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Improved Approximation Algorithms for Maximum Graph Partitioning Problems

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Abstract. We consider the design of approximation algorithms for a number of maximum graph partitioning problems, among others MAX-k-CUT, MAX-k-DENSE-SUBGRAPH, and MAX-k-DIRECTED-UNCUT. We present a new version of the semidefnite relaxation scheme along with a better analysis, extending work of Halperin and Zwick (2002). This leads to an improvement over known approximation factors for such problems. The key to the improvement is the following new technique: It was already observed by Han et al. (2002) that a parameter-driven choice of the random hyperplane can lead to better approximation factors than obtained by Goemans and Williamson (1995). But it remained difficult to find a "good" set of parameters. In this paper, we analyze random hyperplanes depending on several new parameters. We prove that a sub-optimal choice of these parameters can be obtained by the solution of a linear program which leads to the desired improvement of the approximation factors. In this fashion a more systematic analysis of the semidefinite relaxation scheme is obtained.

Keywords: maximum graph partitioning, approximation factor, semidefinite programming

1. Introduction

For a directed graph $G = (V, E)$ with $|V| = n$ and a non-negative weight $\omega_{i,j}$ on each edge $(i, j) \in E$, such that $\omega_{i,j}$ is not identically zero on all edges, and for $0 < \sigma := \frac{k}{n} < 1$ we consider the following problems:

- MAX-k-CUT: determine a subset $S \subseteq V$ of k vertices such that the total weight ω^* of the edges connecting S and $V\backslash S$ or connecting $V\backslash S$ and S is maximized.[1]
- MAX-k-UNCUT: determine a subset $S \subseteq V$ of k vertices such that the total weight ω^* of the edges of the subgraphs induced by S and induced by $V\backslash S$ is maximized.
- MAX-k-DIRECTED-CUT: determine a subset $S \subseteq V$ of k vertices such that the total weight ω^* of the edges connecting S and $V\backslash S$ is maximized.
- MAX-k-DIRECTED-UNCUT: determine a subset $S \subseteq V$ of k vertices such that the total weight ω^* of the edges of the subgraphs induced by S and induced by $V\backslash S$ plus the edge weights connecting $V\backslash S$ and S is maximized.
- MAX-k-DENSE-SUBGRAPH: determine a subset $S \subseteq V$ of k vertices such that the total weight ω^* of the edges of the subgraph induced by S is maximized.
- MAX-k-VERTEX-COVER: determine a subset $S \subseteq V$ of k vertices such that the total weight ω^* of the edges touching S is maximized.

Table 1. Examples for the improved approximation factors.

Problem	σ	Prev.	Our method
MAX-k-CUT	0.3	0.527	0.567
MAX-k-UNCUT	0.4	0.5258	0.5973
MAX-k-DIRECTED-CUT	0.5	0.644	0.6507
MAX-k-DIRECTED-UNCUT	0.5	0.811	0.8164
MAX-k-DENSE-SUBGRAPH	0.2	0.2008	0.2664
MAX-k-VERTEX-COVER	0.6	0.8453	0.8784

As all these problems are NP-hard, we are interested in approximating the optimal solution to these problems within a factor of $0 \leq \varrho \leq 1$. Goemans and Williamson (1995) showed in their pioneer paper that via the semidefinite programming (SDP) relaxation an approximation factor of 0.878 can be proved for the MAX-CUT problem. Stimulated by their work, many authors have considered only one or two of the six above problems (see Ageev and Sviridenko (1999) for MAX-k-CUT, Frieze and Jerrum (1997) and Ye (2001) for MAX-$\frac{n}{2}$-CUT, Ageev et al. (2001) for MAX-k-DIRECTED-CUT, Ye and Zhang (1999) for MAX-$\frac{n}{2}$-UNCUT and MAX-$\frac{n}{2}$-DENSE-SUBGRAPH, Asahiro et al. (2000), Feige et al. (2001), Feige and Seltser (1997), Goemans and Williamson (1995), and Srivastav and Wolf (1998, 1999) for MAX-k-DENSE-SUBGRAPH and Ageev and Sviridenko (1999) for MAX-k-VERTEX-COVER). Feige and Langberg (2001) improved on known special and global approximation factors for the four undirected problems with some new techniques based on semidefinite programming. Their paper contains also a nice summary of known results. Han et al. (2002) also applied semidefinite programming to these four problems and in most cases they managed to obtain better approximation factors than previously known. Halperin and Zwick (2002) used more general methods for the balanced version ($\sigma = \frac{1}{2}$) and in this case achieved substantially improved approximation factors for all six problems above.

In this paper we give an algorithm for the six problems, generalizing the approach of Halperin and Zwick, resp. of Han et al. (2002) by introducing new parameters which enlarge the region of the semidefinite-programming relaxation. This gives a new version of the semidefinite relaxation scheme (Algorithm *Graph Partitioning*, Section 2). We show that the expectation of the approximation factors depend on a set of parameters, which are used in the algorithm, in a *non-linear* way (Lemma 2). The key observation is that a sub-optimal choice of these parameters can be determined by a finite *linear program* (Section 5.1). By discretizing the other parameters, we finally obtain a choice leading to improvements over known approximation guarantees. Here are some examples for which the improvement is significant (Table 1) (comprehensive tables can be found in Section 6).

In summary, we see that our technique of combining the analysis of the random hyperplane with mathematical programming leads to improvements over many previously known approximation factors for the maximization problems considered in this paper. This shows that a more systematic analysis of the semidefinite relaxation scheme gives better approximation guarantees and opens room for further improvements, if better methods for choosing an optimal parameter set can be designed.

Table 2. Parameters for the maximum graph partitioning problems.

Problem	a_1	a_2	a_3	a_4
MAX-k-CUT	0	1	1	0
MAX-k-UNCUT	1	0	0	1
MAX-k-DIRECTED-CUT	0	1	0	0
MAX-k-DIRECTED-UNCUT	1	0	1	1
MAX-k-DENSE-SUBGRAPH	1	0	0	0
MAX-k-VERTEX-COVER	1	1	1	0

2. The algorithm

For $S \subseteq V$ the set of edges E can be divided in the following way:

$$\begin{aligned} E &= S_1 \dot{\cup} S_2 \dot{\cup} S_3 \dot{\cup} S_4, \\ S_1 &= \{(i, j) \mid i, j \in S\}, \\ S_2 &= \{(i, j) \mid i \in S, j \in V \backslash S\}, \\ S_3 &= \{(i, j) \mid i \in V \backslash S, j \in S\}, \\ S_4 &= \{(i, j) \mid i, j \in V \backslash S\}, \end{aligned}$$

As we will see, we distinguish the six problems MAX-k-CUT, MAX-k-UNCUT, MAX-k-DIRECTED-CUT, MAX-k-DIRECTED-UNCUT, MAX-k-DENSE-SUBGRAPH, MAX-k-VERTEX-COVER by four $\{0, 1\}$ parameters a_1, a_2, a_3, a_4. All these problems maximize the sum of a subset of the four edge classes S_1, S_2, S_3, S_4.

For $i = 1, 2, 3, 4$ we define a_i as 1, if the problem maximizes the edge weights of S_i, and 0 otherwise. The following values a_1, a_2, a_3, a_4 lead to the specific problems (see Table 2).

For $F \subseteq E$ we define $\omega(F) = \sum_{(i,j)\in F} \omega_{ij}$ and for $S \subseteq V$:

$$\omega_{a_1,a_2,a_3,a_4}(S) := a_1\omega(S_1) + a_2\omega(S_2) + a_3\omega(S_3) + a_4\omega(S_4).$$

The optimization problem considered in this paper is the following.

General Maximization Problem:

$$\max_{s \subseteq V, |S|=k} \omega_{a_1,a_2,a_3,a_4}(S) \tag{1}$$

Let OPT (a_1, a_2, a_3, a_4) be the value of an optimal solution of (1). Our aim is to design a randomized polynomial-time algorithm which returns a solution of value at least $\varrho \cdot$ OPT $(a_1, a_2, a_3, a_4, \sigma)$, where $\varrho = \varrho(a_1, a_2, a_3, a_4, \sigma)$ is the so-called approximation factor $0 \leq \varrho \leq 1$. In fact, we will show that the expected value of ϱ is large.

In the algorithm we give a formulation of the general maximization problem (1) as a semidefinite program, generalizing Halperin and Zwick (2002).

Algorithm *Graph Partitioning*

Input: A weighted directed graph $G = (V, E)$ with $|V| = n$, $0 < \sigma < 1$, a maximum graph partitioning problem with parameters a_1, a_2, a_3, a_4.
Output: A set S of k vertices with large $\omega_{a_1,a_2,a_3,a_4}(S)$.

1. *Relaxation*
   We solve the following semidefinite program:
   Maximize $\sum_{1\le i\ne j\le n} \frac{1}{4}\omega_{ij}\ [(a_1+a_2+a_3+a_4)+(a_1+a_2-a_3-a_4)X_{i0}+(a_1-a_2+a_3-a_4)X_{j0}+(a_1-a_2-a_3+a_4)X_{ij}]$ with the optimal value ω^* subject to the constraints

   (a) $\sum_{i=1}^{n} X_{i0} = 2k - n$
   (b) $\sum_{1\le i,j\le n} X_{ij} = (2k-n)^2$
   (c) $X_{ii} = 1$ for $i = 0, 1, \ldots, n$
   (d) $X \in \mathbb{R}^{n+1,n+1}$ is positive semidefinite and symmetric
   (e) $X_{ij} + X_{il} + X_{jl} \ge -1$ for $0 \le i, j, l \le n$
   (f) $X_{ij} - X_{il} - X_{jl} \ge -1$ for $0 \le i, j, l \le n$
   From b), c) and d) it follows:
   (g) $\sum_{1\le i<j\le n} X_{ij} = \frac{1}{2}((2k-n)^2 - n)$

   We repeat the following four steps polynomially often and output the best subset.
2. *Randomized rounding*

   - Choose parameters $0 \le \theta, \vartheta \le 1$ and $-1 \le \kappa \le 1$ (note that for every problem and for each σ we choose different parameters).
   - Choose a positive semidefinite symmetric matrix $Y = Z^T Z \in \mathbb{R}^{n+1,n+1}$, depending on $\theta, \vartheta, \kappa$ as follows:
     Put $Y := \theta L + (1-\theta)P$, where we define $L = (l_{ij})_{0\le i,j\le n}$ and $P = (P_{ij})_{0\le i,j\le n}$ by

$$(l_{ij})_{0\le i,j\le n} = \begin{cases} 1 & \text{for } i = j \\ \vartheta X_{0i} & \text{for } i \ne 0,\ j = 0 \\ \vartheta X_{0j} & \text{for } i = 0,\ j \ne 0 \\ \vartheta X_{ij} \text{ or } X_{ij} \text{ or } \vartheta^2 X_{ij} & \text{for } 1 \le i \ne j \le n \end{cases}$$

$$(p_{ij})_{0\le i,j\le n} = \begin{cases} 1 & \text{for } i = j \\ \kappa & \text{for } i = 0,\ j \ne 0 \vee i \ne 0,\ j = 0 \\ \kappa,\ \text{if}\,\kappa \ge 0\,\text{or}\,1\,\text{or}\,\kappa^2 & \text{for } 1 \le i \ne j \le n \end{cases}$$

     We can write the non diagonal elements of Y for $0 \le i \ne j \le n$ as

$$Y_{ij} = \begin{cases} d_1 X_{ij} + e_1, & \text{if } i = 0 \vee j = 0 \\ d_2 X_{ij} + e_2, & \text{otherwise} \end{cases}$$

with

$$d_1 = \theta\vartheta; \tag{2}$$
$$e_1 = (1-\theta)\kappa; \tag{3}$$
$$d_2 = \theta\vartheta, \theta, \theta\vartheta^2; \tag{4}$$
$$e_2 = (1-\theta)\kappa \text{ (if } \kappa \geq 0), 1-\theta, (1-\theta)\kappa^2 \tag{5}$$

Hence: $-1 \leq e_1 \leq 1; 0 \leq d_1, d_2, e_2 \leq 1$.
(It is easy to show that Y is a positive semidefinite symmetric matrix.)
- We choose $\bar{u}$ with $\bar{u}_i \in N(0, 1)$ for $i = 0, 1, \ldots, n$ and let $u = Z\bar{u}$.
- For $i = 1, \ldots, n$ let $\hat{x}_i = 1$, if $u_i \geq 0$ and -1 otherwise and let $S = \{i \geq 1 \mid \hat{x}_i = 1\}$ (see Bertsimas and Ye, 1998).

3. *Linear randomized rounding*
   - Choose a parameter $0 \leq \nu \leq 1$ (again for every problem and for each σ we choose a different parameter).
   - With probability ν we overrule the choice of S made above, and for each $i \in V$, put i into S, independently, with probability $(1 + X_{i0})/2$ and into $V \backslash S$ otherwise.
4. *Size adjusting*
   (a) If the problem is symmetric (MAX-k-CUT or MAX-k-UNCUT):
   - If $k \leq |S| < \frac{n}{2}$, we remove uniformly at random $|S| - k$ vertices from S.
   - If $|S| < k$, we add uniformly at random $k - |S|$ vertices to S.
   - If $\frac{n}{2} \leq |S| < n - k$, we add uniformly at random $n - k - |S|$ vertices to S.
   - If $|S| \geq n - k$, we remove uniformly at random $|S| - n + k$ vertices from S.

   (b) If the problem is not symmetric:
   - If $|S| \geq k$, we remove uniformly at random $|S| - k$ vertices from S.
   - If $|S| < k$, we add uniformly at random $k - |S|$ vertices to S.
5. *Flipping* (only for MAX-$\frac{n}{2}$-DIRECTED-CUT, MAX-$\frac{n}{2}$-DIRECTED-UNCUT, MAX-$\frac{n}{2}$-DENSE-SUBGRAPH and MAX-$\frac{n}{2}$-VERTEX-COVER).

   If $\omega_{a_1,a_2,a_3,a_4}(V \backslash S) > \omega_{a_1,a_2,a_3,a_4}(S)$, we output $V \backslash S$, otherwise S.

3. Computation of the approximation factors

3.1. Main result

The main results are shown in the tables containing the approximation factors for the different problems. Nevertheless, let us state them also in a formal way:

Theorem 1 (Main Theorem). *The expected ratio $\omega_{a_1,a_2,a_3,a_4}(S)$/OPT of the approximation factors for the problems MAX-k-CUT, MAX-k-UNCUT, MAX-k-DIRECTED-CUT,* *MAX-k-DIRECTED-UNCUT, MAX-k-DENSE-SUBGRAPH and MAX-k-VERTEX-COVER is bounded from below by the minimum of the solutions of the linear programs in* (13). *Solving these linear programs lead to the approximation factors shown in the Tables* 6 *to* 13, *Section* 6.

We denote the sets S after the steps 2, 3, 4, 5 of the algorithm by S', S'', S''', S'''' $(= S)$ and define $\delta := \frac{|S''|}{n}$. We want to compute $\varrho := \frac{\omega_{a_1,a_2,a_3,a_4}(S)}{\omega^*}$. For $x_1 \in \mathbb{R}, x_2 \in \mathbb{R}_0^+$ we consider the function:

$$y(x_1, x_2) = \frac{\omega_{a_1,a_2,a_3,a_4}(S'')}{\omega^*} + x_1 \frac{|S''|}{n} + x_2 \frac{|S''|(n - |S''|)}{n^2}.$$

(The case $x_2 < 0$ could also be considered, but as it does not lead to any progress, we omit it.) We estimate the expected values of the three terms. This is done in the following main lemma (proved in Section 4).

Lemma 2 (Main Lemma)**.** *For $n \to \infty$ there are constants $\alpha, \beta^+, \beta^-, \gamma$ with*:
(a) $E[\frac{\omega_{a_1,a_2,a_3,a_4}(S'')}{\omega^*}] \geq \alpha(\theta, \vartheta, \kappa, \nu)$
(b) $E[\frac{|S''|}{n}] \geq \beta^+(\sigma, \theta, \vartheta, \kappa, \nu)$
(c) $E[\frac{|S''|}{n}] \leq \beta^-(\sigma, \theta, \vartheta, \kappa, \nu)$
(d) $E[\frac{|S''|(n-|S''|)}{n^2}] \geq \gamma(\sigma, \theta, \vartheta, \kappa, \nu)$

Remark 3. The variables $\alpha, \beta^+, \beta^-, \gamma$ implicitly also depend on the parameters d_1, e_1, d_2, e_2 and so on the positive semidefinite symmetric matrix Y, as d_1, e_1, d_2, e_2 are functions of the parameters $\theta, \vartheta, \kappa$ (see (2)–(5)).

3.2. *Analyzing the function y(x₁, x₂)*

For $x_1 \geq 0$ define $\beta(\sigma, \theta, \vartheta, \kappa, \nu)$ as $\beta^+(\sigma, \theta, \vartheta, \kappa, \nu)$ and otherwise as $\beta^-(\sigma, \theta, \vartheta, \kappa, \nu)$. As we repeat the steps 2 and 3 of the algorithm polynomially often, the function $y(x_1, x_2)$ is its expected value, up to a factor of $1 - \epsilon$ which can be neglected. By Lemma 2 we get:

$$\begin{aligned}
&\frac{\omega_{a_1,a_2,a_3,a_4}(S'')}{\omega^*} + x_1 \frac{|S''|}{n} + x_2 \frac{|S''|(n - |S''|)}{n^2} \\
&\geq E\left[\frac{\omega_{a_1,a_2,a_3,a_4}(S'')}{\omega^*}\right] + E\left[x_1 \frac{|S''|}{n}\right] + E\left[x_2 \frac{|S''|(n - |S''|)}{n^2}\right] \\
&\geq \alpha(\theta, \vartheta, \kappa, \nu) + x_1 \beta(\sigma, \theta, \vartheta, \kappa, \nu) + x_2 \gamma(\sigma, \theta, \vartheta, \kappa, \nu)
\end{aligned}$$

and so

$$\begin{aligned}
&\frac{\omega_{a_1,a_2,a_3,a_4}(S'')}{\omega^*} \\
&\geq \alpha(\theta, \vartheta, \kappa, \nu) + x_1(\beta(\sigma, \theta, \vartheta, \kappa, \nu) - \delta) + x_2(\gamma(\sigma, \theta, \vartheta, \kappa, \nu) - \delta(1 - \delta)) \\
&=: h(\delta, \sigma, \theta, \vartheta, \kappa, \nu, x_1, x_2) \qquad (6)
\end{aligned}$$

With $\lambda_i := \frac{\omega(S_i'')}{\omega^*}$ for $i = 1, 2, 3, 4$ we obtain:

$$a_1\lambda_1 + a_2\lambda_2 + a_3\lambda_3 + a_4\lambda_4 = \frac{a_1\omega(S_1'') + a_2\omega(S_2'') + a_3\omega(S_3'') + a_4\omega(S_4'')}{\omega^*}$$
$$= \frac{\omega_{a_1,a_2,a_3,a_4}(S'')}{\omega^*} \geq h(\delta, \sigma, \theta, \vartheta, \kappa, \nu, x_1, x_2) \tag{7}$$

So we have by the conditions (e) and (f) of step 1 of the algorithm,

$$\omega(S_1'') + \omega(S_2'') + \omega(S_3'') + \omega(S_4'') = \omega(E) = \sum_{1\leq i\neq j\leq n} \omega_{ij}$$
$$\geq \sum_{1\leq i\neq j\leq n} \frac{1}{4}\omega_{ij}[(a_1 + a_2 + a_3 + a_4) + (a_1 + a_2 - a_3 - a_4)X_{i0}$$
$$+ (a_1 - a_2 + a_3 - a_4)X_{j0} + (a_1 - a_2 - a_3 + a_4)X_{ij}] = \omega^*$$

So we have: $\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 \geq 1$ (8)

(a) If the problem is symmetric, we can assume $\sigma \leq \frac{1}{2}$ w.l.o.g.

Case 1: $\sigma \leq \delta < \frac{1}{2}$ and $k \leq |S''| \leq \frac{n}{2}$, respectively.

Each element from S'' is added to S''' with probability $p := \frac{\sigma}{\delta}$. Then:

$$\begin{pmatrix} E[\omega(S_1''')] \\ E[\omega(S_2''')] \\ E[\omega(S_3''')] \\ E[\omega(S_4''')] \end{pmatrix} = \underbrace{\begin{pmatrix} p^2 & 0 & 0 & 0 \\ p(1-p) & p & 0 & 0 \\ p(1-p) & 0 & p & 0 \\ (1-p)^2 & 1-p & 1-p & 1 \end{pmatrix}}_{M_1(p)} \begin{pmatrix} \omega(S_1'') \\ \omega(S_2'') \\ \omega(S_3'') \\ \omega(S_4'') \end{pmatrix} \tag{9}$$

Case 2: $0 \leq \delta < \sigma$ and $0 \leq |S''| < k$, respectively.

Similarly to case 1 each element from $V\backslash S''$ is added to $V\backslash S'''$ with probability $q := \frac{1-\sigma}{1-\delta}$. Consequently:

$$\begin{pmatrix} E[\omega(S_1''')] \\ E[\omega(S_2''')] \\ E[\omega(S_3''')] \\ E[\omega(S_4''')] \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 1-q & 1-q & (1-q)^2 \\ 0 & q & 0 & q(1-q) \\ 0 & 0 & q & q(1-q) \\ 0 & 0 & 0 & q^2 \end{pmatrix}}_{M_2(q)} \begin{pmatrix} \omega(S_1'') \\ \omega(S_2'') \\ \omega(S_3'') \\ \omega(S_4'') \end{pmatrix} \tag{10}$$

Case 3: $\frac{1}{2} \leq \delta < 1 - \sigma$ and $\frac{n}{2} \leq |S''| < n - k$, respectively.

Each element from $V\backslash S''$ is added to $V\backslash S'''$ with probability $r := \frac{\sigma}{1-\delta}$. We get the same condition as in case 2 with r instead of q.

Case 4: $1-\sigma \le \delta \le 1$ and $n-k \le |S''| \le n$, respectively.

Each element from S'' is added to S''' with probability $s := \frac{1-\sigma}{\delta}$. We get the same condition as in case 1 with s instead of p.

(b) If the problem is not symmetric, the cases 3 and 4 are omitted, and the condition of case 1 is replaced by $\sigma \le \delta \le 1$ and $k \le |S''| \le n$, respectively.

$$\text{With } M(\delta,\sigma) := \begin{cases} M_1\left(\frac{\sigma}{\delta}\right), & \text{if } \sigma \le \delta < \frac{1}{2} \\ M_2\left(\frac{1-\sigma}{1-\delta}\right), & \text{if } 0 \le \delta < \sigma \\ M_2\left(\frac{\sigma}{1-\delta}\right), & \text{if } \frac{1}{2} \le \delta < 1-\sigma \\ M_1\left(\frac{1-\sigma}{\delta}\right), & \text{if } 1-\sigma \le \delta \le 1 \end{cases} \quad \text{in the symmetric case}$$

and

$$M(\delta,\sigma) := \begin{cases} M_1\left(\frac{\sigma}{\delta}\right), & \text{if } \sigma \le \delta \le 1 \\ M_2\left(\frac{1-\sigma}{1-\delta}\right), & \text{if } 0 \le \delta < \sigma \end{cases}$$

in the asymmetric case we obtain:

$$\begin{aligned} &E\left[\frac{\omega_{a_1,a_2,a_3,a_4(S''')}}{\omega^*}\right] \\ &= \frac{a_1 E[\omega(S_1''')] + a_2 E[\omega(S_2''')] + a_3 E[\omega(S_3''')] + a_4 E[\omega(S_4''')]}{\omega^*} \\ &= (a_1\ a_2\ a_3\ a_4) \cdot M(\delta,\sigma) \cdot (\lambda_1\ \lambda_2\ \lambda_3\ \lambda_4)^T \qquad (11) \\ &=: f_{a_1,a_2,a_3,a_4}(\delta,\sigma,\lambda_1,\lambda_2,\lambda_3,\lambda_4) \qquad (12) \end{aligned}$$

As $\lambda_i \ge 0$ holds for $i = 1, 2, 3, 4$, we get with (7) and (8):
The expected approximation factor of Algorithm *Graph Partitioning* is:

$$\min_{0\le\delta\le1} \left[\begin{array}{c} \min \quad z \\ \text{s.t.} \\ a_1\lambda_1 + a_2\lambda_2 + a_3\lambda_3 + a_4\lambda_4 \ge h(\delta,\sigma,\theta,\vartheta,\kappa,\nu,x_1,x_2) \\ \lambda_1+\lambda_2+\lambda_3+\lambda_4 \ge 1 \\ 0 \le \lambda_1,\lambda_2,\lambda_3,\lambda_4 \\ z \ge f_{a_1,a_2,a_3,a_4}(\delta,\sigma,\lambda_1,\lambda_2,\lambda_3,\lambda_4) \\ [\text{for MAX-}\tfrac{n}{2}\text{-DC, MAX-}\tfrac{n}{2}\text{-DU, MAX-}\tfrac{n}{2}\text{-DS, MAX-}\tfrac{n}{2}\text{-VC}: \\ z \ge f_{a_4,a_3,a_2,a_1}(\delta,\sigma,\lambda_1,\lambda_2,\lambda_3,\lambda_4)] \end{array}\right] \qquad (13)$$

Note that for fixed δ, σ and constants $\theta, \vartheta, \kappa, \nu, x_1, x_2$ the last inner minimization problem is a linear program in the variables $z, \lambda_1, \lambda_2, \lambda_3, \lambda_4$.

4. Proof of main Lemma 2

4.1. *Lemma 2, Part (a)*

Define

$$\begin{aligned} &c_{a_1,a_2,a_3,a_4}(\theta, \vartheta, \kappa, \nu, x, y, z) \\ &\quad := (a_1 + a_2 + a_3 + a_4) + (a_1 + a_2 - a_3 - a_4)\left(\nu x + (1-\nu)\frac{2}{\pi}\arcsin(d_1 x + e_1)\right) \\ &\qquad + (a_1 - a_2 + a_3 - a_4)\left(\nu y + (1-\nu)\frac{2}{\pi}\arcsin(d_1 y + e_1)\right) \\ &\qquad + (a_1 - a_2 - a_3 + a_4)\left(\nu xy + (1-\nu)\frac{2}{\pi}\arcsin(d_2 z + e_2)\right) \\ &d_{a_1,a_2,a_3,a_4}(x, y, z) := (a_1 + a_2 + a_3 + a_4) + (a_1 + a_2 - a_3 - a_4)x \\ &\qquad\qquad + (a_1 - a_2 + a_3 - a_4)y + (a_1 - a_2 - a_3 + a_4)z \end{aligned}$$

with $(x, y, z) = (X_{i0}, X_{j0}, X_{ij})$ for $1 \le i \ne j \le n$.

Further we define:

$$\alpha(\theta, \vartheta, \kappa, \nu) := \min_{x,y,z;d>0} \frac{c_{a_1,a_2,a_3,a_4}(\theta, \vartheta, \kappa, \nu, x, y, z)}{d_{a_1,a_2,a_3,a_4}(x, y, z)} \in \mathbb{R}_0^+$$

subject to the constraints $(x, y, z) = (X_{i0}, X_{j0}, X_{ij}$ for $1 \le i \ne j \le n$.

The proof of (a) is the same as in Halperin and Zwick (2002), using a slightly different notation and the more general class of positive semidefinite matrices.

4.2. *Two auxiliary lemmas*

The following lemma, which is also used in Halperin and Zwick (2002), follows straightforward from the definitions of concave and convex.

Lemma 4. *Let $f : [a, b] \to \mathbb{R}$ be convex and concave, respectively. It holds for $x \le x' \le y' \le y$ and $x, y \in [a, b]$ with $x + y = x' + y'$:*

$$f(x') + f(y') \le f(x) + f(y) \quad \textit{and} \quad f(x) + f(y) \le f(x') + f(y'), \ \textit{respectively.}$$

Lemma 5. *Let $(d, e) = (d_1, e_1)$ or $(d, e) = (d_2, e_2)$. It holds for $g_1(x) := \arcsin(dx + e)$ and $g_2(x) := \arccos(dx + e)$:*

(a) *$g_1(x)$ and $g_2(x)$ are defined in $[-1, 1]$.*

(b) *If* $-\frac{e}{d} \leq -1$, $g_1(x)$ *is convex in* $[-1, 1]$.
If $-1 < -\frac{e}{d} < 1$, $g_1(x)$ *is concave in* $[-1, -\frac{e}{d}]$ *and convex in* $[-\frac{e}{d}, 1]$.
If $-\frac{e}{d} \geq 1$, $g_1(x)$ *is concave in* $[-1, 1]$.
(c) *If* $-\frac{e}{d} \leq -1$, $g_2(x)$ *is concave in* $[-1, 1]$.
If $-1 < -\frac{e}{d} < 1$, $g_2(x)$ *is convex in* $[-1, -\frac{e}{d}]$ *and concave in* $[-\frac{e}{d}, 1]$.
If $-\frac{e}{d} \geq 1$, $g_2(x)$ *is convex in* $[-1, 1]$.

Proof:

(a) From the definition of d_1, e_1, d_2, e_2 it follows for $x \in [-1, 1] : -1 \leq dx + e \leq 1$.
(b) We have:

$$g_1'(x) = \frac{d}{\sqrt{1-(dx+e)^2}} = d \cdot (1-(dx+e)^2)^{-\frac{1}{2}}$$

$$g_1''(x) = \frac{1}{2} d \cdot (2d^2x + 2de) \cdot (1-(dx+e)^2)^{-\frac{3}{2}}$$

$$= d^2 \cdot (dx+e) \cdot (1-(dx+e)^2)^{-\frac{3}{2}} = \frac{d^2 \cdot (dx+e)}{\sqrt{1-(dx+e)^2}^3}$$

The assertion follows, as g_1 is convex in x, if and only if $g_1''(x) \geq 0$ and is concave in x, if and only if $g_1''(x) \leq 0$.
(c) Follows from b) with $g_2'(x) = -g_1'(x)$. □

4.3. Lemma 2, Parts (b) and (c)

Lemma 6. *For* $n \to \infty$ *it holds:*

(a)

$$E\left[\frac{|S''|}{n}\right] = (1-\nu)E\left[\frac{|S'|}{n}\right] + \nu\sigma$$

(b) *If* $d_1 = 0$, *we have:* $E[\frac{|S'|}{n}] = \frac{1}{2} + \frac{1}{\pi}\arcsin(e_1)$.

Let $d_1 > 0$. *It holds for* $g_1(x) := \arcsin(d_1x + e_1)$*:*

(c)

$$E\left[\frac{|S'|}{n}\right] \geq \frac{1}{2} + \frac{1}{\pi} \min_{q' \in [-\frac{e_1}{d_1}, 1]} \left(\left(-\frac{2\sigma}{q'+1} + 1\right) \cdot g_1(-1) + \left(\frac{2\sigma}{q'+1}\right) \cdot g_1(q')\right), \quad \text{if} -1 < -\frac{e_1}{d_1} < 1$$

(d)

$$E\left[\frac{|S'|}{n}\right] \leq \begin{cases} \frac{1}{2} + \frac{1}{\pi} \max_{q' \in [-1, -\frac{e_1}{d_1}]} \left(\frac{2\sigma - 2}{q' - 1} \cdot g_1(q') + \left(1 - \frac{2\sigma - 2}{q' - 1}\right) \cdot g_1(1) \right), \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad \textit{if } -1 < -\frac{e_1}{d_1} < 1 \\ \frac{1}{2} + \frac{1}{\pi} \cdot g_1(2\sigma - 1), \quad \text{if } -\frac{e_1}{d_1} \geq 1 \end{cases}$$

We could show a lower bound in c) also for $-\frac{e_1}{d_1} \leq -1$ and $-\frac{e_1}{d_1} \geq 1$ and an upper bound in d) also for $-\frac{e_1}{d_1} \leq -1$, but since these cases do not lead to any progress, we omit them.

Define $\beta^+(\sigma, \theta, \vartheta, \kappa, \nu)$ as the lower bound on $E[\frac{|S''|}{n}]$ resulting from Lemma 6 (a), (b), (c) and define $\beta^-(\sigma, \theta, \vartheta, \kappa, \nu)$ as the upper bound on $E[\frac{|S''|}{n}]$ resulting from Lemma 6 (a), (b), and (d).

Proof:

(a) For $1 \leq i \leq n$ let $\Pr(i)$ be the probability of putting i into S in step 3 of the algorithm (Linear Randomized Rounding). Then,

$$\begin{aligned} E\left[\frac{|S''|}{n}\right] &= (1-\nu)E\left[\frac{|S'|}{n}\right] + \nu\frac{1}{n}\sum_{i=1}^{n}\Pr(i) \\ &= (1-\nu)E\left[\frac{|S'|}{n}\right] + \nu\frac{1}{n}\sum_{i=1}^{n}\frac{1+X_{i0}}{2} \\ &= (1-\nu)E\left[\frac{|S'|}{n}\right] + \nu\sigma \end{aligned}$$

(b), (c), and (d) Let d_1 be arbitrary. Using the new positive semidefinite matrices, we can show as by Halperin and Zwick (2002):

$$E\left[\frac{|S'|}{n}\right] = \frac{1}{n}\sum_{i=1}^{n}\left(\frac{1}{2} + \frac{1}{\pi}g_1(X_{i0})\right) = \frac{1}{2} + \frac{1}{\pi n}\sum_{i=1}^{n} g_1(X_{i0})$$

(b) follows immediately from $d_1 = 0$.

(c) Let $-1 < -\frac{e_1}{d_1} < 1$.
By Lemma 5 b) $g_1(x)$ is concave in $[-1, -\frac{e_1}{d_1}]$ and convex in $[-\frac{e_1}{d_1}, 1]$.
Let X_{i0}, X_{j0} be in the interval $[-1, -\frac{e_1}{d_1}]$ with $X_{i0} \leq X_{j0}$. So, $X_{i0} = -1$ or $X_{j0} = -\frac{e_1}{d_1}$, since otherwise, by Lemma 4 we could decrease the above sum by moving X_{i0} and X_{j0} further apart. So there can only be one summand X_{i0} in the interval $[-1, -\frac{e_1}{d_1}]$ with

$X_{i0} \neq -1$ and $X_{i0} \neq -\frac{e_1}{d_1}$. As each summand makes only a difference of $O(\frac{1}{n})$, we can neglect it.
Let X_{i0} and X_{j0} be in the interval $[-\frac{e_1}{d_1}, 1]$ with $X_{i0} \leq X_{j0}$. So we have $X_{i0} = X_{j0}$, since otherwise, by Lemma 4 we could decrease the above sum by moving X_{i0} and X_{j0} closer together. So all summands X_{i0} in the interval $[-\frac{e_1}{d_1}, 1]$ are equal. So we can find a $p \in [0, 1]$ and a $q \in [-\frac{e_1}{d_1}, 1]$ with

$$E\left[\frac{|S'|}{n}\right] \geq \frac{1}{2} + \frac{1}{\pi n}(p \cdot n \cdot g_1(-1) + (1-p) \cdot n \cdot g_1(q)) \\ = \frac{1}{2} + \frac{1}{\pi}(p \cdot g_1(-1) + (1-p) \cdot g_1(q)). \tag{14}$$

With the above notation we obtain from a) of step 1 of the algorithm:

$$-p \cdot n + (1-p) \cdot n \cdot q = 2k - n$$

which is equivalent to

$$p = -\frac{2\sigma}{q+1} + 1.$$

Using (14) we see that there is a $q \in [-\frac{e_1}{d_1}, 1]$ with:

$$E\left[\frac{|S'|}{n}\right] \geq \frac{1}{2} + \frac{1}{\pi}\left(\left(-\frac{2\sigma}{q+1} + 1\right) \cdot g_1(-1) + \left(\frac{2\sigma}{q+1}\right) \cdot g_1(q)\right) \\ \geq \frac{1}{2} + \frac{1}{\pi} \min_{q' \in [-\frac{e_1}{d_1}, 1]} \left(\left(-\frac{2\sigma}{q'+1} + 1\right) \cdot g_1(-1) \\ + \left(\frac{2\sigma}{q'+1}\right) \cdot g_1(q')\right)$$

(d) *Case 1:* $-1 < -\frac{e_1}{d_1} < 1$.
Let X_{i0}, X_{j0} be in the interval $[-\frac{e_1}{d_1}, 1]$ with $X_{i0} \leq X_{j0}$. By arguing as above we obtain: $X_{i0} = -\frac{e_1}{d_1}$ or $X_{j0} = 1$. Let X_{i0} and X_{j0} be in the interval $[-1, -\frac{e_1}{d_1}]$ with $X_{i0} \leq X_{j0}$. We obtain $X_{i0} = X_{j0}$ and so we can find a $p \in [0, 1]$ and a $q \in [-1, -\frac{e_1}{d_1}]$ with

$$E\left[\frac{|S'|}{n}\right] \leq \frac{1}{2} + \frac{1}{\pi}(p \cdot g_1(q) + (1-p) \cdot g_1(1)) \tag{15}$$

Hence:

$$p \cdot n \cdot q + (1-p) \cdot n = 2k - n$$

which is equivalent to

$$p = \frac{2\sigma - 2}{q - 1}.$$

With (15) it follows for a $q \in [-1, -\frac{e_1}{d_1}]$:

$$E\left[\frac{|S'|}{n}\right] \leq \frac{1}{2} + \frac{1}{\pi}\left(\frac{2\sigma-2}{q-1} \cdot g_1(q) + \left(1 - \frac{2\sigma-2}{q-1}\right) \cdot g_1(1)\right)$$
$$\leq \frac{1}{2} + \frac{1}{\pi} \max_{q' \in [-1, -\frac{e_1}{d_1}]} \left(\frac{2\sigma-2}{q'-1} \cdot g_1(q') + \left(1 - \frac{2\sigma-2}{q'-1}\right) \cdot g_1(1)\right)$$

Case 2: $-\frac{e_1}{d_1} \geq 1$.

From Lemma 5 b) $g_1(x)$ is concave in $[-1, 1]$.
If X_{i0} and Xj_0 are in the interval $[-1, 1]$ with $X_{i0} \leq X_{j0}$, then $X_{i0} = X_{j0}$. So there is a $q \in [-1, 1]$ with

$$E\left[\frac{|S'|}{n}\right] \leq \frac{1}{2} + \frac{1}{\pi} g_1(g). \tag{16}$$

We have $n \cdot q = 2k - n$ and so $q = 2\sigma - 1$. By (16), we get:

$$E\left[\frac{|S'|}{n}\right] \leq \frac{1}{2} + \frac{1}{\pi} g_1(2\sigma - 1)$$

□

4.4. *Lemma 2, Part (d)*

Lemma 7. *For $n \to \infty$ we have:*

(a)

$$E\left[\frac{|S''|(n-|S''|)}{n^2}\right] = (1-\nu)E\left[\frac{|S'|(n-|S'|)}{n^2}\right] + \nu\sigma(1-\sigma)$$

(b) *If $d_2 = 0$, we have:* $E[\frac{|S'|(n-|S'|)}{n^2}] = \frac{1}{2\pi}\arccos(e_2)$.

(c) *Let* $d_2 > 0$. *Then for* $g_2(x) := \arccos(d_2 x + e_2)$,

$$E\left[\frac{|S'|(n-|S'|)}{n^2}\right] \geq \begin{cases} \frac{1}{2\pi}((-2\sigma^2+2\sigma)\cdot g_2(-1) + (1+2\sigma^2-2\sigma)\cdot g_2(1)), & \text{if } -\frac{e_2}{d_2} \leq -1 \\ \frac{1}{2\pi}\min_{q'\in[-1,-\frac{e_2}{d_2}]}\left\{\frac{4\sigma^2-4\sigma}{q'-1}\cdot g_2(q') + \left(1-\frac{4\sigma^2-4\sigma}{q'-1}\right)\cdot g_2(1)\right\} & \text{if } -\frac{e_2}{d_2} > -1 \end{cases}$$

(d) *Let* $\sigma = \frac{1}{2}$ *and* $k = \frac{n}{2}$, *respectively and* $e_2 = 0$. *If*

$$\min_{x\in[-\frac{1}{3},0]}\left\{4g_2(x) - 3g_2\left(\frac{4x-1}{3}\right) - g_2(1)\right\} \geq 0 \tag{17}$$

we have:

$$E\left[\frac{|S'|(n-|S'|)}{n^2}\right] \geq \frac{1}{4\pi}\min_{x\in[-1,-\frac{1}{3}]}\left\{g_2(x) + \frac{3x+3}{4}\cdot g_2\left(-\frac{1}{3}\right) + \frac{1-3x}{4}\cdot g_2(1)\right\}$$

Define $\gamma(\sigma,\theta,\vartheta,\kappa,\nu)$ as the lower bound on $E[\frac{|S''|(n-|S''|)}{n^2}]$ resulting from Lemma 7.

Proof: (a) With the same notation as in the proof of Lemma 6(a),

$$\begin{aligned}
&E\left[\frac{|S''|(n-|S''|)}{n^2}\right] \\
&= (1-\nu)E\left[\frac{|S'|(n-|S'|)}{n^2}\right] + \nu\frac{1}{n^2}\sum_{1\leq i\neq j\leq n}\Pr(i)\cdot(1-\Pr(j)) \\
&= (1-\nu)E\left[\frac{|S'|(n-|S'|)}{n^2}\right] + \nu\frac{1}{n^2}\sum_{1\leq i\neq j\leq n}\frac{1+X_{i0}}{2}\cdot\frac{1-X_{j0}}{2} \\
&= (1-\nu)E\left[\frac{|S'|(n-|S'|)}{n^2}\right] + \nu\frac{1}{n^2}\sum_{i=1}^{n}\frac{1+X_{i0}}{2}\cdot\sum_{j\neq i}\frac{1-X_{j0}}{2} \\
&\geq (1-\nu)E\left[\frac{|S'|(n-|S'|)}{n^2}\right] + \nu\frac{1}{n^2}\left(\frac{n}{2}+\frac{2k-n}{2}\right)\cdot\left(\sum_{j=1}^{n}\frac{1-X_{j_0}}{2}-1\right) \\
&= (1-\nu)E\left[\frac{|S'|(n-|S'|)}{n^2}\right] + \nu\sigma\frac{1}{n}\cdot(n-k-1) \\
&\overset{n\to\infty}{=}(1-\nu)E\left[\frac{|S'|(n-|S'|)}{n^2}\right] + \nu\sigma(1-\sigma)
\end{aligned}$$

(b), (c), and (d) Let d_2 be arbitrary. Using the new positive semidefinite matrices, we can show as in [11]:

$$E\left[\frac{|S'|(n-|S'|)}{n^2}\right] = \frac{1}{\pi n^2}\sum_{1\le i<j\le n} g_2(X_{ij})$$

(b) follows immediately from $d_2 = 0$ and $n \to \infty$.

(c) *Case 1:* $-\frac{e_2}{d_2} \le -1$.

By Lemma 5(c) $g_2(x)$ is concave in $[-1, 1]$.

Let X_{ij}, X_{st} be in the interval $[-1, 1]$ with $X_{ij} \le X_{st}$. Analogously, we obtain $X_{ij} = -1$ or $X_{st} = 1$. So there is a $p \in [0, 1]$ with

$$E\left[\frac{|S'|(n-|S'|)}{n^2}\right] \ge \frac{1}{2\pi}\left(1-\frac{1}{n}\right)(p\cdot g_2(-1+(1-p)\cdot g_2(1)) \tag{18}$$

With the above notation it follows from (g) of step 1 of the algorithm:

$$-p\cdot\frac{n^2-n}{2} + (1-p)\cdot\frac{n^2-n}{2} = \frac{1}{2}((2k-n)^2-n)$$

which is equivalent to

$$1-2p = \frac{4k^2+n^2-4kn-n}{n^2-n}$$

and to

$$p = \frac{-4k\sigma+4k}{2(n-1)} \overset{n\to\infty}{=} -2\sigma^2+2\sigma$$

With (18) we obtain for $n \to \infty$:

$$E\left[\frac{|S'|(n-|S'|)}{n^2}\right] \ge \frac{1}{2\pi}((-2\sigma^2+2\sigma)\cdot g_2(-1) + (1+2\sigma^2-2\sigma)\cdot g_2(1))$$

Case 2: $-\frac{e_2}{d_2} > -1$.

Because of $-\frac{e_2}{d_2} \le 0$ and Lemma 5(c) $g_2(x)$ is convex in $[-1, -\frac{e_2}{d_2}]$ and concave in $[-\frac{e_2}{d_2}, 1]$.

If X_{ij}, X_{st} are in the interval $[-\frac{e_2}{d_2}, 1]$ with $X_{ij} \le X_{st}$, we have $X_{ij} = -\frac{e_2}{d_2}$ or $X_{st} = 1$. If X_{ij} and X_{st} are in the interval $[-1, -\frac{e_2}{d_2}]$ with $X_{ij} \le X_{st}$, we conclude $X_{ij} = X_{st}$. So we can find a $p \in [0, 1]$ and a $q \in [-1, -\frac{e_2}{d_2}]$ with:

$$\begin{aligned} E\left[\frac{|S'|(n-|S'|)}{n^2}\right] &\ge \frac{1}{\pi n^2}\left(p\cdot\frac{n^2-n}{2}\cdot g_2(q) + (1-p)\cdot\frac{n^2-n}{2}\cdot g_2(1)\right) \\ &= \frac{1}{2\pi}\left(1-\frac{1}{n}\right)(p\cdot g_2(q)+(1-p)\cdot g_2(1)) \end{aligned} \tag{19}$$

With the above notation (g) of step 1 of the algorithm gives:

$$
\begin{aligned}
& p \cdot \frac{n^2-n}{2} \cdot q + (1-p) \cdot \frac{n^2-n}{2} = \frac{1}{2}((2k-n)^2 - n) \\
& \quad \Leftrightarrow p \cdot q + (1-p) = \frac{(2k-n)^2-n}{n^2-n} \\
& \quad \Leftrightarrow p \cdot (q-1) = \frac{4k^2+n^2-4kn-n}{n^2-n} - 1 \\
& \quad \Leftrightarrow p = \frac{4k\sigma - 4k}{(n-1)(q-1)} \\
& \quad \overset{n\to\infty}{\Rightarrow} p = \frac{4\sigma^2 - 4\sigma}{q-1}
\end{aligned}
$$

With (19) and $n \to \infty$ there is a $q \in [-1, -\frac{e_2}{d_2}]$ with:

$$
\begin{aligned}
& E\left[\frac{|S'|(n-|S'|)}{n^2}\right] \\
& \quad \geq \frac{1}{2\pi}\left(\frac{4\sigma^2-4\sigma}{q-1} \cdot g_2(q) + \left(1 - \frac{4\sigma^2-4\sigma}{q-1}\right) \cdot g_2(1)\right) \\
& \quad \geq \frac{1}{2\pi} \min_{q' \in [-1, -\frac{e_2}{d_2}]} \left(\frac{4\sigma^2-4\sigma}{q'-1} \cdot g_2(q') + \left(1 - \frac{4\sigma^2-4\sigma}{q'-1}\right) \cdot g_2(1)\right)
\end{aligned}
$$

(d) The claim has been shown by Halperin and Zwick (2002), under the assumption that (17) is true for all $d_2 \geq 0$ (compare footnotes 2 and 3 in Section 6). For the reader's convenience, we sketch their proof:

Define

$$
\begin{aligned}
\mathcal{A} &:= \left\{(i,j) \mid X_{ij} < -\frac{1}{3}, 1 \leq i < j \leq n\right\} \\
\mathcal{B} &:= \left\{(i,j) \mid -\frac{1}{3} \leq X_{ij} < 0, 1 \leq i < j \leq n\right\} \\
\mathcal{C} &:= \left\{(i,j) \mid 0 \leq X_{ij}, 1 \leq i < j \leq n\right\}
\end{aligned}
$$

Because of condition (e) of step 1 of the algorithm, the graph $G' = (V, \mathcal{A})$ is triangle-free. By a theorem of Turán (1941), a triangle-free graph with n vertices has at most $\frac{n^2}{4}$ edges. Thus, $|\mathcal{A}| \leq \frac{n^2}{4}$.

Because of $e_2 = 0$, $g_2(x)$ is convex in $[-1, 0]$ and concave in $[0, 1]$.

Using the convexity condition of $g_2(x)$, it is easy to show that there are $p_1 := |\mathcal{A}|$, $p_2 := |\mathcal{B}|$, $p_3 := |\mathcal{C}| \in [0, \frac{n^2-n}{2}]$ with

$$p_1 + p_2 + p_3 = \frac{n^2 - n}{2} \tag{20}$$

and $q_1 \in [-1, -\frac{1}{3})$, $q_2 \in [-\frac{1}{3}, 0]$ with

$$E\left[\frac{|S'|(n - |S'|)}{n^2}\right] \geq \frac{1}{\pi n^2}(p_1 \cdot g_2(q_1) + p_2 \cdot g_2(q_2) + p_3 \cdot g_2(1)) \tag{21}$$

Clearly, it holds $|\mathcal{A} \cup \mathcal{B}| \geq \frac{n^2}{4}$. If $|\mathcal{A}| < \frac{n^2}{4}$, we could replace the four points q_2, q_2, q_2, q_2 of $\mathcal{B}$ by the points $\frac{4q_2-1}{3}, \frac{4q_2-1}{3}, \frac{4q_2-1}{3}$ of $\mathcal{A}$ and the point 1 of $\mathcal{C}$, using (17). For $n \to \infty$, it follows

$$|\mathcal{A}| = \frac{n^2}{4} \tag{22}$$

Using again the convexity condition of $g_2(x)$, we get

$$q_2 = -\frac{1}{3} \tag{23}$$

With (20), (22), (23) it follows from condition (g) of step 1 of the algorithm:

(I) $\frac{n^2}{4} \cdot q_1 - p_2 \cdot \frac{1}{3} + p_3 \cdot 1 = -\frac{n}{2}$
(II) $\frac{n^2}{4} + p_2 + p_3 = \frac{n^2-n}{2}$

Solving these linear equalities for constant q_1, we conclude:

$$p_2 = \frac{3}{16}n^2 + \frac{3}{16}n^2 \cdot q_1$$
$$p_3 = \frac{1}{16}n^2 - \frac{3}{16}n^2 \cdot q_1 - \frac{n}{2}$$

With (21), we have:

$$E\left[\frac{|S'|(n - |S'|)}{n^2}\right]$$
$$\geq \frac{1}{\pi n^2} \min_{x \in [-1, \frac{1}{3}]} \left\{ \frac{n^2}{4} g_2(x) + \left(\frac{3}{16}n^2 + \frac{3}{16}n^2 \cdot x\right) g_2\left(-\frac{1}{3}\right)\right.$$

$$+\left(\frac{1}{16}n^2 - \frac{3}{16}n^2 \cdot x - \frac{n}{2}\right) g_2(1)\Bigg\}$$

$$\stackrel{n\to\infty}{=} \frac{1}{4\pi} \min_{x\in[-1,-\frac{1}{3}]} \left\{ g_2(x) + \frac{3x+3}{4} g_2\left(-\frac{1}{3}\right) + \frac{1-3x}{4} g_2(1) \right\} \qquad \square$$

5. Maximization of the approximation factors via optimization of the parameters

5.1. The optimization algorithm

We demonstrate our method in this section and also in Section 5.2 for MAX-k-CUT, MAX-k-UNCUT for all σ, and for MAX-k-DIRECTED-CUT, MAX-k-DIRECTED-UNCUT, MAX-k-DENSE-SUBGRAPH and MAX-k-VERTEX-COVER for $\sigma \neq \frac{1}{2}$. The other cases are considered in Section 5.3. In the following let us consider σ as fixed.

The expected approximation factor for the general maximization problem (1) is z, where $z = z(\delta, \sigma, \theta, \vartheta, \kappa, \nu, x_1, x_2)$ is a function depending on the parameters $\delta \in [0, 1], \sigma \in (0, 1), \theta, \vartheta, \nu \in [0, 1], \kappa \in [-1, 1], x_1 \in \mathbb{R}, x_2 \in \mathbb{R}_0^+$ in a complicated way (see (13)). A polynomial-time algorithm for an optimal choice of all parameters in (13) is not known. Thus we choose a hierarchical approach as follows.

1. Fixing the right-hand side.
   Let ϱ_0 be the previously best known approximation factor for the problem in the literature (Halperin and Zwick, 2002; Han et al., 2002; Feige and Langberg, 2001) and put $\varrho := \varrho_0 + k \cdot 0.0001$ for $k = 0, 1, \ldots$. We would like to prove

$$z \geq \varrho \tag{24}$$

   for a k as large as possible.
2. The linear program $LP(\Delta)$.
   For the moment let us fix the parameters $\theta, \vartheta, \kappa, \nu$ and consider them as constants. Let $h = h(\delta, \sigma, \theta, \upsilon, \kappa, \nu, x_1, x_2)$ be the function defined in (6). In Section 5.2, Proposition 10, we will prove that z is a piecewise linear function in h:

$$z = \begin{cases} \omega \cdot h, & \text{if } h \geq 1 \\ \upsilon \cdot (1-h) + \omega \cdot h, & \text{if } 0 \leq h < 1 \\ \upsilon, & \text{if } h < 0 \end{cases} \tag{25}$$

   where υ and ω are constants depending on δ.
   Since h is a linear function in x_1 and x_2 due to (6), we may write:

$$h(x_1, x_2) = f_1(\delta)x_1 + f_2(\delta)x_2 + f_3 \tag{26}$$

suppressing the dependence of h on $\theta, \vartheta, \kappa, \nu$, writing $h(x_1, x_2)$ instead of $h(\delta, \sigma, \theta, \vartheta, \kappa, \nu, x_1, x_2)$ and putting the dependence of h on δ into the coefficients $f_1(\delta)$ and $f_2(\delta)$.

Since by (25), z is only piecewise linear in h, (24) is not a linear equality in h. But in Proposition 10 we can show that (24) is equivalent to the inequality

$$h \geq \min\left\{\frac{\varrho}{\omega}, \frac{\varrho - \upsilon}{\omega - \upsilon}\right\} \quad \text{resp. } h \geq \max\left\{\frac{\varrho}{\omega}, 1\right\} \tag{27}$$

Clearly, this is a linear inequality in x_1 and x_2.

Still, the dependence on δ is an obstacle. We choose a discretization of [0, 1] for the δ's, i.e. define $\Delta := \{k \cdot \frac{1}{10^l}, k = 0, 1, \cdots, 10^l\}$ for $l \in \mathbb{N}$ as large as possible (we will choose $l = 4$). The inequalities in (27) for all $\delta \in \Delta$ form a finite linear program in the variables x_1 and x_2 which we denote by $LP(\Delta)$.

3. Discretization of the other parameters.
   Whether LP(Δ) is solvable or not depends on the choice of the parameters $\theta, \vartheta, \kappa, \nu$. We discretize the ranges of their parameters in finitely many points. For $\theta, \vartheta, \nu \in [0, 1]$, $\kappa \in [-1, 1]$ we take the discretization of both intervals with step size $\frac{1}{10}$ (for some cases we try even the finer discretization with step size $\frac{1}{100}$). We consider all possible values of $\theta, \vartheta, \kappa, \nu$ in this discretization and denote it by the parameter set $\mathcal{P}$. These are about 250,000 possibilities.

The algorithm for finding a good ϱ is the following.

Algorithm *Parameter Set*

1. Choose ϱ as the best previously known approximation factor ϱ_0.
2. Choose $\theta, \vartheta, \kappa, \nu$ from the parameter set $\mathcal{P}$.
3. Given ϱ, solve $LP(\Delta)$ by the simplex algorithm using CPLEX.
4. a) If $LP(\Delta)$ is solvable, increase ϱ by 0.0001 and goto 3.
   b) If $LP(\Delta)$ is not solvable and if not all parameters are tested, goto 2.
5. Output ϱ.

Remark 8. The quality of the approximation depends on the simultaneous optimization of *all* parameters $\theta, \vartheta, \nu, \kappa, x_1, x_2$. But it is not clear at all how to do this. Thus we have chosen a hierarchical way, where we optimize $\theta, \vartheta, \nu, \kappa$ in a preprocessing step and then focus on the mathematical optimization of the x_1, x_2. The main advantage of focusing on the optimization of the parameters x_1, x_2 is that x_1, x_2 live in a large range, i.e. $x_1 \in \mathbb{R}$ and $x_2 \in \mathbb{R}_0^+$, while $\theta, \vartheta, \nu, \kappa$ are only in the relatively small ranges $[-1, 1]$ and $[0, 1]$.

Remark 9. In Section 5.1 we find a good set of parameters $\theta, \vartheta, \kappa, \nu, x_1, x_2$. This step involves the solution of a linear program in the variables x_1 and x_2. The number of inequalities of this linear program which we denote by $LP(\Delta)$ is given by a suitable discretization Δ of the parameter δ in the interval [0, 1]. The careful reader might note that in the algorithm we test whether or not the claimed approximation factor is valid on only a discrete subset from [0, 1] for the δ's. This is not a proof, even if we choose a very fine discretization, as the approximation factor given by (13) is the minimum over the whole range [0, 1] (although in all our examples this discretization seems to be enough). But in Section 5.4 we show how to verify for these fixed parameters $\theta, \vartheta, \kappa, \nu, x_1, x_2$ the claimed approximation factors by solving (13). Note that this verification step and so the correctness of our approximation factors do not depend on the above discretization.

5.2. The linear program

In this section we derive the linear program $LP(\Delta)$ introduced in Section 5.1. For the problems considered in Section 5.1, the flipping step in the Algorithm Graph Partitioning can be omitted. Thus for a fixed δ the minimization problem in (13) has the following form:

$$\left[\begin{array}{c} \min \quad f_{a_1,a_2,a_3,a_4}(\lambda_1,\lambda_2,\lambda_3,\lambda_4)\ (=z) \\ \text{s.t.} \\ a_1\lambda_1 + a_2\lambda_2 + a_3\lambda_3 + a_4\lambda_4 \geq h(\delta,\sigma,\theta,\vartheta,\kappa,\nu,x_1,x_2) \\ \lambda_1+\lambda_2+\lambda_3+\lambda_4 \geq 1 \\ 0 \leq \lambda_1,\lambda_2,\lambda_3,\lambda_4 \end{array}\right] \tag{28}$$

where

$$f_{a_1,a_2,a_3,a_4}(\lambda_1,\lambda_2,\lambda_3,\lambda_4) = b_1\lambda_1 + b_2\lambda_2 + b_3\lambda_3 + b_4\lambda_4$$

with suitable b_1, b_2, b_3, b_4. Before we proceed to solve (28), we have to compute the b_1, b_2, b_3, b_4 for the different problems. We will also need the following numbers v, ω. Define

$$\begin{array}{c} I := \{1,2,3,4\}, \quad I'_{a_1,a_2,a_3,a_4} := \{i \in I \mid a_i = 1\} \\ v := \min_{l \in I}\{b_l\}, \quad w := \min_{l \in I'_{a_1,a_2,a_3,a_4}}\{b_l\} \end{array} \tag{29}$$

Furthermore put $p := \frac{\sigma}{\delta}, q := \frac{1-\sigma}{1-\delta}, r := \frac{\sigma}{1-\delta}, s := \frac{1-\sigma}{\delta}$ as in the cases 1–4 of Section 3.2.

In Tables 3–5 we give the values for (b_1, b_2, b_3, b_4) and v, ω for all δ. Note that according to (11) and (12), (b_1, b_2, b_3, b_4) is given by the vector $(a_1, a_2, a_3, a_4) \cdot M(\delta, \sigma)$, where $M(\delta, \sigma)$ is the matrix defined in (9) and (10). Having computed the b_1, b_2, b_3, b_4, it is straightforward to compute v and ω. Note that we only have to consider $0 < \sigma \leq \frac{1}{2}$ in the cases MAX-k-CUT, MAX-k-UNCUT, MAX-k-DIRECTED-CUT and MAX-k-DIRECTED-UNCUT, because the approximation factors for σ are the same as for $1 - \sigma$ and that the following tables cover all problems, including those of Section 5.3.

For MAX-k-UNCUT we have:

For $\sigma \leq \delta < \frac{1}{2}$ it holds $v = 1 + 2p^2 - 2p$, if $0 \leq \sigma \leq \frac{1}{4}, 2\sigma \leq \delta < \frac{1}{2}$ and $v = 1 - p$ otherwise. Analogously, for $\frac{1}{2} \leq \delta < 1 - \sigma$ it holds $v = 1 + 2r^2 - 2r$, if

Table 3. b_1, b_2, b_3, b_4 and v, w for the asymmetric problems.

Case	$\sigma \le \delta \le 1$	$0 \le \delta < \sigma$
	(b_1, b_2, b_3, b_4) (v, ω)	(b_1, b_2, b_3, b_4) (v, ω)
MAX-k-DS	$(p^2, 0, 0, 0)$ $(0, p^2)$	$(1, 1-q, 1-q, (1-q)^2)$ $((1-q)^2, 1)$
MAX-k-VC	$(2p-p^2, p, p, 0)$ $(0, p)$	$(1, 1, 1, 1-q^2)$ $(1-q^2, 1)$
MAX-k-DC	$(p-p^2, p, 0, 0)$ $(0, p)$	$(0, q, 0, q-q^2)$ $(0, q)$
MAX-k-DU	$(1+p^2-p, 1-p, 1, 1)$ $(1-p, 1+p^2-p)$	$(1, 1-q, 1, 1+q^2-q)$ $(1-q, 1+q^2-q)$

Table 4. b_1, b_2, b_3, b_4 and v, w for the symmetric problems, $\delta < \frac{1}{2}$.

Case	$\sigma \le \delta \le \frac{1}{2}$	$0 \le \delta < \sigma$
	(b_1, b_2, b_3, b_4) (v, ω)	(b_1, b_2, b_3, b_4) (v, ω)
MAX-k-CUT	$(2p-2p^2, p, p, 0)$ $(0, p)$	$(0, q, q, 2q-2q^2)$ $0, q$
MAX-k-UNCUT	$(1+2p^2-2p, 1-p, 1-p, 1)$ $(1-p/1+2p^2-2p, 1+2p^2-2p)$	$(1, 1-q, 1-q, 1+2q^2-2q)$ $(1-q, 1+2q^2-2q)$

Table 5. b_1, b_2, b_3, b_4 and v, w for the symmetric problems, $\delta \ge \frac{1}{2}$.

Case	$\frac{1}{2} \le \delta < 1-\sigma$	$1-\sigma \le \delta \le 1$
	(b_1, b_2, b_3, b_4) (v, ω)	(b_1, b_2, b_3, b_4) (v, ω)
MAX-k-CUT	$(0, r, r, 2r-2r^2)$ $(0, r)$	$(2s-2s^2, s, s, 0)$ $(0, s)$
MAX-k-UNCUT	$(1, 1-r, 1-r, 1+2r^2-2r)$ $(1-r/1+2r^2-2r, 1+2r^2-2r)$	$(1+2s^2-2s, 1-s, 1-s, 1)$ $(1-s, 1+2s^2-2s)$

$0 \le \sigma \le \frac{1}{4}, \frac{1}{2} \le \delta \le 1-2\sigma$ and $v = 1-r$ otherwise. Furthermore:

$$v = w \Leftrightarrow \left(\sigma = \frac{1}{2} \wedge (\delta = 0 \vee \delta = 1)\right) \vee \left(0 \le \sigma \le \frac{1}{4} \wedge 2\sigma \le \delta \le 1-2\sigma\right)$$

Before we start with the calculations, let us state the results in a formal way.

Proposition 10. *For all problems except MAX-$\frac{n}{2}$-DIRECTED-CUT, MAX-$\frac{n}{2}$-DIRECTED-UNCUT, MAX-$\frac{n}{2}$-DENSE-SUBGRAPH, MAX-$\frac{n}{2}$-VERTEX-COVER we have*:

(a)
$$z = \begin{cases} \omega \cdot h, & \textit{if } h \geq 1 \\ v \cdot (1-h) + \omega \cdot h, & \textit{if } 0 \leq h < 1 \\ v, & \textit{if } h < 0 \end{cases} \tag{30}$$

(b) *For $v = \omega$,* (24) *is equivalent to*

$$h \geq \max\left\{\frac{\varrho}{\omega}, 1\right\} \tag{31}$$

(c) *For $v \neq \omega$,* (24) *is equivalent to*

$$h \geq \min\left\{\frac{\varrho}{\omega}, \frac{\varrho - v}{\omega - v}\right\} \tag{32}$$

Proof: (a)

Case 1: $h \geq 1$

In this case, the second inequality $\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 \geq 1$ of (28) is always true, thus redundant. Let i_0 be the index with $b_{i_0} = \omega$. Then we put $\lambda_{i_o} = h$, $\lambda_i = 0$, $i \neq i_0$. It is easily checked that the minimum of (28) is attained for this choice of the $\lambda_1, \lambda_2, \lambda_3, \lambda_4$. So we have $z = \omega \cdot h$.

Case 2: $0 \leq h < 1$

Case 2.1: $v = \omega$

Let i_0 be the index with $b_{i_0} = v = \omega$. Then $\lambda_{i_o} = 1$, $\lambda_i = 0$, $i \neq i_0$ gives a minimum for (28). So we have $z = v = v \cdot (1-h) + \omega \cdot h$.

Case 2.2: $v \neq \omega$

Let i_0 be the index with $b_{i_0} = v$ and i_1 be the index with $b_{i_1} = \omega$. Now $\lambda_{i_0} = 1-h$, $\lambda_{i_1} = h$, $\lambda_i = 0$, $i \neq i_0, i_1$ gives a minimum for (28). Again we have $z = v \cdot (1-h) + \omega \cdot h$.

Case 3: $h < 0$

Because of $\lambda_1, \lambda_2, \lambda_3, \lambda_4 \geq 0$ the first inequality $a_1\lambda_1 + a_2\lambda_2 + a_3\lambda_3 + a_4\lambda_4 \geq h$ is redundant. Let i_0 be the index with $b_{i_0} = v$. Again we put $\lambda_{i_0} = 1$, $\lambda_i = 0$, $i \neq i_0$ and obtain $z = v$.

(b) Let $v = \omega$. One can check that $v < \varrho_0 \leq \varrho$. Thus we can omit the second inequality $v \cdot (1-h) + \omega \cdot h = v \geq \varrho$ and the third inequality $v \geq \varrho$ of (30). Obviously, the first inequality $(z = h \cdot \omega \geq \varrho) \wedge (h \geq 1)$ is equivalent to (31).

(c) Let $v \neq \omega$. As in b), the third inequality in (30) can be omitted. It remains to show

$$(w \cdot h \geq \varrho) \wedge (h \geq 1) \vee (v \cdot (1-h) + w \cdot h \geq \varrho) \wedge (0 \leq h < 1) \tag{33}$$

is equivalent to (32). It is easy to show that (33) is equivalent to

$$\left(h \geq \frac{\varrho}{\omega}\right) \wedge (h \geq 1) \vee \left(h \geq \frac{\varrho - v}{\omega - v}\right) \wedge (0 \leq h < 1) \tag{34}$$

□

We prove the equivalence of (34) and (32).
"$\Rightarrow$" Trivial.
"$\Leftarrow$" It holds:

$$\frac{\varrho}{\omega} \leq \frac{\varrho - v}{\omega - v} \Leftrightarrow \varrho \geq \omega \tag{35}$$

(i) Let $h \geq \frac{\varrho}{\omega}$, i.e. $\frac{\varrho}{\omega} \leq \frac{\varrho-v}{\omega-v}$.

It follows from $\varrho, \omega > 0$ that $h \geq 0$. From (35) it follows $h \geq \frac{\varrho}{\omega} \geq 1$.

(ii) Let $h \geq \frac{\varrho-v}{\omega-v}$, i.e. $\frac{\varrho-v}{\omega-v} \leq \frac{\varrho}{\omega}$.

It follows from $\varrho > v$ and $\omega > v$ that $h \geq 0$.
In the case $h \geq 1$, it holds $h \geq 1 \geq \frac{\varrho}{\omega}$ because of (35).
The case $0 \leq h < 1$ is trivial.
With Proposition 10 (b) and (c) we have defined the relevant inequality for h, and according to Section 5.1, the discretization of the δ's defines the linear program $LP(\Delta)$.

5.3. *The other cases*

Following the argumentation from Section 5.2, we can derive similar expressions for z for the remaining maximization problems, leading to inequalities for h and finally the corresponding $LP(\Delta)$. Since these calculations are technical variations of the argumentation in Section 5.1, we omit the details and list only the inequalities for h.

(1) MAX-$\frac{n}{2}$-DIRECTED-CUT: If $\delta \geq \sigma : h \geq \frac{\varrho}{p}$

$$\text{If} \quad \delta < \sigma : h \geq \frac{\varrho}{q}$$

(2) MAX-$\frac{n}{2}$-DIRECTED-UNCUT:

$$\text{If } \delta \geq \sigma : h \geq \begin{cases} \dfrac{\varrho}{1+p^2-p}, & \text{if } \varrho \geq 1+p^2-p \\ \dfrac{\varrho+p^2-1}{2p^2-p}, & \text{if } \varrho < 1+p^2-p \end{cases}$$

$$\text{If } \delta < \sigma : h \geq \begin{cases} \dfrac{\varrho}{1+q^2-q}, & \text{if } \varrho \geq 1+q^2-q \\ \dfrac{\varrho+q^2-1}{2q^2-q}, & \text{if } \varrho < 1+q^2-q \end{cases}$$

(3) MAX-$\frac{n}{2}$-DENSE-SUBGRAPH: If $\delta \geq \sigma : h \geq \frac{\varrho}{p^2}$

$$\text{If} \quad \delta < \sigma : h \geq \frac{\varrho - (1-q)^2}{2q - q^2}$$

(4) MAX-$\frac{n}{2}$-VERTEX-COVER:

$$\text{If} \quad \delta \geq \sigma : h \geq \begin{cases} \dfrac{\varrho}{1 + p^2 - p}, & \text{if } \varrho \geq 1 + p^2 - p \\ \dfrac{\varrho + p - 1}{p^2}, & \text{if } \varrho < 1 + p^2 - p \end{cases}$$

$$\text{If} \quad \delta < \sigma : h \geq \frac{\varrho + q^2 - 1}{q^2}$$

5.4. Verification of the approximation factors

The approximation factor (13) of Algorithm Graph Partitioning is a minimum over the interval [0, 1]. If we use only a finite discretization of this interval, for example $\Delta := \{k \cdot \frac{1}{10^l},\ k = 0, 1, \ldots, 10^l\}$ with $l = 4$, the computed approximation factor might be inaccurate.

For obtaining good parameters x_1 and x_2 in Section 5.1, this finite discretization is no problem. But for correctness, we have to verify that the parameter setting really gives the claimed approximation factor.

So let the problem parameter σ, the parameters $\theta, \vartheta, \kappa, \nu, x_1, x_2,$, and the claimed approximation factor ϱ be constant. Proposition 10 and the cases (1)–(4) of Section 5.3, respectively, give inequalities which are equivalent to the claim that the approximation factor is at least ϱ.

In the following we demonstrate the correctness proof of the approximation factors, in particular, how to find the minimum of (13) for two of our examples (the proof of the other 196 examples is very similar). In fact, we prove that the inequalities can be verified as they lead to a system of inequalities for polynomials of degree at most 4. For the corresponding parameters we refer to Section 6.

Example 1. MAX-k-DENSE-SUBGRAPH with $\sigma = 0.6$.

Parameters: $\varrho = 0.6753$, $\theta = 0.9$, $\vartheta = 1$, $\kappa = 0.2$, $\nu = 0.2$, case 3 of (5), $x_1 = 0$, $x_2 = 6.39$.

We calculate $\alpha = 0.7678$, $\beta := 0.5648$, $\gamma := 0.2338$.

From Table 3 we get $\upsilon = 0$, $\omega = (\frac{\sigma}{\delta})^2$ for $\delta \in [\sigma, 1]$ and $\upsilon = (1 - \frac{1-\sigma}{1-\delta})^2$, $\omega = 1$ for $\delta \in [0, \sigma]$.

We have (see (6))

$$h(\delta) = \alpha + x_1(\beta - \delta) + x_2(\gamma + \delta^2 - \delta)$$

Then (32) is equivalent to

$$h_1(\delta) := h\omega \geq \varrho \vee h_2(\delta) := h(\omega - v) + v \geq \varrho \tag{36}$$

where for $v = 0$ the two inequalities are the same.

So altogether we have to show the correctness of the following inequalities:

$$(\alpha + x_1(\beta - \delta) + x_2(\gamma + \delta^2 - \delta))\left(\frac{\sigma}{\delta}\right)^2 \geq \varrho \quad \text{for } \delta \in [\sigma, 1] \tag{37}$$

$$\alpha + x_1(\beta - \delta) + x_2(\gamma + \delta^2 - \delta) \geq \varrho \tag{38}$$

$$\vee(\alpha + x_1(\beta - \delta) + x_2(\gamma + \delta^2 - \delta))\left(1 - \left(1 - \frac{1-\sigma}{1-\delta}\right)^2\right) + \left(1 - \frac{1-\sigma}{1-\delta}\right)^2 \geq \varrho \quad \text{for } \delta \in [0, \sigma] \tag{39}$$

If this system of inequalities is valid (for the given range of δ), then by Proposition 10(c) this is equivalent to the fact that the expected approximation factor in (13) is at least ϱ.

All three inequalities can be transformed to polynomial inequalities of degree at most 4 (this holds even for all 198 cases). Using MAPLE, we prove that (37) is true for all $\delta \in [\sigma, 1]$ and (39) is true for all $\delta \in [0, \sigma]$ ((38) is not true for all $\delta \in [0, \sigma]$).

So we have proved the claimed approximation factor of $\varrho = 0.6753$.

Example 2. MAX-k-DIRECTED-UNCUT with $\sigma = 0.5$ (or MAX-$\frac{n}{2}$-DIRECTED-UNCUT).

Parameters: $\varrho = 0.8164$, $\theta = 1$, $v = 0.84$, κ arbitrary, $\nu = 0$, case 3 of (4), $x_1 = 0$, $x_2 = 8.59$.

We calculate $\alpha = 0.8722$, $\beta := 0.486$, $\gamma := 0.2467$.

The inequalities from case 2 of Section 5.3 are relevant for our example. They depend on the solution of the inequalities

$$\varrho \geq 1 + \left(\frac{1}{2 \cdot \delta}\right)^2 - \frac{1}{2 \cdot \delta} \quad \text{for } \delta \in [0.5, 1] \tag{40}$$

$$\varrho \geq 1 + \left(\frac{1}{2 \cdot (1-\delta)}\right)^2 - \frac{1}{2 \cdot (1-\delta)} \quad \text{for } \delta \in [0, 0.5] \tag{41}$$

Let $c_1 \approx 0.6599$. Then (40) is true for $\delta \in [c_1, 1]$ and (41) is true for $\delta \in [0, 1 - c_1]$. So we receive the following system of linear inequalities

$$\alpha + x_1(\beta - \delta) + x_2(\gamma + \delta^2 - \delta) \geq \frac{\varrho}{1 + \frac{1}{4\delta^2} - \frac{1}{2\delta}} \quad \text{for } \delta \in [c_1, 1] \tag{42}$$

Table 6. Approximation factors and Parameters for MAX-k-CUT.

σ	Prev.	Our method	ν	L	P	θ	ϑ	κ	x_1	x_2
0.02	0.5 + c	$0.5-\varepsilon$	1	–	–	–	–	–	$-1.5\cdot 10^7$	$1.56\cdot 10^7$
0.04	0.5 + c	$0.5-\varepsilon$	1	–	–	–	–	–	$-3.59\cdot 10^6$	$3.91\cdot 10^6$
0.06	0.5 + c	$0.5-\varepsilon$	1	–	–	–	–	–	$-1.53\cdot 10^6$	$1.74\cdot 10^6$
0.08	0.5 + c	$0.5-\varepsilon$	1	–	–	–	–	–	$-8.2\cdot 10^5$	$9.77\cdot 10^5$
0.1	0.5 + c	$0.5-\varepsilon$	1	–	–	–	–	–	$-5\cdot 10^5$	$6.25\cdot 10^5$
0.12	0.5 + c	$0.5-\varepsilon$	1	–	–	–	–	–	$-3.3\cdot 10^5$	$4.34\cdot 10^5$
0.14	0.5 + c	$0.5-\varepsilon$	1	–	–	–	–	–	$-2.3\cdot 10^5$	$3.19\cdot 10^5$
0.16	0.5 + c	$0.5-\varepsilon$	1	–	–	–	–	–	$-1.66\cdot 10^5$	$2.44\cdot 10^5$
0.18	0.5 + c	$0.5-\varepsilon$	1	–	–	–	–	–	$-1.23\cdot 10^5$	$1.93\cdot 10^5$
0.2	0.5 + c	$0.5-\varepsilon$	1	–	–	–	–	–	$-9.37\cdot 10^4$	$1.56\cdot 10^5$
0.22	0.5 + c	$0.5-\varepsilon$	1	–	–	–	–	–	$-7.23\cdot 10^4$	$1.29\cdot 10^5$
0.24	0.5 + c	0.5026	0	3	–	1	0.96	–	−2.44	3.32
0.26	0.5 + c	0.5252	0	3	3	0.98	0.97	−0.35	−2.29	3.41
0.28	0.5 + c	0.5467	0	3	3	0.96	0.98	−0.3	−2.17	3.58
0.3	0.527	0.567	0	3	3	0.94	0.99	−0.25	−2.02	3.7
0.32	0.562	0.5864	0	–	3	0.91	1	−0.25	−1.94	4.05
0.34	0.593	0.6045	0	–	3	0.91	1	−0.2	−1.79	4.25
0.36	0.616	0.6218	0	–	3	0.92	1	−0.2	−1.37	3.63
0.38	0.642	0.6451	0	1	–	1	0.99	–	0	0.77
0.4	0.671	0.6727	0	1	–	1	0.99	–	0	1.12
0.42	0.698	0.6994	0	1	–	1	0.98	–	0	1.58
0.44	0.721	0.7216	0	1	–	1	0.96	–	0	2.36
0.46	0.734	0.7351	0	1	1	0.98	0.95	0	0	3.91
0.48	0.725	0.7257	0	1	–	1	0.89	–	0	6.85
0.5	0.7027	0.7016	0	–	1	0.89	1	0	0	7.14

$$\alpha + x_1(\beta-\delta) + x_2(\gamma+\delta^2-\delta) \geq \frac{\varrho + \frac{1}{4\delta^2} - 1}{\frac{1}{2\delta^2} - \frac{1}{2\delta}} \quad \text{for } \delta \in [0.5, c_1] \tag{43}$$

$$\alpha + x_1(\beta-\delta) + x_2(\gamma+\delta^2-\delta) \geq \frac{\varrho}{1 + \frac{1}{4(1-\delta)^2} - \frac{1}{2(1-\delta)}} \quad \text{for } \delta \in [0, 1-c_1] \tag{44}$$

$$\alpha + x_1(\beta-\delta) + x_2(\gamma+\delta^2-\delta) \geq \frac{\varrho + \frac{1}{4(1-\delta)^2} - 1}{\frac{1}{2(1-\delta)^2} - \frac{1}{2(1-\delta)}} \quad \text{for } \delta \in [1-c_1, 0.5] \tag{45}$$

All four inequalities can be transformed to polynomial inequalities of degree at most 4. Using MAPLE, we prove that (42)–(45) are true for their corresponding range of δ.

So we have proved the claimed approximation factor of $\varrho = 0.8164$.

Table 7. Approximation factors and Parameters for MAX-*k*-UNCUT.

σ	Prev.	Our method	ν	L	P	θ	ϑ	κ	x_1	x_2
0.02	0.9608	0.9608	0	–	2	0	–	−1	−46.08	0
0.04	0.9232	0.9232	0	–	2	0	–	−1	−21.16	0
0.06	0.8872	0.8872	0	–	2	0	–	−1	−12.91	0
0.08	0.8528	0.8528	0	–	2	0	–	−1	−8.82	0
0.1	0.82	0.82	0	–	2	0	–	−1	−6.4	0
0.12	0.7888	0.7888	0	–	2	0	–	−1	−4.81	0
0.14	0.7592	0.7592	0	–	2	0	–	−1	−3.77	0
0.16	0.7312	0.7312	0	–	2	0	–	−1	−2.89	0
0.18	0.7048	0.7048	0	–	2	0	–	−1	−2.28	0
0.2	0.68	0.68	0	–	2	0	–	−1	−1.8	0
0.22	0.6568	0.6568	0	–	2	0	–	−1	−1.43	0
0.24	0.6352	0.6352	0	–	2	0	–	−1	−1.13	0
0.26	0.6152	0.6152	0	–	2	0	–	−1	−0.89	0
0.28	0.5968	0.5968	0	–	2	0	–	−1	−0.69	0
0.3	0.58	0.58	0	–	2	0	–	−1	−0.53	0
0.32	0.5648	0.5648	0	–	2	0	–	−1	−0.41	0
0.34	0.5512	0.5512	0	–	2	0	–	−1	−0.3	0
0.36	0.5392	0.5644	0	–	3	0.81	1	−0.35	−2.95	8.52
0.38	0.5288	0.5787	0	–	3	0.81	1	−0.35	−2.43	8.01
0.4	0.5258	0.5973	0	1	–	1	0.97	–	0	1.58
0.42	0.5587	0.6238	0	1	–	1	0.96	–	0	2.51
0.44	0.6013	0.6483	0	1	–	1	0.93	–	0	4.14
0.46	0.6353	0.668	0	1	–	1	0.89	–	0	7.08
0.48	0.6451	0.6737	0	1	–	1	0.82	–	0	13.33
0.5	0.6414	0.6415	0	1	–	1	0.8	–	0	14.93

6. The final approximation results

For all six problems we compute approximation factors as described in Section 5.1. The numbers for L and P, respectively denote the cases, which we use for d_2 and e_2, respectively (see (4), (5)). If parameters are arbitrary, we omit them. We consider $\sigma = 0.02, 0.04, \dots, 0.98$ for MAX-*k*-DENSE-SUBGRAPH and MAX-*k*-VERTEX-COVER and $\sigma = 0.02, 0.04, \dots 0.5$ otherwise, because in these cases the approximation factors for σ are the same as for $1 - \sigma$.

6.1. *MAX-k-CUT*

Using linear programming, Ageev and Sviridenko (1999) proved an approximation factor of 0.5 for arbitrary σ which was improved by Feige and Langberg (2001) to $0.5+c$ for a constant

Table 8. Approximation factors and Parameters for MAX-k-DIRECTED-CUT.

σ	Prev.	Our method	ν	L	P	θ	ϑ	κ	x_1	x_2
0.02	0.5	0.1439	0.4	–	2	0.5	1	−1	−7.06	0.24
0.04	0.5	0.18	0.6	2	–	1	0.9	–	−4.36	0.41
0.06	0.5	0.2211	0.5	2	–	1	0.9	–	−3.51	0.53
0.08	0.5	0.258	0.4	–	3	0.9	1	−0.4	−3.47	1.57
0.1	0.5	0.2916	0.4	–	3	0.9	1	−0.4	−3.3	1.99
0.12	0.5	0.3223	0.4	–	3	0.9	1	−0.3	−3.3	2.55
0.14	0.5	0.351	0.4	–	3	0.9	1	−0.3	−3.28	2.98
0.16	0.5	0.3791	0.3	–	3	0.9	1	−0.3	−3.06	3.03
0.18	0.5	0.4062	0.3	–	3	0.9	1	−0.3	−3.03	3.35
0.2	0.5	0.4321	0.3	–	3	0.9	1	−0.3	−2.95	3.56
0.22	0.5	0.456	0.3	–	3	0.9	1	−0.3	−2.82	3.71
0.24	0.5	0.4779	0.3	–	3	0.9	1	−0.3	−2.68	3.84
0.26	0.5	0.498	0.3	–	3	0.9	1	−0.3	−2.52	3.96
0.28	0.5	0.5165	0.3	–	3	0.9	1	−0.3	−2.38	4.14
0.3	0.5	0.5335	0.3	–	3	0.9	1	−0.3	−2.25	4.36
0.32	0.5	0.5493	0.3	–	3	0.9	1	−0.2	−2.1	4.57
0.34	0.5	0.5644	0.2	–	3	0.9	1	−0.2	−1.86	4.56
0.36	0.5	0.5786	0.2	–	3	0.9	1	−0.2	−1.71	4.84
0.38	0.5	0.5914	0.2	–	3	0.9	1	−0.2	−1.5	4.99
0.4	0.5	0.6029	0.2	–	3	0.9	1	−0.2	−1.31	5.3
0.42	0.5	0.6121	0.2	–	3	0.9	1	−0.1	−1.1	5.65
0.44	0.5	0.6227	0.2	–	3	0.9	1	−0.1	−0.86	5.91
0.46	0.5	0.6305	0.2	–	3	0.9	1	−0.1	−0.6	6.2
0.48	0.5	0.6371	0.2	1	–	1	0.9	–	−0.31	6.53
0.5	0.644	0.6507	0.17	3	–	1	0.95	–	0	7.04

$c > 0$. For 0.3, ... 0.48 Han et al. (2002) got the previously best factors. For 0.24, ... 0.48, we improve these factors. For the case $\sigma = 0.5$ we get the same approximation factor 0.7016 as Halperin and Zwick. Feige and Langberg (2001) improved this factor to 0.7027, using the RPR^2 rounding technique, which additionally analyzes the correction step of changing the sides of so-called misplaced vertices.

We can show the same result as Ageev and Sviridenko, up to $\varepsilon > 0$ arbitrarily small.

Corollary 11. *For arbitrary σ MAX-k-CUT has an approximation factor $0.5-\epsilon$ for $\epsilon > 0$ arbitrarily small.*

Proof: Use $\nu = 1$ (i.e. Linear Randomized Rounding), $x_1 = (2\sigma - 1)\cdot x_2$ and x_2 sufficiently large. (For the proof of $\varrho < 0.5$, x_2 must be chosen $\geq \frac{\varrho^2}{4.(0.5-\varrho)\cdot\sigma^2}$. In Table 6 we prove an approximation factor of 0.49999 for $\sigma = 0.02, 0.04, \ldots, 0.22$.) □

Table 9. Approximation factors and Parameters for MAX-*k*-DIRECTED-UNCUT.

σ	Prev.	Our method	ν	L	P	θ	ϑ	k	x_1	x_2
0.02	–	0.9804	0	–	2	0	–	−1	−50	0
0.04	–	0.9616	0	–	2	0	–	−1	−25	0
0.06	–	0.9436	0	–	2	0	–	−1	−16.67	0
0.08	–	0.9264	0	–	2	0	–	−1	−12.5	0
0.1	–	0.91	0	–	2	0	–	−1	−10	0
0.12	–	0.8944	0	–	2	0	–	−1	−8.33	0
0.14	–	0.8796	0	–	2	0	–	−1	−7.14	0
0.16	–	0.8656	0	–	2	0	–	−1	−6.25	0
0.18	–	0.8524	0	–	2	0	–	−1	−5.56	0
0.2	–	0.84	0	–	2	0	–	−1	−5	0
0.22	–	0.8284	0	–	2	0	–	−1	−4.55	0
0.24	–	0.8176	0	–	2	0	–	−1	−4.17	0
0.26	–	0.8076	0	–	2	0	–	−1	−3.85	0
0.28	–	0.7984	0	–	2	0	–	−1	−3.57	0
0.3	–	0.79	0	–	2	0	–	−1	−3.33	0
0.32	–	0.7824	0	–	2	0	–	−1	−3.13	0
0.34	–	0.7756	0	–	2	0	–	−1	−2.94	0
0.36	–	0.7696	0	–	2	0	–	−1	−2.78	0
0.38	–	0.7644	0	–	2	0	–	−1	−2.63	0
0.4	–	0.7705	0.1	–	3	0.8	1	−0.3	−1.03	4.21
0.42	–	0.7776	0	3	3	0.9	0.9	−0.2	−1.02	5.42
0.44	–	0.785	0	3	3	0.9	0.9	−0.2	−0.86	6.12
0.46	–	0.7919	0	3	3	0.9	0.9	−0.1	−0.64	6.92
0.48	–	0.798	0	3	–	1	0.8	–	−0.45	10.15
0.5	0.811	0.8164	0	3	–	1	0.84	–	0	8.59

6.2. *MAX-k-UNCUT*

For 0.02, ... 0.38, Feige and Langberg (2001) obtained the previously best approximation factor of $1-2\sigma(1-\sigma)$ by independent sampling. For 0.4, ... 0.48, Han et al. (2002) got the best factors. For 0.36, ... 0.48, we improve these factors. For $\sigma = 0.5$ the approximation factor 0.6414[2] can be improved by our algorithm to 0.6415.

We obtain the known global approximation factor of Feige and Langberg.

Corollary 12. *For arbitrary σ MAX-k-UNCUT has an approximation factor of* $1 - 2\sigma(1 - \sigma)$.

Proof: Use $\nu = 0$, L arbitrary, $P = 2$, $\theta = 0$, ϑ arbitrary, $\kappa = -1$, $x_1 \leq 4(1 - \sigma) - \frac{1}{\sigma}$ sufficiently small and $x_2 = 0$. □

Table 10. Approximation factors and Parameters for MAX-*k*-DENSE-SUBGRAPH, $\sigma \leq \frac{1}{2}$.

σ	Prev.	Our method	v	L	P	θ	ϑ	κ	x_1	x_2
0.02	0.02	0.0192	0.96	3	2	0.48	0.99	−1	−48.83	49.08
0.04	0.04	0.0407	0.9	3	2	0.49	0.97	−1	−26.02	26.4
0.06	0.06	0.0604	0.8	3	2	0.5	0.9	−1	−17.23	17.7
0.08	0.08	0.084	0.8	3	–	1	0.9	–	−13.65	14.29
0.1	0.1	0.1123	0.7	3	–	1	0.9	–	−11.7	12.48
0.12	0.12	0.1421	0.7	3	–	1	0.9	–	−10.49	11.52
0.14	0.14	0.1726	0.6	3	–	1	0.9	–	−9.35	10.51
0.16	0.16	0.2027	0.6	3	–	1	0.9	–	−8.58	10.03
0.18	0.18	0.2335	0.5	3	–	1	0.9	–	−7.77	9.36
0.2	0.2008	0.2644	0.5	–	3	0.9	1	−0.4	−6.96	8.58
0.22	0.232	0.295	0.5	–	3	0.9	1	−0.4	−6.47	8.36
0.24	0.2631	0.3248	0.4	–	3	0.9	1	−0.4	−5.95	8.01
0.26	0.2942	0.3548	0.4	–	3	0.9	1	−0.4	−5.55	7.88
0.28	0.3245	0.3833	0.4	–	3	0.9	1	−0.4	−5.17	7.82
0.3	0.3541	0.4061	0.4	–	3	0.9	1	−0.4	−4.81	7.77
0.32	0.3827	0.4359	0.3	–	3	0.9	1	−0.4	−4.35	7.41
0.34	0.4105	0.4619	0.3	–	3	0.9	1	−0.3	−4.05	7.49
0.36	0.4372	0.4864	0.3	–	3	0.9	1	−0.3	−3.73	7.55
0.38	0.4626	0.5092	0.3	–	3	0.9	1	−0.3	−3.39	7.52
0.4	0.4867	0.5305	0.3	–	3	0.9	1	−0.2	−3.11	7.79
0.42	0.5095	0.5505	0.3	–	3	0.9	1	−0.2	−2.76	7.76
0.44	0.531	0.5688	0.3	–	3	0.9	1	−0.2	−2.42	7.78
0.46	0.5511	0.5861	0.2	–	3	0.9	1	−0.1	−2.07	7.56
0.48	0.5697	0.6031	0.2	–	3	0.9	1	−0.1	−1.73	7.53
0.5	0.6221	0.6223	0.21	3	1	0.92	0.97	0	−1.39	8.57

6.3. *MAX-k-DIRECTED-CUT*

Ageev et al. (2001) showed an approximation factor of 0.5 for arbitrary σ. For 0.28, ... 0.48, we substantially improve this factor. For the case $\sigma = 0.5$ we improve the approximation factor 0.644 of Halperin and Zwick to 0.6507.

6.4. *MAX-k-DIRECTED-UNCUT*

For 0.02, ... 0.48, the approximation factors have not been considered until now. For $\sigma = 0.5$ the approximation factor 0.811^3 can be improved by our algorithm to 0.8164.

We get a global approximation factor for MAX-*k*-DIRECTED-UNCUT.

Table 11. Approximation factors and Parameters for MAX-k-DENSE-SUBGRAPH, $\sigma > \frac{1}{2}$

σ	Prev.	Our method	ν	L	P	θ	ϑ	κ	x_1	x_2
0.52	0.6022	0.6339	0.2	3	–	1	0.9	–	−0.94	9.94
0.54	0.6161	0.6471	0.2	3	–	1	0.9	–	−0.49	10.25
0.56	0.6287	0.6585	0.2	3	–	1	0.9	–	0	10.64
0.58	0.6402	0.6667	0.2	–	3	0.9	1	0.2	−0.11	7.34
0.6	0.6488	0.6753	0.2	–	3	0.9	1	0.2	0	6.39
0.62	0.6539	0.6807	0.2	–	3	0.9	1	0.3	0	5.31
0.64	0.6563	0.685	0.2	–	3	0.9	1	0.3	0	4.53
0.66	0.66	0.6888	0.2	–	3	0.9	1	0.4	0	3.94
0.68	0.68	0.6927	0.2	–	3	0.9	1	0.4	0.02	3.54
0.7	0.7	0.6976	0.16	–	3	0.91	1	0.44	0	3.13
0.72	0.72	0.7024	0.16	–	3	0.9	1	0.5	0	2.84
0.74	0.74	0.7068	0.2	–	3	0.9	1	0.6	0	2.6
0.76	0.76	0.7266	0.6	–	1	0	–	1	0.56	3.38
0.78	0.78	0.7491	0.6	–	1	0	–	1	0.82	3.6
0.8	0.8	0.7714	0.6	–	1	0	–	1	1.16	3.91
0.82	0.82	0.7934	0.6	–	1	0	–	1	1.5	4.19
0.84	0.84	0.8152	0.6	–	1	0	–	1	1.97	4.63
0.86	0.86	0.8367	0.6	–	1	0	–	1	2.47	5.07
0.88	0.88	0.858	0.6	–	1	0	–	1	3.22	5.78
0.9	0.9	0.8806	0.5	–	1	0	–	1	3.82	6.25
0.92	0.92	0.9048	0.5	–	1	0	–	1	5.2	7.6
0.94	0.94	0.9288	0.5	–	1	0	–	1	7.22	9.55
0.96	0.96	0.9527	0.5	–	1	0	–	1	11.68	13.98
0.98	0.98	0.9764	0.5	–	1	0	–	1	24	26.22

Corollary 13. *For arbitrary σ MAX-k-DIRECTED-UNCUT has an approximation factor of* $1 + \sigma^2 - \sigma$.

Proof: Use $v = 0$, L arbitrary, $P = 2$, $\theta = 0$, ϑ arbitrary, $\kappa = -1$, $x_1 = -\frac{1}{\sigma}$ and $x_2 = 0$. □

6.5. *MAX-k-DENSE-SUBGRAPH*

For 0.2, … 0.48 and 0.52, … 0.64, the previously best approximation factors were given by Han et al. (2002) for 0.5 by Halperin and Zwick (2002) and in the other cases by Feige and Langberg (2001). Our improvement is for 0.04, … 0.68.

Table 12. Approximation factors and Parameters for MAX-k-VERTEX-COVER, $\sigma \leq \frac{1}{2}$

σ	Prev.	Our method	v	L	P	θ	ϑ	κ	x_1	x_2
0.02	0.75 + c	$0.75 - \epsilon$	1	–	–	–	–	–	$-3.37 \cdot 10^7$	$3.52 \cdot 10^7$
0.04	0.75 + c	$0.75 - \epsilon$	1	–	–	–	–	–	$-8.09 \cdot 10^6$	$8.79 \cdot 10^6$
0.06	0.75 + c	$0.75 - \epsilon$	1	–	–	–	–	–	$-3.44 \cdot 106$	$3.91 \cdot 10^6$
0.08	0.75 + c	$0.75 - \epsilon$	1	–	–	–	–	–	$-1.85 \cdot 10^6$	$2.2 \cdot 10^6$
0.1	0.75 + c	$0.75 - \epsilon$	1	–	–	–	–	–	$-1.12 \cdot 10^6$	$1.41 \cdot 10^6$
0.12	0.75 + c	$0.75 - \epsilon$	1	–	–	–	–	–	$-7.42 \cdot 10^5$	$9.77 \cdot 10^5$
0.14	0.75 + c	$0.75 - \epsilon$	1	–	–	–	–	–	$-5.17 \cdot 10^5$	$7.17 \cdot 10^5$
0.16	0.75 + c	$0.75 - \epsilon$	1	–	–	–	–	–	$-3.74 \cdot 10^5$	$5.49 \cdot 10^5$
0.18	0.75 + c	$0.75 - \epsilon$	1	–	–	–	–	–	$-2.78 \cdot 10^5$	$4.34 \cdot 10^5$
0.2	0.75 + c	$0.75 - \epsilon$	1	–	–	–	–	—-	$-2.11 \cdot 10^5$	$3.52 \cdot 10^5$
0.22	0.75 + c	$0.75 - \epsilon$	1	–	–	–	–	–	$-1.63 \cdot 10^5$	$2.91 \cdot 10^5$
0.24	0.75 + c	$0.75 - \epsilon$	1	–	–	–	–	–	$-1.27 \cdot 10^5$	$2.44 \cdot 10^5$
0.26	0.75 + c	$0.75 - \epsilon$	1	–	–	–	–	–	$-9.98 \cdot 10^4$	$2.08 \cdot 10^5$
0.28	0.75 + c	$0.75 - \epsilon$	1	–	–	–	–	–	$-7.89 \cdot 10^4$	$1.79 \cdot 10^5$
0.3	0.75 + c	$0.75 - \epsilon$	1	–	–	–	–	–	$-6.25 \cdot 10^4$	$1.56 \cdot 10^5$
0.32	0.75 + c	$0.75 - \epsilon$	1	–	–	–	–	–	$-4.94 \cdot 10^4$	$1.37 \cdot 10^5$
0.34	0.75 + c	$0.75 - \epsilon$	1	–	–	–	–	–	$-3.89 \cdot 10^4$	$1.22 \cdot 10^5$
0.36	0.75 + c	$0.75 - \epsilon$	1	–	–	–	–	–	$-3.04 \cdot 10^4$	$1.09 \cdot 10^5$
0.38	0.75 + c	0.7538	0.2	–	3	0.9	1	−0.3	−1.92	2.06
0.4	0.75 + c	0.7684	0.2	–	3	0.9	1	−0.3	−1.79	2.04
0.42	0.7518	0.7819	0.2	–	3	0.9	1	−0.3	−1.67	2.06
0.44	0.7687	0.7947	0.1	–	3	0.8	1	–0.2	−1.63	2.75
0.46	0.7844	0.8082	0.1	–	3	0.8	1	–0.2	–1.47	2.65
0.48	0.7987	0.8209	0.1	–	3	0.8	1	–0.1	–1.34	2.99
0.5	0.8452	0.8454	0.08	2	1	0.89	0.96	0	–1	1.33

6.6. *MAX-k-VERTEX-COVER*

Using linear programming, Ageev and Sviridenko (1999) proved an approximation factor of 0.75 for arbitrary σ which was improved by Feige and Langberg (2001) to $0.75 + c$ for a constant $c > 0$. For $0.02, \ldots, 0.4$, this was the previously best approximation factor. For $0.42, \ldots, 0.48$ and $0.52, \ldots, 0.6$, it was found by Han et al. (2002) and for 0.5 by Halperin and Zwick (2002). For $0.62, \ldots, 0.98$, the previously best approximation factor was $2\sigma - \sigma^2$ which was received by independent sampling (Feige and Langberg, 2001). Our improvement is for $0.38, \ldots, 0.74$.

We can verify the results of Ageev and Sviridenko, up to $\epsilon > 0$ arbitrarily small, and of Feige and Langberg:

Table 13. Approximation factors and Parameters for MAX-k-VERTEX-COVER, $\sigma > \frac{1}{2}$

σ	Prev.	Our method	ν	L	P	θ	ϑ	κ	x_1	x_2
0.52	0.822	0.843	0	–	3	0.8	1	−0.1	−1.06	2.62
0.54	0.8307	0.8532	0	3	1	0.9	0.9	0	−0.86	3.35
0.56	0.8377	0.8625	0	3	1	0.9	0.9	0	−0.71	3.31
0.58	0.8425	0.8707	0	1	–	1	0.8	–	−0.75	2.15
0.6	0.8453	0.8784	0	1	–	1	0.8	–	−0.69	1.89
0.62	0.8556	0.886	0	1	3	0.8	0.9	0.1	−0.62	1.81
0.64	0.8704	0.8934	0	–	3	0.7	1	0.1	−0.63	1.47
0.66	0.8844	0.9008	0	–	3	0.7	1	0.1	−0.63	1.23
0.68	0.8976	0.9081	0	–	3	0.7	1	0.1	−0.67	0.96
0.7	0.91	0.916	0	–	3	0.6	1	0.1	−0.71	0.75
0.72	0.9216	0.9241	0	1	3	0.6	0.9	0.1	−0.77	0.53
0.74	0.9324	0.9328	0	1	–	1	0.3	–	−0.95	0.14
0.76	0.9424	0.9424	0	–	2	0	–	−1	−2	0
0.78	0.9516	0.9516	0	–	2	0	–	−1	−2	0
0.8	0.96	0.96	0	–	2	0	–	−1	−2	0
0.82	0.9676	0.9676	0	–	2	0	–	−1	−2	0
0.84	0.9744	0.9744	0	–	2	0	–	−1	−2	0
0.86	0.9804	0.9804	0	–	2	0	–	−1	−2	0
0.88	0.9856	0.9856	0	–	2	0	–	−1	−2	0
0.9	0.99	0.99	0	–	2	0	–	−1	−2	0
0.92	0.9936	0.9936	0	–	2	0	–	−1	−2	0
0.94	0.9964	0.9964	0	–	2	0	–	−1	−2	0
0.96	0.9984	0.9984	0	–	2	0	–	−1	−2	0
0.98	0.9996	0.9996	0	–	2	0	–	−1	−2	0

Corollary 14. *For arbitrary σ MAX-k-VERTEX-COVER has an approximation factor $0.75 - \epsilon$ for $\epsilon > 0$ arbitrarily small and at least $2\sigma - \sigma^2$.*

Proof: (a) Use $\nu = 1$ (i.e. Linear Randomized Rounding), $x_1 = (2\sigma - 1) \cdot x_2$ and x_2 sufficiently large. (For the proof of $\varrho < 0.75$, x_2 must be chosen $> \frac{\varrho}{4\cdot(0.75-\varrho)\cdot\sigma^2}$. In Table 12 we prove an approximation factor of 0.74999 for $\sigma = 0.02, 0.04, \ldots, 0.36$.)

(b) Use $\nu = 0$, L arbitrary, $P = 2$, $\theta = 0$, ϑ arbitrary, $\kappa = -1$, $x_2 = -2$ and $x_2 = 0$. □

Remark 15. (a) Algorithm Graph Partitioning with the parameter settings of Corollary 14 (a) is equivalent to Linear Randomized Rounding with Size Adjusting. The approximation factor of Corollary 14(a) can also be obtained by standard concentration results on the size of $|S''|$ without using the Algorithm Graph Partitioning. The same holds for Corollary 11.

(b) Algorithm Graph Partitioning with the parameter settings of Corollary 14(b) is equivalent to independent sampling. The approximation factor of Corollary 14(b) can also be obtained by analyzing independent sampling without using the Algorithm Graph Partitioning. The same holds for Corollary 12 and Corollary 13.

7. Questions and open problems

- Can we improve the approximation factors by combining the RPR^2 rounding technique (Feige and Langberg, 2001) with our methods?
- Is there a technique to find "good" parameters θ, ϑ, κ, ν?

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Notes

1. In some literature MAX-k-CUT denotes the problem of partitioning the set of vertices into subsets $S_1, \ldots, S_k$, so that the total weight of the edges connecting S_i and S_j for $1 \leq i \neq j \leq k$ is maximized.
2. The approximation factor of 0.6436 of Halperin and Zwick (2002) seems to be incorrect. Halperin and Zwick in section 4.1 claim that (17) holds for $d_2 \geq 0$. But for $d_2 = 0.81$ (their parameter for MAX-k-UNCUT) and $x = -\frac{1}{3}$ we have: $4 \arccos(-\frac{1}{3} \cdot 0.81) - 3 \arccos(-\frac{7}{9} \cdot 0.81) - \arccos(0.81) < 0$ (so the assumption of Lemma 7 (d) does not hold). Using Lemma 7 (c) of this paper and $d_2 = 0.81$, we get an approximation factor of 0.6414.
3. Again the approximation factor of 0.8118 of Halperin and Zwick seems to be incorrect, as for $d_2 = 0.74$ and $x = -\frac{1}{3}$ we have: $4 \arccos(-\frac{1}{3} \cdot 0.74) - 3 \arccos(-\frac{7}{9} \cdot 0.74) - 3 \arccos(0.74) < 0$. Using again Lemma 7 (c) instead of Lemma 7 (d), the approximation factor becomes 0.811.

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