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Mathematical knowledge for teaching and mathematics didactic knowledge: a comparative study

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Abstract

This paper compares and contrasts two approaches that are widely used in the English- and German-speaking discourse on mathematics teacher knowledge: 'mathematical knowledge for teaching' and 'mathematics didactic knowledge'. It is proposed that these constructs are based on distinct theoretical and conceptual positions and origins. Mathematical knowledge for teaching is viewed as a utilitarian-pragmatic approach rooted in English-speaking traditions as it focuses on its use in teaching and represents a practice-based conceptualization of knowledge domains required for mathematics teaching. Mathematics didactic knowledge, on the other hand, is considered normative-descriptive as it is formulated based on didactic principles and broader theoretical perspectives, providing a theory-driven conceptualization of knowledge domains rooted in traditions of German-speaking didactics of mathematics. The paper further highlights similarities and differences in these two constructs through an examination of two central knowledge domains: specialized content knowledge (part of mathematical knowledge for teaching) and subject matter didactic knowledge (part of mathematics didactic knowledge).

Keywords Elementarization · Didactics of mathematics · Mathematical knowledge for teaching · Mathematics didactic knowledge · Teacher knowledge · Unpacking mathematics

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Introduction

In recent decades, there has been a considerable surge in research on mathematics teacher knowledge (see Ball et al., 2001; Boero et al., 1996; Kaiser & König, 2019; Ponte & Chapman, 2006; Rowland & Ruthven, 2011; Sullivan & Wood, 2008). This is attributed, in part, to Shulman's (1986, 1987) proposition that teachers possess a specialized knowledge base for teaching. This notion triggered an ongoing quest to delineate the professional knowledge base for mathematics teaching, leading to a variety of conceptualizations and interpretations of mathematics teacher knowledge (e.g., Buchholtz et al., 2013; Kaiser et al., 2017; Neubrand, 2018; Rowland et al., 2005; Schoenfeld, 2020; Silverman & Thompson, 2008). Yet, the nature of mathematics teacher knowledge remains a subject of ongoing debate and exploration (see Scheiner et al., 2019).

This paper delves into the prevalent discourse on mathematics teacher knowledge, which has been shaped by influential approaches with diverse theoretical orientations and practical implications for teacher education and professional development. Within this discourse, two approaches have become salient with their own cultural prominence: 'mathematical knowledge for teaching', proposed by Ball and colleagues (e.g., Ball & Bass, 2000, 2003; Ball et al., 2005, 2008) and widely recognized in English-speaking circles,[1] and 'mathematics didactic knowledge' (*mathematikdidaktisches Wissen*), originating from German-speaking didactics of mathematics and empiricial research on mathematics teacher knowledge within German-speaking contexts (e.g., Baumert & Kunter, 2006; Blömeke et al., 2008; Buchholtz et al., 2014; Krauss et al., 2008).[2]

The purpose of this paper is to compare and contrast these two approaches—'mathematical knowledge for teaching' and 'mathematics didactic knowledge'. The reasons for carrying out this comparative study are threefold: Firstly, these approaches are prevalent in both the English-speaking and the German-speaking academic discourse, significantly influencing the research on, and the education of, mathematics teachers. Yet, despite their extensive usage, a comprehensive comparison of these approaches is noticeably absent. Secondly, their respective constructs have been the basis of extensive empirical studies aiming to measure mathematics teacher knowledge, including notable projects such as the *Mathematics Teaching and Learning to Teach* project (Ball et al., 2005; Hill et al., 2005), the *Cognitive Activation in the Classroom* project (Baumert et al., 2010; Kunter et al., 2013), and the *Teacher Education and Development Study in Mathematics* (Blömeke et al., 2014; Tatto et al., 2012). Lastly, these approaches, although bearing similarities, present unique characteristics that warrant a more detailed examination.[3]

[1] Despite its origin in the USA, this approach has been widely adopted and further developed in numerous other countries, including Ghana, Indonesia, Ireland, Malawi, Norway, and South Korea.

[2] The term 'mathematikdidaktisches Wissen' is often translated as 'mathematical pedagogical content knowledge'. This translation may stem from the common practice of translating 'pedagogical content knowledge' as 'fachdidaktisches Wissen', and vice versa. However, this practice may be misleading, given the distinct theoretical positions and origins of these two concepts (for a discussion, see Scheiner & Buchholtz, 2022). Therefore, in this paper, we opt for the term 'mathematics didactic knowledge' as a more precise translation of 'mathematikdidaktisches Wissen', ensuring the nature of the concept is more accurately conveyed.

[3] Despite the prominence of the constructs of 'mathematical knowledge for teaching' and 'mathematics didactic knowledge', they are not necessarily emblematic of the understandings of mathematics teacher knowledge in their respective cultural regions (English-speaking and European countries). Alternative frameworks, such as the *Knowledge Quartet* (Rowland, 2009; Rowland et al., 2005), *Didactic Mathematical Knowledge* (Godino et al., 2017; Pino-Fan et al., 2018), and *Mathematics Teacher Specialized Knowledge* (Carrillo-Yañez et al., 2018), have also gained substantial attention. Unique cultural perspectives, includ-

This paper will therefore compare and contrast these two approaches, while acknowledging their roots in specific historical, intellectual, and linguistic contexts.[4] Through this comparison, we aim to gain insights into the cultural influences that shape teacher knowledge in various contexts and a more nuanced understanding of how teacher knowledge is situated within different traditions. Furthermore, this comparison may stimulate further research on synthesizing approaches to conceptualize teacher knowledge—its nature and its development—inspiring others to undertake similar comparisons and learn from diverse cultural perspectives.

The paper is structured into three main sections. In the first section, the constructs of 'mathematical knowledge for teaching' and 'mathematics didactic knowledge' are outlined and then compared and contrasted in terms of their theoretical foundations, the way the knowledge bases are conceptualized, and the nature of their approaches. The second section highlights the differences between the two constructs in terms of their theoretical and conceptual positions, examining two pertinent knowledge domains: specialized content knowledge (part of mathematical knowledge for teaching) and subject matter didactic knowledge (part of mathematics didactic knowledge). The third section delves into their significance as cognitive resources for mathematics teachers in professional practice, comparing the 'unpacking of mathematics' proposed by Ball and Bass with the 'elementarization of mathematics' in German-speaking tradition of subject matter didactics.

To this end, an integrative review of relevant literature was conducted to compare and contrast the constructs of mathematical knowledge for teaching and mathematics didactic knowledge. Integrative reviews provide a comprehensive overview of the current state of knowledge on a particular research topic and aim to critically evaluate and synthesize existing research, thereby paving the way for new theoretical frameworks and perspectives (Snyder, 2019). The review focused on papers that articulated fundamental positions within each approach, with the aim of comparing and contrasting these positions. Following this, we considered subsequent work based on these key contributions, as well as more recent, influential, and highly regarded contributions in the field to ensure broader relevance.

Approaches to mathematical knowledge for teaching and mathematics didactic knowledge

In the following, the approaches to mathematical knowledge for teaching and mathematics didactic knowledge are outlined and then compared and contrasted in order to highlight similarities and differences between these approaches.

Footnote 3 (continued)

ing the French *didactique* and the Chinese *bianshi* method, add to the rich tapestry of interpretations on mathematics teacher knowledge (Rowland, 2020), creating invaluable opportunities for understanding the traditions shaping the teaching profession and teacher education internationally (Kaiser & Blömeke, 2014).

[4] Prediger et al. (2008) noted in their 'networking of theories' approach that comparing (i.e., identifying similarities and differences) and contrasting (i.e., emphasizing disparities) are strategies often applied to assess opposing or competing theoretical approaches. However, in this paper, the approaches of 'mathematical knowledge for teaching' and 'mathematics didactic knowledge' are not considered competitors. Instead, exploring their distinct characteristics and contrasts provide opportunities for stimulating further reflection and advancing theoretical development (for a discussion, see Scheiner, 2020).

The approach to mathematical knowledge for teaching: a focus on the work of teaching mathematics

Following Shulman's (1986, 1987) conviction that teaching requires a special kind of content knowledge, Ball and colleagues (e.g., Ball & Bass, 2000, 2003; Ball et al., 2005, 2008) developed a research program for specifying the kind of mathematical knowledge entailed by and used in teaching mathematics. They began with the observation that in the work of teaching mathematics, teachers are confronted with a variety of tasks that pose mathematical problems that they must solve in practice, for example, when they look at students' mathematical work to find out what students think and know. Teachers' engagement in this kind of mathematical problem-solving, Ball and colleagues suggested, requires the use of specific mathematical knowledge.

The central concern of Ball and colleagues' research program was to study the mathematical work—or, more precisely, the mathematical demands—of teaching mathematics and to identify the mathematical knowledge needed for specific tasks that teachers are engaged in doing this work. The central questions under consideration were: "What mathematical knowledge is needed to teach elementary school mathematics well? How must it be understood and held so that it is available for use?" (Ball & Bass, 2000, p. 89). Later, the wording of these questions was slightly modified to include, "What do teachers do in teaching mathematics, and in what ways does what they do demand mathematical reasoning, insight, understanding, and skill?" (Ball et al., 2005, p. 17). The way these questions were framed placed the emphasis on the use of knowledge in and for teaching rather than on the teachers themselves. That is, instead of investigating these questions by examining the curriculum or asking expert mathematicians and mathematics educators to identify the mathematical knowledge that teachers should have, Ball and colleagues focused on the professional work teachers do in teaching mathematics—to identify what mathematical knowledge is needed in teaching and how that knowledge is used in practice.

They described their approach as a kind of 'job analysis' of the work of teaching mathematics, aimed at locating and analyzing mathematics as it is used in practice (e.g., Ball & Bass, 2000, p. 89). This included analyzing not only the actual interactive work of teaching in the classroom, but also the tasks that arise in the course of this work, such as lesson planning and evaluating student work. The purpose of this 'job analysis' was to identify the fundamental tasks of teaching mathematics and to analyze the mathematical demands of these tasks in order to better understand the use of mathematics in the work of teaching.[5]

This 'job analysis' formed the basis for what has been described as a practice-based conceptualization of mathematical knowledge for teaching (e.g., Ball & Bass, 2003) and the development and validation of measures of mathematical knowledge for teaching (e.g., Ball et al., 2005; Hill et al., 2004, 2005). Mathematical knowledge for teaching has been conceptualized as a multidimensional construct consisting of several—more or less empirically separable—domains of knowledge that can be assigned to Shulman's (1987) subject

[5] This 'job analysis' involved extensive qualitative analyses of the work of mathematics teaching. Detailed records of classroom instruction were analyzed, including video and audio recordings of lessons, transcripts, student work samples, curriculum materials, and teacher notes (Ball & Bass, 2000, 2003). These analyses helped to identify what mathematical knowledge is used in the teaching of mathematics. Based on these findings, hypotheses about the nature of mathematical knowledge for teaching were developed and tested, by creating specific measures of mathematical knowledge for teaching and then linking those measures to growth in student mathematical achievements (Hill et al., 2004, 2005).

matter knowledge or pedagogical content knowledge.[6] Table 1 provides an overview of these domains of mathematical knowledge for teaching and gives examples of tasks or situations in which they are used in mathematics teaching.

It should be noted that Ball and colleagues did not attempt to provide a list or catalog of what mathematics teachers need to know. Instead, their research program was intended to provide a conceptual orientation and a set of analytic distinctions that would focus attention on the nature and types of mathematical knowledge required in a variety of tasks and settings in mathematics teaching.

The approach to mathematics didactic knowledge: a concretization of specialized knowledge gained in didactics of mathematics

What exactly constitutes mathematics didactic knowledge is rather open and an ongoing endeavor, especially since German-speaking didactics of mathematics draws from various traditions (see e.g., Biehler et al., 1994; Jahnke & Hefendehl-Hebeker, 2019; Sträßer, 2019). In the following, we will concentrate on three prominent, yet complementary, traditions that are widely acknowledged: the subject matter didactics tradition, the psychological-sociological tradition, and the educational-theoretical tradition.

The subject matter didactics tradition (*Stoffdidaktik*) has been a rather prominent and predominant approach in German-speaking didactics of mathematics, with an explicit focus on the subject matter (for a discussion; see Hefendehl-Hebeker, 2016; Hußmann et al., 2016). Subject matter didactics, in its more traditional view, has been concerned with the presentation of the subject matter in terms of teaching (Drenckhahn, 1952/1953). One of its primary tasks has been to develop approaches to presenting the subject matter that are appropriate to students' cognitive abilities and personal experiences without compromising the mathematical substance, leading to the postulation of a central principle of didactics of mathematics: to simplify the subject matter without distorting it (*Vereinfachen, ohne zu verfälschen*; see Kirsch, 1977). Simplification of the subject matter should make the mathematics in question accessible to students; it should be 'intellectually honest' and 'upwardly compatible' (Kirsch, 1977, 1987). Accordingly, it has been central to subject matter didactics to generate knowledge about how to make central ideas, notions, and concepts of the subject, as well as central working methods, accessible to students. Preparing mathematics for teaching and making it accessible to students has been carried out in particular through specific theoretical didactic analyses of the subject matter (see e.g., Griesel, 1972; Kirsch, 1987).

What the mathematical content is about and how it should be introduced into the classroom cannot be decided by mathematical analyses alone, but should be based on epistemological and historical analyses of the genesis of the subject matter. Therefore, it has been crucial to generate knowledge about how mathematics instruction should be designed with respect to the genesis of the natural epistemological processes of mathematics and the

[6] Previous work on mathematical knowledge for teaching has found mixed empirical evidence for the distinction between different domains of knowledge, with recent work challenging the multidimensionality of 'mathematical knowledge for teaching' (e.g., Charalambous et al., 2020; Copur-Gencturk et al., 2019; Hill, 2010).

Table 1 Overview of domains of mathematical knowledge for teaching and examples of tasks or settings in which they are used (based on Ball & Bass, 2009; Ball et al., 2005, 2008; Hill & Ball, 2009)

Dimension of mathematical knowledge for teaching	Domain of mathematical knowledge for teaching	Description of knowledge domain	Examples of tasks or settings in which knowledge domain is used or needed
Subject matter knowledge	Common content knowledge	Mathematical knowledge used in a variety of settings, including those other than teaching (likely not unique to teachers)	Understanding the mathematics in the curriculum; solving mathematical problems; recognizing the correctness or incorrectness of a student answer
	Specialized content knowledge	Mathematical knowledge specific to the tasks of teaching (not typically needed for purposes other than teaching)	Presenting, exemplifying, or representing mathematical ideas; modifying mathematical tasks; evaluating the plausibility of students' claims
	Horizon content knowledge	Knowledge of the larger mathematical landscape that teaching requires (incl. awareness of how mathematical topics are related over the span of mathematics included in the curriculum)	Making connections to earlier or later mathematical ideas; making judgments about mathematical importance; hearing mathematical significance in what students are saying
Pedagogical content knowledge	Knowledge of content and students	Knowledge that combines knowing about mathematics and knowing about students (incl. familiarity with what students often think or do in regard to a particular mathematical idea or procedure)	Anticipating students' mathematical thinking and knowing students' common conceptions; choosing interesting and motivating examples for students; interpreting students' emerging thinking as expressed in ways pupils use language
	Knowledge of content and teaching	Knowledge that combines knowing about mathematics and knowing about teaching (incl. familiarity with pedagogical principles for teaching a particular mathematical content)	Sequencing particular mathematical content for instruction; evaluating the instructional (dis-) advantages of representations or examples for a particular mathematical idea; Identifying instructional affordances of different methods and procedures
	Knowledge of content and curriculum[a]	Knowledge of indications and contraindications for the use of particular curriculum materials in specific circumstances	Using particular curriculum materials developed on specific topics at a specific educational level; dealing with the variety of instructional materials available

[a]The description of the domain 'knowledge of content and curriculum' and the examples of tasks or settings in which it is used refer to Shulman's (1986, p. 10) description of curriculum knowledge

historical development of the subject, which is strongly reflected in the genetic principle of didactics of mathematics (see e.g., Schubring, 1978).[7]

The psychological-sociological tradition, on the other hand, considers psychological and social aspects of mathematics teaching and learning (Cobb & Bauersfeld, 1995). Case studies, especially classroom studies, have been used to reconstruct different aspects of teaching and learning (see Gellert & Krummheuer, 2019). For example, interaction analysis, which combines a sociological and a mathematical perspective, has been used to reconstruct social processes related to the negotiation of meaning and the social constitution of shared knowledge in a particular classroom interaction (Bauersfeld, 1980; Krummheuer & Voigt, 1991). These studies have contributed to building a body of knowledge about central aspects of teaching and learning mathematics, particularly in relation to the principle that mathematical learning is a participatory developmental process (Krummheuer, 2007).

The educational-theoretical tradition is more concerned with the educational significance of mathematics and mathematics teaching (Winter, 1975; Wittmann, 1975). In this tradition, an important goal of mathematics and mathematics teaching is to promote the personal and cultural formation of students. When discussing the promotion of personal and cultural growth, researchers have provided justifications for the inclusion of mathematics in the curriculum (Heymann, 2003). An important guideline has been that mathematics education should aim to provide various basic experiences (*Grunderfahrungen*), such as perceiving and understanding phenomena of the world through a mathematical lens, developing an appreciation and understanding of mathematical concepts as intellectual creations, and acquiring problem-solving skills that extend beyond the realm of mathematics itself (Winter, 1995, p. 37). These theoretical, philosophical, and critical considerations, which established an important body of knowledge of the role of mathematics and mathematics education in society, have also influenced contemporary discussions of mathematical literacy and competencies (see Neubrand, 2003).

Although other traditions within didactics of mathematics could also be mentioned, the three traditions outlined here—the subject matter didactics, the psychological-sociological, and the educational-theoretical—form the core of German-speaking didactics of mathematics. What is then called mathematics didactic knowledge results from specific considerations and working methods that are typical for these respective traditions. Thus, by means of different working methods specialized knowledge is generated, which is then concretized into mathematics didactic knowledge. Mathematics didactic knowledge is thus a concretized body of specialized knowledge that is essentially based on didactic work as well as on theoretical, philosophical, and critical reflections on mathematics and on the teaching and learning of mathematics.

It should be noted that mathematics didactic knowledge is not a systematized canon of specialized knowledge that could be taught to (prospective) teachers in the form of a predefined curriculum. Rather, mathematics didactic knowledge is exemplary knowledge that refers to some fundamental principles of didactics of mathematics (see Steinbring, 2011). It is knowledge that is public and collectively held, that is, it has been articulated (e.g., in research literature or academic circles) and is shared and understood by a scholarly community of professionals.

[7] For a discussion on the relevance of epistemological considerations in German-speaking didactics of mathematics, especially with regard to teachers' classroom practices, see Prediger and Hefendehl-Hebeker (2016).

Table 2 provides a selection of domains of mathematics didactic knowledge, chosen because they illustrate the central traditions of didactics of mathematics mentioned above. It should be noted that the aim here is not to establish a canon of binding domains of mathematics didactic knowledge, but to exemplify some domains of mathematics didactic knowledge that reflect some of the more fundamental principles and working methods in German-speaking didactics of mathematics.

Comparative discussion

The outline above highlights important similarities and differences between the approaches of mathematical knowledge for teaching and mathematics didactic knowledge. Notably, both of these approaches differ from mathematical knowledge per se and pedagogical knowledge, with the latter encompassing a wide range of generic aspects from teaching methodology and classroom management to diagnostics and performance assessment (e.g., Baumert & Kunter, 2006; König et al., 2018). The primary focus of both the mathematical knowledge for teaching approach and the mathematics didactic knowledge approach lies in generating or identifying knowledge for the purpose of teaching mathematics rather than in generating or identifying mathematical knowledge for scientific mathematical purposes. Moreover, they center on the knowledge inherent in the teaching profession, encompassing both the collectively understood and shared knowledge among experts, as well as the individual teacher's actions in the classroom. Both approaches are closely connected to both mathematics itself and the learners, generating knowledge about the teaching of mathematics in an 'intellectually honest' way in the sense of Bruner (1960). This means adhering to a "twin imperative" (Ball, 1993, p. 374), which demands equal responsiveness to the learners' capabilities while ensuring mathematically accurate teaching.

However, there are also important differences between mathematical knowledge for teaching and mathematics didactic knowledge. Mathematical knowledge for teaching refers more to the mathematical knowledge required by teachers to perform the professional work of teaching mathematics. It is the knowledge needed to meet the mathematical demands of the tasks that arise in practice and in particular mathematics teaching situations. Mathematics didactic knowledge, on the other hand, refers more to the collectively held, concretized knowledge of the discipline of didactics of mathematics, which is gained through various types of analyses (mathematical, didactic, epistemological, historical) and critical reflections on the educational value of mathematics and mathematics education.

The two approaches, mathematical knowledge for teaching and mathematics didactic knowledge, have distinct ways of conceptualizing the respective knowledge bases. Mathematical knowledge for teaching is viewed as a practice-based knowledge base, defined in terms of its use in performing specific tasks of mathematics teaching. Mathematics didactic knowledge, on the other hand, is seen as a theory-based knowledge base, formulated in light of fundamental principles of didactics of mathematics. The conceptualization of mathematical knowledge for teaching as a practice-based knowledge base arose from considering mathematical knowledge from the perspective of practice and examining the actual work of teaching. Conversely, the conceptualization of mathematics didactic knowledge as a theory-based knowledge base resulted from the elaboration of fundamental principles in German-speaking didactics of mathematics that serve as guidelines and directives for teaching and teacher education. This approach is more concerned with "producing knowledge for teachers to use" (Fenstermacher, 1994, p. 9), rather than defining knowledge domains based on what teachers know or should know.

Table 2 Exemplary selection of domains of mathematics didactic knowledge within central traditions and aligned with fundamental principles and working methods in German-speaking didactics of mathematics

Tradition	Domain of mathematics didactic knowledge	Description of knowledge domain	Fundamental principle	Working method
Subject matter didactics	Knowledge of making mathematics accessible	Knowledge of how to make mathematics accessible to students (including ways of explaining and representing mathematical content) (e.g., Kunter et al., 2013)	Simplifying subject matter without falsifying it (e.g., Kirsch, 1977)	Subject matter analyses (e.g., Griesel, 1972; Kirsch, 1987)
	Knowledge of the epistemic and historical genesis of mathematics	Knowledge of epistemological and historical processes (including epistemological constraints in teaching and learning mathematics (e.g., Steinbring, 1998)	Genetic principle (e.g., Schubring, 1978)	Structure-genetic didactic analysis (e.g., Wittmann, 2021); epistemological analysis (Steinbring, 1997)
Psychological-sociological	Knowledge of teaching and learning mathematics	Knowledge of psychological and social aspects of teaching and learning mathematics (including the development of student knowledge in classroom interaction) (e.g., Buchholtz et al., 2014)	Mathematical learning as a participatory developmental process (e.g., Krummheuer, 2007)	Interaction analysis (e.g., Bauersfeld, 1980)
Educational-theoretical	Knowledge of the role of mathematics and mathematics education in society	Knowledge of the educational value of mathematics and ways of promoting the personal and cultural formation of students (e.g., Jahnke, 2019)	Basic experiences (*Grunderfahrungen*; e.g., Winter, 1995)	Philosophical and theoretical reflections on the educational value of mathematics and mathematics teaching (e.g., Wittmann, 1975)

The two approaches also reveal different positions. The approach to mathematical knowledge for teaching can be characterized as of *utilitarian-pragmatic* nature, guided by questions such as, “What mathematical knowledge is needed for specific tasks of teaching mathematics?”, and *effective-empirical*, guided by questions such as, “What mathematical knowledge matters for effective teaching and does it impact students’ mathematical achievement?”. The focus on the actual work of teaching was linked to the intention to increase the relevance of the identified knowledge to practice and to focus on the effective application of this knowledge (see Ball et al., 2008). The approach to mathematics didactic knowledge, on the other hand, can be characterized as of *normative-descriptive* nature, guided by questions such as, “What knowledge is significant in disciplinary, epistemological, and cultural-historical terms?”, and *critical-didactic*, guided by questions such as, “What is the educational value of mathematics and how should it be taught to promote students’ personal and cultural formation?”).[8]

The two approaches, mathematical knowledge for teaching and mathematics didactic knowledge, also differ in both their object of analysis and their method of inquiry. The approach to mathematical knowledge for teaching analyzed the practices of teaching mathematics, and based on the analysis of the mathematical demands of these practices, identified specific domains of mathematical knowledge necessary for effective teaching. This approach can be viewed as an empirically, inductive method for identifying specific domains of mathematical knowledge relevant to the work of mathematics teaching, that places it in the tradition of the ‘empirical scientist’ (Bishop, 1992), whose goal is to analyze practice and generate explanatory models based on empirical evidence.

In contrast, the approach to mathematics didactic knowledge focused in particular on the subject matter content (*den Stoff*), analyzing its mathematical, epistemological, historical, and educational-theoretical significance. Through different types of analysis, mathematical knowledge that is both epistemologically critical and educationally valuable could be formulated and concretized into specific domains. This approach aligns with the ‘scholastic-philosopher tradition’ (Bishop, 1992) and is a more theoretical method that reflects a rigorously argued theoretical reflection process. The epistemological goal in this tradition is to develop a justified position on theory-driven (including didactically motivated mathematical) research questions and curriculum development that is grounded in mathematical insight and logical rigor.

Table 3 provides an overview of the central aspects of contrast between mathematical knowledge for teaching and mathematics didactic knowledge. It should be noted that this overview is not exhaustive but highlights the fundamental distinctions that exist between these two approaches.

[8] These differences are likely to be historically and culturally influenced and can be found in similar forms elsewhere in the context of mathematics teaching. For example, Kaiser (1999) has shown how the empiricist-utilitarian approach in the English education system, with its emphasis on pragmatism, contrasts with the humanist approach in the German education system, with its emphasis on general education (*Allgemeinbildung*), and how they influence different understandings of mathematics teaching and classroom practice. For a brief comparison of these two different educational systems and their positions, see Kaiser (2002).

Table 3 An overview of central differences between mathematical knowledge for teaching and mathematics didactic knowledge

	Mathematical knowledge for teaching	Mathematics didactic knowledge
Guiding question for identifying knowledge base	What mathematical knowledge is needed in teaching mathematics? (Or, more precisely, what mathematical knowledge is involved in the work of teaching mathematics?)	What knowledge about mathematics and mathematics teaching is significant? (Or, more precisely, what knowledge about mathematics and mathematics teaching is significant from a disciplinary, epistemological, historical, and educational-theoretical point of view?)
Type of analysis used to identify knowledge base	Job analysis (analysis of the mathematical work in mathematics teaching and identification of the mathematical knowledge required for this work)	Various types of analysis, such as didactically oriented subject matter analysis (analysis of the mathematical content and identification of its core), epistemic-genetic analysis (analysis of the epistemological structures of the mathematical content), didactic analysis (analysis of the content in terms of its educational relevance and significance)
Type of conceptualization of knowledge base	Practice-based conceptualization (mathematical knowledge framed in terms of its use to perform particular tasks of mathematics teaching)	Theory-based conceptualization (mathematics didactic knowledge framed in terms of fundamental principles of didactics of mathematics)
Type of approach	Utilitarian-pragmatic (knowledge needed to perform specific tasks of mathematics teaching) and effective-empirical (knowledge that matters for effective mathematics teaching)	Normative-descriptive (knowledge significant in disciplinary, epistemological, and cultural-historical ways) and critical-didactic (knowledge of the educational value of mathematics and ways of promoting the personal and cultural formation of students)

In-depth comparison of the two approaches: using specialized content knowledge and subject matter didactic knowledge as examples

In this section, the two approaches to mathematical knowledge for teaching and mathematics didactic knowledge are compared in greater depth by focusing on two central knowledge domains: specialized content knowledge and subject matter didactic knowledge. These two knowledge domains are of interest because they highlight different practices in modifying mathematics for teaching and learning.

The following practices are of specific relevance for the two approaches: unpacking mathematics, central to specialized content knowledge, and elementarizing mathematics, central to subject matter didactic knowledge. The examination of these practices is important because it reveals the fundamentally different assumptions about how mathematics is modified to be useful or meaningful in teaching and learning. These practices are also key aspects in discussions within the respective scholarly communities (for a discussion, see Scheiner et al., 2022).

Specialized content knowledge and the practice of unpacking mathematics

The research program by Ball and colleagues has yielded a plethora of tasks in the work of teaching that are mathematical in nature (see Table 1). Of particular interest are those tasks with significant mathematical demands that require mathematical knowledge for teaching, especially "a specialized form of pure subject matter knowledge" (Ball et al., 2008, p. 396):

> 'pure' because it is not mixed with knowledge of students or pedagogy and is thus distinct from the pedagogical content knowledge identified by Shulman and his colleagues and 'specialized' because it is not needed or used in settings other than mathematics teaching. (Ball et al., 2008, p. 396)

Ball and colleagues referred to this as 'specialized content knowledge,' a kind of mathematical knowledge that is necessary for teaching but is not part of pedagogical content knowledge.

> What distinguishes this sort of mathematical knowledge from other knowledge of mathematics is that it is subject matter knowledge needed by teachers for specific tasks of teaching … but still clearly subject matter knowledge. These tasks of teaching depend on mathematical knowledge, and, significantly, they have aspects that do not depend on knowledge of students or of teaching. These tasks require knowing how knowledge is generated and structured in the discipline and how such considerations matter in teaching… (Ball et al., 2008, p. 402)

Specialized content knowledge is thus seen as a domain of mathematical knowledge unique to the work of teaching and distinct from the common content knowledge needed by teachers and non-teachers alike. This domain of mathematical knowledge for teaching has been described as particularly characteristic of the distinctive work of teachers, which requires a kind of mathematical thinking and reasoning that most adults do not need on a regular basis. As such, mathematical knowledge for teaching "must be detailed in ways unnecessary for everyday functioning" (Ball et al., 2008, p. 396) in order to be appropriate for student learning. To this end, teachers must *unpack* the mathematics under consideration into more basic forms:

> … one needs to be able to deconstruct one's own mathematical knowledge into less polished and final form, where elemental components are accessible and visible. … Paradoxically, most personal knowledge of subject matter, which is desirably and usefully compressed, can be ironically inadequate for teaching. … Indeed, its polished, compressed form can obscure one's ability to discern how learners are thinking at the roots of that knowledge. Because teachers must be able to work with content for students in its growing, not finished, state, they must be able to do something perverse: work backward from mature and compressed understanding of the content to unpack its constituent elements. (Ball & Bass, 2000, p. 98)

In this regard, unpacking mathematics is not only a crucial practice that mathematics teachers must use in their work (i.e., working with mathematics in its more elemental and basic form), but it is also distinct from the mathematical work of mathematicians. While the development of mathematics, and thus the work of mathematicians, is characterized by increasing abstraction and compression of mathematical ideas, mathematics teaching requires the reverse process: compressed forms must be unpacked because these compressed forms of knowledge can impede student learning.

According to Ball and colleagues, much of the mathematical work of mathematics teaching involves "an uncanny kind of unpacking of mathematics that is not needed—or even desirable—in settings other than teaching. Many of the everyday tasks of teaching are distinctive to this special work … these tasks demand unique mathematical understanding and reasoning." (Ball et al., 2008, p. 400). In this view, teachers work with mathematics in its decompressed or unpackaged form:

> Teaching involves the use of decompressed mathematical knowledge that might be taught directly to students as they develop understanding. However, with students the goal is to develop fluency with compressed mathematical knowledge. In the end, learners should be able to use sophisticated mathematical ideas and procedures. Teachers, however, must hold unpacked mathematical knowledge because teaching involves making features of particular content visible to and learnable by students. (Ball et al., 2008, p. 400)

Subject matter didactic knowledge and the practice of elementarizing mathematics

Subject matter didactics has traditionally played an important role in German-speaking didactics of mathematics, mainly because of its emphasis on the essential structures of mathematics and the subject-specific ways of thinking that are crucial for preparing mathematics for teaching. In particular, subject matter didactic working methods have been important tools of mathematics didactic research, resulting from the desire for a solid subject foundation and pursued with the goal of preparing the subject matter in a way that is compatible with the standards of the subject and at the same time appropriate to the learners and the demands of teaching.

One of the central working methods in subject matter didactics is the 'didactically oriented subject matter analysis' (*didaktisch orientierte Sachanalyse*), introduced by Griesel (1972), whose aim is to clarify the mathematical substance—especially the mathematical core (*mathematischen Kern*)—of a particular subject matter with the help of mathematical methods.[9]

> The research methods in this area are identical to those used in mathematics, so that outsiders have sometimes gained the impression that mathematics (especially elementary mathematics) and not mathematics education is being pursued here. … The aim of 'didactically oriented subject matter analysis,' which is essentially based on mathematical methods, is to create a better basis for the formulation of content-related learning objectives and for the development, definition, and application of a

[9] The working method is 'didactically oriented' in that the choice of the mathematical object under consideration is based on didactic decisions (Griesel, 1972, pp. 79–80). It is worth noting that this working method can involve a reconstruction of meanings of mathematical concepts, which has become an important didactic task (Biehler, 2005). This reconstruction of meanings is also present in the English-speaking discourse, such as in Silverman and Thompson's (2008) work on 'key developmental understandings' of particular mathematical ideas (i.e., new mathematical knowledge with pedagogical potential). This underscores Thompson's (2016) assertion, "The mathematical knowledge that matters most for teachers resides in the mathematical meanings they hold" (p. 437).

differentiated set of methodological tools. (Griesel, 1974, p. 118; translated by the first author).

Didactically oriented mathematical analyses form the core of the scientific approach to the development of new subject matter didactic knowledge. These are analyses that do not only deal with individual mathematical topics, but axiomatically analyze a mathematical content, dissect it, and examine it with regard to its cognitive learning requirements in order to derive recommendations for action in teaching. This is done by interlocking mathematical analysis, teaching experience, and didactic considerations, which then require empirical validation (Griesel, 1972).

Concentrating on the mathematical core does not mean trivializing or thinning out the subject matter, but rather specifying the subject matter to elementary aspects (*elementare Aspekte*). This specification of the subject matter to elementary aspects is already given in Klein's (1933/2016) 'Elementary Mathematics from a Higher Standpoint' (*Elementarmathematik vom höheren Standpunkte aus*), where the elementary is not the subject matter in a simple or basic form, but a concretization and embodiment of the essential meaning inherent in the subject matter (for a discussion, see Schubring, 2016).

Accordingly, elementarization (*Elementarisierung*) does not simply mean a reduction of subject matter, but a concentration of what is of fundamental significance—from a disciplinary, epistemological, and/or educational-theoretical point of view. The function of the elementary is to open up what is mathematically, epistemologically, and culturally fundamental and worthy of education.

The practice of elementarization and the identification of the elementary have fostered the development of central concepts in German-speaking subject matter didactics, especially the concepts of 'fundamental ideas' (*fundamentale Ideen*) and 'basic ideas' (*Grundvorstellungen*). Fundamental ideas describe the underlying principles or essence of a subject area, such as the idea of approximation, algorithmization, induction, linearization, or symmetry (Schreiber, 1983; Schweiger, 1992). In a sense, they are 'vertical fibers' in the curriculum, in that they explain and focus essential content, ways of thinking, and ways of working in mathematics (see Vohns, 2016).

Basic ideas are more local than fundamental ideas; they are normative interpretations of a mathematical object, such as the ideas of 'taking away' and 'determining the difference' for the subtraction of natural numbers (vom Hofe, 1995; vom Hofe & Blum, 2016). Basic ideas describe, in a sense, adequate interpretations for dealing with mathematical objects and function as pedagogical guidelines that pursue a specific educational goal. They also serve to provide constructive insights for research on the teaching and learning of mathematics, especially in comparison to students' individual conceptions, representations, and explanatory models.

Subject matter didactic knowledge is then the knowledge that belongs to the body of specialized knowledge of subject matter didactics. It includes knowledge about ways to elementarize subject matter and make it accessible to students. In particular, it consists of knowledge about the elementary, such as knowledge about certain fundamental ideas and basic ideas in mathematics.

Comparative discussion

Underlying both knowledge domains—specialized content knowledge and subject matter didactic knowledge—is the recognition that mathematical content cannot be directly transposed into teaching. Instead, it must be modified to become teachable by teachers and learnable by students.

Specialized content knowledge is based on the recognition that mathematics teachers must perform a special task, unpacking mathematics, which sets it apart from the practices of other practitioners of mathematics (e.g., mathematicians). The notion of unpacking mathematics has been used to mark the specificity of a central practice of the work of mathematics teaching. It has also been used to give legitimacy to the documentation of a particular kind of mathematical knowledge needed for teaching that is distinct from other kinds of mathematical knowledge used in a variety of settings other than teaching. For Ball et al. (2008), this particular task in the work of mathematics teaching requires "a body of mathematical knowledge specialized to teaching" (p. 401). Some scholars have further posited that the 'essence' of mathematics teacher knowledge is based on an explicit recognition of unpacking mathematics, while doing mathematics requires only an implicit recognition of such unpacking (Hodgen, 2011, pp. 34–35).

German-speaking tradition of subject matter didactics, on the other hand, posits that mathematics must be elementarized to make it worthwhile for teaching and accessible to students. Subject matter didactic analyses focus on identifying the mathematical core of the subject matter under consideration and enriching it through additional analyses and reflections to make it not only mathematically significant but also of epistemological, cultural, and educational value.

Although the elementarization of mathematics can be read as something like a decomposition of mathematics, and thus could be seen as closely related to the unpacking of mathematics, it refers instead to an intensification of mathematics to its elementary aspects, a concretization and embodiment of what is mathematically, epistemologically, and culturally fundamental and worthy of education. This normative and top-down orientation of the elementarization of mathematics differs from the more psychological and bottom-up orientation of unpacking mathematics. The unpacking of mathematics is typically a practice of an individual teacher or a collective of teachers; that is, unpacking can be seen as a process that takes place in the teacher's mind and is asserted by collective agreement (see the notion that a teacher needs "to deconstruct one's own mathematical knowledge into less polished and final form, where elemental components are accessible and visible"; Ball & Bass, 2000, p. 98). As Scheiner and Bowers (2023) suggested, such an understanding is closely related to cognitivism and the individualization of the teacher in preparing mathematics for teaching. Elementarization, on the other hand, is a complex undertaking that often requires the collaborative work of mathematicians, mathematics educators, curriculum designers, and mathematics teachers. The ability to elementarize mathematics, then, is not a property of an individual teacher, but of social and cultural systems that organize the ways in which mathematics is prepared for teaching, in which particular kinds and forms of knowledge are valued based on history and tradition as well as social needs and cultural conventions.[10]

Table 4 provides an overview of central aspects of contrast between unpacking mathematics and elementarization of mathematics that underlie and inform the approaches of specialized content knowledge and subject matter didactic knowledge.

[10] For a reflection on the idea that the individual teacher is responsible for preparing and possessing knowledge for teaching in English-speaking discourse, and an account of alternatives from French and German traditions of didactics, see Scheiner (2022).

Table 4 A comparative overview of unpacking mathematics and elementarizing mathematics

	Unpacking mathematics	Elementarizing mathematics
Objective	To deconstruct the mathematical knowledge possessed by a teacher, making it more understandable for students	To make disciplinary mathematical knowledge accessible and enhance its mathematical, epistemological, and educational significance by identifying its elementary aspects
Orientation	Psychological and bottom-up (unpacking is a somewhat mental process performed by an individual teacher)	Normative and top-down (elementarization is a process primarily performed by mathematics educators and guided by normative guidelines)
Application/implication	Unpacking highlights the distinctiveness of a central practice in mathematics teaching and establishes the need for specialized content knowledge as a domain of mathematical knowledge for teaching	Elementarization is a central practice in subject matter didactics and provides insights for generating specialized mathematical knowledge that is referred to as part of subject matter didactic knowledge

Conclusion

In this paper, the focus has been on two constructs widely used in the English- and German-speaking discourse on mathematics teacher knowledge: mathematical knowledge for teaching and mathematics didactic knowledge. Reviewing central publications on the two constructs and comparing and contrasting them showed that while they share several similarities, they are based on different theoretical and conceptual positions and origins. Mathematical knowledge for teaching refers to the body of mathematical knowledge required for mathematics teaching and can be characterized as utilitarian-pragmatic as it refers to the use of teachers in their daily classroom work. It thus articulates a bottom-up, practice-based conceptualization of the domains of knowledge required for mathematics teaching. Mathematics didactic knowledge, on the other hand, refers to the concretized body of knowledge of the discipline of German-speaking didactics of mathematics and can be characterized as normative and top-down as it is formulated along didactic principles and broader theoretical considerations. It thus articulates a theory-driven conceptualization of knowledge domains held collectively by a group of professionals.

These differences have to some extent historical and cultural roots. Mathematical knowledge for teaching is grounded in the English-speaking tradition of pedagogical-psychological classroom research and teacher effectiveness research, which focuses on a descriptively oriented reconstruction of effective classroom practice. It was a consequence of Shulman's (1986, 1987) attempt to bring together the rather separate streams of curriculum research in the United States, one concerned with content and the other with pedagogy. Like Shulman (1986, 1987), the relationship between knowledge and practice is fundamental to mathematical knowledge for teaching. The focus of mathematical knowledge for teaching was on the mathematical-pedagogical thinking that teachers need in their work of mathematics teaching.

Mathematics didactic knowledge, on the other hand, is grounded in German-speaking tradition of didactics of mathematics and reflects its central concern to make the subject matter accessible without distorting it. Starting from the subject matter (*den Stoff*), analyses and reflections have been carried out in German-speaking didactics of mathematics that ask about its subject-specific, epistemological-genetic, and educational-theoretical relevance. Through these working methods, principles were derived and knowledge bases generated, which then led to the concretization of mathematics didactic knowledge.

These fundamental differences between mathematical knowledge for teaching and mathematics didactic knowledge could be further explored by examining two domains of their knowledge bases: specialized content knowledge and subject matter didactic knowledge. It was established that the psychological and bottom-up orientation is evident in the ‘unpacking of mathematics’ (i.e., the teacher’s practice of breaking down mathematics into elemental components), underlying specialized content knowledge, a domain of mathematical knowledge for teaching. The normative and top-down orientation is evident in ‘elementarization of mathematics’ (i.e., bringing out the elementary and the educational value of the mathematics in question), underlying subject matter didactics, a domain of mathematics didactic knowledge.

Though the comparison of mathematical knowledge for teaching and mathematics didactic knowledge presented in this paper is not exhaustive, it highlights the central differences in their underpinnings and positions, which have not received much attention in the literature. Based on this comparison, the following three recommendations for future research on mathematics teacher knowledge are suggested.

Firstly, the approaches of ‘mathematical knowledge for teaching’ and ‘mathematics didactic knowledge’ have much to offer each other. For instance, studies conducted under the ‘mathematical knowledge for teaching’ framework could benefit from adopting didactic analyses (Pansell, 2023), such as those common within the ‘mathematics didactic knowledge’ sphere. Specifically, such research could concentrate on a specific mathematical topic, aiming to uncover its various aspects for teaching and learning. Conversely, those working within the realm of ‘mathematics didactic knowledge’ might delve into how this knowledge is implemented by teachers in their day-to-day teaching activities. This exploration might bring light into unrecognized key activities in mathematics teaching and move beyond the division between knowledge and practice (Brodie, 2004).

Secondly, given the profound differences between the constructs of ‘mathematical knowledge for teaching’ and ‘mathematics didactic knowledge’, it is essential that these differences are taken into account when comparing outcomes of studies that have been conducted with one of the traditions. This recognition will allow for more meaningful comparisons, averting potential misunderstandings arising from a failure to acknowledge these inherent construct disparities.

Thirdly, the international discourse on mathematics teacher knowledge could benefit from additional comparisons of various approaches. These comparisons are crucial, as they provide a deeper understanding of individual constructs and their practical applications. Such a process can steer the field towards a more synergistic understanding of different constructs or frameworks, highlighting their unique strengths and showcasing how they can be interconnected to reveal fresh opportunities for research and practice in mathematics teacher education. However, this necessitates an ongoing, reciprocal process of reflection on differing research traditions.

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