\}$. Clearly, I_s is affinely independent to any of the previously introduced solutions (wherein q is never used), and it satisfy (16) at equality.

Step 3. We assume that our inductive hypothesis holds for $|V| = 4, \ldots, s$, $b = 3, \ldots, t$, for $t \leq s - 1$, and prove that it holds for $|V| = s + 1$ and $b = t + 1$. Recall $I_1, \ldots, I_{s-1}$ defined in Step 2. First, consider the solution $I'_s = I_1 \cup \{k\}$, for any $k \in V \backslash I_1$, (hence $|I'_s| = t + 1$, and is affinely independent to $(y, z)^0$, $(y, z)^1$ and $(y, z)^2$, bear in mind that node i is not selected in these three sets of solutions). Then, we define $I'_l = I_l \cup \{q\}$, for all $l = 1, \ldots, s - 1$, and thus obtain $s - 1$ affinely independent feasible solutions each selecting $t + 1$ nodes. These are affinely independent to $(y, z)^0$, $(y, z)^1$, $(y, z)^2$, and I'_s due to the use of node q. Thus completes the proof.

□

4. Polyhedral results for the QSTS polytope

In this section, we present our polyhedral results for the QSTS polytope, $\tilde{P}_{QS}^{nb}$. (Recall that this concerns the formulation without the y variables). We first present the dimension of $\tilde{P}_{QS}^{nb}$ and establish the links between $\tilde{P}_{QS}^{nb}$ and P_{QS}^{nb}. We then prove that five classes of constraints are facet-defining for $\tilde{P}_{QS}^{nb}$. The first class of constraints concerns the relationship between x_e and z_e; the second class of constraints is a lifted version of the subtour elimination constraints (5); and the last three classes of constraints are facet-defining constraints for the QK polytope, except that herein we use $\frac{1}{2}\sum_{e\in\delta(i)} x_e$ in place of y_i.

In what follows, we use incidence vectors $(x, z) \in \{0, 1\}^{2|E|}$, for $x, z \in \{0, 1\}^{|E|}$ to represent our solutions. We also define $(\lambda, \mu) \in \mathbb{R}^{2|E|}$, for $\lambda, \mu \in \mathbb{R}^{|E|}$, with each element in λ and μ representing an edge $e \in E$. Furthermore, when we refer to p-cycles, we refer to cycles in G that contain p nodes. In this section, we consider the 0-cycles as feasible solutions, however the 1-cycles and the 2-cycles are not feasible solutions.We will use the following result frequently.

Proposition 4.1. *Given an undirected graph $G = (V, E)$, $|V| = 5$, let M be the matrix generated by incidence vectors of all 3- and 4-cycles in G (i.e. each row of M is a vector of $\{0, 1\}^{2|E|}$ that represents a solution for the QSTSP). Under the assumption that G is complete, if $M(\lambda, \mu)^T = 0$, then $\lambda_e = \mu_e = 0$ for all $e \in E$.*

Proof: It can be verified that M is of rank $2|E| = 20$, hence the result follows immediately. □

Theorem 4.1. *Given any QSTSP defined on an undirected graph $G = (V, E)$, with $|V| \geq 5$ and $4 \leq b \leq |V|$, under the assumption that G is complete, the dimension of the QSTS polytope, $\tilde{P}_{QS}^{nb}$, is $2|E|$.*

Proof: We show this by contradiction. We first assume that $\tilde{P}_{QS}^{nb}$ is not full-dimensional, and hence there must be at least one equality constraint, $\lambda \cdot x + \mu \cdot z = \lambda_0$, satisfied by all feasible solutions in the polytope. Then we establish that this is true only when $\lambda_e = \mu_e = \lambda_0 = 0$, for all $e \in E$, thus implying that there is no equality constraint satisfied by all feasible solutions in the polytope and hence the polytope is full dimensional.Consider the 0-cycle defined by $(x, z) = 0$. We have $\lambda \cdot 0 + \mu \cdot 0 = \lambda_0$. Hence we get $\lambda_0 = 0$. To show that $\lambda_e = \mu_e = \lambda_0 = 0$, for all $e \in E$, consider any arbitrary subgraph $\tilde{G} = (\tilde{V}, \tilde{E})$ for $\tilde{V} \subseteq V$, $|\tilde{V}| = 5$, and $\tilde{E} = E(\tilde{V})$. Under the assumption that G is complete, $\tilde{G}$ is also complete. Now, consider a matrix M generated by the incidence vectors of all the 3-cycles and the 4-cycles in $\tilde{G}$. Since $\lambda_0 = 0$, by result of Proposition 4.1, we have $\lambda_e = \mu_e = 0$ for all $e \in \tilde{E}$. (Note that since we must use 4-cycles in here, b must be at least four). As $\tilde{G}$ is arbitrary in G, we have that $\lambda_e = \mu_e = 0$, for all $e \in E$. Hence the theorem is proved. □

Next, we discuss the relation between $\tilde{P}_{QS}^{nb}$ and P_{QS}^{nb}. Essentially, we show that the two polytopes represent the same set of feasible solutions, and that facets found for one are facets for the other (with slight modifications). Hence, all facets of $\tilde{P}_{QS}^{nb}$ we propose in this paper are also facets for P_{QS}^{nb}. These results are echos of similar results of Bauer et al. (2002) for the CCCP.

Proposition 4.2. *For any QSTSP defined on $G = (V, E)$ where $|V| \geq 5$, and $4 \leq b \leq |V|$, we have that $dim(\tilde{P}^{nb}_{QS}) = dim(P^{nb}_{QS})$.*

Proof: Each incidence vector $(x, z) \in^{2|E|} \cap \tilde{P}^{nb}_{QS}$ can be represented by an incidence vector $(x, y, z) \in \mathbb{R}^{2|E|+|V|} \cap P^{nb}_{QS}$ simply by setting $y_i = \frac{1}{2}\sum_{e\in\delta(i)} x_e$ for all $i \in V$. For any set of $2|E| + 1$ affinely independent incidence vectors that spans $\tilde{P}^{nb}_{QS}$, we can get $2|E| + 1$ affinely independent incidence vectors in P^{nb}_{QS}. Thus $dim(P^{nb}_{QS}) \geq 2|E|$. As the rank of the degree constraints, (2), is $|V|$, clearly $dim(P^{nb}_{QS}) \leq 2|E| + |V| - |V|$, and thus $dim(P^{nb}_{QS}) = 2|E|$. □

Remark 1. Since $dim(\tilde{P}^{nb}_{QS}) = dim(P^{nb}_{QS})$, and each incidence vector $(x, z) \in \mathbb{R}^{2|E|} \cap \tilde{P}^{nb}_{QS}$ can be represented by an incidence vector $(x, y, z) \in \mathbb{R}^{2|E|+|V|} \cap P^{nb}_{QS}$, the two polytopes describe the same set of feasible solutions for the QSTSP.

Proposition 4.3. *For any QSTSP defined on $G = (V, E)$ where $|V| \geq 5$, and $4 \leq b \leq |V|$, if $ax + bz \leq a_0$ defines a facet for $\tilde{P}^{nb}_{QS}$, then it also defines a facet for P^{nb}_{QS}.*

Proof: By result of Proposition 4.2 and the fact that the same $2|E|$ affinely independent incidence vectors $(x, z) \in \mathbb{R}^{2|E|} \cap \tilde{P}^{nb}_{QS}$ that satisfy $ax + bz \leq a_0$ at equality can be converted to $2|E|$ affinely independent incidence vectors $(x, y, z) \in \mathbb{R}^{2|E|+|V|} \cap P^{nb}_{QS}$, the proposition is proved. □

Proposition 4.4. *For any QSTSP defined on $G = (V, E)$ where $|V| \geq 5$, and $4 \leq b \leq |V|$, if $\alpha x + \beta y + \gamma z \leq \alpha_0$ defines a facet for P^{nb}_{QS}, then $\tilde{\alpha} x + \gamma z \leq \alpha_0$ also defines a facet for $\tilde{P}^{nb}_{QS}$, where $\tilde{\alpha}_{ij} = \alpha_{ij} + \frac{1}{2}(\beta_i + \beta_j)$.*

Proof: Suppose $\Omega = \{(x^1, y^1, z^1), \ldots, (x^{|E|}, y^{|V|}, z^{|E|})\}$ defines $2|E|$ affinely independent feasible solutions that satisfy $\alpha x + \beta y + \gamma z \leq \alpha_0$ at equality,then $\tilde{\Omega} = \{(\tilde{x}^1, 0, z^1), \ldots, (\tilde{x}^{|E|}, 0, z^{|E|})\}$, where $\tilde{x}_{ij} = x_{ij} + \frac{1}{2}(y_i + y_j)$ for all $(i, j) \in E$, (which is obtained from Ω by simple linear row operations), are also affinely independent. Hence the result. □

Proposition 4.5. *Given any QSTSP defined on an undirected graph $G = (V, E)$, with $3 \leq b \leq |V| \leq 5$, the constraints given below, are valid for the QSTS polytope, $\tilde{P}^{nb}_{QS}$.*

$$x_e \leq z_e, \qquad \forall e \in E. \qquad (17)$$

This is obviously true by the definitions of the x- and z-variables.

Theorem 4.2. *Given any QSTSP defined on an undirected graph $G = (V, E)$, with $|V| \geq 5$ and $4 \leq b \leq |V|$, Constraints (17) are facet-defining for the QSTS polytope, $\tilde{P}^{nb}_{QS}$.*

Proof: We show that the result holds for $|V| \geq 6$ and $b \geq 4$. (For $|V| = 5$ and $5 \geq b \geq 4$, one can easily prove this by enumerating feasible points that satisfy (17) at equality and verify that there are $2|E|$ affinely independent feasible points). First we show that $\tilde{P}^{nb}_{QS} \cap \{x_e = z_e\}$ defines a proper face. Let $e = (i, j)$. Consider a 4-cycle given by (l, i, m, j), for l, m, i, j distinct, clearly $x_e = 0$ and $z_e = 1$, hence the constraint is not satisfied at equality. Now consider a 3-cycle given by (l, i, j), for l, i, j distinct, clearly $x_e = z_e = 1$ and the constraint is satisfied at equality.

Now, using Theorem 3.6 in Part I.4 of Nemhauser and Wolsey (1988), we need to show that if $\lambda \cdot x + \mu \cdot z = \lambda_0$ for all $x \in \tilde{P}_{QS}^{nb} \cap \{x_e = z_e\}$, then

$$\lambda_f = \begin{cases} \alpha, & \text{if } f = e, \\ 0, & \text{otherwise;} \end{cases} \quad \lambda_0 = 0; \quad \text{and} \quad \mu_f = \begin{cases} -\alpha, & \text{if } f = e, \\ 0, & \text{otherwise;} \end{cases}$$

for some $\alpha \in \mathbb{R}$.

By considering the 0-cycle, we obtain $\lambda_0 = 0$. Let $e = (i, j)$. Consider any arbitrary subgraph $\tilde{G} = (\tilde{V}, \tilde{E})$ for $\tilde{V} \subseteq V \backslash \{i\}$, $|\tilde{V}| = 5$ and $\tilde{E} = E(\tilde{V})$. As $i \notin \tilde{V}$, $e \notin \tilde{E}$, hence (17) holds with equality for all cycles in $\tilde{G}$. By Proposition 4.1, we have $\lambda_f = \mu_f = 0$, for all $f \in \tilde{E}$. Now, consider any arbitrary distinct $j, k, l \in V \backslash \{i\}$, and compare 3-cycles (i, j, k), (i, k, l), and (i, j, l), we get $\lambda_e + \mu_e = 0$. Let $\lambda_e = \alpha$ for some $\alpha \in \mathbb{R}$, we have $\mu_e = -\alpha$ and thus the theorem is proved. □

To eliminate subtours for the QSTSP, we propose a class of constraints lifted from (5), given as:

$$\sum_{e \in \delta(S)} x_e \geq 2z_{kl}, \quad \forall \emptyset \subset S \subset V, k \in S, l \notin S. \qquad (18)$$

Proposition 4.6. *Given any QSTSP defined on an undirected graph $G = (V, E)$, with $3 \leq b \leq |V|$, the constraints given by (18), are valid for $\tilde{P}_{QS}^{nb}$.*

This is just the classic subtour elimination constraint.

Theorem 4.3. *Given any QSTSP defined on an undirected graph $G = (V, E)$, with $|V| \geq 10$, $|V| - 5 \geq |S| \geq 5$ and $4 \leq b \leq |V|$, the constraints given by (18), are facet-defining for the QSTS polytope, $\tilde{P}_{QS}^{nb}$.*

Proof: $\tilde{P}_{QS}^{nb} \cap \{\sum_{e \in (S, \bar{S})} x_e = 2z_{kl}\}$ defines a proper face, since (18) holds with equality for the 0-cycle while it does not for the 3-cycle (k, p, q), for $p, q \in \bar{S} \backslash \{l\}$, $p \neq q$ (as $z_{kl} = 0$ but $\sum_{e \in \delta(S)} x_e = 2$).

Now, we are left to show that if $\lambda \cdot x + \mu \cdot z = \lambda_0$ for all $x \in \tilde{P}_{QS}^{nb} \cap \{\sum_{e \in (S, \bar{S})} x_e = 2z_{kl}\}$, then

$$\lambda_e = \begin{cases} \alpha, \text{ if } e \in (S, \bar{S}), \\ 0, \text{ otherwise;} \end{cases} \qquad \lambda_0 = 0; \text{ and } \mu_e = \begin{cases} -2\alpha, & \text{if } e = (k, l), \\ \;\;0, & \text{otherwise;} \end{cases}$$

for some $\alpha \in \mathbb{R}$.

By considering the 0-cycle, we have $\lambda_0 = 0$. Now, consider any arbitrary subgraph $\tilde{G} = (\tilde{S}, \tilde{E})$ for $\tilde{S} \subseteq S$, $|\tilde{S}| = 5$, and $\tilde{E} = E(\tilde{S})$. It can be easily verified that Constraints (18) hold with equality for all cycles in $\tilde{G}$. Thus, by Proposition 4.1, we have $\lambda_f = \mu_f = 0$, for all $f \in \tilde{E}$. As $\tilde{G}$ is arbitrary, we have $\lambda_f = \mu_f = 0$, for all $f \in E(S)$. Analogously it can be obtained that $\lambda_f = \mu_f = 0$, for all $f \in E(\bar{S})$.

Now we obtain values for all the remaining elements in (λ, μ), i.e., we find λ_e and μ_e for all $e \in (S, \bar{S})$, by comparing 3- or 4-cycles such that (18) holds with equality. Assume that we have arbitrary distinct i, j, m, for $i, j \in S\backslash\{k\}$, and $m \in \bar{S}\backslash\{l\}$. Let (x^1, z^1) and (x^2, z^2) be the incidence vectors of the 4-cycle defined by (k, i, j, l) and the 3-cycle defined by (k, i, j) respectively. We get:

$$\lambda \cdot x^1 + \mu \cdot z^1 - (\lambda \cdot x^2 + \mu \cdot z^2) = \lambda_{jl} + \lambda_{kl} - \lambda_{jk} + \mu_{kl} + \mu_{il} + \mu_{jl} = 0. \qquad (19)$$

Note that $\lambda_{jk} = 0$ since $k, j \in S$. Analogously let (x^3, z^3) be the incidence vectors of the 4-cycle defined by (k, j, i, l). We get:

$$\lambda \cdot x^3 + \mu \cdot z^3 - (\lambda \cdot x^2 + \mu \cdot z^2) = \lambda_{kl} + \lambda_{il} - \lambda_{ik} + \mu_{kl} + \mu_{il} + \mu_{jl} = 0. \qquad (20)$$

Note that $\lambda_{ik} = 0$ since $k, i \in S$. By comparing (19) with (20), we get $\lambda_{il} = \lambda_{jl}$. Let $\lambda_{il} = \alpha$, by symmetry, we get $\lambda_{il} = \alpha$ for all $i \in S\backslash\{k\}$. Now by comparing the 3-cycle (k, j, l) with (19) it follows that $\mu_{il} = 0$ for all $i \in S\backslash\{k\}$.

Comparing the 4-cycle (k, i, l, j) with the 3-cycle (k, i, j), we get $\mu_{kl} = -2\alpha$ and by comparing the 3-cycle (k, j, l) with the 4-cycle (k, j, l, i), we get $\lambda_{kl} = \alpha$. Given this and by symmetry, $\lambda_{km} = \alpha$ and $\mu_{km} = 0$ for all $m \in \bar{S}\backslash\{l\}$.

By comparing the 3-cycle (i, l, k) and the 4-cycle (i, l, m, k), we get $\mu_{im} = 0$ for all $i \in S\backslash\{k\}$ and all $m \in \bar{S}\backslash\{l\}$. Last of all, by comparing the 3-cycle (k, i, l) and the 4-cycle (k, i, m, l), we obtain $\lambda_{im} = \alpha$, for all $i \in S\backslash\{k\}, m \in \bar{S} \setminus \{l\}$. $\square$

Theorem (4.3) does not hold for $7 \leq |V| \leq 9$, but it actually holds for $|V| = 6$, $|S| = 5$ and $4 \leq b \leq |V|$ (and for $|S| = 1$ which is the symmetric case). This can be verified by generating $2|E|$ affinely independent feasible points that satisfy (18) at equality.

Proposition 4.7. *Given any $G = (V, E)$, $3 \leq b \leq |V|$, the constraints, given by:*

$$\sum_{e \in (i,S)} z_e \leq \frac{1}{2} \sum_{e \in \delta(i)} x_e + \sum_{e \in E(S)} z_e, \quad \forall i \in V, S \subset V\backslash\{i\}, \qquad (21)$$

are valid for $\tilde{P}_{QS}^{nb}$.

Constraint (21) is obtained by replacing y_i by $\frac{1}{2} \sum_{e \in \delta(i)} x_e$ in (14) and is a generalization of (3).

Theorem 4.4. *Given any $G = (V, E)$, $|V| \geq 6$, $b \geq 4$, Constraints (21) are facet-defining for $\tilde{P}_{QS}^{nb}$ if $1 \leq |S| \leq |V| - 5$.*

Proof: $\tilde{P}_{QS}^{nb} \cap \{\sum_{e \in (i,S)} z_e = \frac{1}{2} \sum_{e \in \delta(i)} x_e + \sum_{e \in E(S)} z_e\}$ defines a proper face since the 0-cycle satisfies the constraint at equality whereas the 3-cycle (i, p, q), for $p, q \in \bar{S}\backslash\{i\}, p \neq q$, does not. Now, we need to show that if $\lambda \cdot x + \mu \cdot z = \lambda_0$ for all $x \in \tilde{P}_{QS}^{nb} \cap \{\frac{1}{2} \sum_{e \in \delta(i)} x_e + \sum_{e \in E(S)} z_e = \sum_{e \in (i,S)} z_e\}$, then

$$\lambda_e = \begin{cases} \frac{1}{2}\alpha, & \text{if } e \in \delta(i), \\ 0, & \text{otherwise;} \end{cases} \quad \lambda_0 = 0; \quad \text{and} \quad \mu_e = \begin{cases} \alpha, & \text{if } e \in E(S), \\ -\alpha, & \text{if } e \in (i, S), \\ 0, & \text{otherwise;} \end{cases}$$

for some $\alpha \in \mathbb{R}$.

By considering the 0-cycle we get $\lambda_0 = 0$. W.l.o.g. let R, k be arbitrary for $R \subseteq \bar{S}\backslash\{i\}$, $|R| = 4$ and $k \in S$. Consider the subgraph $\tilde{G} = (\tilde{V}, \tilde{E})$, $\tilde{V} \subseteq V$, $\tilde{V} = R \cup \{k\}$ and $\tilde{E} = E(\tilde{V})$. Constraint (21) holds with equality for all cycles in $\tilde{G}$, hence by Proposition 4.1, $\lambda_e = \mu_e = 0$ for all $e \in \tilde{E}$. Since R and k are arbitrary, $\lambda_e = \mu_e = 0$ for all $e \in E(\bar{S}\backslash\{i\}) \cup (S, \bar{S}\backslash\{i\})$.

Let $k \in S$ and $p, q \in \bar{S}\backslash\{i\}$, $p \neq q$ be arbitrary. By comparing the cycles (k, i, p, q) and (k, i, p), we obtain $\lambda_{pq} + \lambda_{kq} - \lambda_{kp} + \mu_{kq} + \mu_{iq} + \mu_{pq} = 0$. Since $\lambda_{pq} = \lambda_{kq} = \lambda_{kp} = \mu_{kq} = \mu_{pq} = 0$, $\mu_{iq} = 0$. Since k, p, and q are arbitrary, $\mu_{ip} = 0$ for all $p \in \bar{S}\backslash\{i\}$. By comparing the cycles (k, p, i, q) and (k, p, i), we obtain $\lambda_{kq} + \lambda_{iq} - \lambda_{ki} + \mu_{kq} + \mu_{pq} + \mu_{iq} = 0$. Since $\lambda_{kq} = \mu_{kq} = \mu_{pq} = \mu_{iq} = 0$, $\lambda_{iq} = \lambda_{ki}$ are constant and let the constant be $\frac{1}{2}\alpha$. As k, p and q are arbitrary, $\lambda_e = \frac{1}{2}\alpha$ for all $e \in \delta(i)$. Consider the cycle (i, k, p) to obtain $\mu_{ik} = -\alpha$ for all $k \in S$. For cases where $|S| \geq 2$, let l be arbitrary in S, and $l \neq k$. By comparing the cycles (k, l, i, p) and (k, i, l, p), we obtain $\lambda_{kl} + \lambda_{ip} = \lambda_{ki} + \lambda_{lp}$. Since $\lambda_{lp} = 0$ and $\lambda_{ip} = \lambda_{ki} = \frac{1}{2}\alpha$, $\lambda_{kl} = 0$. As k, l, and p are arbitrary, $\lambda_e = 0$ for all $e \in E(S)$. Finally, consider the cycle (i, k, l) to obtain $\mu_{kl} = \alpha$. Since k and l are arbitrary, $\mu_e = \alpha$ for $e \in E(S)$. □

Proposition 4.8. *Given any $G = (V, E)$, $3 \leq b \leq |V|$, the constraints, given by:*

$$\sum_{e \in E(S)} z_e + 1 \geq \sum_{e \in E(S)} x_e + \frac{1}{2} \sum_{e \in \delta(S)} x_e, \quad \forall S \subset V,\ 1 \leq |S| \leq |V|, \tag{22}$$

are valid for $\tilde{P}_{QS}^{nb}$.

Constraint (22) is obtained by replacing y_i by $\frac{1}{2}\sum_{e \in \delta(i)} x_e$ in (15). Note that (4) is a special case of (22).

Theorem 4.5. *Given any $G = (V, E)$, $|V| \geq 5$, $b \geq 5$, Constraints (22) are facet-defining for $\tilde{P}_{QS}^{nb}$ if $2 \leq |S| \leq |V| - 3$.*

Proof: $\tilde{P}_{QS}^{nb} \cap \{\sum_{e \in E(S)} z_e + 1 = \sum_{e \in E(S)} x_e + \frac{1}{2}\sum_{e \in \delta(S)} x_e\}$ defines a proper face since the 3-cycle (i, j, k), $i \in S$, $j, k \in \bar{S}$ satisfies the constraint at equality and the 0-cycle does not.

Now, we need to show that if $\lambda \cdot x + \mu \cdot z = \lambda_0$ for all $x \in \tilde{P}_{QS}^{nb} \cap \{\sum_{e \in E(S)} z_e + 1 = \sum_{e \in E(S)} x_e + \frac{1}{2}\sum_{e \in \delta(S)} x_e\}$, then

$$\lambda_e = \begin{cases} -\alpha, & \text{if } e \in E(S), \\ -\frac{1}{2}\alpha, & \text{if } e \in \delta(S), \\ 0, & \text{otherwise;} \end{cases} \quad \lambda_0 = \alpha; \quad \text{and} \quad \mu_e = \begin{cases} \alpha, & \text{if } e \in E(S), \\ 0, & \text{otherwise;} \end{cases}$$

for some $\alpha \in \mathbb{R}$.

W.l.o.g. let $R \subseteq S$, $|R| = 2$ and $T \subseteq \bar{S}$, $|T| = 3$ be arbitrary. Consider the subgraph $\tilde{G} = (\tilde{V}, \tilde{E})$, $\tilde{V} \subseteq V$, $\tilde{V} = R \cup T$, (so $|\tilde{V}| = 5$) and $\tilde{E} = E(\tilde{V})$. Let $\lambda_0 = \alpha$. Let the matrix M be generated by the incidence vectors of all the cycles in $\tilde{G}$ for which (22) holds with equality. M is found to be of rank $2|\tilde{E}| = 20$, thus $M(\lambda, \mu)^T = \alpha$ has an unique solution. The solution is $\lambda_e = -\alpha$ for all $e \in E(R)$, $\lambda_e = -\frac{1}{2}\alpha$ for all $e \in \delta(R)$, and $\lambda_e = 0$ for all $e \in E(T)$; $\mu_e = \alpha$ for all $e \in E(R)$ and $\mu_e = 0$ for all $e \in \delta(R) \cup E(T)$. Since R is arbitrary in S, T is arbitrary in $\bar{S}$, we get $\lambda_e = -\alpha$ for all $e \in E(S)$, $\lambda_e = -\frac{1}{2}\alpha$ for all $e \in \delta(S)$ and $\lambda_e = 0$ for all $e \in E(\bar{S})$, $\mu_e = \alpha$ for all $e \in E(S)$ and $\mu_e = 0$ for all $e \in \delta(S) \cup E(\bar{S})$. □

The following constraints are found to be very effective in practise when solving QSTSPs using a branch-and-cut method (see Stidsen and Thomadsen, 2005):

$$\sum_{e\in\delta(i)} z_e \le \frac{b-1}{2}\sum_{e\in\delta(i)} x_e, \quad \forall i \in V. \qquad (23)$$

Proposition 4.9. *Given any QSTSP defined on an undirected graph $G = (V, E)$, with $3 \le b \le |V|$, the constraints given by (23) are valid for the QSTS polytope, $\tilde{P}^{nb}_{QS}$.*

Constraint (23) is obtained from Constraint (16) by replacing y_i with $\frac{1}{2}\sum_{e\in\delta(i)} x_e$.

Theorem 4.6. *Given any QSTSP defined on an undirected graph $G = (V, E)$, with $|V| \ge 6$ and $4 \le b \le |V| - 1$, the constraints given by (23) are facet-defining for $\tilde{P}^{nb}_{QS}$.*

Proof: A 3-cycle (i, j, k), for distinct $i, j, k \in V$, does not satisfy the constraint at equality (since $b \ge 4$) whereas the 0-cycle does. Hence $\tilde{P}^{nb}_{QS} \cap \{\sum_{e\in\delta(i)} z_e = \frac{b-1}{2}\sum_{e\in\delta(i)} x_e\}$ defines a proper face.

Now, we are left to show that if $\lambda \cdot x + \mu \cdot z = \lambda_0$ for all $x \in \tilde{P}^{nb}_{QS} \cap \{\sum_{e\in\delta(i)} z_e = \frac{b-1}{2}\sum_{e\in\delta(i)} x_e\}$, then

$$\lambda_e = \begin{cases} \dfrac{\alpha(b-1)}{2}, & \text{if } e \in \delta(i), \\ 0, & \text{otherwise;} \end{cases} \qquad \lambda_0 = 0; \quad \text{and} \quad \mu_e = \begin{cases} -\alpha, & \text{if } e \in \delta(i), \\ 0, & \text{otherwise;} \end{cases}$$

for some $\alpha \in \mathbb{R}$.

By considering the 0-cycle, we get $\lambda_0 = 0$. Now, consider any arbitrary subgraph $\tilde{G} = (\tilde{V}, \tilde{E})$ for $\tilde{V} \subseteq V\backslash\{i\}$, $|\tilde{V}| = 5$, and $\tilde{E} = E(\tilde{V})$. Since all cycles in $\tilde{G}$ satisfies Constraint (23) at equality, it follows from Proposition 4.1 that $\lambda_f = \mu_f = 0$, for all $f \in \tilde{E}$. As $\tilde{G}$ is arbitrary, we have $\lambda_f = \mu_f = 0$, for all $f \in E(V\backslash\{i\})$.

Let $\{j_1, \ldots, j_{b-1}\} \subseteq V\backslash\{i\}$ be arbitrary. Now compare the two b-cycles $(i, j_1, j_2, j_3, \ldots, j_{b-1})$ and $(i, j_2, j_1, j_3, \ldots, j_{b-1})$. This gives $\lambda_{ij_1} + \lambda_{j_2j_3} = \lambda_{ij_2} + \lambda_{j_1j_3}$. Since $\lambda_{j_2j_3} = \lambda_{j_1j_3} = 0$, λ_{ij_a} is constant for $a = 1, \ldots, b-1$ and let the constant be $\frac{\alpha(b-1)}{2}$. Since $\{j_1, \ldots, j_{b-1}\} \subseteq V\backslash\{i\}$ is arbitrary, $\lambda_{ij} = \frac{\alpha(b-1)}{2}$ for all $j \in V\backslash\{i\}$. Finally compare the b-cycle $(i, j_1, j_2, j_3, \ldots, j_{b-1})$ with the $(b-1)$-cycle $(j_1, j_2, j_3, \ldots, j_{b-1})$ to obtain $\lambda_{ij_1} + \lambda_{ij_{b-1}} - \lambda_{j_1j_{b-1}} + \sum_{k=1}^{b-1}\mu_{ij_k} = 0$. Since $\lambda_{j_1j_{b-1}} = 0$ $\lambda_{ij_1} = \lambda_{ij_{b-1}} = \frac{\alpha(b-1)}{2}$, and that by symmetry $\mu_{ij_1} = \mu_{ij_k}$ for all $k = 2, \ldots, b-1$, $\mu_{ij_a} = -\alpha$ for all $a = 1, \ldots, b-1$. As $\{j_1, \ldots, j_{b-1}\} \subseteq V\backslash\{i\}$ is arbitrary, $\mu_{ij} = -\alpha$ for all $j \in V\backslash\{i\}$ and the theorem is proved. □

5. Conclusion

In this paper, we studied the polyhedra of the Quadratic Knapsack Problem and the Quadratic Selective Travelling Salesman Problem. For each of these polytopes, we established its dimension, identified a number of strong constraints, and proved that these constraints are indeed facet-defining cuts. Various mathematical techniques were used in proving these results.

These results are of great significance in the implementation of a branch-and-cut method for obtaining exact solutions. The benefit of using such facet-defining cuts is that it improves the quality of the linear programming relaxation bounds.

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References

Balas E (1989) The prize collecting traveling salesman problem. Networks 19:621–636

Balas E (1995) The prize collecting traveling salesman problem. II. Polyhedral results. Networks 25:199–216

Bauer P (1997) The circuit polytope: Facets. Mathematics of Operations Research 22:110–145

Bauer P, Linderoth JT, Savelsbergh MWP (2002) A branch and cut approach to the cardinality constrained circuit problem. Mathematical Programming Ser A 91:307–348

Billionnet A, Calmels F (1996) Linear programming for the 0-1 quadratic knapsack problem. European Journal of Operational Research 92:310–325

Caprara A, Pisinger D, Toth P (1999) Exact solution of the quadratic knapsack problem. INFORMS Journal on Computing 11:125–137

Erkut E (1990) Discrete p-dispersion problem. European Journal of Operational Research 16:48–60

Fischetti M, Salazar Gonzalez JJ, Toth P (1995) The symmetric generalized traveling salesman polytope. Networks 26:113–123.

Fischetti M, Salazar Gonzalez JJ, Toth P (1997) A branch-and-cut algorithm for the symmetric generalized traveling salesman problem, Operations Research 45:378–394

Fischetti M, Salazar Gonzalez JJ, Toth P (1998) Solving the orienteering problem through branch-and-cut. INFORMS Journal on Computing, 10:133–148

Gendreau M, Labbe M, Laporte G (1995) Efficient heuristics for the design of ring networks. Telecommunication Systems—Modeling, Analysis, Design and Management 4:177–188

Gendreau M, Laporte G, Semet F (1998) A branch-and-cut algorithm for the undirected selective traveling salesman problem. Networks 32:263–273

Gouveia L, Manuel Pires J (2001) Models for a Steiner ring network design problem with revenues. European Journal of Operational Research 133:21–31

Johnson EL, Mehrotra A, Nemhauser GL (1993) Min-cut clustering. Mathematical Programming 62:133–151

Laporte G, Martello S (1990) The selective travelling salesman problem. Discrete Applied Mathematics 26:193–207

Nemhauser GL, Wolsey LA (1988) Integer and combinatorial optimization, Wiley & Sons

Pisinger D (1999) Exact solution of p-dispersion problems. DIKU-rapport 14

Stidsen T, Thomadsen T (2005) Hierarchical ring networks using branch-and-price. Telecommunication Systems 29(1):61–76