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Primary teachers' preferred fraction models and manipulatives for solving fraction tasks and for teaching

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Abstract

When teaching fractions, teachers make instructional decisions about if, when, and how to use the many different types of fraction models and manipulatives. In this study, we sought insights into their pedagogical reasoning with fraction representations via their preferences, both for solving tasks themselves and for teaching (in general and for specific fraction concepts and operations). Nearly 200 practising Australian primary teachers participated in an online survey and we drew on a Fraction Schemes theorisation to analyse quantitative and qualitative data. A majority of teachers indicated a personal preference for the set model for four out of five schemes; for one scheme most teachers preferred the circle model. Their reasons suggested that the nature of each task in a scheme and the specific fractions involved, played a role in influencing their preferences. With respect to teaching fractions, the teachers also indicated a high level of preference for teaching with the set model in general, and secondly for the rectangle model. Their preferences, except for number lines, were not found to be associated with the teachers' nominated year level. We found that a high personal preference for a set model was associated with a preference for teaching with the same model in general, but not for teaching with the matching manipulative (counters or chips). The teachers indicated a high level of preference for teaching with the fraction bars manipulative for several fraction concepts, but this was not associated with a personal preference for linear models. Implications for further research are discussed.

Keywords Fractions · Models · Manipulatives · Fraction Schemes · Teacher reasoning · Primary education

A deep conceptual understanding of rational number is arguably one of the most foundational mathematics learning goals for students at all levels of schooling and beyond (Elias

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et al., 2020). Importantly, an early competence with fractions has been shown to uniquely predict later mathematics achievement (Bailey, et al., 2012; Siegler, et al., 2012).

There are many studies in the research literature on student difficulties and misconceptions with fractions (e.g. Aksoy & Yazlik, 2017; Hansen et al., 2017; Lewis & Perry, 2017; van den Heuvel-Panhuizen, 2010). Many difficulties, particularly with fraction computation, have been related to students' lack of development of key conceptions or 'Schemes' (e.g. Norton & Wilkins, 2012; Steffe, 2002, 2010). Students' progression through these schemes is often determined by the way in which they interpret and construct fractions through the use of various models. Despite differing views on which meanings and models to introduce to students, and at what developmental stage (e.g. Cramer & Wyberg, 2009), there is widespread agreement that students need opportunities to make connections across different constructs and visual representations of rational number.

An inability to model or represent fractions can be an indicator of a lack of conceptual understanding (Lamon, 2001). Students' understanding of fractions can be advanced through learning with continuous and discrete representations to model fractions (e.g. Behr et al., 1988; Martin, et al., 2012; Soni & Okamoto, 2020). Furthermore, students need to be exposed to multiple models and to develop understandings that enable them to transition between different forms of representations involving fractions (Behr, et al., 1988; Lesh et al., 1987; Tsai, 2017; Zhang et al., 2015). However, students are often exposed to only a limited number of models (Clarke et al., 2011). Martin et al. (2012) in their study of primary students found that the use of manipulatives sped up the learning compared to static models (pictures) because the children could experiment with arranging the materials. Nevertheless, manipulatives did not inherently convey fraction concepts but needed facilitated interpretation.

Numerous curriculum documents advocate the learning and teaching of fractions with multiple representations. For example, the NCTM Principles and Standards states 'representations should be treated as essential elements' (National Council of Teachers of Mathematics, 2000, p. 67). The Australian Curriculum: Mathematics (Australian Curriculum, Assessment and Reporting Authority [ACARA], 2017) refers to a range of models and manipulatives for teaching fractions, such as materials, objects, sets, shapes, areas, lengths, number lines, paper sheets, and paper strips. The Rational Number Project curriculum (Behr et al., 1992; Cramer et al., 2002) advocates students solving the same task using two different fraction models and critiquing their strengths and weaknesses for that task. Student selection and justification of their own models are also a feature of the curriculum.

Given the importance of a robust fraction understanding for students, studies have also highlighted the need to consider teachers' understanding of fractions concepts and representations. For example, there is evidence of considerable reliance on procedural knowledge among prospective teachers (Lovin et al., 2018). A study of 109 US prospective teachers found fewer than half evidencing the higher-level Fraction Schemes (Lovin et al., 2018). Recent research has validated a theorisation of five developmental Fraction Schemes with both student and prospective teacher cohorts and in different curriculum contexts (e.g. Lovin et al., 2018; Norton et al., 2018; Stevens et al., 2020). Yet little is known about practising teachers' knowledge, particularly if and why they select visual representations in their teaching practice or if they rely on procedural approaches.

Given the importance of both using and connecting different visual representations for progressing beyond the lower-level Fraction Schemes, research exploring the pedagogical reasoning of practising teachers is warranted. This article discusses the findings from an online anonymous survey of nearly 200 practising primary teachers on teachers' preferences and critiques of different representations of fractions—both for solving fraction tasks themselves and for teaching fractions concepts and operations. The following section presents details on the theoretical background and contextual information on the survey that informed its design and data analysis.

Background and context

There are different theoretical conceptualisations of rational number, each of which provide insight into the complexity of learning and teaching fractions. In the following three subsections, we overview in turn: different conceptualisations of rational number, critiques in the literature of different representations for learning fractions, and prior research on teachers' use of fraction models.

Theorisations of fractions

A well-known and highly regarded framework for rational numbers is Kieren's (1976) seven interpretations or subconstructs: fractions which can be compared, added, subtracted, etc.; decimal fractions; equivalence classes of fractions; ratios in the form $\frac{p}{q}q \neq 0$; multiplicative operators; elements of an infinite ordered quotient field where $x = \frac{p}{q}$ and $qx = p$; and measures or points on a number line. Kieren (1976) argued that 'rational numbers, from the point of view of instruction, must be considered in all of the interpretations' (p. 127). Each interpretation involves several related teaching strategies, which in turn employ numerous physical and symbolic models, discrete or continuous. Kieren cautioned that teachers need to select models carefully so that they do not conflict cognitively with a particular rational number concept. For example, a continuous model supports repeated and infinitely varied subdivision of a referent unit, whereas a discrete model supports counting but with less obvious emphasis on the referent unit. He gave the example of a number line model supporting the interpretation of rationals as a measure but not the concept of multiplication of rationals. He also suggested that the number line model might conflict cognitively with an area model for generating multiplicative ideas. Kieren (1988) argued that an overemphasis on a part-whole interpretation with static part-whole models—typically seen in primary school—could result in children not developing a powerful measure model of comparison to a unit (but relying instead on a double count of parts).

Another theorisation for fractions considers them a cognitive 'synthesis' of schemes of conceptual operations—that 'fractional reasoning develops by interrelating several conceptual schemes often not associated with fractions', such as multiplication, division, and measurement (Thompson & Saldanha, 2003, p. 12). Thompson and Saldanha defined such schemes as 'stable ways of thinking that entail imagining, connecting, inferring, and understanding situations in particular ways' (p. 13). They argued that fraction reasoning is a type of multiplicative reasoning grounded in a deep understanding of proportionality. As with Kieren (1988), they highlighted the importance of children coming to see fractions in terms of the image of relative size, for example of 'A is $\frac{m}{n}$ of B' as 'A is m times as large as $\frac{1}{n}$ of B'—i.e. multiplicatively—rather than only the part-whole meaning, 'A is some fraction or subset of B'. Otherwise, improper fractions won't make sense because A is thought of as a subset of B. These theorisations suggest that using only static models (drawings) of proper fractions and an overemphasis on part-whole meanings can cause cognitive difficulties with understanding other interpretations of rational number.

A framework that similarly theorises the development of fraction knowledge in terms of conceptual schemes, termed 'Fraction Schemes' (Steffe & Olive, 2010) attends to mental actions such as partitioning, disembedding, and iterating (Norton & Hackenburg, 2010). Partitioning involves the mental action of breaking a continuous whole into equal pieces. Disembedding is the mental action of taking a part from a whole without destroying the whole. Iterating involves making connected copies of a part (Lovin et al., 2018). Steffe (2010) described research into young children's development of different levels of fragmenting that correspond to their construction of whole number counting. It was found that children who can establish 'figurative quantity' of discrete objects are also able initially to share continuous linear objects into two or three parts (p. 4). Steffe expressed surprise that linear (physical strings) rather than area models ('cakes') appeared to be more compatible initially with children's counting operations. Sensitivity to the *equality* of parts was found to develop after the 'sense of twoness' (p. 4). The researcher's earlier reorganisation hypothesis was evidenced by children's quantitative operations emerging in both continuous and discrete cases in the same time frame and in quite similar ways. Steffe (2010) argued for *not* reserving fractional language notational systems for continuous models and whole number counting language for discrete quantities. As also emphasised by others' perspectives (e.g. Kieran, 1988; Thompson & Saldanha, 2003), children need to not only see four sevenths as 'four parts out of seven equal parts' but also as 'four of one seventh'—an iteration of a unit fraction.

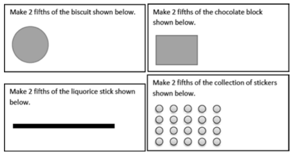
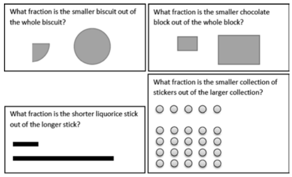
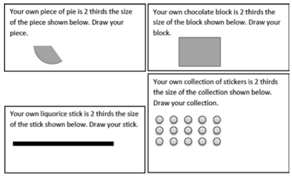
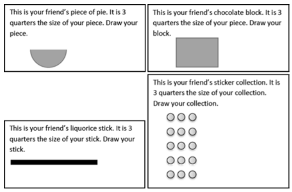
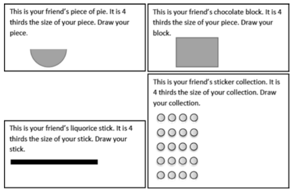
In this study, we drew on the recently validated Fraction Schemes theorisation (Stevens et al., 2020). In focusing on numerical operations, it highlights the need for learners to transcend part-whole reasoning, by developing partitive conceptions and with both proper and improper fractions (Norton & Hackenberg, 2010). The Fraction Schemes learning trajectory has been validated in American and Chinese education contexts (see Norton et al., 2018). They have also been used for assessing US pre-service teachers (see Lovin et al., 2018) by seeking evidence of using a particular conceptual scheme with more than one type of model (e.g. linear and area tasks). The Fraction Schemes are presented and defined in Table 1 (Sect. 2.1) along with illustrative questions from this study's survey.

Perspectives on representations of fractions for learning and teaching

The mathematics education literature highlights the importance of students learning with visual representations across all mathematics domains and making connections among them (e.g. Arcavi, 2003; Presmeg, 2006; Zazkis et al., 1996). Fischbein (1987, p. 104) argued that 'a visual image not only organises the data at hand in meaningful structures, but it is also an important factor guiding the analytical development of a solution; visual representations are an essential anticipatory device'. Zoltan P. Dienes is famous for championing the use of concrete materials for learning mathematics visually. He was the inventor of a range of manipulatives and arguably 'sowed the seeds of contemporary uses of manipulative materials in instruction' (Sriraman & Lesh, 2007, p. 59), including fractions.

Models for fractions have been categorised as continuous or discrete (Kieran, 1976). Continuous models include linear, area, and volume models, where lengths, two-dimensional regions, or three-dimensional objects define the referent whole and parts, e.g. long thin rectangles, number lines, circles, rectangles, shapes drawn on grid or dot paper, spheres, and prisms. Discrete models include representations of sets or collections. Concrete manipulatives for teaching and researching fractions include fraction kits of pre-partitioned circles and squares, fraction bars, geoboards, paper sheets and strips, Cuisenaire

Table 1 Fraction Schemes (adapted from Stevens et al., 2020) and illustrative task versions used in the study's survey for teachers' personal preferences for solving tasks

Fraction scheme	Task versions used in survey
1. Part-Whole Scheme (PWS) Producing $\frac{m}{n}$ by partitioning a continuous whole into n equal pieces and removing m of those pieces (while still being aware of the size of the whole)	
2. Partitive Unit Fraction Scheme (PUFS) Determining the size of a unit fraction relative to a given unpartitioned whole, by iterating the unit fraction (making connected copies of it) to produce a continuous partitioned whole	
3. Partitive Fraction Scheme (PFS) Determining the size of a proper fraction relative to a given unpartitioned whole, by partitioning the whole to produce a unit fraction and iterating the unit fraction to reproduce the proper fraction and the whole	
4. Reversible Partitive Fraction Scheme (RPFS) Reproducing the whole from a proper fraction of it by partitioning the fraction to produce a unit fraction and iterating that unit fraction the appropriate number of times	
5. Iterative Fraction Scheme (IFS) Reproducing the whole from an improper fraction of it by partitioning the fraction to produce a unit fraction and iterating that unit fraction the appropriate number of times	

rods, pattern blocks, and discrete counters (chips) (e.g. Cramer et al., 2008). Behr et al. (1992) drew on the previously mentioned work of Dienes with concrete materials and rational number concepts to research and develop fraction learning sequences. They incorporated a range of visual experiences including: hands-on tasks with circular and rectangular manipulative pieces (colour-coded); model comparisons; paper folding; use of Cuisenaire rods; tasks with sets of chips or counters (discrete model); critique of different models for solving a particular task; and tasks with a number line.

Kamii and Clark (1995) highlighted the difficulties upper primary students experienced when reasoning about equivalent fractions with paper rectangles folded in different ways. They attributed these difficulties to students' lack of experience with physical manipulatives, as opposed to pictures of already partitioned area models, and their lack of experience with improper fractions. Cramer et al. (2008) researched upper primary students' thinking when adding and subtracting fractions. They found that concrete representations of the operations, with estimation and visualisation experiences, supported students in moving to symbolic representations.

Circles have traditionally been used for early fractions teaching (Moss, 2005) but there are differing opinions evident in the literature on their efficacy for learning. These differences seem related to how circular representations are used in instruction. For example, Moss (2005) argued that children's additive thinking is reinforced—that they treat the 'pieces of pie' as discrete objects. 'The four pieces into which a pie is cut is just four pieces' (p. 321). Yet Cramer et al. (2008) emphasised that such concerns about the potential for such thinking with circle representations applied to most models for fractions. They researched several types of representations and found that the fraction circle model, as physical pre-partitioned pieces of varying sizes, was actually the most effective for building students' mental images for fractions. They argued that changing the referent whole in the learning tasks (e.g. a semicircle or quarter circle could be the referent whole) mediated the likelihood of whole number thinking. Rectangular areas drawn on dot paper and discrete sets of counters were found to reinforce incorrect fraction addition strategies, such as adding numerators and denominators together. They argued that nothing is obvious in either dot paper or sets of counters that demonstrates visually the need for finding common denominators when adding fractions. Pattern blocks were also found to be problematic because of the varying types of shapes; the students struggled to find equivalent shapes for adding fractions. Cramer and Wyberg (2009) contrasted the differing shapes of pattern blocks with segments of a circle, which remain the same general shape. Circular (pre-cut) segments were found to support students in finding equivalent fractions for adding and subtracting fractions. They also suggested deferring the use of discrete sets for addition/subtraction until students have developed a rationale for why common denominators are needed.

Researchers have advocated the use of number lines for helping students recognise fractions as numbers in their own right and that there is an infinite number of fractions between any two distinct numbers (e.g. Clarke et al., 2008). English (1997) conceptualised number lines as a metaphorical representation of our number system: a complex representation that requires the integration of visual and symbolic information. She argued that ideas, such as the density of rational numbers, are difficult for students to abstract from number lines. Students tend to see number lines as having stepping stones with space in between them (English, 1997). Cramer and Wyberg (2009) also highlighted the difficulty students have in identifying the referent unit on a number line. Students may see the whole line segment as the unit when trying to locate a fraction on it. Izsák et al. (2008) argued that it is a teacher's purpose for using a particular model that influences their students' opportunities to learn particular concepts. They illustrated this with student difficulties in trying to make sense of equivalent fractions and referent units with number lines. They suggested that teachers can tend to see and use linear models, like fraction strips and number lines, as 'temporary aides for visualising "amounts"' (p. 52) and not for developing students' understanding of the mental operations underlying those representations. English (1997) emphasised the need for representations to be the source or vehicle for students' learning rather than the target. Otherwise, students' reasoning with fractions is disconnected from meaningful experience.

Fraction walls—a vertical arrangement of several 'wholes' partitioned differently in each row (often from larger to smaller size pieces)—are another representation used for teaching fractions. Cramer and Wyberg (2009) found that static drawn walls (rather than manipulative linear bars) supported students' construction of the inverse relationship between the denominator and size of fraction parts, but not the coordination of the iterative role of the numerator. They advocated for teachers learning to critique the efficacy of different models for showing a fraction concept clearly, and also learning to draw on multiple models with students, rather than preference one.

Prior research on teachers' practice with fraction models and manipulatives

In the past few decades, researchers have investigated prospective and practising elementary teachers' use of representations for solving fraction problems, but very few studies were found on teachers' use of representations in their own practice. Various studies have sought insights into prospective teachers' (PSTs) conceptual understanding of different representations for fractions. One study focused on fraction division and if or how 71 PSTs transferred from one type of representation to another (Biber, 2014). It was found that a verbal or symbolic representation (written operation) was needed alongside other representations for expressing fraction division. There was a tendency for PSTs to show the results of fraction division on the representation after calculating the answer with a written algorithm, rather than use the representation for reasoning. The PSTs themselves indicated a preference for symbolic representation above other types, such as area models or number lines. Jansen and Hohensee (2016) researched 17 elementary PSTs' understanding of partitive fraction division. They found difficulties in translating between representations when the divisor was a proper fraction (e.g. $24 \div \frac{1}{4}$ or $4 \div \frac{2}{3}$) and surmised a lack of experience with the conceptions of division required for such tasks, including partitioning and iterating.

Boyce and Moss (2017) investigated elementary PSTs' constructed Fraction Schemes with discrete models (dots), linear models (bars), and area models (circles). They looked for scheme internalisation and PSTs' perception of difficulty. A key finding was that the PSTs could correctly solve tasks with dots (discrete model) but not structurally identical tasks with bars and circles. They surmised that the PSTs were using proportional reasoning with the discrete model rather than Fraction Schemes and operations. They also found that tasks for the *Partitive Fraction* Scheme (PFS; Scheme #3 in Table 1) were answered correctly more frequently with circles than with bars, suggesting that perception of the referent unit was clearer with the circle. At the *Iterative Fraction* Scheme level—where whole circles were not the referent unit—the PSTs perceived a higher level of difficulty with circles. Boyce and Moss argued for the need to consider both the structure of tasks and the representations used for assessing fraction knowledge.

Lovin et al.'s (2018) study of 109 PSTs' constructed Fraction Schemes also highlighted that representations seemed to be used after obtaining the answer first procedurally—as a means of illustrating the answer rather than finding the answer. As with other researchers (e.g. Izsák, 2008; 2012), they found that a relatively low number of PSTs evidenced constructing the most advanced schemes (*Reversible Partitive* (RPFS) and *Iterative* Fraction (IFS) Schemes; see Table 1), suggestive of difficulties in moving beyond part-whole reasoning and in coordinating multiple levels of units. These difficulties are also implicated in issues working with improper fractions and fraction multiplication and division. Stevens et al. (2020) demonstrated that a shift away from part-whole to iterative language, and an emphasis on improper fractions with area, linear, and discrete models, supported 82 Pre K-8 PSTs in learning to coordinate the three levels of units—the referent whole, the unit fraction, and the referent part (see Lovin et al., 2018)—and construct the higher-level Fraction Schemes. They emphasised the vital importance of teachers being able to work conceptually with all types of fraction representations, including circular models, to discourage reliance on algorithmic procedures.

Some studies of practising teachers' use of models for solving fractions tasks were found. Lee et al. (2011) in their research on 12 middle-grade mathematics teachers' reasoning with area models and number lines found that the teachers struggled more with number lines than with area models. Flexibility with the referent unit was found to be the key to being able to reason conceptually with the representations rather than knowledge of written algorithms. They argued that teachers need to know when and why representations should be used as well as how students might use (or misuse) them. Izsák and colleagues' (2012) study explored 14 middle-grade mathematics teachers' use of drawn representations for fraction operations before and after a 42-h professional development course. They also highlighted the need for teachers to be able to reason with the previously mentioned three levels of units for being able to use drawn representations conceptually in teaching fraction operations.

Ma (2020) compared US and Chinese teachers' knowledge of fraction division as well as their ability to generate a model or story to represent the operation. Diverse levels of responses were found, and a lack of conceptual knowledge was implicated in difficulties with creating a model or story that gives meaning to $1\frac{3}{4} \div \frac{1}{2}$. More recently, Copur-Gencturk and Doleck (2021) researched 350 elementary teachers' responses to multistep fraction problems. They found that those teachers with stronger strategic competence were more likely to use pictorial or direct modelling, particularly for dealing with unknown quantities. Unlike PSTs in other studies, these practising teachers frequently used partitioning and iterating—evidence of having constructed (some of) the Fraction Schemes.

We did find one study on a teacher's actual practice with fraction representations—at upper primary level and with the number line for teaching fraction addition (Izsák et al., 2008). It was found that only a short time was spent on using the number line before moving to the written algorithm. When the students experienced difficulties, the teacher asked them to imagine (but not draw) fraction bars. Rather than making connections between the number line and the algorithm, the teacher explicitly said that her intent was for students 'to be able to use the number line and be able to do the algorithm independently of each other' as this shows that 'they truly understand both' (p. 53). Izsák et al. (2008) argued for further research on the purposes for which teachers use fraction representations in their teaching practice.

The research questions for this study were (1) How do practising teachers reason about their fraction representation preferences, for solving tasks and for teaching students? and (2) What is the relationship between teachers' personal representation preferences for solving fraction tasks and for teaching fractions? The following section overviews the design of the study to gain insights into how teachers critique different (static) models and manipulatives for the purpose of solving tasks themselves or teaching fractions.

Research design

In a mixed methods study, we investigated practising primary teachers' personal preferences when solving fraction tasks with different types of models, and also their purposes for using particular models and manipulatives for teaching fraction ideas relevant to their curriculum context. We sought insights into their pedagogical reasoning with fraction representations—an aspect of teacher knowledge for teaching mathematics about which we found few studies in the literature. We quantitatively analysed teachers' responses to a variety of Likert scale items and qualitatively analysed numerous open response items. The findings are intended to inform the future development of teacher professional learning. In the following, three subsections we overview contextual information for the study: survey design considerations, the teacher participants, and data collection and analysis.

Design considerations for the survey: the Fraction Schemes and the teachers' curriculum context

In designing the survey used in the research, we drew on the previously mentioned Fraction Schemes (Steffe & Olive, 2010; Stevens et al., 2020) and the prescribed curriculum content on fractions (ACARA, 2017). Kieran's (1976) subconstructs provide a comprehensive map for understanding rational number from a mathematician's perspective (Elias et al., 2020) but it has been argued that they are less useful for designing learning sequences for students (Thompson & Saldanha, 2003). The Fraction Schemes were developed from multiple research studies investigating the actual processes for learning fractions and therefore were considered useful for exploring teachers' personal preferences for representations when solving tasks themselves. In our survey, firstly to gain insights into teachers' preferences in solving tasks themselves, we presented teachers with various versions of a task at each of the five scheme levels, each of which included a different model. They are presented in Table 1. We asked teachers to select their preferred task version for each scheme and to give a written explanation for their choice. The original design was a paper-based questionnaire that also elicited teachers' written solutions to the various fraction tasks, but the survey had to be modified for online completion during Melbourne's lockdown over several months.

We also sought insights into teachers' model and manipulative preferences for teaching fractions. We included survey questions both on assigning a level of preference for each of a list of models and manipulatives (in general) and on selecting a preferred model and manipulative for each of seven aspects of teaching fractions, drawn from the prescribed national curriculum content of the study's participants (ACARA, 2017). The seven fraction ideas and the lists of models and manipulatives are presented in Fig. 1. More detailed content descriptions from the teachers' national curriculum are presented in Appendix.

Study participants and context

The fraction representation preferences survey was completed by 198 primary teachers or leaders in mid-2021. They were practising teachers in Catholic schools from the state of Victoria, Australia, and constituted a convenience sample. (Nearly 40% of Victorian students attend non-government schools.) The teachers were participants in various professional development opportunities organised by Melbourne Archdiocese Catholic Schools (MACS). The survey was completed by volunteers prior to or outside the professional learning (PL) workshop. No surveys were completed following PL about fractions; hence, their PL was not considered an influence on the teachers' survey responses.

Tables 2 and 3 outline the year levels of the teacher participants and the number of years they had been teaching. Overall, the majority were not novice teachers, as 74% had more than 5 years teaching experience. Also 75% had taught either Grade 5 or 6 at some stage during their teaching career, indicating that for many, the teaching of fractions was likely to

Fraction idea	Preferred model for teaching fraction idea	Preferred manipulative for teaching fraction idea
1. Comparing fractions ('Which is bigger'?)	o Circles	o Pre-made circles
2. Finding equivalent fractions	o Squares	o Pre-made squares
3. Finding a fraction between two other fractions	o Rectangles	o Fraction bars
4. Working with improper fractions (e.g., $\frac{4}{3}$)	o Number lines	o Pattern blocks
5. Adding and subtracting fractions	o Sets / collections	o Cuisenaire rods
6. Multiplying and dividing fractions	o I prefer not to use models to teach this big idea	o Geoboards
7. Partitioning a whole or fractions into unequal parts (e.g., $\frac{1}{2} = \frac{1}{6} + \frac{1}{3}$)		o Counters o I prefer not to use manipulatives to teach this big idea

Fig. 1 Curriculum aspects used in the survey, for each of which teachers indicated their model and their manipulative preferences

Table 2 Frequency and percentage of participants teaching F-2, 3–4, 5–6 and other, respectively

Grade level	Foundation- 2	3–4	5–6	Other	Total
	67 (34%)	57 (29%)	48 (24%)	26 (13%)	198 (100%)

Table 3 Frequency and percentage of participants' years of teaching experience

Years teaching experience	1–5 years	6–10 years	11–20 years	> 20 years	Not specified	Total
	51 (26%)	31 (16%)	43 (22%)	70 (35%)	3 (2%)	198 (100%)

be part of their pedagogical content knowledge. There were more female participants than male (86 and 12%, respectively), and 2% declined to answer this question.

Data collection and analysis

Quantitative and qualitative data were analysed. The quantitative data included demographic data, and Likert scale items. The demographic data related to the participants (years of teaching, year level currently teaching, whether they had ever taught Grade 5 or 6, and gender) have been reported in an earlier section (i.e., study participants). Likert scale items, which were analysed quantitatively, are discussed throughout the Results section. The percentages of participants provide an indication of the spread of preferences for each item in the survey.

For each open-response item in which the participants provided an explanation for their personal choice of model for solving fraction tasks, the data were sorted by the participants' selection of a model (i.e., circle, rectangle, line, or set) for each scheme. Their written explanations for the choice of a particular model were then analysed using qualitative line-by-line coding as outlined by Braun and Clarke (2012). The first author followed the following stages: (1) familiarisation with the data, (2) generating initial codes, (3) searching for themes, (4) reviewing themes, (5) defining and naming themes, and (6) producing a report and coding framework. The second author also double-coded the Scheme #1 (*Part-Whole*) data for the discrete model preferences (see Table 3 in the Results section), and subsequent refinements were made before Scheme #2 (*Partitive Unit*) data were also double-coded by both researchers and discussed to reach consensus. The same coding framework was used for the open-response items related to the teachers' preferences for a model for each fraction scheme, regardless of the scheme or type of model. The coding framework and percentages of teachers for the qualitative analysis are presented in Tables 4 and 5 in the Results section. Some of the teachers' written responses were related to more than one code and therefore were included in each of them, so the percentages of teachers refer to those who referenced a particular idea. For example, the response 'I can see a $\frac{1}{5}$ straight away with the visual. I know that by using the array I am going to be accurate' was coded as *About seeing part of the whole* and *About accuracy, exactness*.

Results

To address the first research question on teacher representation preferences for solving tasks themselves and for teaching students, the first two subsections present results on teachers' preferences for a particular model for each of the five Fraction Schemes (see Table 1), and for various models and manipulatives in their teaching (in general and for specific fraction ideas from their curriculum). To address the second research question on relating teachers' personal and teaching practice preferences, the third subsection analyses similarities and differences in the teachers' preferences and patterns of reasoning about particular representations.

Teachers' personal model preferences for solving tasks for each Fraction Scheme

To gain insights into teachers' model preferences for solving fraction tasks, the participants were asked to both select their preferred task version for each of the Fraction Schemes (see Table 1) and to give a reason. Figure 2 presents the percentages of teachers who selected a particular model for each of the five schemes. (Note that some participants did not make any selection for a particular scheme so the number of responses is also given—the number of responses decreased as the scheme type number increased.)

Overall, it can be seen that except for Scheme #2 (*Partitive Unit*), most of the teachers preferred a set (discrete) model. For Scheme #2, a vast majority preferred the circle model. Across the five schemes, there were teachers who preferred each model type, providing evidence of diverse individualised preferences.

To examine the teachers' reasons for selecting the discrete model for Scheme #1 (*Part-Whole*), we coded the reasons (prompted by an open-response item), which are presented in Table 4, along with illustrative examples.

Table 4 highlights that more than half of the teachers evidenced preferring the set model for the Part-Whole Scheme (#1) because of its discrete nature—that they were exact amounts, or columns or groups, so that there was no need to estimate the size of the part of the whole, in this case fifths. In Table 1, the set model diagram was an array

Table 4 Teachers' reasons for choosing discrete model for the Part-Whole Scheme

Categorisation ($n=146$)	Percentage of teachers (%)	Illustrative examples
About discrete nature	**58***	
About number, groups, columns	29	'Easier to section off into groups' 'I don't have to make an educated guess on the size of the fraction. I have been given a number instead of a shape or line. I am able to find a fraction of a whole number in this case'
About accuracy, exactness	17	'It is the easiest way to get an accurate answer, whereas the other ones require you to "eye ball" when you are dividing into fifths' 'Because I can make an exact and correct answer, rather than an approximate answer'
About no need for measuring	8	'Does not require exact measuring' 'Easy to visualise, doesn't require a ruler to measure equal parts'
About 'array', 'multiplication'	4	'Link fractions to an array model' 'Representation of the array makes clear connections to my prior knowledge of multiplication and division'
About continuous nature	–	
About visualising or seeing	**33**	
In general (visual, clear, easy to see)	18	'Clear to see visual representation' 'I think it's a visual thing for me'
About seeing the whole	1	'Visually able to see the collection as a whole and distinguish 2 parts out of the whole'
About seeing part of the whole	14	'It is easy to see the parts—easy to mark 2/5' 'Clearly shows the fifths'
About ease or easier	**42**	
In general (easy, easier)	8	'Easier to understand' 'Easy to do quickly'
About 'easy to partition'	22	'Easier to divide a set of dots that are countable' 'To me it feels like I could be more accurate in breaking it into 1/5 s initially as they are more easily identified'
About 'easy to iterate'	12	'Visually it is easier to see what 1/5 is and then to find 2/5' 'The array shown makes it easy to identify one fifth, so two fifths would be two columns'
About teaching	4	'Students tend to understand arrays—rows and columns—better and have the option of counting individual units to help with calculations' 'Best visual representation model for students to grasp'

Table 4 (continued)

Categorisation ($n=146$)	Percentage of teachers (%)	Illustrative examples
Unclear/irrelevant response	4	'It is what I am used to seeing' 'I think this because it mentions the word collection and this was a typical representation'
No written response	1	

* Percentages in bold are subtotals for the subcodes underneath

Table 5 Teachers' reasons for choosing a circle model for the Partitive Unit Fraction Scheme

CATEGORISATION ($n = 113$)	Percentage of teachers (%)	Illustrative examples
About discrete nature	–	
About number, groups, columns	–	
About accuracy, exactness	–	
About no need for measuring	–	
About 'array', 'multiplication'	–	
About continuous nature	–	
About visualising or seeing	**74***	
In general (visual, clear, easy to see)	32	'Easier to see' 'Easy to see visually'
About seeing the whole	7	'Clear to see where it fits on the large circle in order to work out the fraction' 'It's a commonly seen representation of the fractions 1 and 1/4 and would be recognisable by many straight away'
About seeing part of the whole	35	'It is easier to see the 1/4' 'It is easy to see 1/4 of the round shape'
About ease or easier	**14**	
In general (easy, easier)	4	'Easier to make the comparison' 'Just so easily evident'
About 'easy to partition'	4	'It is easier to see the circle divided into quarters' 'Circles are easy to divide into halves, quarters, eighths…'
About 'easy to iterate'	6	'I know that is 1 quarter because 4 quarters make the whole biscuit' 'It looks more obvious—you can easily see that you would need four of these to make a whole'
About teaching	5	'Relative to the shape of a biscuit makes it realistic for juniors' 'Quarters are very obvious in circles; students have experience of segments of food and telling the time'
Unclear / irrelevant response	12	'I like it better' 'I like to think of quarters as food examples'
No written response	5	

*Percentages in bold are subtotals for the subcodes underneath

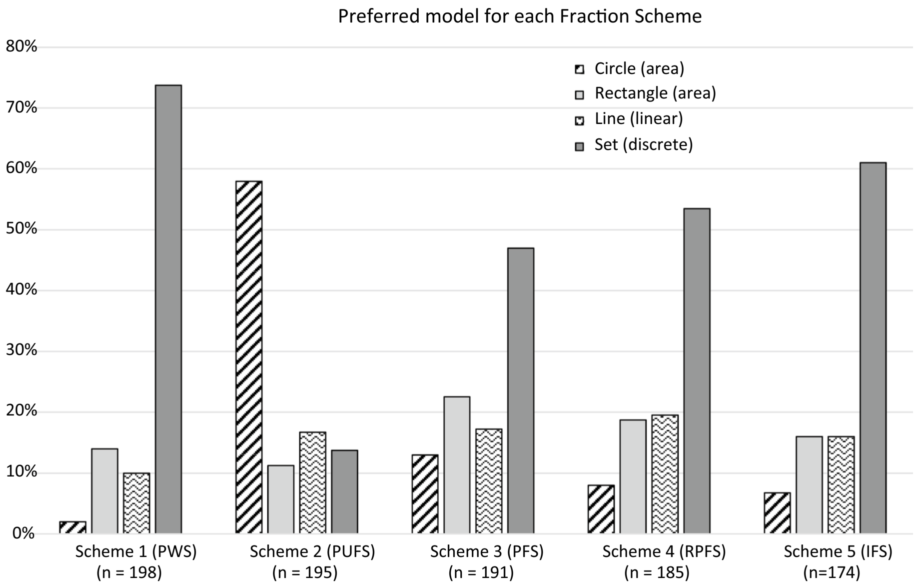


Fig. 2 Preferred model for solving each Fraction Scheme task (percentage of respondents for each survey item)

of five columns and four rows. Several of the teachers explained that this arrangement made seeing parts of the whole easier. Nearly one third of the teachers referenced the idea of being able to visualise or see. It is likely that a random arrangement of counters, or a different array—rather than an array matching the task's required partitioning—may have elicited different responses.

Similar explanations for the teachers' set model preferences across Schemes #3 to #5 (*Partitive*, *Reversible Partitive*, and *Iterative*) were also found—about its being easier to visualise part of the whole and only needing to count, not measure or estimate.

To examine the teachers' reasons for preferring a circle model for Scheme #2 (*Partitive Unit*), we coded the responses using the same categories as Scheme #1 (*Part-Whole*), and the results are presented in Table 5.

Table 5 shows that more than two thirds of the teachers referred to finding the circle model easier to visualise or see for Scheme #2, in this case a quarter of a circle. Many teachers referred to the concept of clearness or ease of seeing in general terms, but a slightly larger percentage described specifically the ease of seeing a quarter as part of a whole circle.

To investigate the participants' range of preferences across the five schemes, we also counted the number of models preferred by each individual teacher, presented in Fig. 3.

It can be seen that nearly three quarters of the teachers selected either two or three model types, suggesting that the nature of the task, in terms of fraction scheme, and the fractions involved, play a role in teachers' solving preferences, rather than teachers having a stable model preference across all contexts. That said, the overall pattern of these results highlights the teachers' noticeable preference for set (discrete) models.

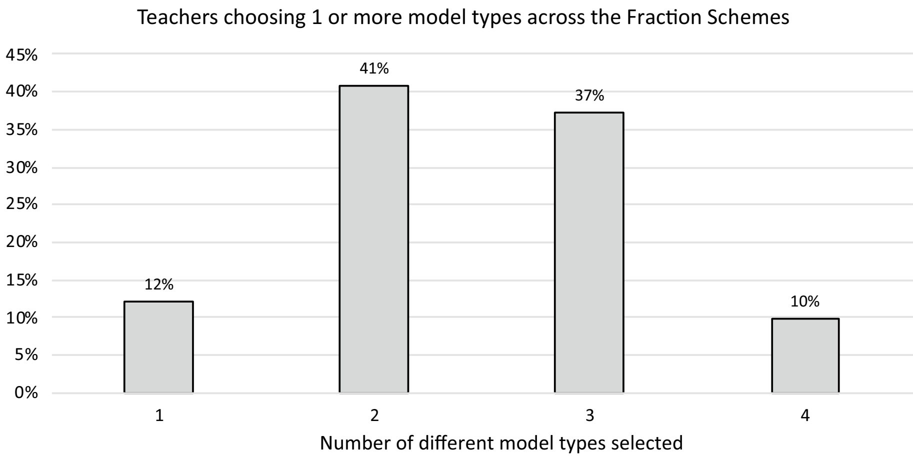


Fig. 3 Number of model types selected by teachers across the Fraction Schemes

Teachers' model and manipulative preferences for their teaching practice

We now turn to the teachers' responses regarding their preferences for teaching fractions (rather than solving tasks themselves). Prior to selecting their model preferences for teaching fractions, the participants were asked to nominate a year level for which they have the most experience teaching fractions and to consider this year level when choosing a preferred model or manipulative. Two subsections present data, firstly on teachers' preferences in general, and secondly for teaching particular fraction ideas—for a range of models and manipulatives.

Model and manipulative preferences for teaching in general

Figure 4 presents the teachers' level of preference (on a five-point Likert scale from 'do not prefer' to 'prefer a great deal') for teaching with each model in general.

We also examined these data by nominated year level, presented in Fig. 5.

The data highlight a full range of levels of preference for each model type, with the majority of teachers preferring a particular model type 'a lot' or 'a great deal'. The highest percentage of preference was for the set (discrete) model (69.4%) and then the rectangle (area) model (69.3%). The other model types were around 50% (for prefer a lot or a great deal). Overall, the teachers across nominated teaching year levels indicated a high preference for all model types except the number line, which was more preferred by those teaching Years 4 to 6.

A different main preference emerged from the teachers' data about teaching with hands-on manipulatives, as seen in Fig. 6. Nearly 80% of the teachers highly preferred fraction bars—a linear representation—whereas only around 50% highly preferred counters (chips; a discrete representation).

For each manipulative type, there was a full range of levels of preferences made by the teachers, including those teachers who reported not having taught with a particular

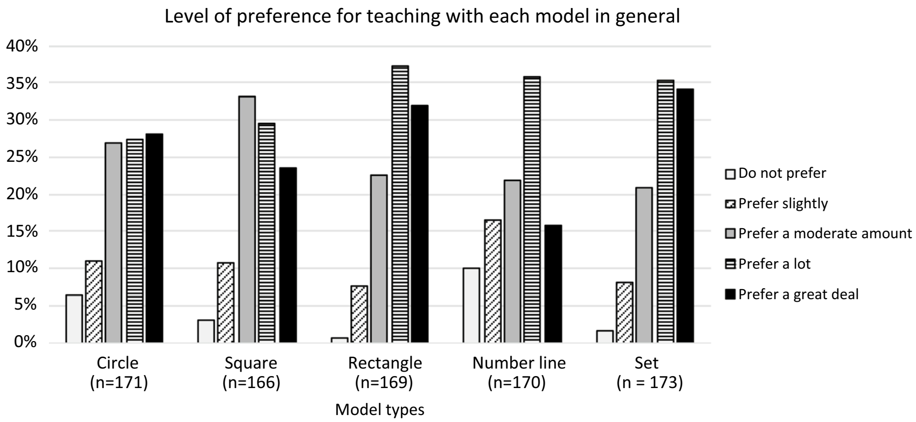


Fig. 4 Level of preference for teaching with each model type in general

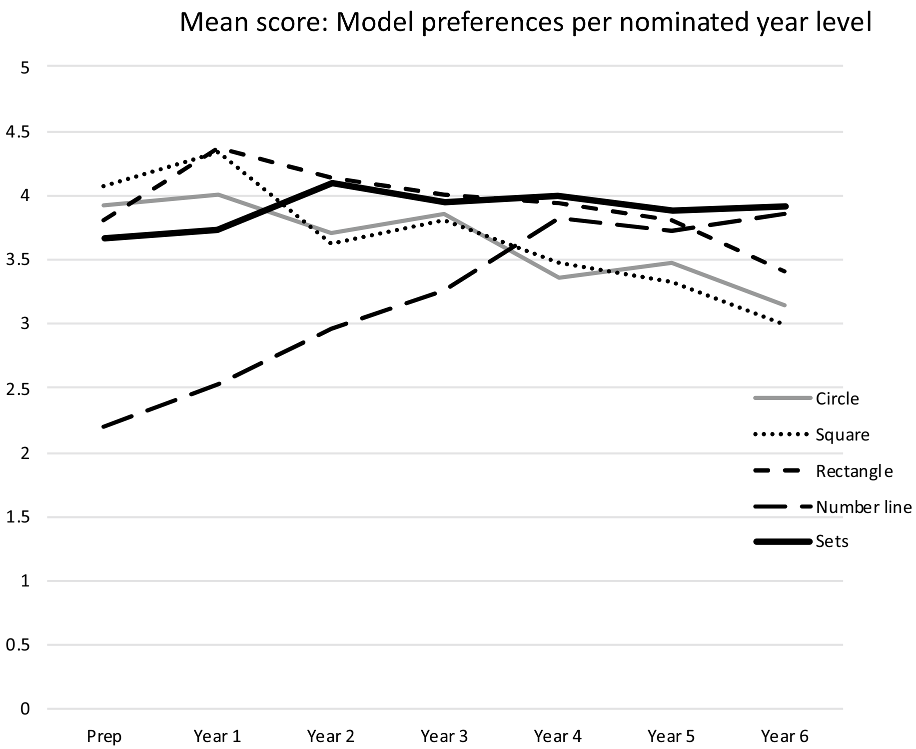


Fig. 5 Level of preference for teaching with each model in general across year levels

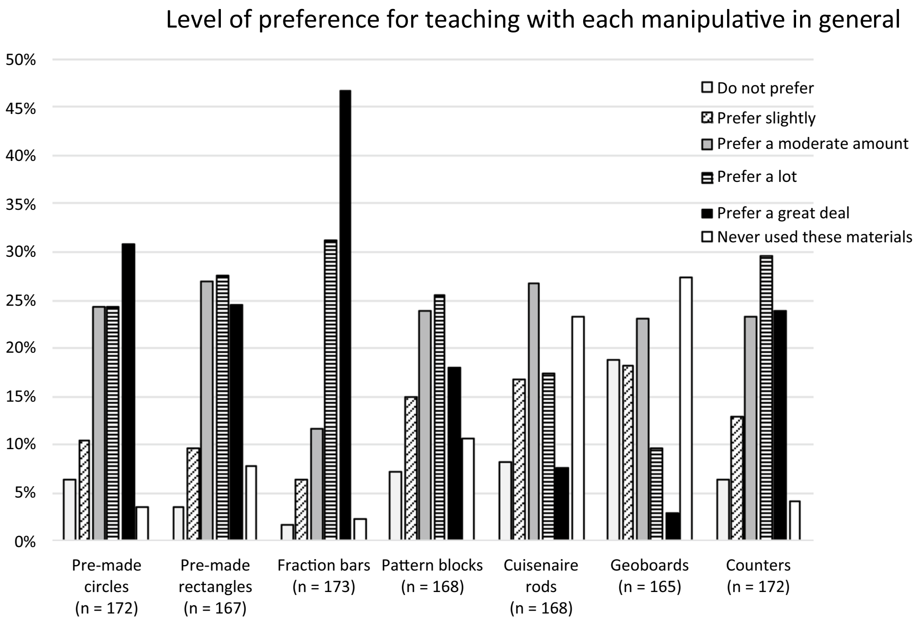


Fig. 6 Level of preference for teaching with each manipulative in general

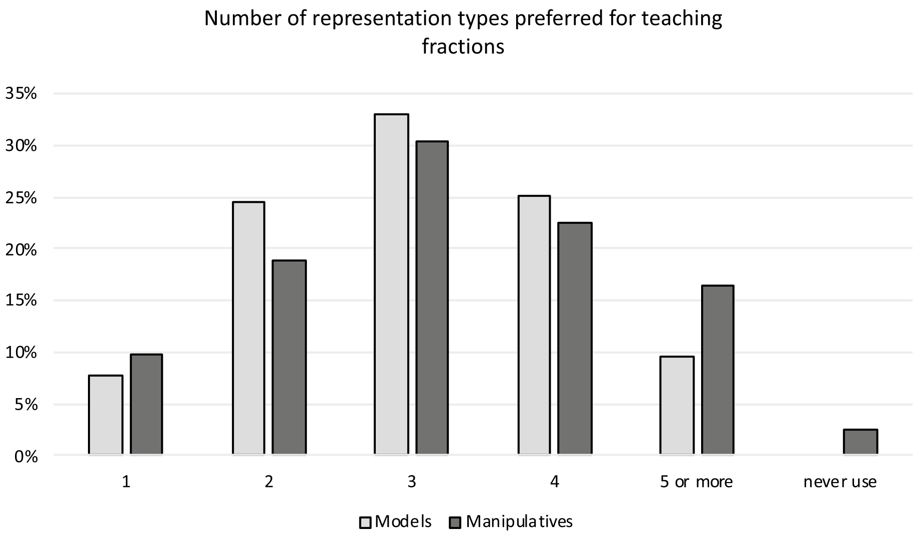


Fig. 7 Number of model types and manipulatives preferred by teachers (prefer a great deal / a lot)

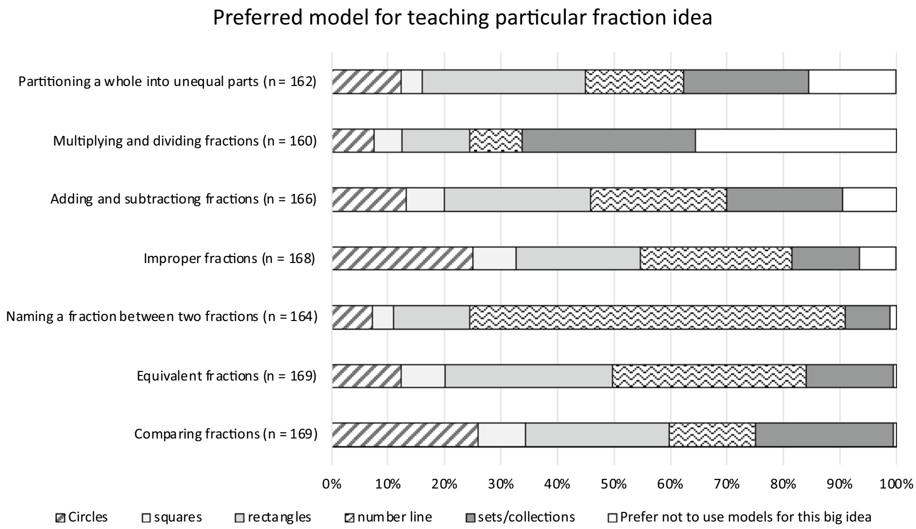


Fig. 8 Model types preferred by teachers for teaching a particular fraction idea (percentage of respondents for each idea)

manipulative before. This suggests individualised preferences for teaching fractions with manipulatives in general.

Figure 7 presents percentages of teachers who, across their teaching preferences, selected a high level of preference (a lot or a great deal) for one or more types of models and manipulatives.

The data indicate that the vast majority of the teachers reported teaching with both models and manipulatives. Around half of the teachers indicated a high level of preference for three or four models and three or four manipulatives. Overall, these differences in levels of preference for teaching with models and manipulatives for the different types of representations (area, linear, or discrete) suggest that these teachers prefer teaching with area and discrete static models and with linear manipulatives (fraction bars).

Model and manipulative preferences for teaching specific fraction ideas

Figure 8 presents the teachers' most preferred model for teaching each previously mentioned fraction idea (or 'big idea'; see Fig. 1).

Each fraction concept elicited the full range of model types as well as those teachers who reported preferring not to use any model for teaching it. In other words, each model type was preferred by at least some teachers for teaching a particular fraction idea. Number lines were favoured by the majority of teachers for teaching: the 'naming of fractions between two fractions', 'improper fractions', and 'equivalent fractions.' A rectangle model was the next most frequently preferred for teaching these concepts. A circle model was the most favoured for teaching fraction comparison.

Figure 9 presents data on the teachers' most preferred manipulative for teaching each fraction concept.

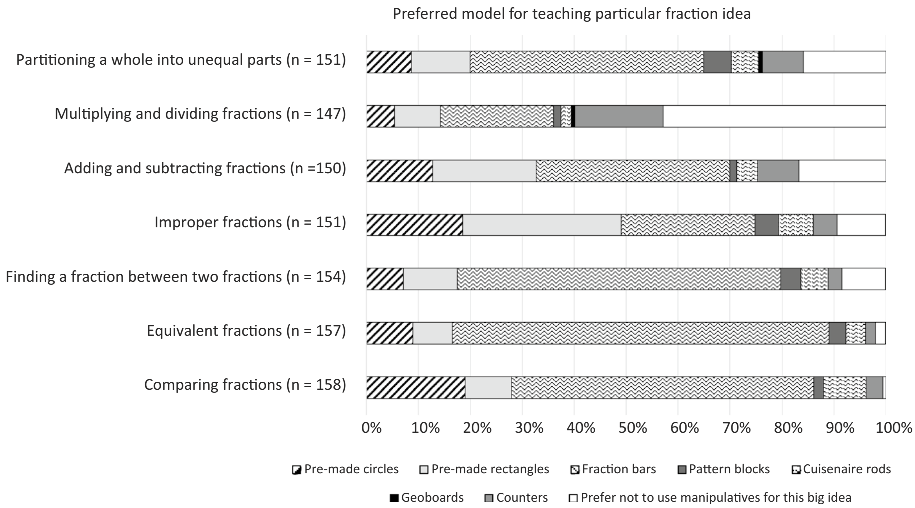


Fig. 9 Manipulatives preferred by teachers for teaching a particular fraction idea (percentage of respondents for each idea)

As with model types, the teachers' preferences can be seen across the full range of manipulative types for each concept. Noticeably, fraction bars were the most preferred manipulative for teaching all of the fraction concepts except 'improper fractions', where pre-made rectangles are slightly more preferred. Some illustrative examples of the written reasons the teachers gave regarding their preference for fraction bars (including their nominated year level in parenthesis) are:

> The fraction bars are easy for students to compare fraction amounts and identify equivalent fractions. Easy to make and manipulate. (Yr 2)
> The fraction bars are a great visual for the students to see how the same length can be divided into many different fractions. (Yr 3)
> Fraction bars help students' understanding of equivalence and I feel it also helps with linking to a linear model of teaching fractions. (Yr 4)
> Students can see which fractions are 'bigger' in an immediate look when all are together. Can also be used to look for equivalent fractions. (Yr 5)

These teachers identified the importance of the fraction bars for understanding the equivalence of fractions, part-whole relationships, the relative size of fractions, and interestingly also the connection to other linear representations, such as number lines.

It is of note that a considerable percentage of the teachers indicated that for teaching 'multiplying and dividing fractions' they prefer not to use a model (more than 30%) or a manipulative at all (more than 40%). From our data, we were not able to ascertain possible underlying reasons for this result, as our open-response items prompted teachers to explain their reasons for preferring their chosen model/manipulative, rather than for not preferring to use them. However, the reason for not using a model to assist students' understanding of fraction division may be a consequence of teachers' lack of conceptual understanding, as identified by Ma (2020).

Overall, these responses indicate that although the teachers reported diverse individual preferences regarding model types for teaching each of their fraction concepts, fraction

Table 6 Cross-tabulation of teachers' discrete model preferences for solving fraction tasks and manipulative preferences for teaching ($n = 198$)

Solving own tasks		Teaching in general		Teaching specific ideas	
Discrete (sets) model preferred		Model—Sets	Manipulative—Counters	Model—Sets	Manipulative—Counters
Num. schemes	Subtotal ($n = 198$) (%)	High preference (%)	High preference (%)	Preferred for at least 2 ideas (%)	Preferred for at least 2 ideas (%)
0 schemes	**12.1**	2.5	2.5	1.0	–
1 scheme	**20.2**	10.1	9.6	2.0	1.0
2 schemes	**19.2**	10.1	7.6	5.1	2.5
3 schemes	**23.2**	19.2	16.7	10.6	4.0
4 schemes	**18.2**	12.6	8.1	5.6	1.5
5 schemes	**7.1**	6.1	5.6	4.5	1.0
	Subtotal	**60.6**	**50.0**	**28.8**	**10.1**

bars were clearly the most preferred manipulative (with pre-made rectangles being preferred for teaching improper fractions). This result is perhaps surprising, given that multiple wholes are not considered as easily visualised with rectangles, compared to circles, and compared to fraction bars, each of which is a whole (Cramer & Wyberg, 2009). Fraction bars can be arranged in a long line to connect to a number line representation, an important link between representations which one of the previously mentioned teachers referenced.

Relating teachers' personal and practice-based preferences

Given that a set (discrete) model was most preferred by these teachers for solving tasks in all of the five schemes except for #2 (*Partitive Unit*; See Fig. 2), we created cross-tabulations to investigate their preference for discrete models when solving fraction tasks and their preferences for teaching with discreet representations (the set model and counters manipulative). These are presented in Table 6.

In the second column of Table 6, it can be seen that nearly two thirds of the teachers preferred a set model for solving tasks in two or more schemes (out of five). For teaching in general (Columns 3 & 4), there were larger proportions of teachers indicating a high level of preference for sets (models) compared to counters (manipulatives). This same pattern can be seen in Columns 5 and 6 for teaching specific fraction concepts. This suggests that static discrete representations are more likely to be used by teachers than counters (chips) as such. Prior research has pointed to the effectiveness of manipulatives compared to static models for learning fractions (Martin et al., 2012). In this study, we did not ask teachers to critique models versus manipulatives, but future research on their preferences and if, how, and when they use manipulatives instead of static models would be worthwhile.

Given that many teachers described the arrangement of groups or columns or arrays as the reason for their personal preference for solving with a set model (Table 3), yet do not also use counters for teaching fractions, it seems likely that if teachers are using arrays in their teaching, it is not necessarily for fractions per se. In the Year 2 curriculum content, students are to ‘Recognise and represent multiplication as repeated addition, groups and arrays’ (ACMNA031; ACARA, 2017). The array representation, which the teachers themselves preferred to use for solving fraction tasks, seems more likely to be used for teaching multiplication.

Similarly, it can be seen that a majority of the teachers who showed a strong preference for solving fraction tasks with the discrete model (three or more schemes out of five) also indicated a strong preference for using them in their teaching. Out of the 25% who selected the discrete model for four or five schemes, nearly three quarters indicated a high preference for teaching with sets as well. When we calculated the Pearson’s correlation coefficient for solving with a (set) discrete model (0 to 5 schemes) and level of preference for teaching fractions with the set model in general (score of 0 to 4; don’t use to prefer greatly), we found a medium positive correlation (0.356). When we calculated the Pearson’s correlation coefficient for solving with a (set) discrete model (0 to 5 schemes) and number of fraction ideas for which a set model is preferred for teaching (0 to 7 ideas), we also found a medium positive correlation (0.379). When we calculated the Pearson’s correlation coefficient for solving with a discrete model and number of ideas where a discrete manipulative (counters or chips) is preferred for teaching (0 to 7 ideas), we found a negligible positive correlation (0.149). These results suggest that teachers who preferred the (static) set model for solving also indicated a preference for teaching with it in general and for specific ideas, but not for teaching with the matching manipulative counters (chips). There was noticeable association between not selecting a discrete model for any of the schemes (12.1% of teachers) and a low level of preference for teaching with either a discrete model or a discrete manipulative.

Given that the teachers demonstrated a noticeable preference for the fraction bars across most of the specific fraction concepts (Fig. 9), we also created cross-tabulations to relate this preference both to teaching in general and to solving with a linear model. These are presented in Table 7.

We calculated the Pearson’s correlation coefficient for number of fraction ideas for which fraction bars are preferred for teaching (0 to 7 ideas) and level of preference for solving with a linear model (0 to 5 schemes) and we found a negligible positive correlation (0.104). This result suggests that those teachers who preferred fraction bars for teaching specific fraction concepts do not necessarily prefer a linear model for their own solving of fraction tasks. Cuisenaire rods are also a linear manipulative, and interestingly, 41% of teachers overall indicated a high preference for them across the range of levels of preference for fraction bars.[1]

Some illustrative examples of the written reasons teachers gave regarding their preference for Cuisenaire rods (with their nominated year level in parentheses) are:

> Great for kids to discover by trial and error which rods are ‘half’ and ‘one quarter’ the length of others. (Foundation)
> Cuisenaire rods for a linear model—the whole can be different lengths (unlike the fraction bars which are usually named). (Yr 2)
> Cuisenaire rods are easy for students to compare fractions and show fractions of something. (Yr 4)
> Cuisenaire rods—great to visualise for students to manipulate and move. (Yr 5)

[1] An Australian company produced Cuisenaire rod kits from the 1970s and the government resourced primary schools with them; the kits continue to be readily available to teachers.

Table 7 Cross-tabulation of teachers' fraction bars preference for teaching specific fraction ideas and preferences for teaching (in general) and solving with linear representations ($n = 198$)

Teaching specific ideas		Teaching in general			Solving own tasks					
Fraction bars manipulative preferred		Model – Number line	Manip. – Fraction bars	Manip. – Cuisena-ire rods	Linear model preferred					
Num. fraction ideas	Subtotal ($n = 198$) (%)	High preference (%)	High preference (%)	High preference (%)	0 Schemes (%)	1 Scheme (%)	2 Schemes (%)	3 Schemes (%)	4 Schemes (%)	5 Schemes (%)
0 ideas	**28.8**	6.1	11.6	6.6	19.2	5.6	3.5	0.5	–	–
1 ideas	**10.1**	5.6	7.6	5.6	4.5	3.0	2.0	0.5	–	–
2 ideas	**13.1**	6.6	9.6	7.1	7.1	2.0	4.0	–	–	–
3 ideas	**15.7**	9.1	13.6	7.6	6.6	5.1	1.5	1.5	0.5	0.5
4 ideas	**12.6**	6.1	10.1	7.6	5.1	5.1	1.0	0.5	1.0	–
5 ideas	**7.6**	4.0	7.6	3.0	5.1	2.5	–	–	–	–
6 ideas	**5.6**	3.5	4.5	0.5	3.5	0.5	1.5	–	–	–
7 ideas	**6.6**	3.5	5.6	3.0	4.0	0.5	1.0	–	0.5	0.5
Subtotal		**44.4**	**70.2**	**40.9**	**55.1**	**24.2**	**14.6**	**3.0**	**2.0**	**1.0**

These responses highlight the perceived value of Cuisenaire rods' manipulability in the process of making sense of fractions, for representing unit fractions, for comparing fractions, and for making connections to linear models such as the number line. There seemed to be less emphasis on showing equivalence, compared to the fraction bar responses. Interestingly, the Year 2 teacher in the second quote above compared Cuisenaire rods to fraction bars and highlighted the flexibility of the rods in being able to change the size of the wholes, unlike commercial fraction bars which are typically labelled with the fraction name (as part of the longest 'whole' bar).

More than half of the teachers (55.1%) did not select a linear model at all for solving fraction tasks and yet their preferences for teaching with fraction bars were spread across the full range of fraction concepts (from 0 to 7 concepts). This suggests that many teachers may consider fraction bars a valuable manipulative even though they prefer not to use linear models for their own solving. Several teachers commented negatively on the need to use estimation or measuring for solving the given tasks, and it seems probable that because fraction bars are *pre-cut* into unit fractions, they are viewed as different to unpartitioned linear representations (we used in the scheme tasks). If we had also pre-partitioned the linear bars in the solving tasks, the responses may have been different. It is worth noting that the developers of the five Fraction Schemes framework highlighted the importance of learners being able to partition into equal parts themselves as evidence of having constructed the schemes, rather than always being presented with pre-partitioned representations (Stevens et al., 2020).

In reverse, perhaps surprisingly, nearly 30% of the teachers did not choose fraction bars for teaching any of the fraction concepts, even those for whom fraction bars are recommended, such as 'equivalent fractions' and 'adding and subtracting fractions' (Clarke & Roche, 2014). Nearly two thirds of these teachers also did not select a linear model for solving any of the scheme tasks, suggestive of a relationship between not solving with a particular model and not teaching with it.

Discussion and conclusion

There is widespread agreement that for learning fraction concepts, students need opportunities to make connections across different constructs and visual representations of rational number. This study sought to investigate practising teachers' knowledge for teaching fractions through their preferences, both for solving fraction tasks themselves and for teaching. This study contributes to the literature on primary teachers' pedagogical reasoning with fractions through their critiques of different models and manipulatives.

Teachers' personal preferences

In this study, we found evidence that teachers' preference for a particular model was at least in part task-specific. Nearly 90% did not choose the same type of model across all five Fraction Schemes and their reasons suggested that the nature of each task's requirements, and the specific fractions involved, influenced their preferences. Nearly three quarters of the teachers selected two or even three model types across the tasks, which was not suggestive of a stable model preference independent of context. Nevertheless, a majority of teachers in this study indicated a preference for the set (discrete) model for all five schemes, with the exception of Scheme #2 (*Partitive Unit*), for which most teachers preferred a circle model. Their reasons were most frequently related to the discrete (rather than continuous) nature of the set representation that made it visually easy for them to see parts of the whole without having to measure or estimate equal parts. We surmise that the teachers' preference was at least partially influenced by our decision to include sets arranged in an array where the columns match the unit fraction needed for the task. This likely enabled 'calculating-by-structuring' rather than 'calculating-by-counting' (Venkat et al., 2021) which would have been necessary with an 'ad hoc' set of dots. A different array (where columns didn't match the unit fraction) or a random arrangement would have certainly required more work to partition the set and iterate parts. The research on Fraction Schemes in the literature (Lovin et al., 2018; Norton et al., 2018; Stevens et al., 2020) emphasised the importance of students experiencing partitioning into equal parts, i.e. not providing learners with pre-partitioned representations all the time. In one sense the arrays in the tasks for this study were pre-partitioned, unlike the other three representations—circle, rectangle, and linear. Pre-partitioning these models as well may have also produced different results.

The finding that most teachers preferred the circle model for Scheme #2 (*Partitive Unit*)—which involved identifying part of a whole—also confirmed the likelihood of teachers' preferences being at least partially task-specific. Their reasons highlighted that the quarter circle shape was easily recognised in comparison to the whole circle. If we had chosen a different unit fraction to one quarter for this item, whose mental representation was less familiar—such as one seventh—the teachers may have preferred another representation where iterating parts would have been easier, such as the linear model. Nevertheless, Cramer and Wyberg (2009) argued that for comparing unit fractions to the whole (needed for Scheme #2) the circle is clearer than the set (discrete) model. This could explain the teachers' circle preference with only this scheme.

In their study of pre-service teachers (PSTs), Boyce and Moss (2017) found evidence of the PSTs using sets for proportional reasoning with whole numbers and not with Fraction Schemes as such. They were found to solve fraction tasks correctly with a set model but not structurally identical tasks with linear or circle models. In our study with practising teachers, we did find references to proportional reasoning or multiplicative thinking, but there was also evidence of teachers partitioning and iterating with the set model. To delve further into the teachers' noticeable preference for the discrete model, future research involving set models not arranged into arrays, and pre-partitioned models for all types, would be worthwhile. Further research could also explore the influence of choice of fraction on teacher preferences and also with hands-on manipulatives (rather than with only static models).

Regarding the circle model in Schemes #3 to 5 (*Partitive*, *Reversible Partitive*, and *Iterative*), we had included tasks where the referent whole was *not* a whole circle, and we surmise that this is also likely to have influenced teachers' preferences. Boyce and Moss (2017) found that PSTs ranked such tasks with circles as the most difficult compared to similar tasks with other representations. Yet flexibility, in being able to coordinate the referent whole, is considered critical to the development of Scheme #5 (*Iterative*), particularly for understanding improper fractions conceptually (Stevens et al., 2020). As previously mentioned, our online survey did not ask teachers to include their solutions for the task versions (as we felt this was too onerous for them in an online context compared to the originally intended hardcopy format which did prompt for solutions). More fine-grained research and teachers solving the same task type with various models and explaining their pedagogical reasoning with them would be valuable.

Teachers' preferences for teaching

To gain multiple sources of data about teachers' representation preferences for teaching fractions, the survey prompted for levels of preference about specific models and manipulatives in general (see Figs. 4 and 6), and also choice of a model and manipulative for teaching specific fraction concepts (see Figs. 8 and 9). Overall, the teachers indicated most frequently a high level of general preference for the set (discrete) model and secondly for the rectangle model. These preferences did not evidence being associated with teachers' nominated year level (Fig. 5). The only exception was their preference for number lines, which increased with increasing year level. This is perhaps unsurprising given the teachers' curriculum context. Their prescribed national curriculum explicitly refers to a 'number line' for fractions in Years 4 to 6 (ACARA, 2017; see Appendix for the actual content descriptions). 'Shapes and collections' are mentioned in Year 2, and the direction to 'model and represent' is prescribed for Year 3, but otherwise the teachers' national curriculum does not specify fraction models or representations. In research with practising teachers, Lee et al. (2011) associated difficulties in reasoning about fractions with number lines (compared to area models) and with a lack of flexibility in coordinating the referent whole. This issue could also be implicated in the lower levels of preference for number lines at lower year levels found in our study.

The teachers' noticeable preference for set models, for solving the scheme tasks and for teaching in general, was not found to match their preferences for teaching with different types of manipulatives, either in general or for teaching specific concepts. Instead of counters (chips), they demonstrated a strong preference for fraction bars, which are a linear representation. This was evidenced in general and across several specific concepts. Their written reasons for such a preference were related to the fraction bars' usefulness for teaching partitioning and comparing fractions, and pervasively about demonstrating the equivalence of different fractions. It is worth considering that similar to set (discrete) models arranged in an array, that the teachers personally preferred fraction bars, are also pre-partitioned, so again there is no need for estimating or measuring. This could be the link between the teachers' model preference for solving tasks and teaching in general (discrete sets) and their manipulative preference (linear fraction bars), even though the type of model is different. Although the teachers also evidenced positive views about Cuisenaire rods, they were not as popular as fraction bars. As previously mentioned, one teacher remarked on (commercial) fraction bar pieces being labelled as a particular type (e.g. one half, one quarter etc. of the longest bar), and preferred Cuisenaire rods because the referent whole could be changed. This suggests that teachers' preferences may be less about a particular type of representation (area, linear, discrete) but perhaps more about the perceived level of salience of parts and the whole. Cuisenaire rods are also linear manipulatives suitable for learning about equivalence and flexibility with referent units, but they are not labelled. Future research on the issues of partitioning and referent wholes with different types of representations would be worthwhile to learn more about their role in teachers' decision-making.

We found evidence that some practising teachers may not be using models or manipulatives at all for teaching operations with fractions, and especially multiplication and division. More than 30% of teachers in our study indicated that they prefer not to use a model and more than 40% that they don't use manipulatives for multiplication and division, but it was not clear from our data if they found representations problematic for students' learning or if they personally preferred written procedures. This finding is also evident in previous research in the literature on PSTs' (e.g. Jansen & Hohensee, 2016; Lovin et al., 2018) and on practising teacher knowledge of representations for fraction operations (e.g. Copur-Gencturk & Doleck, 2021; Ma, 2020). Ma (2020) in research on teacher knowledge for division by a fraction found that a lack of conceptual meaning for the procedure itself hindered teachers' ability to generate an appropriate representation for the operation. Further research on teacher reasons for preferring not to use representations for teaching the operations would be worthwhile.

Cramer and Wyberg (2009) highlighted students' difficulties with, and incorrect use of discrete and rectangle representations when learning to add fractions, which reinforce whole number thinking (adding the numerators and adding the denominators). They suggested that the ease of losing a sense of the referent whole was problematic with these particular representations. They argued that the circle representation is more supportive when teaching fraction operations because the need for a common denominator is particularly salient. Our study found evidence that few teachers seem to be utilising circle representations in this way, possibly because of conflicting opinions about the efficacy of circles for teaching these more complex concepts beyond part-whole ideas (e.g. Moss, 2005). Recent research on practising teachers' responses to multistep fraction problems highlighted that stronger strategic competence was associated with the use of fraction representations (rather than abstract algorithms), and in ways that evidence the Fraction Scheme operations—partitioning, iterating, and coordinating the three levels of units (Copur-Gencturk & Doleck, 2021). More research on teachers' reasoning with different types of representations and for teaching fraction operations would be worthwhile.

Relating teachers' preferences for solving fraction tasks and for teaching

In our cross-tabulation comparing teachers' representation preferences personally for solving fraction tasks and for teaching, we found that a high personal preference for a set (discrete) model was associated with a preference for teaching with the same model in general and for teaching specific fraction concepts, but not for teaching with the matching manipulative (counters or chips). This result suggests that teachers may have responded positively to the static representation of sets, and as arrays, not just the 'exactness' of numbers of objects inherent in a discrete representation. Evidence, however, was found of very high or very low personal preferences being mirrored in teachers' teaching preferences, but the relatively small (and convenience) sample ($n = 198$) precludes statistical generalisation to other teachers and contexts. Further research on the influences on teachers' decision-making for using (or not using) models and manipulatives in their teaching practice would be worthwhile. Potential influences include teachers' own prior learning and teaching experience (including professional learning), familiarity with particular representations, and use of certain curriculum and teacher resources. Researchers in the literature have emphasised the importance of teachers developing their knowledge for when and why fraction representations are best used so that they teach concepts and not only procedures (e.g. Izsák et al., 2012; Lee et al., 2011).

In another cross-tabulation, we found that although teachers favoured the use of fraction bars for teaching, this was not associated with a personal preference for a linear model when solving fraction tasks. Fraction bars are pre-partitioned, whereas the linear models we used in the scheme tasks were not. We surmise that whether a representation is pre-partitioned may play a role in influencing teachers' personal representation preferences. Several teachers explained that they preferred the set (discrete) model because they did not need to estimate or measure parts of the whole, suggestive of disliking the 'messiness' of having to partition continuous models. Coles (2021) argued that recent research in various contexts has provided compelling evidence that curriculum needs to move away from counting discrete objects (and numbers as discrete) as a primary basis and instead draw much more on a non-numeric exploration of relations, such as between lengths, areas, and other measurement attributes, as advocated historically by Davydov. Further research into teachers' reasoning and critique of continuous and discrete representations of rational number for teaching fractions is considered important for developing effective professional learning opportunities for them.

Appendix 1

See Table 8.

Table 8 Fraction-related content descriptions from the teachers' prescribed *Australian Curriculum: Mathematics* (ACARA, 2017)

Level	Curriculum content descriptions for fractions
Foundation	—
Year 1	Recognise and describe one half as one of two equal parts of a whole. (ACMNA016)
Year 2	Recognise and interpret common uses of halves, quarters and eighths of **shapes** and **collections.** (ACMNA033)
Year 3	**Model** and **represent** unit fractions including 1/2, 1/4, 1/3, 1/5 and their multiples to a complete whole. (ACMNA058)
Year 4	Investigate equivalent fractions used in contexts. (ACMNA077) Count by quarters halves and thirds, including with mixed numerals. Locate and represent these fractions on a **number line**. (ACMNA078) Recognise that the place value system can be extended to tenths and hundredths. Make connections between fractions and decimal notation. (ACMNA079)
Year 5	Compare and order common unit fractions and locate and represent them on a **number line**. (ACMNA102) Investigate strategies to solve problems involving addition and subtraction of fractions with the same denominator. (ACMNA103) Recognise that the place value system can be extended beyond hundredths. (ACMNA104) Describe, continue, and create patterns with fractions, decimals and whole numbers resulting from addition and subtraction. (ACMNA107)
Year 6	Compare fractions with related denominators and locate and represent them on a **number line**. (ACMNA125) Solve problems involving addition and subtraction of fractions with the same or related denominators. (ACMNA126) Find a simple fraction of a quantity where the result is a whole number, with and without digital technologies. (ACMNA127) Make connections between equivalent fractions, decimals, and percentages. (ACMNA131) Continue and create sequences involving whole numbers, fractions, and decimals. Describe the rule used to create the sequence. (ACMNA133)

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