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ORIGINAL PAPER

Quantum-like behavior without quantum physics I

Kinematics of neural-like systems

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Abstract Recently there has been much interest in the possible quantum-like behavior of the human brain in such functions as cognition, the mental lexicon, memory, etc., producing a vast literature. These studies are both empirical and theoretical, the tenets of the theory in question being mainly, and apparently inevitably, those of quantum physics itself, for lack of other arenas in which quantum-like properties are presumed to obtain. However, attempts to explain this behavior on the basis of actual quantum physics going on at the atomic or molecular level within some element of brain or neuronal anatomy (other than the ordinary quantum physics that underlies everything), do not seem to survive much scrutiny. Moreover, it has been found empirically that the usual physics-like Hilbert space model seems not to apply in detail to human cognition in the large. In this paper we lay the groundwork for a theory that might explain the provenance of quantum-like behavior in complex systems whose internal structure is essentially hidden or inaccessible. The approach is via the logic obeyed by these systems which is similar to, but not identical with, the logic obeyed by actual quantum systems. The results reveal certain effects in such systems which, though quantum-like, are not identical to the kinds of quantum effects found in physics. These effects increase with the size of the system.

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1 Introduction

The observation that certain brain functions seem to obey quantum rules goes back at least to 1949, when Heinz von Foerster drew a compelling parallel between human memory function and decay in quantum systems (6th Macy Conference on Cybernetics, 1949, proceedings published 1950, republished in [1]). More recently there has been a resurgence of interest in the possible quantum-like behavior of the human brain in such functions as cognition, the mental lexicon, memory, etc., producing a vast literature. These studies are both empirical and theoretical, the tenets of the theory in question being mainly, and apparently inevitably, those of quantum physics itself, for lack of other arenas in which quantum-like properties are presumed to obtain. (Some exceptions are discussed below.) For comprehensive recent treatments see [2] and [3]. For an application in the spirit of the present paper, see [4].

The elephant-like conundrum in the room is the provenance of this quantum-like behavior. The obvious suspect, namely some actual quantum physics going on at the atomic or molecular level within some element of brain or neuronal anatomy (other than the ordinary quantum physics that underlies everything), does not seem to survive much scrutiny [5]. Moreover, it has been found empirically that the usual physics-like Hilbert space model seems not to apply in detail to human cognition in the large [6] and in fact our approach here is not Hilbert space based. For other non-Hilbertian models completely different from ours, see [7] and for a review of these matters see [8]. (For a probabilistic model more general than ordinary Hilbertian quantum mechanics see [3]. For an argument in favor of a classical, or non-quantum mechanical, effect via interference and synchronicity see [9]. We shall return to these notions in the sequel.) As far as fundamental reasons for the existence of this behavior are concerned, the matter seems to rest: the empirical results are generally regarded as justifying the adoption of the tenets of standard quantum physics despite the fact that these tenets seem not to apply in detail. So a mystery remains.

Classical systems, which do not exhibit quantum-like behavior, follow ordinary Boolean logic. The systems we study, which may include neural systems that exhibit quantum-like behavior, have states that we call "confusable". These are states that are similar to one another but are such that their small differences may affect the system's behavior in certain ways not necessarily apparent to external systems. We call systems with confusable states *discriminating* systems; we call other (classical) systems *non-discriminating* systems. Discriminating systems and their quantum-like behavior can be described using a special non-classical logic.

We shall argue that the logic intrinsic to such systems requires a small adjustment to, or deformation of, the usual Boolean logic of non-discriminating systems, where here non-discriminating means "confusable iff identical." For such a non-discriminating system, this logic, namely the collection of all possible propositions concerning the system, is the Boolean lattice of all subsets of the set of states of the system. This Boolean lattice of propositions is replaced in the "discriminating" cases of interest here with a different kind of lattice of subsets. These lattices differ in only one respect from the Boolean case, namely, they are not distributive: the meet does not distribute over the join, nor the join over the meet, an equivalent condition in any lattice. Such lattices are called *ortholattices*, the involution taking the place of complementation in the Boolean case being called in this case the *orthocomplement*. As we shall argue, this single difference, namely the non-distribution of meet over join, is sufficient to explain most if not all of the quantum-like behaviors which seem so anomalous to classical thinkers. Just as ordinary propositional calculus (**PC**) is modeled by Boolean lattices, so there is a logic modeled by ortholattices. It is called *orthologic* (**OL**) and was first studied by R. Goldblatt [10]. This is the logic that emerges as the correct replacement for **PC** in the models of interest, and we shall exploit various forms of its model theory to reveal quantum-like attributes of these systems. We argue that certain of these models already exhibit, in the total absence of physical trappings, such standard quantum-like classically anomalous behaviors as "quantum parallelism" (as in the fable of Schrödinger's cat) and "quantum interference" (*á la* the double slit experiment), though these phenomena are not independent, both stemming from the peculiarities of quantum-like disjunction (Section 4). As examples of such models we posit the sets of states of drastically simplified versions of a "network" of the kind mentioned above. Namely, we shall, for the purpose of this paper, except in the the simplest cases of Boolean or classical networks discussed in Subsection 6.1, ignore the details of the network itself, returning to it in the sequel. We are left with the state spaces of clusters of nodes, considered as discriminating systems, whose appropriate logic is **OL**. We shall find that, in analogy with the case of aggregates of non-interacting physical quanta, our logical requirements impose quantum-like behavior on such clusters, though apparently in a different form from actual quantum mechanics (Section 6.3).

We emphasize that our considerations here refer to the kinematics of the possible spaces of states involved: that is to say, the states of affairs before the systems are "observed" or "measured." Thus the correspondent here to the problematic phenomenon known in ordinary quantum theory as the "collapse of the wave-function" does not arise in this paper. It will be addressed in the sequel.

The layout of the paper is as follows. Section 2 discusses the kind of model (or Kripke frame) of interest which determines the logic **OL** as that which is modeled by its lattice of propositions. Section 3 is a recital of the known model theory of **OL** and various closely related topics which will form the foundation for what is to follow. Section 4 traces the logical manifestations of quantum-like behavior in these models, namely *parallelism* and *interference*, as mentioned above. In Section 5 we state a theorem and three of its corollaries, which enables us to make precise the distinction between the presence of quantum-like behavior and its absence. Its proof, along with other distracting mathematical material, is relegated to an Appendix A. In Subsection 6.2 we discuss various examples both non quantum-like and quantum-like. The case of a single node produces non quantum-like behavior as expected but nevertheless the logic constrains this classical behavior in a possibly surprising way, and one which is entirely consistent with the behavior of a biological neuron.

The last Subsection 6.3 considers the case of nodal clusters in which quantum-like behavior is to be expected in the light of the theorem mentioned above, and we indicate how our conclusions differ from the case of the ordinary quantum theory of collections of quanta. In Section 7 we summarize our conclusions.

There are two appendices: Appendix A is the mathematical appendix mentioned above, and Appendix B describes in very simple terms the structure and function of biological neurons.

2 Proximity spaces and ortholattices

Our interest is in systems whose sets of states may be characterized as *proximity spaces*. A proximity space $\langle W, \approx \rangle$ is a set W with a relation $\approx$ on it having the properties: reflexivity (for $w \in W$, $w \approx w$) and symmetry (for $w, v \in W$, $w \approx v$ iff $v \approx w$). Such a relation, called a *proximity*, is not generally transitive. Note that identity, $=$, is a proximity on any set. A proximity relation has the informal intuitive reading: $v \approx w$ iff the state v is *confusable* with the state w in the absence, say, of a direct observation or measurement, though the states are discernibly different in general (except of course in the case of identity). In other words, in the absence of any interference (such as a measurement), if such a system can be surmised to be in the state v, it may as well, for all intents and purposes, be surmised to be in the state w, and conversely. (For other similar interpretations and examples, see [11].)

This confusability of elements of W will now affect the practical operational matter of assembling subsets of W. If the proximity were identity, $=$, then a subset E is trivially assembled from its elements as

$$\begin{aligned} E &= \bigcup_{v \in E} \{v\} && (1)\\ &= \bigcup_{v \in E} \{w \in W : w = v\} && (2)\\ &= \{w \in W : \exists\, v \in E \text{ such that } w = v\}. && (3) \end{aligned}$$

In the case of a general proximity, $\approx$, this must be replaced by

$$\{w \in W : \exists\, v \in E \text{ such that } w \approx v\} =: \Diamond E. \tag{4}$$

This is the set constructible out of the elements of E to within the limits of confusability. Within the operational dictates of confusability, this is the closest to the set E one can get by assembling its elements. Note that $E \subseteq \Diamond E$.

Similarly, since in the case of the identity proximity (and denoting set complementation by the superscript c) we have

$$E^c = \{w \in W : \exists\, v \in E \text{ such that } w = v\}^c \tag{5}$$

this becomes, in the general case, $(\Diamond E)^c$ and so, within the limits of confusability, we obtain, as the proper generalization of $(\ )^c$, the subset

$$(\Diamond E)^* := \Diamond(\Diamond E)^c. \tag{6}$$

For a proximity space $\langle W, \approx \rangle$ the sets of the form $\Diamond E$, for $E \subseteq W$, were shown by J. L. Bell [11, 12] to constitute a complete ortholattice, with join being the ordinary set union, the concomitant meet of two elements $\Diamond E$ and $\Diamond F$ being the largest subset of the form $\Diamond(\ )$ contained within $\Diamond E \cap \Diamond F$, and the complement given by $(\ )^*$. Bell calls this lattice the lattice of "parts" of $\langle W, \approx \rangle$ and we shall follow him in denoting it by Part W since the proximity relation will never be ambiguous. (See Subsection A.3 for more on the meet in this lattice.)

The upshot is that the proper logic of propositions that deal "operationally" with the systems we are interested in, which generalizes **PC** in the Boolean case, is **OL**. Consequently we shall adopt **OL** as our overarching logic when dealing with the systems of interest to us here.

We shall therefore need first to rehearse the relevant material on this logic and its model theory.

3 Orthologic, modal logic and the Goldblatt completeness theorem

As mentioned above, orthologic (**OL**), which is in fact the core logic underlying *echt* quantum logic, is a weakening of ordinary **PC**: one assumption is dropped, namely that conjunction distributes over disjunction. In 1974 Goldblatt [10] introduced a deductive system for **OL** and proved completeness theorems for it in terms of certain models. For example, ordinary **PC** may be characterized by morphisms of formulas into Boolean lattices: a formula is valid, or provable, in **PC** if and only if its image under any morphism into any Boolean lattice is the top element. There is a similar completeness theorem for intuitionistic logic (**IL**) with Boolean lattices being replaced by Heyting lattices. Goldblatt proved a similar completeness theorem for **OL**, the target lattices in this case being ortholattices. (Since Boolean lattices are ortholattices it follows immediately from the respective completeness theorems that any orthotheorem is also a theorem of **PC**, but clearly not conversely: **OL** is strictly weaker than **PC**). In the **OL** case the model theory bifurcates in the sense that there is another kind of model that also characterizes **OL**, namely a Kripkean one. From the existence of such models, one finds a different sort of semantics arising solely from the peculiarities of disjunction, and it is this semantics—entirely absent in the classical Boolean case—that mimics quantum behavior. This is because in the case of the slightly stronger quantum logic itself, disjunction is exactly "quantum" superposition, the nexus of most if not all of the puzzlements classical thinkers experience when confronted with quantum theory. We remark that in the case of the Boolean models of **PC** these two types of model collapse into one. The existence of these Kripkean models of **OL** led Goldblatt to realize that **OL** itself may be embedded into a well-studied modal system, namely the B (for "Brouwersche") modal system of Becker [13], which well predates the advent of quantum logic in 1936 [14]. The same result was obtained almost simultaneously but independently by Dishkant [15]: see also [16]. The associated Kripke models for this B-system also provide a semantics for probing the anomalies of disjunction and it is these we shall focus on first, since this system is well known, and reveals the quantum-like behavior clearly and simply. In this section we give also a brief account of Goldblatt's Kripkean completeness theorem for **OL** (theorem 3): the models here are closer in their properties to actual quantum theory.

It will suffice for our immediate purpose here to rehearse the modal embedding theorem for **OL**. Fairly complete though slightly simplified accounts of these matters, with proofs and references, may be found in any of [17–19].

Goldblatt's realization (ibid.) of **OL** as a deductive system may be described as follows. The atoms, or primitive symbols, are

(i) a denumerable collection Φ_0 of propositional variables $a_1, a_2, \ldots$;
(ii) the connectives $\sim$ ("negation") and $\sqcap$ ("conjunction"); and
(iii) parentheses.

The set Φ of well-formed *orthoformulas* is constructed from these in the usual manner. Elements of Φ will be denoted by lower case Greek characters α, β, ..., usually taken from the beginning of the alphabet. We shall generally try to reserve characters at the end of the Latin alphabet for elements of sets of various kinds.

Since there is no implication sign in Φ, a formal deductive calculus is based on *sequents* involving at most single formulas and written in the form

$$\alpha \vdash \beta \tag{7}$$

for $\alpha, \beta \in \Phi$, the reading of which is that β may be inferred from α. Certain sequents are designated as *axioms*, and there are three *rules of inference,* namely, for any formulas α, β:

AXIOMS
O1. $\alpha \vdash \alpha$
O2. $\alpha \sqcap \beta \vdash \alpha$
O3. $\alpha \sqcap \beta \vdash \beta$
O4. $\alpha \vdash \sim\sim\alpha$
O5. $\sim\sim\alpha \vdash \alpha$
O6. $\alpha \sqcap \sim\alpha \vdash \beta$
INFERENCE RULES
O7. $\dfrac{\alpha \vdash \beta \quad \beta \vdash \gamma}{\alpha \vdash \gamma}$
O8. $\dfrac{\alpha \vdash \beta \quad \alpha \vdash \gamma}{\alpha \vdash \beta \sqcap \gamma}$
O9. $\dfrac{\alpha \vdash \beta}{\sim\beta \vdash \sim\alpha}$

A disjunctive connective may be introduced through the following definition

$$\alpha \sqcup \beta := \sim((\sim\alpha) \sqcap (\sim\beta)) \tag{8}$$

and dual forms of O2, O3, O6 and O8 follow.

A string $s_1; s_2; \ldots; s_n$ of sequents is called a *proof* of its last member s_n if each s_i is either an axiom or follows from some preceding sequent through the use of one of the rules of inference. If there exists a proof of a sequent $\alpha \vdash \beta$ then we write

$$\alpha \vdash_{\mathbf{O}} \beta \tag{9}$$

and say the β is *deducible from* α *in orthologic*. If $\alpha \vdash_{\mathbf{O}} \beta$ for *any* formula α we say that β is a *theorem of orthologic* or an *orthotheorem*, and we write

$$\vdash_{\mathbf{O}} \beta. \tag{10}$$

(Note that this is equivalent to $\alpha \sqcup \sim\alpha \vdash_{\mathbf{O}} \beta$.)

The main result we shall utilize initially is the Modal Translation Theorem [10, 15, 16]. This requires a brief account of the modal system involved, namely the "Brouwersche" or B-system of Becker (cf. for example [20–22]). To describe this system we introduce the ordinary Boolean connectives $\neg$ (negation), $\wedge$ (conjunction), and the modal operator $\Box$ (necessity). Material implication $\rightarrow$, and the possibility operator $\Diamond$ are introduced through the usual definitions (for example $\Diamond := \neg\Box\neg$). The axioms and inference rules include the usual ones for **PC** with *modus ponens*, and the modal additions (for formulas α and β):

$$\Box(\alpha \rightarrow \beta) \rightarrow (\Box\alpha \rightarrow \Box\beta) \tag{11}$$

$$\Box\alpha \rightarrow \alpha \tag{12}$$

$$\alpha \rightarrow \Box\Diamond\alpha \quad \text{(the "Brouwersche" axiom)} \tag{13}$$

If α is deducible then $\Box\alpha$ is deducible. (Necessitation) (14)

We denote the set of modal formulas by Φ_M. The theoremhood of a formula α is defined as usual and we write

$$\vdash_{\mathbf{B}} \alpha \tag{15}$$

if α is a theorem of the B-modal system.

(The origin of the odd nomenclature in (13) is that $\Box\Diamond := \Box\neg\Box\neg$ is like a strong form of double negation and the rule in (13) is then similar to the rule $p \rightarrow \neg\neg p$ in **IL**, the form of logic favored by L. E. J. Brouwer, whose converse is invalid in that system.)

The completeness theorem for this system involves models of the following type. A *(Kripke) B-model* is a triple $\mathscr{B} = \langle W, \approx, \vartheta\rangle$ where W is a set of "worlds" (later to be renamed), $\approx$ is a proximity on W, and ϑ is a function $\vartheta\colon \Phi_M \times W \to \mathbf{2}$, where $\mathbf{2}$ denotes the two element Boolean lattice, satisfying:

V1. For each $w \in W$, $\vartheta(\ , w)\colon \Phi_M \to \mathbf{2}$ is a Boolean valuation with respect to $\neg$ and $\wedge$. That is:
$\vartheta(\neg\alpha, w) = \neg\vartheta(\alpha, w)$ where $\neg$ on the right denotes complementation in $\mathbf{2}$ and
$\vartheta(\alpha \wedge \beta, w) = \vartheta(\alpha, w) \wedge \vartheta(\beta, w)$ where $\wedge$ on the right denotes the meet in $\mathbf{2}$;
V2. For any modal formula α, $\vartheta(\Box\alpha, w) = 1$ iff $\vartheta(\alpha, x) = 1 \forall x$ such that $x \approx w$. It follows that:
V3. $\vartheta(\alpha \vee \beta, w) = \vartheta(\alpha, w) \vee \vartheta(\beta, w)$ where $\vee$ on the right denotes the join in $\mathbf{2}$, and
V4. For any modal formula α, $\vartheta(\Diamond\alpha, w) = 1$ iff $\exists\, x \approx w$ (i.e. $\exists\, x$ with $x \approx w$) such that $\vartheta(\alpha, x) = 1$.

A modal formula α is said to be:
true at the world w in the B-model $\mathscr{B}$, written $w \models_{\mathscr{B}} \alpha$, iff $\vartheta(\alpha, w) = 1$;
true on the set $E \subseteq W$, written $E \models_{\mathscr{B}} \alpha$, iff $w \models_{\mathscr{B}} \alpha$ for all $w \in E$;
true in the B-model $\mathscr{B}$ iff $W \models_{\mathscr{B}} \alpha$;
B-valid, written $\models \alpha$, iff it is true in *all* B-models.
These models characterize the B-system:

Theorem 1

$$\vdash_{\mathbf{B}} \alpha \text{ iff } \models \alpha$$

This is proved in the references cited. (Note that if we consider the ensemble of B-models for which the proximity relation is taken to be that of identity, then the modalities collapse and the logic so characterized is just ordinary **PC**. In this sense, the quantum-like behavior to be discussed in the next section arises from deformations of the identity relation into more general proximity relations in B-models.)

As noted, a translation of **OL** into the B-system was found by Goldblatt and independently Dishkant. The translation recursively assigns to each orthoformula $\alpha \in \Phi$ a modal formula $\alpha^\circ \in \Phi_M$ as follows:

T1. For atomic formulas a_i: $a_i^\circ = \Box\Diamond a_i$
T2. $(\alpha \sqcap \beta)^\circ = \alpha^\circ \wedge \beta^\circ$
T3. $(\sim\alpha)^\circ = \Box\neg\alpha^\circ$

Then we have the following Modal Translation Theorem.

Theorem 2 *For $\alpha \in \Phi$*

$$\vdash_{\mathbf{O}} \alpha \text{ iff } \vdash_{\mathbf{B}} \alpha^\circ$$

For proofs see the references already cited.

The translation of **OL** into the modal B-system is interesting and instructive, as we shall see in Section 4, but of course the B-system is presumably a stronger logic. It will be essential for later applications to state a completeness theorem for **OL** itself, which has a somewhat similar Kripkean model theory, but yields a rather different semantics. As mentioned above, this is due to Goldblatt ([10]: see also [15, 16]). To state the result, some terminology is required.

An *orthogonality space* $F = \langle W, \perp \rangle$ comprises a set W and a binary relation $\perp$ which is an *orthogonality*: namely, it is *irreflexive* ($x \not\perp x$) and *symmetric* ($x \perp y$ iff $y \perp x$). For $x \in W, Y \subseteq W$ we write $x \perp Y$ iff $x \perp y\ \forall y \in Y$ and define $Y^{\perp} := \{x \in W : x \perp Y\}$. Note that each proximity space $\langle W, \approx \rangle$ determines an orthogonality space $\langle W, \perp \rangle$ where $x \perp y$ iff $x \not\approx y$, and conversely each orthogonality space $\langle W, \perp \rangle$ determines a proximity space $\langle W, \approx \rangle$ where $x \approx y$ iff $x \not\perp y$. It is easy to show that for any $E, F \subseteq W$, we have $E \subseteq E^{\perp\perp}$, and $(E \cup F)^{\perp} = E^{\perp} \cap F^{\perp}$. Note also that $\emptyset^{\perp} = W$ and $W^{\perp} = \emptyset$. (cf. Section A).

In Goldblatt's terminology $Y \subseteq W$ is said to be *regular* if

$$Y^{\perp\perp} = Y. \tag{16}$$

For reasons to appear, we shall call such subsets *propositions*.

The class $R(\langle W, \perp \rangle)$ of propositions of W constitutes a complete ortholattice under the partial order given by set inclusion and $\perp$ as orthocomplement. Note that W and $\emptyset$ are both propositions.

A *Kripke orthomodel* $\mathscr{M} = \langle W, \approx, \varrho \rangle$ is a proximity space $\langle W, \approx \rangle$ and a *valuation* $\varrho : \Phi \to R(\langle W, \perp \rangle)$ satisfying:

$$\varrho(\sim\alpha) = \varrho(\alpha)^{\perp} \tag{17}$$

$$\varrho(\alpha \sqcap \beta) = \varrho(\alpha) \cap \varrho(\beta). \tag{18}$$

We will say that a formula α is:

true at the "world" $w \in W$, and write $w \Vdash_{\mathscr{M}} \alpha$, iff $w \in \varrho(\alpha)$;

true on a set $E \subseteq W$, and write $E \Vdash_{\mathscr{M}} \alpha$, iff $w \in \varrho(\alpha)$ for all $w \in E$—i.e. iff $E \subseteq \varrho(\alpha)$;

true in the Kripke orthomodel $\mathscr{M}$ iff $W \Vdash_{\mathscr{M}} \alpha$;

Kripke valid, and write $\Vdash \alpha$, iff it is true in *all* Kripke orthomodels.

Then the Goldblatt completeness theorem is as follows:

Theorem 3

$$\vdash_{\boldsymbol{O}} \alpha \textit{ iff } \Vdash \alpha$$

For proofs we refer to the references cited.

Now, returning to Section 2 in light of this model theory, our proximity spaces of states are seen to provide Kripkean models for **OL** in which the valuations ϱ take values in the lattice $R(\langle W, \perp \rangle)$ of propositions. But our argument in Section 2 would seem to have identified the lattice Part W as the correct generalization of the lattice of Boolean propositions. Fortunately, these notions coincide thanks to the following theorem due to J. L. Bell [11, 12]. Here $\Diamond(\ )$ is as defined in Section 2, and $\Box(\ )$ is as defined in (64).

Theorem 4 *Given an orthogonality space $\langle W, \perp \rangle$:*

$$\Diamond : R(\langle W, \perp \rangle) \to \mathit{Part}\, W \tag{19}$$

is an isomophism of complete ortholattices. Its inverse is

$$\Box : \mathit{Part}\, W \to R(\langle W, \perp \rangle). \tag{20}$$

A complete proof may be found in [17], section 2.4; see also [11]. We shall now generally relinquish the lattice Part W in favor of the completely isomorphic lattice $R(\langle W, \perp \rangle)$ in those places where the Kripkean models are invoked. However, it will be used again in the proof of theorem 5 given in Subsection A.3.

4 Quantum–like behavior in the models

With theorems 2, 3 and 4 in hand we can now investigate some semantics of orthotheorems in these models.

By way of preamble, let us note some properties of actual quantum mechanics. In this theory (which has passed every conceivable test over the course of nearly a century) a "quantum" (such as a particle like an electron or a photon) is not an object in the classical macroscopic sense since it does not have objective properties or attributes. These will generically depend upon the mode of obtaining information about them. The observation, or determination, of one property (for instance, the momentum of an electron, say) will render another property indefinite (the electron's position in this case) and vice versa. This of course flies in the face of classical tenets about objects. This particular example epitomizes the Heisenberg Uncertainty Principle, whose real import is that quanta do not have objective states of being. Instead, certain repertoires of actions may be specified as possible to carry out upon a quantum, and it is the ensembles of these actions—which occupy a complex Hilbert space—that determine all possible propositions that can be asserted about the quantum. These Hilbert space elements may be thought of as descriptors of those possible elementary experiments that can be done on the quantum, and as such they are not inherent to the quantum, but rather inherent to the *episystem*, which is everything involved in experimenting upon the system which is not the system itself, including the experimenter, apparatus, etc. (cf. [23]). "Observers" correspond to different sets of elements and that which is observed now depends on these sets, and are generically different among observers. (Technically, an "observer" corresponds to an algebra of commuting Hermitian operators on the Hilbert space in question, each such algebra determining an eigenbasis common to each of its elements.) This is how the non-objective nature of the quantum manifests itself in the formalism. The operational upshot of this is that a quantum can only be characterized per "observer" or experimenter or episystem, by certain combinations, called *superpositions*, of sets of measurable acts (namely, the eigenbasis mentioned above) which are in a sense simultaneously potential, or in *parallel*. (This is the property that is exploited in the hopeful science of quantum computing.) Only when a measurement or intervention occurs may the superposition collapse onto a suitable eigenvector representing that act corresponding to the measurement or intervention (whose eigenvalue is the value of the outcome). (This operational view is expounded for instance in [23]. See also [18], and for a treatment in the same spirit [24].)

This realization by Heisenberg of the non-objective nature of quanta constituted a radical departure from the ontology of macroscopic classical physics. For the systems of interest to us here, we do not posit such a radical ontology. For these systems, the states are actual, though they may not be directly accessible to "observers," by which we mean possible episystems, or other components of a family of systems. Thus, they reveal themselves, as in the case of actual quantum mechanics, in terms of the orthological version of superposition, namely *orthodisjunction*, via the modal translation theorem (Theorem 2). As we have noted, this is how orthodisjunction is interpreted in actual quantum logic, whose models are certain lattices of closed subspaces of complex Hilbert spaces, which we do not discuss here. It is the strangely non-classical behavior of this disjunction that lies at the root of the discomfort experienced by classical thinkers when confronted with quantum mechanics, as we have noted. These non-classical properties are preserved in **OL** and its models, entirely in the absence of any physical trappings, as we shall now demonstrate.

Thus suppose $\mathscr{B} = \langle W, \approx, \vartheta \rangle$ is a B-model and α, β are two orthoformulas. We wish to simplify the modal translation of the formula $\alpha \sqcup \beta$ (Theorem 2). It will prove convenient to employ the formalism of *truth sets*. Namely, we write for each modal formula α:

$$\|\alpha\|_{\mathscr{B}} := \{w \in W : w \models_{\mathscr{B}} \alpha\}, \tag{21}$$

called the *truth set* of α relative to $\mathscr{B}$. Then, dropping the subscripts, and noting the definitions in Subsection A.1, the following assertions are easily proved:

$$\|\alpha \wedge \beta\| = \|\alpha\| \cap \|\beta\|, \tag{22}$$

$$\|\neg\alpha\| = \|\alpha\|^c, \tag{23}$$

$$\|\alpha \vee \beta\| = \|\alpha\| \cup \|\beta\|, \tag{24}$$

$$\|\Box\alpha\| = \Box\|\alpha\|, \tag{25}$$

$$\|\Diamond\alpha\| = \Diamond\|\alpha\|. \tag{26}$$

Then (from T1–T3 in Section 3)

$$\|(\alpha \sqcup \beta)^\circ\| = \|(\sim[(\sim\alpha) \sqcap (\sim\beta)])^\circ\| \tag{27}$$

$$= \|\Box\neg[(\sim\alpha)^\circ \wedge (\sim\beta)^\circ]\| \tag{28}$$

$$= \|\Box\neg[\Box\neg\alpha^\circ \wedge \Box\neg\beta^\circ]\| \tag{29}$$

$$= \|\Box[\Diamond\alpha^\circ \vee \Diamond\beta^\circ]\| \tag{30}$$

$$= \Box\Diamond(\|\alpha^\circ\| \cup \|\beta^\circ\|) \quad \text{from M1 etc. of proposition 1 in Subsection A.1} \tag{31}$$

$$= \Box\Diamond(\|\alpha^\circ \vee \beta^\circ\|) \tag{32}$$

$$= \|\Box\Diamond(\alpha^\circ \vee \beta^\circ)\|. \tag{33}$$

Thus, for $w \in W$,

$$w \models_{\mathscr{B}} (\alpha \sqcup \beta)^\circ \qquad \text{iff} \qquad w \models_{\mathscr{B}} \Box\Diamond(\alpha^\circ \vee \beta^\circ). \tag{34}$$

The "semantics" is now given by just unfolding the definitions in the last formula, namely:

In any state $w \in W$: $\forall u \approx w\ \exists x \approx u$ such that $x \models_{\mathscr{B}} \alpha^\circ$ or $x \models_{\mathscr{B}} \beta^\circ$.

The points to notice are:

– The validity or otherwise of either α° or β° may not be definite nor even defined in every state w. These may only be determined at at least one unknown daughter (i.e. w-proximal) state, since $w \approx w$, and unknown granddaughter states. (Since the proximity relation is not transitive, these granddaughter states are not proximal, i.e. not confusable with the current state w.) This indefiniteness of validity in the current state is the characteristic feature of (quantum-like) *superposition*. In the case of actual quantum mechanics, the system would be said to be *in a superposition* (of states).
– The conditions for the validity or otherwise of *both* branches of the disjunction are carried over to at least one unknown daughter state and many unknown granddaughter states and must, to a classical thinker, therefore be somehow available to them. So, to such a thinker, these potentialities must seem to coexist in some form over the span of pairs of states proximally connected to the current state w. This interpretation of superposition, in actual quantum theory, goes by the name *parallelism*.

After inspecting two examples we shall, in Section 5, delineate more precisely what we mean by quantum-like behavior in a model of this type.

4.1 Example 1. Schrödinger's cat

Our first application will be to Schrödinger's cat Ketzi. Here we take an atomic formula a to represent the proposition *Ketzi is alive.* The appropriate orthotheorem is then $\vdash_{\mathbf{O}} a \sqcup \sim a$. By theorem 2 this translates into the B-theorem $\vdash_{\mathbf{B}} (a \sqcup \sim a)^{\circ}$. Our task is to simplify this translation and investigate its behavior vis-à-vis B-models.

From (33) we have in any B-model:

$$\begin{aligned}
\|(a \sqcup \sim a)^{\circ}\| &= \|\Box\Diamond(a^{\circ} \vee (\sim a)^{\circ}))\| && (35)\\
&= \|\Box\Diamond(\Box\Diamond a \vee \Box\neg\Box\Diamond a)\| && (36)\\
&= \|\Box\Diamond(\Box\Diamond a \vee \neg\Diamond\Box\Diamond a)\| && (37)\\
&= \|\Box\Diamond(\Box\Diamond a \vee \neg\Diamond a)\| \quad \text{from M9} && (38)\\
&= \|\Box(\Diamond\Box\Diamond a \vee \Diamond\neg\Diamond a)\| \quad \text{from M1} && (39)\\
&= \|\Box(\Diamond a \vee \Diamond\Box\neg a)\| \quad \text{from M9} && (40)\\
&= \|\Box\Diamond(a \vee \Box\neg a)\| \quad \text{from M1.} && (41)
\end{aligned}$$

We now consider the elements of W to represent "states" of a system in which the interpretation is carried out, and choose any $w \in W$. Then, from the last equation,

$$w \models_{\mathscr{B}} (a \sqcup \sim a)^{\circ} \qquad \text{iff} \qquad w \models_{\mathscr{B}} \Box\Diamond(a \vee \Box\neg a). \tag{42}$$

As before, the semantics is given by just unfolding the definitions in the last formula, namely:

In any state $w \in W$: $\forall u \approx w\ \exists x \approx u$ such that $x \models_{\mathscr{B}} a$ or $x \models_{\mathscr{B}} \Box\neg a$.

or

In any state $w \in W$: $\forall u \approx w\ \exists x \approx u$ such that $x \models_{\mathscr{B}}$ *Ketzi is alive* or $x \models_{\mathscr{B}} \Box$ *Ketzi is dead*

If we interpret W as the set of states of Ketzi's abuser's box, in the absence of any external interference, such as releasing the poison or opening the box, then the above interpretation may be read:

> In a given state w, $\forall u \approx w\ \exists x \approx u$ such that Ketzi is alive in state x or Ketzi is dead in state x and remains dead at x-proximal states (since $z \models_{\mathscr{B}} \neg a$ *for all* $z \approx x$). This occurs at every state w.

Note here, as before, the appearance of the characteristic feature of "quantum" superposition: at some states w of the system the truth or otherwise of the proposition a may not be definite nor even defined, as far as an episystem, or "observer," is concerned. To a classical thinker the truth value at such states would seem to hover between the two alternative possibilities. This value is potential, not actual in such states, only determined at some unknown daughter and many unknown granddaughter states, never at the state in question. Moreover, the apparent retention of both branches of a decision tree of potential outcomes over two generations, seems to classical thinkers to imply some sort of inherent joint existence or storage of these two branches. As noted above, this is exactly ("quantum") *parallelism*. So to classical thinkers Ketzi seems to be dead and alive simultaneously in such states since to them both outcomes seem to be somehow stored and the truth value of the proposition a is always in abeyance, at least until the box is opened.

The reader may note that these conclusions have been reached just by working through the ineluctable algebra of modal logic, and that no metaphysical contortions concerning the actual state of Ketzi as an object, her de Broglie wavelength, etc., were required.

The phenomena of "quantum" parallelism, and another quantum-like feature called *interference*, may be more plainly seen in our second example.

4.2 Example 2. The double slit experiment

In this case we concoct another orthotheorem in the form of a disjunct. For instance suppose the atomic formula a_i, $i = 1, 2$, denotes the proposition *the electron passes through slit i*. Then $\vdash_{\mathbf{O}} a_i \sqcup \sim a_i$ is the orthotheorem representing the proposition *the electron passes through slit i or it does not* and $\vdash_{\mathbf{O}} (a_1 \sqcup \sim a_1) \sqcup (a_2 \sqcup \sim a_2)$ is the orthotheorem (its orthotheoremhood following from the dual forms of either O2 or O3) representing the proposition *the electron passes through slit 1 or the electron passes though slit 2 or the electron passes through neither slit*. Writing this as $\vdash_{\mathbf{O}} \alpha_1 \sqcup \alpha_2$ it is an easy exercise using M1 and M9, and (33) and (41), to show that for an arbitrary B-model $\mathscr{B}$ and $w \in W$:

$$w \models_{\mathscr{B}} (\alpha_1 \sqcup \alpha_2)^\circ \quad \text{iff} \quad w \models_{\mathscr{B}} \Box\Diamond((a_1 \vee \Box\neg a_1) \vee (a_2 \vee \Box\neg a_2)) \tag{43}$$

As before, this unfolds as

> For any state $w \in W$: $\forall u \approx w\ \exists x \approx u$ such that $x \models_{\mathscr{B}} a_1 \vee \Box\neg a_1$ or $x \models_{\mathscr{B}} a_2 \vee \Box\neg a_2$

which can be read:

> For any state $w \in W$: $\forall u \approx w\ \exists x \approx u$ such that in the state x: (the electron goes through slit 1 or stays away from it in all x-proximal states) or (the electron goes though slit 2 or stays away from it in all x-proximal states).

As before, there may be states w at which nothing concerning the truth or otherwise of the proposition can be determined or is defined. Moreover since in general $(\alpha_1 \sqcup \alpha_2)^\circ \neq \alpha_1^\circ \vee \alpha_2^\circ$ we can have $w \models_{\mathscr{B}} (\alpha_1 \sqcup \alpha_2)^\circ$ without either $w \models_{\mathscr{B}} \alpha_1^\circ$ or $w \models_{\mathscr{B}} \alpha_2^\circ$. That is to say, it might be the case that the proposition [*(the electron goes though slit 1 or does not) or (the electron goes though slit 2 or does not)*] is valid at w without it being the case that **either** the proposition *(the electron goes though slit 1 or does not)* is valid at w **or** the proposition *(the electron goes though slit 2 or does not)* is valid at w. Thus, if the electron could go through either slit independently (i.e."classically"), the outcome would in general be different at such a state—namely the latter ordinarily disjunctive case above—from the former case. That is to say, the outcome for one slit is affected by the presence of the other slit. It would appear to a classical thinker that the advent of the second slit *interferes* with the state of affairs when the electron faces only one slit. This is "quantum" *interference*, which should not be confused with the entirely classical phenomenon of the interference of waves, but most often is. Although wave-like interference patterns emerge in experiments upon quanta like electrons in the two slit experiment, they are not the result of the interference of actual waves.

This fundamental non-classical property—namely the non truth–functionality of orthodisjunction, so disconcerting to classical thinkers—emerges more clearly in the

Kripkean models appearing in theorem 3. Namely, suppose we have an orthodisjunction of the form $\alpha \sqcup \beta$. Then in a Kripkean model $\mathscr{M}$:

$$\begin{aligned}
\varrho(\alpha \sqcup \beta) &= \varrho(\sim(\sim\alpha \sqcap \sim\beta)) && (44)\\
&= \varrho(\sim\alpha \sqcap \sim\beta)^{\perp} && (45)\\
&= (\varrho(\alpha)^{\perp} \cap \varrho(\beta)^{\perp})^{\perp} && (46)\\
&= (\varrho(\alpha) \cup \varrho(\beta))^{\perp\perp} \quad \text{from M10} && (47)\\
&\supseteq \varrho(\alpha) \cup \varrho(\beta) \quad \text{from M5.} && (48)
\end{aligned}$$

Thus, there may be states $w \in \varrho(\alpha \sqcup \beta)$ that are not in $\varrho(\alpha) \cup \varrho(\beta)$, or, in other words, there may be states w such that $w \Vdash_{\mathscr{M}} \alpha \sqcup \beta$ but neither $w \Vdash_{\mathscr{M}} \alpha$ nor $w \Vdash_{\mathscr{M}} \beta$. (Cf. Section 5.)

It is this property that puts paid to any attempt to build a Curry–Howard type correspondence for **OL** (or quantum logic).

It may be seen that the modal translation formalism used above, rather than the Kripke orthomodel (which requires an interpretation in the family of propositional subsets rather than locally at states) is useful for providing a local semantics relative to the proximity space involved.

4.3 The hallmarks of quantum-like behavior

We digress briefly to recapitulate the characteristic signatures of quantum and quantum-like behavior.

- Superposition. Firstly, there is the primary ontological revolution instigated by Heisenberg: namely, the non-objective nature of quantum entities or *quanta*. Their states of being are not objective attributes. This manifests in actual quantum mechanics as the superpositional nature of the states of a quantum. The "actual" state of being is indefinite until a measurement operation or experiment is performed upon the quantum. In the logical context here, superposition is achieved by the orthodisjunct, whose apparently anomalous properties have been discussed.
- Parallelism. One interpretation or side-effect of superposition goes by the name of *parallelism*, discussed above: this paradigm has been found useful, particularly in application to quantum computing, though it is essentially just another name for superposition.
- Interference. Yet again, superposition is responsible for the phenomenon known as *interference* as in the double slit example. Here the situation obtaining for one slit is apparently altered by the advent of another slit. The orthodisjunct responsible in this interpretation is the one coming between other orthodisjuncts in the leftmost assertion in (43).

5 Non quantum-like behavior vs. quantum-like behavior in the models

Although we have adopted **OL** as our overarching logic on the basis of the structure of propositional lattices in general proximity spaces, there are cases in which this lattice is in fact Boolean. In these cases the modality collapses on the lattice of propositions but not necessarily on the entire set of subsets, so there might be a non-trivial local modal semantics (as in the examples in Section 4) even in these cases. However, in these cases the resulting semantics do not exhibit the orthodisjunctional anomalies to which we have attributed quantum-like behavior. In a sense, the Booleanness of the propositional lattice in a model constrains the local modality just enough to prevent quantum-like behavior.

To be specific, we have the following.

Theorem 5 *For an orthogonality space $\langle W, \perp \rangle$ the following are equivalent:*

(1) The lattice $R(\langle W, \perp \rangle)$ is Boolean;
(2) For every $E \subseteq W$, $(\Diamond E)^{\perp} = E^{\perp}$;
(3) The modalities collapse on $R(\langle W, \perp \rangle)$. That is, for any proposition E, $\Diamond E = \Box E = E$, and $\sqcup = \cup$ on $R(\langle W, \perp \rangle)$ so that it is a Boolean sublattice of the Boolean lattice $\mathbf{2}^W$ of subsets of W.

This is proved in Subsection A.3.

Corollary 1 *If $R(\langle W, \perp \rangle)$ is Boolean, then as operators on subsets of W, $\Box\Diamond = \Diamond$.*

This is proved in Subsection A.4.

Corollary 2 *Suppose $R(\langle W, \perp \rangle)$ is not Boolean. Then there exists an orthotheorem of the form $\alpha_1 \sqcup \alpha_2$, a Kripkean model $\mathscr{M} = \langle W, \approx, \varrho \rangle$ and a state $w \in W$ such that*

$$w \Vdash_{\mathscr{M}} \alpha_1 \sqcup \alpha_2 \quad \textit{but} \quad w \nVdash_{\mathscr{M}} \alpha_1 \quad \textit{and} \quad w \nVdash_{\mathscr{M}} \alpha_2.$$

Conversely, if $\mathscr{M} = \langle W, \approx, \varrho \rangle$ is a Kripkean model and there exists an orthotheorem of the form $\sqcup_i^n \alpha_i$ such that for some $w \in W$ we have $w \nVdash_{\mathscr{M}} \alpha_i$ for all i, then $R(\langle W, \perp \rangle)$ is not Boolean.

This is proved in Subsection A.5.

Corollary 3 *Suppose $R(\langle W, \perp \rangle)$ is not Boolean. Then there exists a modal theorem of the form $(\alpha_1 \sqcup \alpha_2)^\circ$, a B-model $\mathscr{B} = \langle W, \approx, \vartheta \rangle$ and a state $w \in W$ such that*

$$w \models_{\mathscr{B}} (\alpha_1 \sqcup \alpha_2)^\circ \quad \textit{but} \quad w \nvDash_{\mathscr{B}} \alpha_1^\circ \quad \textit{and} \quad w \nvDash_{\mathscr{B}} \alpha_2^\circ.$$

Conversely, if $\mathscr{B} = \langle W, \approx, \vartheta \rangle$ is a B-model and there exists a modal theorem of the form $(\sqcup_i^n \alpha_i)^\circ$ such that for some $w \in W$ we have $w \nvDash_{\mathscr{B}} \alpha_i^\circ$ for all i, then $R(\langle W, \perp \rangle)$ is not Boolean.

This is proved in Subsection A.6.

(Of course the assertions on the left in both of the last two corollaries are redundant since the formulas to the right of the turnstile are assumed to be theorems in the respective logics, but we state them anyway to emphasize the non-classical nature of the disjunction in these cases.)

The dichotomy is now clear. We shall attribute quantum-like behavior to those systems any of whose states validate a finitely disjunctive orthotheorem as in corollary 2 (or the modal version as in corollary 3). These are precisely the systems whose proximity spaces of states have non-Boolean proposition lattices. We shall call a state validating some orthotheorem which is an orthodisjunct of two or more orthoformulas a *superpositional state* or just a *superposition*. So a system admitting a superpositional state of the above type—namely one satisfying the negated assertions shown above for some disjunctional orthothereom—will be taken to exhibit quantum-like behavior. A proximity space with a Boolean proposition lattice cannot have any superpositional states of this kind, and therefore exhibits no quantum-like behavior.

These results will be borne out in examples shown below (Subsection 6.2).

6 Neural-like networks

Brains are collections of many systems of extremely complex neuronal networks interacting in complex dynamical ways. Considered ontologically in terms of states of these subsystems, or sub-networks, such subsystem's states are inaccessible and confusable until and in case some effect is registered by other subsystems. It is these subsystems we wish to model here, postponing a treatment of their possible interactions and integration to the sequel. (While it is true that certain aspects of brain states may be measured *in vivo* from the outside, like EEGs and fMRIs, our formalism must accommodate also such "observers" or "measuring devices" as other subsystems of the brain or nervous system. For a careful discussion of the problems of measuring brain function, see [25] chapter 3.)

Thus, our main example is modeled loosely on the idea of a *neural network*, though omitting most of the assumptions traditionally ascribed to them. We shall consider networks of nodes which are supposed to be capable of holding real values within certain ranges depending on the node in question, perhaps transiently. We initially frame no hypotheses concerning the network topology nor its possible physical attributes. The upshot is that for a finite, though arbitrarily large, such network, of n nodes, say, we may characterize the internal "states" of the system via the sets of allowed values in the nodes organized into vectors in a certain region of real n-space, once an ordering of the nodes has been chosen. We note, for use in subsequent work, that the associated network topology may be encoded as an incidence or adjacency matrix and included in the state description. (Conclusions should then be examined for dependency upon this ordering). We assume the system to be possibly volatile and dynamically changing and to take into account growth, pruning, change of topology, etc., within a large but finite domain of variation, we may adopt as our model of the space of possible states for a single such network, an appropriate subset of a real vector space of sufficiently high (but finite) dimension. Having done so, the question arises as to what proximity to choose on these spaces of states. Here we note that two normalized vectors in a Euclidean space of any dimension are identical if parallel and most dissimilar when orthogonal with respect to the inner product determined by the nodes. So our orthogonality relation among normalized vectors is just the one given by ordinary inner product, denoted in what follows by a dot. That is, for non zero states $\mathbf{x}$, $\mathbf{y}$ we have $\frac{\mathbf{x}}{\|\mathbf{x}\|} \perp \frac{\mathbf{y}}{\|\mathbf{y}\|}$ iff $\mathbf{x}.\mathbf{y} = 0$ so that $\mathbf{x} \approx \mathbf{y}$ iff $\mathbf{x}.\mathbf{y} \neq 0$.

In this article we shall confine our attention initially to the less volatile situation of a fixed number of nodes, indicating in the sequel how the logic may be externalized to deal with combinations of, and interactions among, different systems of this type, which will yield a formalism to handle the waxing and waning of network size and complexity.

In de-emphasizing the usual details, our initial view of these structures can be thought of as a coarse meta-level engagement with systems of daunting—indeed dismaying—complexity: cf. the brain networks discussed in [25]. It is a view subject to almost arbitrary refinement however, though our immediate aim is to reveal possible effects that may be regarded as quantum–like.

We shall start the discussion with the "digital" case, and some other kinds of restrictions on the node values, in which the we do not expect to find quantum–like behavior. These examples bear out the results in Section 5.

6.1 Some Boolean circuits

By "digital" we mean a network of the above type in which the nodal values are taken from a finite set or alphabet. In the case of ordinary digital computers, this alphabet is generally taken to consist of the bits, 1 and 0. Although an observation or intervention is still required to "read" or ascertain the contents of an internal register, say, the fact that we know that the alphabet is finite allows the conclusion that the only distinctions that can be made between nodal settings are those between the settings that are completely different. If two settings differ even in only one position, then they are completely different. That is to say, two such settings are confusable iff they are identical, so identity is the appropriate proximity. A Kripkean model $\mathscr{M} = \langle W, =, \varrho\rangle$ has in this case the family of propositions $R(\langle W, \neq\rangle) = \mathbf{2}^W$ since for any subset $E \subseteq W$, $E^\perp = E^c$, so $E = E^{\perp\perp} = E^{cc}$ so E is a proposition. Thus, the logic so modeled is ordinary **PC**, there is neither parallelism nor interference since the modalities completely collapse on all subsets and the now ordinary disjunct satisfies

$$\varrho(\alpha \vee \beta) = \varrho(\alpha) \cup \varrho(\beta) \tag{49}$$

from (26), and the situation is completely classical.

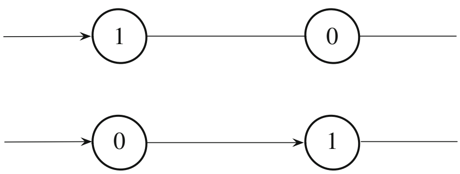
We shall consider first the simplest non-trivial case of a network of two nodes, each capable of being in one and only one of two states, namely ON or OFF, denoted by the attribution respectively of the bit 1 or 0 to the node. Then we have $W = \{00, 01, 10, 11\}$ in an obvious notation, and there is not much more that can be said without further semantic restrictions. Thus we now assume the simplest network configuration as shown in either of the two diagrams in Fig. 1.

We postulate a pulsed flow of some discrete kind along the edge shown, from left to right, perhaps a spike of current or a particle or charge. We may think of the values in a node being reset periodically back to 0 (as would be associated with some physically transient quantity). Then a single pulse consists of a 1 being sent along the edge (or channel) shown. If we find the 1 in the leftmost node, after such a pulse, then the rightmost node must be still at 0 (the upper diagram in Fig. 1). If on the other hand we find a 0 in the the leftmost node, then the 1 must have gone all the way and turned ON the rightmost node (the lower diagram in Fig. 1). Thus, by restricting the possible states to the proposition $\{01, 10\}$ of W we have produced the table for the Boolean circuit or gate known as the *inverter*.

Thus, let an atomic formula a stand for the proposition *the current configuration conflicts with the flow model* and put $\varrho(a) = \{00, 11\}$. Then the now classical Kripkean model has

$$\Vdash_{\mathscr{M}} a \vee \neg a \tag{50}$$

Fig. 1 Inverter



and $w \Vdash_{\mathscr{M}} \neg a$ iff $w \in \{10, 01\}$. So the interpretation of $\neg a$ (*a pruned*) in the model $\mathscr{M}$ describes exactly the table of a Boolean inverter gate. Note that the flow model semantics unequivocally "chooses" the negated branch of the now classical disjunct. There is no parallelism and no interference.

The next example, though apparently almost as trivial, may in fact be regarded as telling the whole story of Boolean gates, and therefore most of classical computer theory, in this context. Namely, we consider the network depicted in Fig. 2.

Here again, from rest or reset, we pulse 1s simultaneously through both channels. If we find a 1 in one left node and a 0 in the other, then the other 1 must have made it all the way though so there must be a 1 in the right node. If we find a 1 in each left node then there must be a 0 in the right node. If there are 0s in both left nodes, it means that both 1s have made it to the right node turning it ON so that it is ascribed the value 1. Thus, the states consistent with the flow model may be written $\{011, 101, 001, 110\}$. This is the table for the Boolean NAND gate—when we interpret the left nodes in the obvious way as inputs and the right node as output—and is well known to be universal among Boolean gates: any other may be obtained by linking NAND gates together appropriately. A similar logical interpretation applies here but of course remains a rather sterile exercise in the presence of the ordinary Boolean disjunct: as before there is no parallelism or interference.

If we were to suppose that these particular configurations were selectively advantageous in an environment of dynamically changing network topologies, growth and pruning, flows, chemical baths, sources, sinks, etc., then these configurations may be selected and stabilized, and the essential elements of Boolean computation may be seen to evolve. This is not to suggest that these specific gate-like elements actually did evolve in biology, though other network configurations, or *motifs* [25], certainly did, though very probably not for computational purposes. (For instance, the 3 neuron motif called the "dual dyad" by Sporns [25], which seems to proliferate in certain areas of mammalian brains and were probably selected because they seem to promote connectivity among subnetworks, may be obtained from our NAND gate example by arranging the two input edges on the left to emanate from the node on the right.)

6.2 Other Boolean situations and the case of a single node

The "digital" case, in which the states can be represented by bit strings, say, is not the only case in which the lattice of propositions in a Kripkean model is Boolean. We illustrate this phenomenon in the completely generalizable case of $n = 2$. Thus we suppose that the allowable node values are real and non-negative, so that the set of possible vectors of values comprises a product of intervals $[0, t_1] \times [0, t_2]$, with $t_i > 0$ for $i = 1, 2$. As above, let us adopt as proximity relation the one arising from the Euclidean inner product on $\mathbb{R}^2$: namely for $W := [0, t_1] \times [0, t_2] \setminus \{(0, 0)\}$ and $\mathbf{v}, \mathbf{w} \in W$, $\mathbf{v} \approx \mathbf{w}$ iff $\mathbf{v}.\mathbf{w} \neq 0$. Now it is a simple

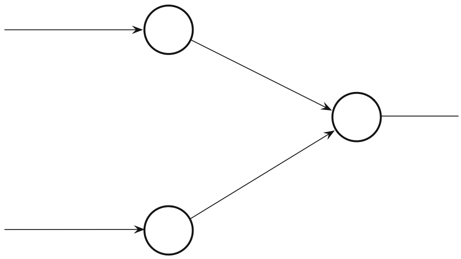


Fig. 2 NAND gate

matter to determine the lattice of propositions $R(\langle W, \perp\rangle)$. Choose an arbitrary vector in W, $\mathbf{w} = (w_1, w_2)$ say. Then if $w_1 > 0$ and $w_2 > 0$, there are no vectors $\mathbf{v} \in W$ such that $\mathbf{v}.\mathbf{w} = 0$, so that $\{\mathbf{w}\}^\perp = \emptyset$ and $\{\mathbf{w}\}^{\perp\perp} = W$. But $\{\mathbf{w}\}^{\perp\perp}$ is the smallest proposition containing $\{\mathbf{w}\}$ by the remark following Proposition 2 (Appendix A). So the only proposition containing any such positive vector is W.

On the other hand, if $w_1 = 0$, the set of vectors $V := \{(\lambda, 0) \colon 0 < \lambda \leq t_1, \lambda \in \mathbb{R}\}$ is exactly $\{\mathbf{w}\}^\perp$ in W and $\{\mathbf{w}\}^{\perp\perp} = \{(0, \mu) \colon 0 < \mu \leq t_2, \mu \in \mathbb{R}\}$, namely the vectors parallel to $\mathbf{w}$ lying in W. Consequently, the only propositions are W, V, $V^\perp$,and $\emptyset$ with the lattice structure as depicted in Fig. 3.

This is isomorphic with the Boolean lattice $\mathbf{2}^2$, the lattice of subsets of the two element set. The generalization to n nodes, with the corresponding $W = \prod_{i=1}^{n}[0, t_i\,]\setminus\{(0, \ldots, 0)\}$, leads in a similar way to a Boolean proposition lattice isomorphic with $\mathbf{2}^n$. But note that in these cases the modalities do not collapse on the entire set of subsets, since, for instance, $\Diamond\{\mathbf{w}\} = W \neq \{\mathbf{w}\}$.

In light of corollary 2 to theorem 5 the attempt in these cases to perform a modal semantic analysis via the modal translation theorem (as in Section 4) is doomed to reduce to the case of mere non-quantum-like indeterminacy.

It is clear from the geometry that if the node values are constrained in certain ways as above, then the proposition lattices will remain Boolean. On the other hand, if the node value ranges are differently disposed, then a non-Boolean lattice of propositions will result, as we show below (Subsection 6.3).

This is not the case for $n = 1$: in this case, the associated lattice of propositions is always Boolean. For, let us choose as the range of node values any interval $I \subseteq \mathbb{R}$. Then $W_1 := I\setminus\{0\}$ and for $x, y \in W_1$, $x \approx y$ iff $xy \neq 0$. But then for any $x \in W_1$, $\{x\}^\perp = \emptyset$ and $\{x\}^{\perp\perp} = W_1$ so there are only two propositions W_1 and $\emptyset$ and the lattice structure is isomorphic to the simplest Boolean lattice $\mathbf{2}$, as is to be expected in this apparently trivial case. (Note that in this case, for any $x, y \in W_1$, we have $x \approx y$).

However, in this case it is interesting to perform a modal analysis as in the examples in Section 4. This can be done because in this case—as in the Boolean $\mathbf{2}^n$ case considered above—although the lattice of propositions is Boolean, the modality does not collapse on the whole of $\mathbf{2}^{W_1}$. (For instance, for any non-empty $E \subset W_1$, $\Diamond E = W_1 \neq E$.)

We emphasize that in the examples to follow, an "observer" cannot just read the values in the nodes, since nothing is supposed to emanate from them until some event transpires, such as a firing or turning ON of the node. Please see below.

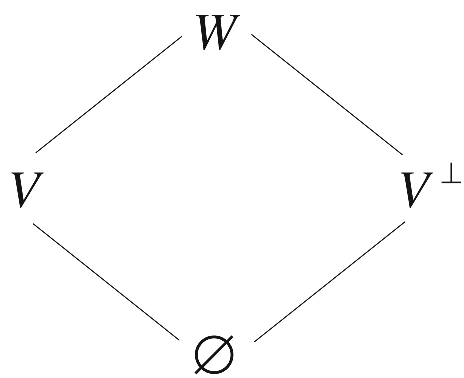


Fig. 3 An isomorph of $\mathbf{2}^2$

So in this case of a single node, we can take further the semantics in the conclusion of corollary 1 (cf. Subsection A.4) for the case of the single atomic formula as used in the Schrödinger's cat example in Section 4.1. The conclusion is, perhaps surprisingly, not completely trivial.

Thus suppose the atomic formula a stands, for example, for the proposition *the node fires* (or, as in the case of an actual neuron, *reaches the threshold potential*: cf. Section B. We could as well say *the node turns ON*). Then the theorem $\vdash_{\mathbf{O}} a \sqcup \sim a$ translates as in Subsection 4.1 as $\vdash_{\mathbf{B}} (a \sqcup \sim a)^{\circ}$. Now suppose that $\mathscr{B} = \langle W_1, \approx, \vartheta \rangle$ is some B-model. Since $R(\langle W_1, \perp \rangle)$ is Boolean it follows from the converse to corollary 3 of theorem 5 that for all $w \in W_1$:

$$w \models_{\mathscr{B}} \alpha^{\circ} \qquad \text{or} \qquad w \models_{\mathscr{B}} (\sim a)^{\circ} \tag{51}$$

Thus

$$w \models_{\mathscr{B}} \Box\Diamond a \qquad \text{or} \qquad w \models_{\mathscr{B}} \Box\Diamond\Box\neg a, \tag{52}$$

and from corollary 1 of theorem 5 on the left and M8 (Section A) on the right we obtain

$$w \models_{\mathscr{B}} \Diamond a \qquad \text{or} \qquad w \models_{\mathscr{B}} \Box\neg a. \tag{53}$$

But in this case the $\Box$ can be removed on the right since $x \approx y$ for all $x, y \in W_1$. So, finally, we have

$$w \models_{\mathscr{B}} \Diamond a \qquad \text{or} \qquad W_1 \models_{\mathscr{B}} \neg a, \tag{54}$$

and this unfolds as in Subsection 4.1 to give:

In any state $w \in W_1$:

$$\exists x \in W_1 \text{ such that } x \models_{\mathscr{B}} a \quad \text{or} \quad W_1 \models_{\mathscr{B}} \neg a, \tag{55}$$

or

In any state $w \in W_1$:

$\exists x \in W_1$ such that *the node fires at the non-zero value x* or *the node does not fire at all.* (56)

Of course, in certain respects this result is not exactly surprising: if the node fires at all it must do so at at least one value, but it is perhaps worth stating it in the modal language. Namely, we note that in any state (w), or at any time, say, if the node is firable at all, there is some generically different state or *non-zero* value (x) at which this happens. Our formalism has forbidden inclusion of the 0 state, since it is not confusable with itself. (At another state w the node's condition may be different. Without further restrictions a node may be firable at one moment but not firable at the next.) This is of course classical behavior though the value of x is not determined in the current state. This is indeterminacy, not parallelism, and bears out corollaries 2 and 3 of Section 5 which assert that there can be no quantum-like behavior if the proposition lattice is Boolean. Of course, the value of x (the *threshold*), may be independent of w as it is in the case of actual neurons (Appendix B).

We might summarize this non-quantum-like, or *classical*, 1-node case by saying that the node is generally in a state of potentially firing or not firing, i. e. potentially turning ON or not turning ON.

This behavior surprisingly mimics the similar all-or-nothing operation of an actual biological neuron: it can fire at some value of its membrane potential, though this value cannot be determined until the neuron is probed and/or or stimulated. And when these cells are broken into, probed and stimulated, as they have been in laboratories over the course of centuries, it is found that the threshold potential value is indeed non-zero, in consonance with our model, though of course the actual potential does pass through the zero value during the cell's firing phase. In an Appendix B, we give a vastly simplified account of the functioning of real neurons, and other excitable cells.

Although the behavior of a single node is classical as expected, the behavior of a cluster of more than one is not, as we will see in Subsection 6.3.

6.3 Non–Boolean examples: quantum-like behavior in nodal clusters

As remarked, if the node value ranges are appropriately disposed, then a non-Boolean lattice of propositions will result. Again, the easily generalizable case of $n = 2$ will suffice to demonstrate this. We now consider node values of the form $[-s_i, t_i]$, $s_i > 0$, $t_i > 0$, $i = 1, 2$, so that $W_2 := [-s_1, t_1] \times [-s_2, t_2] \setminus \{(0, 0)\}$. Then any vector $\mathbf{w} \in W_2$ has an associated orthogonal 1-dimensional subspace $\{\mathbf{w}\}^\perp$ of $\mathbb{R}^2$ which now intersects W_2. Moreover, $\{\mathbf{w}\}^{\perp\perp}$ is the line containing $\mathbf{w}$ so the intersection of this line with W_2 is the smallest proposition containing $\{\mathbf{w}\}$. Each intersection of W_2 with a line of slope $\tan\theta$ going through the origin—call it V_θ—is thus a proposition, and these are the only ones lying between $\emptyset$ and W_2. Thus, the lattice structure in this case is that of an infinite Chinese lantern as partially depicted in flattened out form in Fig. 4, with $\theta \in [0, \pi)$ and $V_{(\theta+\frac{\pi}{2})} = V_\theta^\perp$, with the meridian thought of as lying on a circle, identifying the leftmost subspace with the rightmost one. This is a non-Boolean ortholattice (indeed, it is orthomodular).

In this case we may expect to find quantum-like phenomena and indeed we do. With the set of states of the form $W_2 = [-s_1, t_1] \times [-s_2, t_2] \setminus \{(0, 0)\}$ as above let us now denote by the atomic formula a_i the proposition *the node i fires* (or *the node i is ON*) for $i = 1, 2$. Now the circumstances for each component node has changed since $R(\langle W_2, \perp\rangle)$ is non-Boolean. We can define a valuation ϱ on the atoms a_i as follows

$$\varrho(a_1) = \{(\lambda, 0) \colon \lambda \in \mathbf{R}\} \cap W_2 \tag{57}$$

$$\varrho(a_2) = \{(0, \mu) \colon \mu \in \mathbf{R}\} \cap W_2 \tag{58}$$

with any propositions assigned to other atoms, and extend it inductively to all formulas.

Then it is obvious that there are an infinite number of $\mathbf{w} \in W_2$ such that in the Kripkean model $\mathscr{M} = \langle W_2, \approx, \varrho\rangle$

$$\mathbf{w} \Vdash_{\mathscr{M}} a_i \sqcup {\sim} a_i \quad \text{but} \quad \mathbf{w} \nVdash_{\mathscr{M}} a_i \quad \text{and} \quad \mathbf{w} \nVdash_{\mathscr{M}} {\sim} a_i$$

for $i = 1, 2$. So each node now exhibits quantum-like behavior and can languish in Limbo-like, unobservable superpositions of firing (or being ON) and not firing (or being OFF), as in the case of Ketzi, as far as episystems are concerned. This is to be contrasted with the case of only one node considered above, which is entirely different: classical though indeterminate. The presence of the other node has made available more degrees of freedom, that is to say more superpositional states, and has therefore induced quantum-like behavior which had been entirely absent from the single node in isolation.

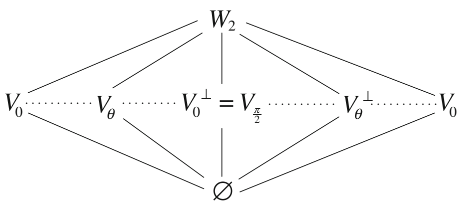


Fig. 4 A Chinese lantern

To see this more clearly, let us introduce a numerical measure into the situation by asking what the probability could be of a state $\mathbf{w}$ in one of our W_ns (in an obvious notation) above being proximal to—or confusable with—a state $\mathbf{v} \in W_n$. We shall denote this by prob($\mathbf{w} \approx \mathbf{v}$), and note that we should have $0 \leq \text{prob}(\mathbf{w} \approx \mathbf{v}) \leq 1$, $\text{prob}(\mathbf{w} \approx \mathbf{w}) = 1$, $\text{prob}(\mathbf{w} \approx \mathbf{v}) = \text{prob}(\mathbf{v} \approx \mathbf{w})$ and $\text{prob}(\mathbf{w} \perp \mathbf{v}) = 0$. There is an obvious choice in our examples, namely

$$\text{prob}(\mathbf{w} \approx \mathbf{v}) := \left(\frac{\mathbf{w}.\mathbf{v}}{\|\mathbf{w}\| \|\mathbf{v}\|} \right)^2 \tag{59}$$

where the norms denote Euclidean lengths. (An analogous identity obtains in actual quantum mechanics but there it is not obvious. It goes by the name Born's Law and was at first added as an axiom. However, more recently it has been realized to be deducible from the other axioms of quantum theory [18].)

Let us test this in the case of a single node, with the model as above. Here clearly we have $\text{prob}(x \approx y) = 1$, which is consistent with the fact that it is certain that $x \approx y$ for every pair. This is a reflection of some observer's ignorance of the contents of the node until the node does something, like fire. (Of course in the case of an actual neuron, observers such as humans, equipped with appropriate instruments, may detect intracellular or membrane potentials when the cell is not firing, but the formalism must allow for all observers including, for example, other networks in the same brain. The same is true in the case of actual quantum mechanics.) Before such an event or observation, all possible non-zero values are mutually confusable.

Let us now choose a general state $\mathbf{w} = (w_1, w_2)$ in the last 2-node example, and ask the question: what is the probability that $\mathbf{w}$ is confusable with a state in which the first node appears to behave as if it were alone, namely, as in the statement (56) summarized by saying that the first node is in its classical state of being potentially firable or not, that is, potentially ON or not? (Note that the W_1 there is $I \setminus \{0\}$ which in the present example lies along the axis in $\mathbf{R}^2$ determined by the first component of the $\mathbf{w}$s). This is tantamount to calculating $\text{prob}(\mathbf{w} \approx (w, 0))$, to wit

$$\text{prob}(\mathbf{w} \approx (w, 0)) = \left(\frac{(w_1, w_2).(w, 0)}{\|\mathbf{w}\| \|(w, 0)\|} \right)^2 \tag{60}$$

$$= \left(\frac{w_1 w}{(w_1^2 + w_2^2)^{\frac{1}{2}} |w|} \right)^2 \tag{61}$$

$$= \frac{w_1^2}{w_1^2 + w_2^2}, \tag{62}$$

a result which is independent of w. Thus, in a general state $\mathbf{w}$ of the 2-node system, the first node's apparent behavior in isolation, though not quantum in itself, has only a certain probability (< 1) of occurring. Likewise the second node. A similar effect obtains in a system of n nodes, of course, and we notice that the probability of any particular node appearing to behave in its isolated fashion when the system is in this general superpositional state $\mathbf{w}$ decreases as n increases, since the denominator in (62) will increase with n. This is certainly quantum-like behavior, and it increases with the number of nodes. Namely, the apparent classical behavior of each node in a general state of an n-node system becomes less probable with increasing n.

Put another way, as n increases, the number of superpositional possibilities, or families, increases at least exponentially, namely at least as $2^{n-1} - 1$: this is shown in Subsection A.7. There is an increasing repertoire of states for the system to appear to be hovering among as the number of nodes goes up, while the probability of any one constituent node appearing to behave as it would by itself, decreases: we note that a human brain is estimated to have 10^{11} neurons. The upshot is that this quantum-like behavior becomes more apparent as complexity increases.

It may be noted also, from (62), that the probability in general that at least one node is in a classical state is unity, since this probability is the sum over the node index of those shown on the right hand side of that equation. (It is a simple matter to compute the probability of a subset of k nodes to appear to behave classically in a general state, and the result is generally < 1.) So it is certain that such a system, in any state, will have at least one (unknown) classically behaving node while in that state, so that this node will either fire at a certain non-zero value or not fire at any value. This situation would seem evanescent, since the classically behaving node and its firing value could change with $\mathbf{w}$. But what we have been describing here is merely the kinematical template of all the possible outcomes and would require in reality the imposition of some sort of external principle or principles, as in the examples discussed in Subsection 6.1. For instance, it is known that in the case of biological neurons, the firing or threshold value (x in (56)) would be constant across many species (namely -55 mV), a finding not in conflict with our statement (56). (In actual physics, it is usually a dynamical principle that confines a system's actual states to subsets of its state space. This is true of quantum physics also, but in this case it is the nature of the information determined by the state that is different from the classical case.)

It may seem counterintuitive that two nodes are quantum-like while one node is not. However, even a small cluster of neurons, say, embedded within a mass of other neurons, is still a black box as far as our initial assumptions concerning these models are concerned. The analogy here is not to a cluster of neurons sitting on a laboratory bench hooked up to gauges and probes, but rather to a cluster of neurons embedded in a living brain. To predict the actual statistical behavior forthcoming from our assumptions will require the development of an appropriate theory of measurement, which is work for the future.

This multiple node case is similar to, but apparently not identical with, the situation in actual physics in which there is a collection of non-interacting quanta: despite the lack of mutual interaction, the quantum assumptions enforce certain statistics upon the quanta. This raises the question of how to distinguish among such systems—for example a family of n one node systems vs. an n-node system—which will be addressed in the sequel.

All these considerations (in the quantum-like cases) have been conducted in the absence of any interaction posited between the nodes which would be effected by assumptions concerning how they are networked and what characteristics the networks may have. In the sequel we will follow up these arguments and consider problems we have not begun, or only just begun, to address here. Namely: the nature of possible dynamics, entanglement of states of different networks, network topology, and interaction of networks.

7 Conclusions

We have found the following kind of quantum-like behavior in clusters of n nodes of postulated "networks" under the presumption of confusability of internal states. Namely, in a general state of the system:

– At least one node is certain to appear to behave classically—which is to say it is in a state of potential firing or not firing, not in a superposition—but which node it is is not determined by the system state.

– The classical behavior of a specific node becomes less probable as n increases.

It would seem that as n increases the effect of massive superposition obliterates the influence of individual nodes, until some sort of measurement event or interaction intervenes to destroy the superposition. And we note that the repertoire of such interventions is rather limited: a node fires or does not fire.

Although this behavior is apparently quantum-like, it is unlike any actual physical quantum system. In a general state of a nodal cluster, each node appears to play the role of a qubit, in that it is generally in a superposition of two states, but an actual physical qubit in isolation never exhibits the classical behavior of a single node of our type. It remains in a superposition until a measurement operation is performed, whereupon the superposition collapses onto one of its two constituent states. Nor do we seem to have an analog of quantum *entanglement* since the states here are generally not tensorial. (The entanglement phenomenon might be expected when we have more than one cluster in some kind of juxtaposition and this will be considered in the sequel.)

One upshot of these differences is the lesson that one should tread carefully when applying the paraphernalia of ordinary quantum theory to these systems. It would seem that a new quantum-like theory needs to be developed *ab initio*, and we hope to take this up in further work.

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Compliance with Ethical Standards

Conflict of interests The authors declare that they have no conflicts of interest.

Informed Consent The authors declare that there were no human or animal participants involved in this purely theoretical study.

Research Data No data was used or generated during the course of this study.

Appendix A: Mathematical results

A.1 Modal identities

It will be convenient to record some *modal identities*. Let $\langle W, \approx\rangle$ denote a proximity space, with $\langle W, \perp\rangle$ denoting the associated orthogonality space. We have defined in Section 2 for each subset $E \subseteq W$:

$$\Diamond E := \{w \in W : \exists\, v \in E \text{ such that } w \approx v\} \tag{63}$$

and noted that $E \subseteq \Diamond E$. Dually, we define $\Box E$ as

$$\Box E := \left(\Diamond E^c\right)^c . \tag{64}$$

Then we have:

Proposition 1 *For $\langle W, \approx\rangle$ as above and $E, F \subseteq W$:*

M1. $\Diamond(E \cup F) = \Diamond E \cup \Diamond F$

M2. *For a family $\mathscr{F}$ of subsets of W,* $\Diamond(\bigcap_{F\in\mathscr{F}} F) \subseteq \bigcap_{F\in\mathscr{F}}(\Diamond F)$

M3. $\Diamond E = (E^{\perp})^c := E^{\perp c}$
M4. $\Box E = E^{c\perp} = \{w \in W : S_w \subseteq E\}$
M5. $E \subseteq E^{\perp\perp}$
M6. $E \subseteq F$ *implies* $F^{\perp} \subseteq E^{\perp}$
M7. $E^{\perp\perp\perp} = E^{\perp}$
M8. $\Box\Diamond\Box E = \Box E$
M9. $\Diamond\Box\Diamond E = \Diamond E$
M10. $(E \cup F)^{\perp} = E^{\perp} \cap F^{\perp}$
M11. $E^{\perp} \cup F^{\perp} \subseteq (E \cap F)^{\perp}$

The proofs are elementary and may be found in the references cited (or [17, 18]).

A.2 Propositions in Kripke orthomodels

For a proximity space as above Dalla Chiara et al. [16] define a *proposition* to be a subset $X \subseteq W$ satisfying:

$$\text{if } x \in W \text{ is such that } \forall\, y \approx x,\ \exists\, z \in X \text{ such that } y \approx z, \text{ then } x \in X. \tag{65}$$

Equivalently, X is a proposition iff

$$S_x \subseteq \Diamond X \text{ implies } x \in X. \tag{66}$$

Then we have:

Proposition 2 1. *For all* $X \subseteq W$, $\Box\Diamond X = X^{\perp\perp}$ *and* X *is a proposition iff* $\Box\Diamond X = X^{\perp\perp} = X$.
2. X *is a proposition iff* $x \notin X$ *implies* $\exists\, y \approx x$ *with* $y \perp X$.
3. *For any* $Y \subseteq W$, $Y^{\perp}$ *is a proposition.*
4. *If* $\mathscr{C}$ *is a family of propositions, then* $\bigcap \mathscr{C}$ *is a proposition.*
5. *If* Y *is a proposition, then* $X \subseteq Y$ *iff* $\Diamond X \subseteq \Diamond Y$.

Thus, from the first statement above, the two definitions of propositions (cf. (16)) are equivalent. Note also that it follows from M5 and M6 that for a subset $E \subseteq W$, $E^{\perp\perp}$ is the smallest proposition containing E.

For proofs, see the references cited (or [17, 18]).

A.3 Proof of Theorem 5, section 5

(1) implies (2). Assume the lattice $R(\langle W, \perp\rangle)$ is Boolean. Then in view of theorem 4, Section 3, Part W is also Boolean, hence distributive. The join in Part W is just set union and we shall exploit its distributivity in this case. First we derive an expression for the meet in this lattice which we shall write as $\sqcap$. Thus, for any subsets E, K of W:

$$\Diamond E \sqcap \Diamond K = ((\Diamond E)^* \cup (\Diamond K)^*)^* \tag{67}$$
$$= ((\Diamond(\Diamond E)^c) \cup (\Diamond(\Diamond K)^c))^* \tag{68}$$
$$= ((\Diamond E^{\perp}) \cup (\Diamond K^{\perp}))^* \qquad \text{from M3} \tag{69}$$

$$= (\Diamond(E^{\perp} \cup K^{\perp}))^{*} \qquad \text{from M1} \tag{70}$$
$$= \Diamond(\Diamond(E^{\perp} \cup K^{\perp}))^{c} \tag{71}$$
$$= \Diamond((E^{\perp} \cup K^{\perp})^{\perp}) \qquad \text{from M3} \tag{72}$$
$$= \Diamond(E^{\perp\perp} \cap K^{\perp\perp}) \qquad \text{from M10.} \tag{73}$$

(It is not hard to show that this meet is indeed the largest subset of $\Diamond E \cap \Diamond K$ of the form $\Diamond(\ )$: that is to say, if $\Diamond L \subseteq \Diamond E \cap \Diamond K$ then $\Diamond L \subseteq \Diamond(E^{\perp\perp} \cap K^{\perp\perp}) \subseteq \Diamond E \cap \Diamond K$.) Meet distributes over join in any lattice iff the converse also holds, so if Part W is Boolean, hence distributive, the following identity would obtain for any subsets E, F, G of W:

$$\Diamond E \cup (\Diamond F \sqcap \Diamond G) = (\Diamond E \cup \Diamond F) \sqcap (\Diamond E \cup \Diamond G). \tag{74}$$

The left hand side of this last equation is

$$\Diamond E \cup (\Diamond F \sqcap \Diamond G) = \Diamond E \cup \Diamond(F^{\perp\perp} \cap G^{\perp\perp}) \qquad \text{from (73)} \tag{75}$$
$$= \Diamond(E \cup (F^{\perp\perp} \cap G^{\perp\perp})) \qquad \text{from M1} \tag{76}$$
$$= \Diamond((E \cup F^{\perp\perp}) \cap (E \cup G^{\perp\perp})). \tag{77}$$

The right hand side is, from (73):

$$(\Diamond E \cup \Diamond F) \sqcap (\Diamond E \cup \Diamond G) = \Diamond((\Diamond E \cup \Diamond F)^{\perp\perp} \cap (\Diamond E \cup \Diamond G)^{\perp\perp}) \tag{78}$$
$$= \Diamond((\Diamond(E \cup F))^{\perp\perp} \cap (\Diamond(E \cup G))^{\perp\perp}). \tag{79}$$

If the the left hand side equals the right hand side for any subsets, take $E = \emptyset$ and $G = F$. Then equality gives

$$\Diamond(F^{\perp\perp}) = \Diamond((\Diamond F)^{\perp\perp}) \tag{80}$$

or

$$F^{\perp\perp\perp c} = F^{\perp c\perp\perp\perp c} \tag{81}$$

or, from M7

$$F^{\perp c} = F^{\perp c\perp c} \tag{82}$$

So

$$\Diamond F = \Diamond\Diamond F \tag{83}$$

or

$$F^{\perp} = (\Diamond F)^{\perp}. \tag{84}$$

This proves the assertion (2).

(2) implies (3). First we note that for any subset $E \subseteq W$

$$(\Diamond E)^{c} = E^{\perp} \tag{85}$$

from M3. So if (84) holds we have

$$(\Diamond E)^{c} = (\Diamond E)^{\perp}. \tag{86}$$

So if E is a proposition

$$E = \Box\Diamond E \tag{87}$$
$$= (\Diamond E)^{c\perp} \qquad \text{from M3} \tag{88}$$
$$= (\Diamond E)^{\perp\perp} \qquad \text{from (86).} \tag{89}$$

But then

$$\Diamond E \subseteq (\Diamond E)^{\perp\perp} \qquad \text{from M5} \tag{90}$$
$$= E \qquad \text{from (89)} \tag{91}$$
$$\subseteq \Diamond E. \tag{92}$$

Thus $E = \Diamond E$ if E is a proposition. But then, since E is a proposition

$$E = \Box\Diamond E = \Box E. \tag{93}$$

Since, for any proposition E,

$$E = \Diamond E = E^{\perp c} \tag{94}$$

we have

$$E^c = E^{\perp}. \tag{95}$$

Then the join ($\sqcup$) in $R(\langle W, \perp\rangle)$ is given by

$$E \sqcup F = (E^{\perp} \cap F^{\perp})^{\perp} \tag{96}$$

$$= (E^c \cap F^c)^c \quad \text{by the above and proposition 2(4), subsection A.2} \tag{97}$$

$$= E \cup F \tag{98}$$

so that $R(\langle W, \perp\rangle)$ is just a Boolean lattice of subsets of W.

This proves (3).

Assertion (1), hence the theorem, follows immediately.

A.4 Proof of Corollary 1, section 5

If $R(\langle W, \perp\rangle)$ is Boolean then we have shown above (95) that for any proposition E, we have $E^{\perp} = E^c$. Then, since for any $F \subseteq W$, $F^{\perp}$ is a proposition (proposition 2(3)), we have

$$F^{\perp\perp} = F^{\perp c} \tag{99}$$

which was to be proved.

A.5 Proof of Corollary 2, section 5

First note again that for elements E, F of $R(\langle W, \perp\rangle)$, from M10,

$$E \sqcup F = (E^{\perp} \cap F^{\perp})^{\perp} = (E \cup F)^{\perp\perp} \supseteq E \cup F. \tag{100}$$

Now suppose that for all propositions E in $R(\langle W, \perp\rangle)$ we have $E \sqcup E^{\perp} = E \cup E^{\perp}$. Since $E \sqcup E^{\perp} = W$ this entails $E^{\perp} = E^c$. Then it follows as in the proof of theorem 5(3) that $\sqcup = \cup$ on the whole of $R(\langle W, \perp\rangle)$ so that the latter lattice is a Boolean sublattice of $\mathbf{2}^W$, contradicting our assumption. Consequently there exists a proposition, F, say, such that

$$W = F \sqcup F^{\perp} \neq F \cup F^{\perp} \tag{101}$$

so that the inclusion $F \cup F^{\perp} \subset F \sqcup F^{\perp} = W$ is strict and F cannot be empty (since otherwise (101) would not hold). Now consider the orthotheorem $a \sqcup \sim a$ where a is an atomic formula, and define $\varrho(a)$ to be F with any other assignments to the other atoms, extending this assignment to formulas inductively in the usual way.

Then

$$W = \varrho(a \sqcup \sim a) \tag{102}$$

$$= \varrho(a) \sqcup \varrho(a)^{\perp} \tag{103}$$

$$= F \sqcup F^{\perp} \tag{104}$$

$$\supset F \cup F^{\perp} \tag{105}$$

$$= \varrho(a) \cup \varrho(a)^{\perp}. \tag{106}$$

So, there exists a $w \in W$ such that

$w \Vdash_{\mathscr{M}} a \sqcup \sim a$ but $w \nVdash_{\mathscr{M}} a$ and $w \nVdash_{\mathscr{M}} \sim a$.

Conversely, if there exists an orthotheorem of the form $\sqcup_i^n \alpha_i$ such that $\bigsqcup_i^n \varrho(\alpha_i) \neq \bigcup_i^n \varrho(\alpha_i)$ then $R(\langle W, \perp\rangle)$ cannot be Boolean by theorem 5(3).

This proves the corollary.

A.6 Proof of Corollary 3, section 5

From corollary 2 there exists a modal theorem of the form $(\alpha_1 \sqcup \alpha_2)^\circ$, by theorem 2, and a Kripkean model $\mathscr{M} = \langle W, \approx, \varrho\rangle$. Define, for atomic formulas a_i, and $w \in W$

$$\vartheta(a_i, w) := \begin{cases} 1 \text{ if } w \in \varrho(a_i) \\ 0 \text{ if } w \notin \varrho(a_i) \end{cases} \tag{107}$$

and inductively extend it to all modal formulas via V1–V4 in Section 3. Then $\mathscr{B} = \langle W, \approx, \vartheta\rangle$ is a B-model and it is easily proved that for any orthoformula α

$$\|\alpha^\circ\|_{\mathscr{B}} = \varrho(\alpha) \tag{108}$$

(cf. [17], proposition 2.5.1). Consequently (dropping the $\mathscr{B}$ subscripts),

$$\begin{aligned} \|(\alpha_1 \sqcup \alpha_2)^\circ\| &= \varrho(\alpha_1 \sqcup \alpha_2) & (109)\\ &= \varrho(\alpha_1) \sqcup \varrho(\alpha_2) & (110)\\ &\supset \varrho(\alpha_1) \cup \varrho(\alpha_2) & (111)\\ &= \|\alpha_1^\circ\| \cup \|\alpha_2^\circ\| & (112) \end{aligned}$$

since from corollary 2 there exists a $w \in W$ such that $w \in \varrho(\alpha_1) \sqcup \varrho(\alpha_2) = \|(\alpha_1 \sqcup \alpha_2)^\circ\|$ but $w \notin \varrho(\alpha_1) \cup \varrho(\alpha_2)$ and thus $w \models_{\mathscr{B}} (\alpha_1 \sqcup \alpha_2)^\circ$ but $w \not\models_{\mathscr{B}} \alpha_1^\circ$ and $w \not\models_{\mathscr{B}} \alpha_2^\circ$ which was to be proved.

Conversely, suppose $\mathscr{B} = \langle W, \approx, \vartheta\rangle$ is a B-model, that there exists a modal theorem of the type $(\sqcup_i^n \alpha_i)^\circ$ and a $w \in W$ satisfying the stated condition. Define ϱ on atoms a_i by

$$\varrho(a_i) := \|a_i^\circ\|_{\mathscr{B}} = \|a_i\|_{\mathscr{B}}^{\perp\perp} \tag{113}$$

and extend it inductively to all formulas. Then, for any formula α it follows from an easy induction on complexity that:

$$\varrho(\alpha) = \|\alpha^\circ\|_{\mathscr{B}}. \tag{114}$$

Then $\mathscr{M} = \langle W, \approx, \varrho\rangle$ is a Kripkean model for **OL**, and since it is assumed that $\vdash_{\mathbf{B}} (\sqcup_i^n \alpha_i)^\circ$ we have also $\vdash_{\mathbf{O}} \sqcup_i^n \alpha_i$ by theorem 2. Now it is clear that for all i, $w \not\Vdash_{\mathscr{M}} \alpha_i$, for otherwise there would be an i_0 say, such that $w \in \varrho(\alpha_{i_0}) = \|\alpha^\circ{}_{i_0}\|$, i.e. $w \models_{\mathscr{B}} \alpha^\circ{}_{i_0}$ contrary to our hypotheses. The non-Booleanness of $R(\langle W, \perp\rangle)$ now follows from the converse to corollary 2.

This proves the corollary.

A.7 Counting certain families of superpositions

A proposition in W_n is its intersection with a unique non-zero subspace of $\mathbf{R}^n$. Each proper non-zero subspace of $\mathbf{R}^n$ determines an orthogonal complement which is also proper and non-zero. So each proper non-zero subspace of $\mathbf{R}^n$ determines a proposition in W_n along with its orthogonal complement, and each such pair determines an infinite family of superpositions, namely elements of W_n not contained in either the proposition or its complement. (Note that in W_2 there is only one such family, depicted as the meridian in Fig. 4, since each member of any pair of linearly independent vectors is expressible as a linear combination of any other independent pair, so any pair of orthogonal propositions determines the same family of superpositions.) These pairs of propositions may of course be used to represent an orthotheorem of the form $a \sqcup \sim a$ in any Kripkean model based on W_n. Different families of associated superpositions will now correspond two different pairs of such propositions. Thus, to count the number of such families in the case of this special orthotheorem, we need only determine the number of non-zero proper subspaces of $\mathbf{R}^n$ and halve it, to avoid a double count of each pair. But each k-dimensional subspace of $\mathbf{R}^n$ corresponds to a unique one-dimensional subspace of $\bigwedge^k \mathbf{R}^n$ and there are $\dim(\bigwedge^k \mathbf{R}^n) = \binom{n}{k}$ such independent subspaces. So the number we seek is

$$\frac{1}{2}\sum_{k=1}^{n-1}\binom{n}{k} = \frac{1}{2}\left(\sum_{k=0}^{n}\binom{n}{k} - \binom{n}{0} - \binom{n}{n}\right) \tag{115}$$

$$= \frac{1}{2}(2^n - 2) \tag{116}$$

$$= 2^{n-1} - 1. \tag{117}$$

Note the for $n = 2$ we do indeed get just one family of superpositions.

Appendix B: Neuron structure and function

We give a very brief and impressionistic account of neuron structure and function. Although we will use the word "neuron," much of what is described applies to other types of cell such as nerve cells. (For this section see [26]. For a deep mathematical study of neuron dynamics, and to get some idea of the complexity of neuronal internal states, see [27]). A neuron is a specialized cell that functions in a node-like manner, is networked with other neurons, and communicates with them via the chemical mediation of electrical impulses in a manner to be described. A neuron generally has very many input channels, but only one signal is output, though perhaps to many recipients. The inputs are branched projections, called *dendrites*, of the neuron body, or *soma*, that conduct electrochemical signals into the body of the neuron from other cells and neurons. The outgoing branch is a single pseudopod-like projection, usually on the side opposite to the dendrites, called the *axon*. The active site where the axon connects to the soma is called the *hillock*. The axon conducts a single electrochemical output pulse or train of pulses, but may have many branching outputs at its end, called the axon *terminals*. These then provide the input signals to the dendrites of other neurons. (The axon can be immensely long relative to the size of the cell, as in nerve cells.)

The electrochemistry underlying this activity is extremely complex. In simple terms, electrical potentials are formed across the cell membranes via the bidirectional flows through the membrane of various types of ion: in this case mainly sodium and potassium ions, though there is a multitude of other ions in this environment. These ion flows are controlled by clusters of proteins embedded in the cell membranes (as they are in all cells) called *ion channels*, which act upon ion flows like gates, pumps and/or valves. There is a multiplicity of varieties of ion channel.

In its "resting" state, a neuron holds a potential of about −70 mV (millivolts) reflecting a steady state of polarization between the internal and external ion flow states. If input signals arrive from the dendrites, or groups of them, the cell depolarizes and its potential goes up. The cell may then rapidly repolarize without issue if the signal is not strong enough, in which case the signal does not penetrate very far into the soma. But if the stimulus is strong enough to enable a threshold potential to be reached (at about −55 mV) then positive feedback kicks in and there is a rapid depolarization and concomitant rise in the potential to a peak of about 60 mV. At some point during this rise, the signal penetrates the soma and reaches the hillock, and the potential profile, or a train of copies of it, begins to be conducted along the axon via a spontaneously choreographed succession of opening and closing ion channels along the body of the axon. This is the point at which the neuron is said to *fire*. Meanwhile, the cell body rapidly repolarizes and the *action potential* now plunges down to around −90 mV, which has the salutary effect of preventing the signal from being conducted back up the axon toward the soma. After a refractory period in this territory, the potential rises again to the resting value of −70 mV. The time scales are roughly as follows: the duration of the "spike" is about 0.5 ms (milliseconds); the entire duration of the action potential, including the refractory period, between the two rest states, has a duration of about 5 ms. It should be mentioned that it is only the frequency of the output signal train that is determined by the strength of the input signal, not the amplitude, which is independent of the input signal strength.

(A couple of remarks are in order. Firstly, a set of equations describing the progress of the action potential, called the Hodgkin–Huxley model, was proposed in 1952 and was considered a great advance in biophysics, winning for its authors two thirds of the Nobel Prize in Physiology and Medicine in 1963. Secondly, the mode of transmission of a signal from an axon terminal of one neuron, to a dendrite of another neuron, is not by direct contact but rather through a complex intermediary mechanism involving *synapses* (presynaptic bodies at the ends of axon terminals, with postsynaptic receptors at the ends of dendrites) and *neurotransmitters* which are complex molecules of a large variety of types, including serotonin, dopamine, tryptophan, histamine, etc. The elucidation of this mechanism, by J. C. Eccles, garnered for its author the other third of the 1963 Nobel Prize in Physiology and Medicine.)

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