Journal of Mathematics Teacher Education (2022) 25:403–428
https://doi.org/10.1007/s10857-021-09497-1

Conceptualizing content-related PD facilitator expertise

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Accepted: 30 March 2021 / Published online: 21 April 2021

Abstract
Studies of facilitators of professional development (PD) for mathematics teachers have been increasing in order to improve their preparation for conducting PD. However, specifications of what facilitators should learn often lack a conceptualization that captures facilitators' expertise for different PD content. In this article, we provide a framework for facilitator expertise that is in line with current conceptualizations but makes explicit the content-related aspects of such expertise. The framework for content-related facilitator expertise combines cognitive and situated perspectives and allows unpacking different components at the PD level and the classroom level. Using two illustrative cases of different PD content (probability education in primary school and language-responsive mathematics teaching in secondary school), we exemplify how the framework can help to analyze facilitators' practices in content-related ways in a *descriptive mode.* This analysis reveals valuable insights that support designers of facilitator preparation programs to specify what facilitators should learn in a *prescriptive mode*. We particularly emphasize the importance of working on content-related aspects, unpacking the PD content goals into the content knowledge and pedagogical content knowledge elements on the classroom level and developing facilitators' pedagogical content knowledge on the PD level (PCK-PD), which includes curricular knowledge, as well as knowledge about teachers' typical thinking about a specific PD content. Situated learning opportunities in facilitator preparation programs can support facilitators to activate these knowledge elements for managing typical situational demands in PD.

Keywords PD facilitator · Content-related expertise · PCK

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Introduction

Since the early work of Zaslavsky and Leikin (1999), the leaders of professional development programs (PD) for mathematics teachers have attracted increasing research attention: We call them PD facilitators (other names for them are multipliers, teacher educators, teacher leaders, etc.). This increasing research attention has been evidenced, for example, by two recent special issues in the *International Journal of STEM Education* (Tekkumru-Kisa & Stein, 2017a) and the *International Journal of Science and Mathematics Education* (Berry et al., 2021).

The work of PD facilitators has been described as challenging (Borko et al., 2014; Borko et al., 2014; Even, 2008, 2014; Jacobs et al., 2017) but crucial for the success of PD programs (Beisiegel et al., 2018; Roth et al., 2017). To improve facilitator preparation programs, researchers and designers have started to examine what exactly facilitators need to learn (Borko et al., 2014a, 2021; Borko, et al., 2014; Lesseig et al., 2017). Some studies have focused on facilitators' *knowledge* as the foundation of their actions (Borko et al., 2014; Elliott et al., 2009; Jacobs et al., 2017; Wilhelm et al., 2019), whereas other studies have emphasized the important role of facilitators' *practices* while preparing and conducting PD (Lesseig et al., 2017; Zaslavsky & Leikin, 1999). However, the scope of research has rarely been broadened to take into account knowledge and practices simultaneously (with the exception of Even, 2008). Thus, a comprehensive framework for facilitator expertise that considers a facilitator's knowing and acting while allowing a *content-related* view of PD has not yet been available. This raises the question: *In which framework can the expertise of PD facilitators be captured so that the interplay of situated practices and underlying cognitive components can be explained in a content-related way?*

In this paper, we suggest a framework for content-related facilitator expertise that lifts an existing content-related framework from the teacher level (Bromme, 1992; Prediger, 2019) to the facilitator level. The aims of our article are, first, to introduce the framework and connect it to the existing literature and, second, to illustrate by two qualitative cases how the framework can be used to (a) analyze and explain facilitators' practices with respect to content-related and generic aspects of the underlying expertise in *descriptive modes* and (b) conclude what facilitators should learn in a *prescriptive mode* that can inform the design of facilitator preparation programs by elaborating on content-related and generic learning goals. In particular, we argue why the content of the preparation program must be unpacked carefully in descriptive and prescriptive modes and why the content-relatedness adds to the existing research.

After outlining the theoretical background and existing approaches, we introduce the framework of content-related facilitator expertise in the first section. In the empirical second and third sections, we present two illustrative cases to show how the framework can help to analyze content-related facilitators' practices in a *descriptive mode.* For this, we refer to two different areas of PD content, probability education in primary school and language-responsive mathematics teaching in secondary school. Finally, the last section exemplifies how to draw consequences in a *prescriptive mode* for specifying what facilitators should learn.

Theoretical background and introduction of the framework

Our framework for facilitator expertise is inspired by important research on the teacher level, which is reported in the first subsection. Existing perspectives on facilitators are presented in the second subsection and form the theoretical base for the introduction of our framework in the third subsection.

Existing perspectives on teacher expertise and a framework for content-related teacher expertise

Research focusing particularly on mathematics *teachers'* expertise has favored either a cognitive or a situated perspective (Depaepe et al., 2013; Stahnke et al., 2016). Studies taking a *cognitive perspective* have focused on teachers' knowledge and analytically distinguishable facets of this knowledge while considering this knowledge to be the foundation of expertise (Ball et al., 2008; Baumert & Kunter, 2013; Kaiser et al., 2014). Studies in a *situated perspective*[1] have often analyzed teachers' practices by means of classroom videos, video vignettes or discussions of exemplary student work in order to reconstruct teachers' expertise (e.g., Jacobs et al., 2010; Kersting, 2008; van Es & Sherin, 2002). Recent studies have combined advantages of both perspectives by developing video-based instruments to investigate teachers' situation-specific expertise (e.g., Gasteiger et al., 2020; Kaiser et al., 2017). Studies from both perspectives have aimed at describing the expertise of both novice and expert teachers in terms of underlying knowledge and by investigating how expertise can be enhanced.

A conceptualization of teacher expertise combining cognitive and situated perspectives has already been suggested by Bromme (1992). He started from analyzing typical situational demands that teachers have to face (in brief, *jobs*) and identified the *categories* and generic *orientations* that teachers activate when managing these jobs. In this conceptualization, *orientations* subsume beliefs and pedagogical attitudes toward the content (in this case, mathematical) and its teaching and learning (e.g., the conviction that productive struggle is relevant for learning mathematical reasoning). The *categories* comprise non-propositional (i.e., conceptual) knowledge that serves a specific function, namely, filtering teachers' perceiving, thinking and evaluating. For example, some teachers rely on highly differentiated categories from their pedagogical content knowledge (PCK; e.g., allowing deep analysis of students' conceptual knowledge) when evaluating the success of their teaching, whereas other teachers might only refer to surface categories from their pedagogical knowledge (PK; e.g., discipline or motivation). Although they might theoretically know the relevant PCK aspects, they do not use them for their evaluation practices. This construct of categories allowed Bromme to include knowledge aspects from a mere cognitive perspective into the situated perspective in a conceptualization of knowledge-based noticing and reasoning. He thereby acknowledged that expert knowledge is often implicit (as elaborated by Gasteiger et al., 2020) and that inert knowledge is useless unless being really activated for perceiving and thinking, as conceptual tools for enacting certain practices. Grossman et al. (1999) connected these conceptual tools to pedagogical tools, emphasizing the instrumental character of both. Bromme's categories correspond to Grossman et al. (1999) conceptual tools.

Bromme's (1992) conceptualization of teacher expertise is in line with Schoenfeld's (2010) ROG framework for teachers' decision-making in which he explained what teachers do and decide against the background of situational demands. Schoenfeld referred to teachers' knowledge resources (R; including categories), their orientations (O; beliefs and attitudes) and the situative goals (G) that they pursue in a certain situation.

[1] By "situated" or "situatedness" we refer to the idea that teacher expertise or teacher learning should be captured and enhanced in specific situations of practice instead of focusing on cognitive aspects that can be more removed from practice. This definition differs from the understanding of situatedness by Lave and Wenger (1991), who focused on social aspects of learning and immersion.

These two conceptualizations have been combined and adapted into a *framework for content-related teacher expertise* (Prediger, 2019). The framework states that teachers' *practices* for managing specific *jobs* can be characterized by the *categories* they (implicitly or explicitly) activate for perceiving, decision-making and thinking; their underlying *orientations*; their situative *goals*, which foreground specific orientations in specific situations; and finally, the *pedagogical tools* applied during teaching (e.g., tasks or teacher moves). The main difference that this framework has from Bromme's (1992) and Schoenfeld's (2010) pioneering work is the stronger emphasis on *content-relatedness* going beyond generic categories and orientations. For instance, the framework does not only point to the general distinction between PK and PCK but lists particular categories from PCK that are relevant for a particular area of mathematical content, such as particular mental models for a certain mathematical concept. With respect to an area of PD content such as learning probability by simulations, the content-specific categories would comprise, for example, the typical risks of dealing with simulations. If activated by the teacher while noticing students' processes, the teacher can perceive and overcome the risks.

Within the content-related framework, orientations are also considered not only in terms of generic beliefs about mathematics or mathematics learning, but in much more specific ways, such as the conviction that simulations are highly relevant for students' inquiry processes in probability (further examples in Prediger, 2019).

The framework for content-related teacher expertise is used in *descriptive* and *prescriptive* modes: In a descriptive mode, the framework helps to examine factors underlying teachers' enacted practices by a content-related job analysis applying qualitative research methods (e.g., Schoenfeld, 2010; Prediger, 2019). In a prescriptive mode, the empirical findings can be used to specify what teachers should learn in order to develop the expertise (Bass & Ball, 2004).

Existing perspectives on PD facilitator expertise

Specifying mathematics PD facilitator expertise means asking what facilitators need to know in order to cultivate rich and content-related learning opportunities for teachers. The question is prominent in the emerging field of research on PD facilitation since the early work of Zavlavsky and Leikin (1999). Recently, several studies have used so-called lifting and nesting strategies that link the classroom and the PD levels (Prediger et al., 2019): The *lifting* strategy utilizes existing frameworks from the classroom level to the PD level, based on structural analogies in phenomena of teaching and learning on both levels. For example, the didactical triangle between teacher, student and classroom content can be lifted to a triangle for PD between facilitator, teacher as learner, and PD content (see Fig. 1 for two prominent examples).

However, the PD content is more complex and of a different nature than the classroom content as it also includes mathematics education aspects. This is why several researchers have emphasized that the PD content comprises the whole complexity of the classroom level by nesting the classroom triangle into it as depicted in Fig. 1 (Ball, 2012; Luft & Hewson, 2014; Wood & Turner, 2015; Zavlasky & Leikin, 1999, 2004). Prediger et al. (2019) termed this strategy a *nesting strategy* and suggested using it in the prescriptive and descriptive modes. In a prescriptive mode, facilitator preparation programs need to address the complexities in teaching and learning mathematics on both the classroom and the PD levels, as classroom issues are nested in the PD content. In a descriptive mode, Karsenty et al. (submitted), for example, have shown that a facilitator's decision-making

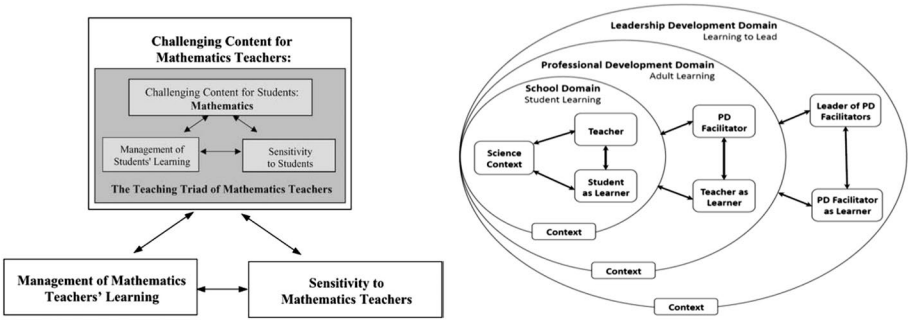


Fig. 1 Nested didactical triangles (left: Zavlasky & Leikin, 2004; right: Luft & Hewson, 2014)

can be better understood while unpacking the nested structure into the components on different levels.

Similar to research about teachers, research about facilitators has adopted different degrees of purely cognitive approaches or also included situated approaches with a focus on facilitators' practices.

Examining the *knowledge categories* needed for facilitating mathematics PD, many researchers have adopted lifting and nesting strategies. That is, constructs well established to analyze teacher knowledge were lifted to the facilitator level, such as mathematical knowledge for teaching (Ball et al., 2008), sometimes split into content knowledge (CK: Borko et al., 2014b) and pedagogigcal content knowledge (PCK: Jacobs et al., 2017; Lesseig et al., 2017; Wilhelm et al., 2019) and generic pedagogical knowledge (GPK: Wilhelm et al., 2019, following Baumert & Kunter, 2013 and Shulman, 1986).

Lifting such well-researched knowledge facets has thereby served both purposes: researching facilitators' knowledge and designing facilitator preparation programs for them. Regarding the latter, Schueler and Roesken-Winter (2018), for instance, took the knowledge facets as a starting point to develop video scenes for facilitator preparation programs to engage prospective facilitators in noticing teacher thinking.

In order to consider the introduced distinctions of facilitators' knowledge and their nested structure, we add the suffixes "-C" for the classroom level and "-PD" for the PD level (see Fig. 2 for an overview of facilitators' knowledge facets). At the same time, we follow Wilhelm et al. (2019) and the work of Zavlasky and Leikin (2004; see Fig. 1) in conceptualizing content knowledge on the PD level (CK-PD) as including all knowledge categories on the classroom level, in other words, generic pedagogical knowledge (GPK-C), pedagogical content knowledge (PCK-C) and content knowledge (CK-C).

As Fig. 2 depicts, categories from facilitators' content knowledge (CK-PD) include content knowledge categories on the classroom level (CK-C) in terms of a sophisticated understanding of the relevant mathematical concepts (e.g., fraction addition), pedagogical content knowledge categories (PCK-C) regarding students' mathematics learning or curriculum aspects (e.g., typical student misconceptions or mistakes regarding fraction addition) and generic pedagogical knowledge (GPK-C) such as classroom management and other mere pedagogical issues. Hence, a facilitators' content knowledge on the PD level corresponds to all knowledge facets relevant for teachers on the classroom level. Specific for facilitators are then the categories of pedagogical content knowledge on the PD level and generic pedagogical knowledge on the PD level: categories that are

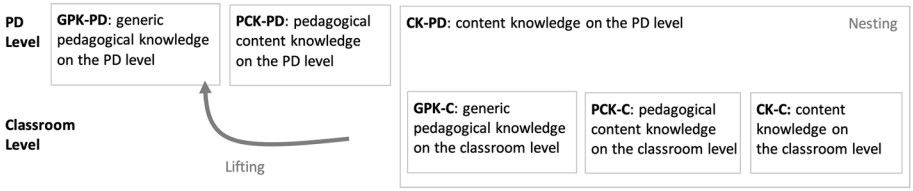


Fig. 2 Components of facilitator knowledge categories on PD and classroom levels and their nestedness

structurally analogical to those on the classroom level. Pedagogical content knowledge on the PD level (PCK-PD) comprises all aspects of teacher thinking and teacher learning about the specific PD content. Jacobs et al. (2017) characterize PCK-PD as "the ability to engage teachers in purposeful activities and conversations about those mathematical concepts and relationships and to help teachers gain a better understanding of how students are likely to approach related tasks" (p. 3). Generic pedagogical knowledge on the PD level (GPK-PD) includes generic knowledge on how to manage and instruct a PD course, for instance, dealing with teacher resistances or motivational aspects, going beyond the subject of mathematics.

Taking a more situated perspective, other researchers have focused on facilitators' PD *practices* as the key element of their expertise and developed frameworks that specify facilitators' general jobs, which are independent from the PD content in view (Coles, 2019; González et al., 2016; Roth et al., 2017; van Es et al., 2014). Van Es et al. (2014) developed a list of facilitation moves that are linked to four sub-jobs in facilitating video-based PD: orienting the group to the task, sustaining an inquiry stance, maintaining a focus on the video and the mathematics, and supporting group collaboration. Tekkumru-Kisa and Stein (2017b) lifted five practices for conducting productive discussions from the classroom level (Stein et al., 2008) to the PD level, thereby also including the jobs of noticing and lesson preparation. Acknowledging the important role of noticing, Borko et al. (2017) identified relevant knowledge categories for facilitators' noticing in the descriptive and prescriptive modes, also combining a cognitive and situated perspective. Thus, Even's (2008) call for integrating knowledge and practice, which she emphasized by using the term *knowtice,* is also relevant for the facilitator level. However, the synthesizing construct knowtice does not allow disentangling how knowledge and practices are connected.

A framework for explaining facilitator practices that combines situated and cognitive perspectives by nesting and lifting is the ROGI framework proposed by Karsenty et al. (submitted; see Fig. 3): This framework lifts Schoenfeld's (2010) ROG framework for teachers' decision-making to the facilitator level and uses the constructs resources (intellectual, material, contextual, social resources, etc., on which the person draws), orientations and goals (as well as identity) to examine the background underlying facilitators' visible actions and to describe facilitator practices in two PD programs. The authors explored the analogy to research on teachers and emphasized that facilitators draw upon resources, orientations and goals both on the PD and on the classroom levels. However, they ended their case studies by emphasizing that the construct of resources needs to be further refined, including with respect to the specificity of different PD content and classroom content.

This overview on various conceptual frameworks shows that the researchers and facilitator preparation designers labeled their constructs differently but share some features:

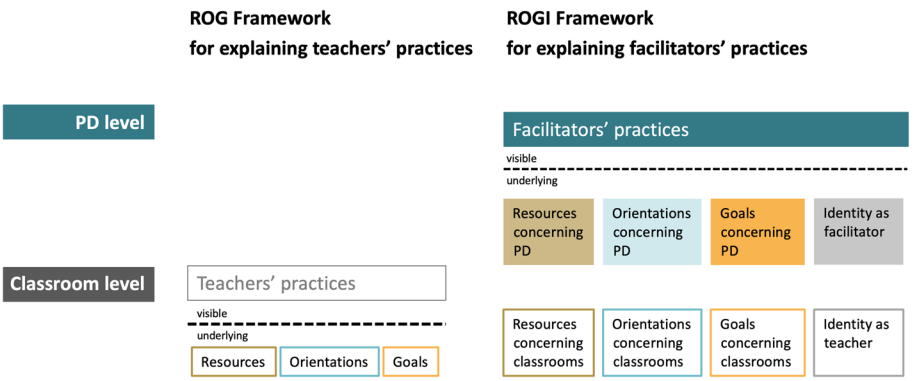


Fig. 3 Constructs for disentangling the background underlying facilitators' practices (Karsenty et al., submitted)

- *Lifting strategy:* Frameworks from the classroom level can be transferred to the PD level.
- *Nesting strategy:* Designs for facilitator preparation must take into account the nested structure in which the classroom complexities occur nested within the PD complexities.
- *Combining situated and cognitive perspectives:* For conceptualizing teacher expertise, it is useful to combine both perspectives and relate facilitators' practices to the underlying knowledge categories, orientations, pedagogical tools and goals.

Research on facilitator expertise and the design of facilitator preparation programs is a rather young field that started by focusing on generic aspects that consider PD a single entity without considering specific content-related aspects. Recently, some studies have taken into account such content-specificities of the PD content in view (e.g., Borko et al., 2021; Wilhelm et al., 2019). Accordingly, Prediger et al. (2019) called for a third strategy: *unpacking the nestedness* to also reveal *content-related aspects* besides the generic issues. Such research goes beyond, for instance, revealing the benefits of facilitators' reflections during a preparation course per se but reveals which activities need intensive reflection to focus on their teachers' mathematical thinking during PD.

In the following subsection, we build upon this state of research and propose a framework for content-related facilitator expertise that allows for considering generic and content-specific questions and supports the unpacking of the PD content and the mathematical content in view.

Introducing the framework for content-related facilitator expertise

Based on the literature review in the previous subsections, we propose our *framework for content-related facilitator expertise* that lifts the framework for content-related *teacher* expertise (Prediger, 2019) to the facilitator level and thereby combines cognitive and situated perspectives as suggested by previous work (Borko et al., 2014; Borko et al., 2017; Karsenty et al., submitted).

The framework consists of the following components:

- *Jobs:* Typical and often complex situational demands of facilitating PD that are most relevant to the PD program in view.
- *Practices:* Recurrent patterns of facilitator's utterances and actions for managing the jobs. Facilitators' practices can be characterized by the underlying categories, pedagogical tools, orientations and situative goals on which the facilitator implicitly or explicitly draws.
  - *Pedagogical tools:* Concrete and visible tools are applied to manage the job (e.g., facilitation moves enacted, activities used, pedagogies initiated, videos used or other didactical artifacts).
  - *Categories:* Non-propositional knowledge that filters and focuses the categorial perception and the thinking of the facilitator and comprises aspects of the five knowledge areas shown in Fig. 2: generic pedagogical knowledge (GPK-PD), pedagogical content knowledge (PCK-PD) and content knowledge on PD level (CK-PD; consisting of GPK-C, PCK-C, CK-C).
  - *Orientations:* Generic or content-related beliefs and pedagogical attitudes (e.g., about teachers' thinking or about the PD content) that implicitly or explicitly guide the facilitators' perceptions and prioritization of jobs (e.g., participant orientation).
  - *Situative goals:* The goals that the facilitators pursue in a respective situation can directly refer to PD content learning goals (in brief, PD *learning goals*), can address process qualities (e.g., cognitive activation, briefly, *process goals*), or can be of an atmospheric nature (briefly *atmospheric goals*).

Figure 4 provides a first illustration of typical components of facilitator expertise that have turned out to be important in several projects for three different areas of PD content (language-responsive mathematics teaching in Prediger & Pöhler, 2019; probability education in Schueler & Roesken-Winter, 2018; dealing with mathematics difficulties in Wilhelm et al., 2019).

In the following sections, we show that the *categories* in particular reflect the content-relatedness, whereas *orientations* and (*sub-*)*jobs* can be treated in a generic way. Thus, for each area of PD content, the specific content-related *categories* have to be identified empirically (see frameworks realized for probability education PD content in Case 1 and for language-responsive mathematics teaching PD content in Case 2).

Substantiating facilitator expertise in content-related ways: two illustrative cases

In the following, we exemplify the described facilitator expertise framework with two areas of PD content: probability and probability education and language-responsive mathematics teaching. We briefly introduce the methods of data gathering and data analysis in the first subsection. The analyses in the second and third subsections identify generic orientations and content-related categories on which the facilitators draw to fulfill these jobs. Thereby, we show how the framework can help to analyze facilitators' practices in content-related ways in the descriptive mode. After that, we will compare both cases in the next section. Against this background, we adopt the prescriptive mode

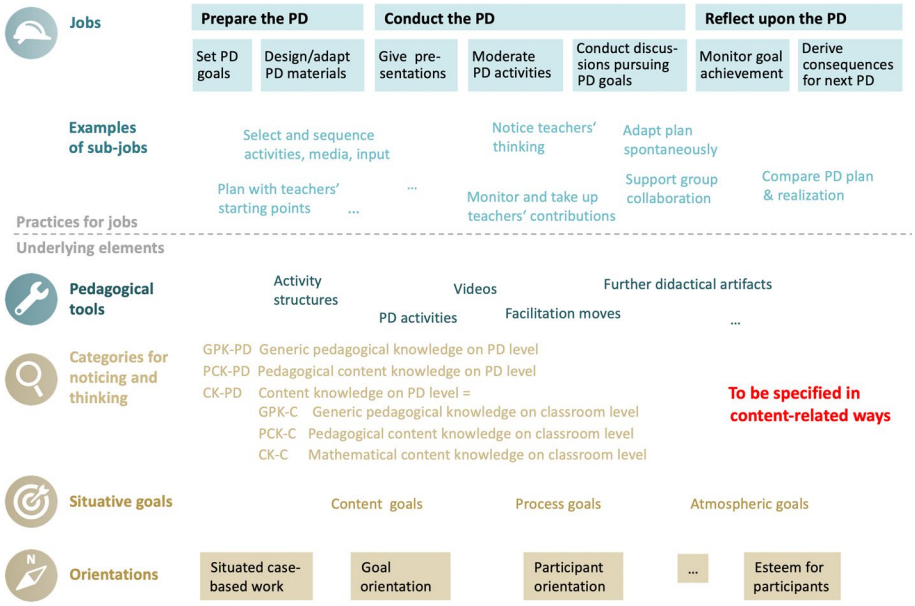


Fig. 4 Framework for content-related facilitator expertise: generic and content-related parts (which will be analytically connected later on)

in the final section and specify what facilitators need to learn in order to achieve PD learning goals for each of the areas of PD content.

Methods for data gathering and data analysis

The data corpora used for the two illustrative case studies consist of video data gathered in PD sessions in two different PD contexts in Germany. We chose these two contrasting cases as they focus on similar jobs in different PD contexts: Case 1 stems from a PD with a CK-C focus (i.e., foregrounding the teachers' learning of mathematical aspects), and Case 2 stems from a PD with a PCK-C focus (i.e., foregrounding teachers' learning of aspects of mathematics education). We briefly present the sample in the beginning of each illustrative case. In both video data corpora, we selected and transcribed episodes in which the facilitators tackled the jobs of *introducing and moderating PD activities* and *conducting discussions*. The data corpus also comprises the PD course material in terms of slides, tasks for teachers and the facilitator manual. Additionally, we draw on interview data collected by one researcher in which the facilitator was interviewed, based on the PD transcript or PD videos, regarding her choices at different moments.

The qualitative data analysis for each illustrative case relies on the framework for content-related facilitator expertise and, in the descriptive mode, aims at scrutinizing the facilitators' practices for managing the selected jobs and the underlying content-independent and content-related categories, orientations and goals. The analysis starts with specifying the PD content goals in articulated CK-C, PCK-C and GPK-C categories (from the interview data), as this allows identification of how the facilitators acted

with these categories. The PCK-PD categories and orientations were inferred inductively from the analysis of the video and interview data.

Illustrative case 1: facilitator expertise for probability PD content (education)

Background and sample of the episode from a PD session on probability

Our illustrative Case 1 stems from a PD course for primary teachers (Grades 1–6) in the domain of probability. Overall, four PD sessions took place during one semester with an experienced facilitator whom we call Erin. Between the four sessions, teachers tried new tasks in their classrooms (Binner & Rösken-Winter, 2019). Teachers' prior knowledge about probability was heterogeneous: While some teachers did have knowledge about this topic, other participants had not learned about probability during their teacher education program or were even teaching out of field (i.e., teaching mathematics without a certification or prior teacher education).

The episode comes from the third PD session on probability in which Erin worked toward the following PD goals (as noted in the PD manual):

CK-C1 Immersing teachers in typical probability ideas

- *CK-C1a* Identifying the sample space
- *CK-C1b* Generating a structured list of combinations
- *CK-C1c* Determining the probability of events

PCK-C1 Anticipating and identifying students' difficulties with CK-C1
PCK-C2 Considering the learning trajectory and typical learning pathways toward CK-C1

With respect to teachers' heterogeneous and low prior knowledge, Erin mainly pursued CK-C goals in the PD course. However, she always presented the CK-C learning goals linked to the respective PCK goals, meaning that she often used tasks from the classroom level and embedded them in more complex activities that fostered teachers CK-C (by asking them to solve the tasks and reflect on them mathematically) and PCK-C (by asking them to think about possible student solutions and learning difficulties).

Presenting the episode from a PD

The task shown on the left in Fig. 5 immerses the teachers in typical probability ideas CK-C1a to CK-C1c. The following moments are decisive for later discussing the facilitator's expertise:

(1) The facilitator Erin starts the PD session by engaging the teachers in a classroom task (see Fig. 5, upper left box). Erin pursues the above-mentioned overarching PD content goal CK-C1, as well as PCK goals PCK-C1 and PCK-C2 for this activity. She prepares the teachers to anticipate students' difficulties and solutions by having them solve the task themselves, thus encountering their own difficulties.
(2) Some teachers try experimenting by throwing non-distinguishable coins. One teacher asks whether it would make a difference if the four coins were thrown at once. A

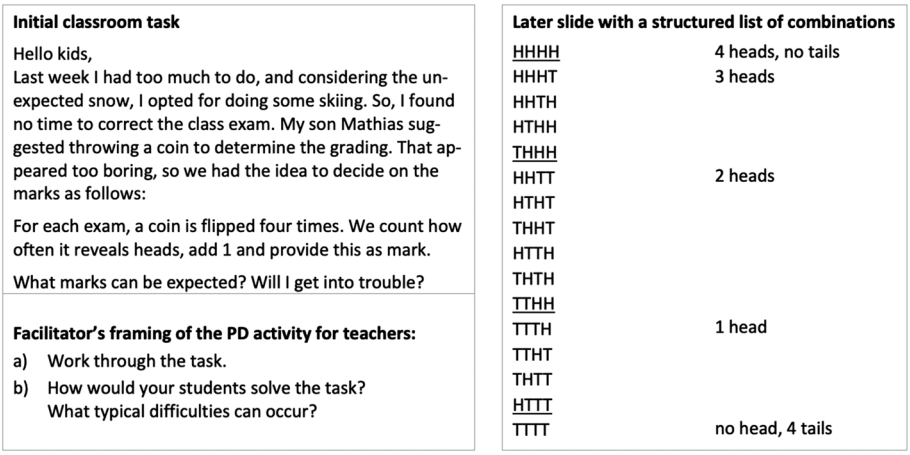


Fig. 5 Left: PD activity, based on a rich classroom task to first experience and reflect probability ideas. Right: Slide shown during the later whole-group discussion for determining the probability of the events

discussion among the teachers emerges. Erin does not provide an explanation at this moment; she leaves the group of teachers free to decide whether they throw the coins one after the other or all at once.

(3) When the groups have developed their ideas, Erin initiates a joint reflection. She shows a structured list of possible outcomes (see Fig. 5, right box) and asks the teachers how primary students' solutions would differ. Not all teachers have produced a structured list themselves, but all anticipate their students' approaches to be less structured. Together, they elaborate on the need to progress from an unstructured toward a structured list.

(4) Subsequently, the teachers return to discussing the one coin or four coins options, and one teacher asks:

Teacher 1 Is that right, to get that result, you threw one coin? Because we threw four coins at once, and we always started with T [abbr. for tail]: TTTH, THHH, then we do not write down THTH. We look to see how many Ts there are and how many Hs [abbr. for head]. And, of course, this combination comes [refers to THTH]. But the kids, they just write down one after the other. That is the difference.

(5) The facilitator stresses how important the question is, and that it needs clarification before the task is given to the students. Otherwise, they will have trouble documenting their simulation in a structured way. Five minutes later, again, some teachers still doubt whether throwing one or four coins is equivalent. Erin then explains that students can throw four coins at once but need to mark the coins in order to make them distinguishable. Only marking the coins clearly enables a structured approach to note the different combinations. One teacher concludes:

Teacher 2 Otherwise, they [the students] are fooled by thinking that two heads, TTHH, do not come as often as it does. One then thinks that two tails, TTHH, come as often as one tail, THHH. And then, misconceptions occur. Hence, I need to mark the single coins.

(6) The facilitator then summarizes:

Erin In the classroom, when you do not clarify this issue in the beginning, you really have to pay attention to students' performances during the documenting phase. You need to talk to the students and ask for the first, the second, the third and the fourth coin. When the students do not correctly document their ways [i.e., the ways that they have done things], they will have problems analyzing their data.

(7) This facilitator statement elicits the following objection by one teacher:

Teacher 3 Under these circumstances, I think it would be much better to just use one coin in the classroom, also to avoid any misunderstanding.

The facilitator provides the following answer:

Erin That is why you get a lot of discussion on the number of coins: whether the same coins are used, in what order and so on. It is important that the experiment is well documented: If I throw one coin, that means that this is the first, the second, the third or the fourth throw. If I throw four coins, you note which one is the first, the second, the third, the fourth. Do you get different results? Actually not—hence, from a mathematical viewpoint, you are doing the same.

That is, Erin contradicts the teacher by emphasizing the benefit of discussing the different approaches together with the students.

Analysis of the episode with respect to the facilitator expertise in a descriptive mode

The analysis of the episode builds on the different components of facilitator expertise in order to characterize the facilitators' practices for mastering the jobs *introducing and moderating PD activities* and *conducting discussions*.

As a *pedagogical tool*, the facilitator uses a task from the classroom level (in Moment 1; see above) and performs facilitation moves such as *connecting ideas* (*raised during the discussion*) or *standing back* to enable teachers to discuss the task without being embarrassed. Different sub-jobs became salient when scrutinizing Moments 1–7: The facilitator was mainly concerned with *noticing teacher thinking*. Confronted with the ideas the teachers brought in, Erin *pursued her plan in a goal- and participant-oriented way*. How she *monitored and took up teachers' contributions* is related to her PCK-PD about the PD content *goals* and teachers' learning pathways. To understand the background of her practices in depth, an analysis of the activated orientations, situative goals and categories for noticing and thinking is crucial, as we show in the following.

As mentioned above, the researchers got access to the facilitator's activated orientations (marked by < … >), goals and categories through the facilitator's reflection with one researcher after the PD session. She said that her *practices* were influenced by the following *orientations* and her *knowledge* about the target group's typical learning pathways (PCK-PD): She knew that the group of out-of-field primary teachers was vulnerable due to emerging mathematics anxiety and resistance to isolated CK-C. Hence, she drew upon the orientation < situated case-based work > and pursued the PD content goal CK-C1 by embedding it in didactical situations that seemed relevant to the participants. Due to her < esteem for participants >, every action needed to contribute to her central *atmospheric goal* of avoiding any face-threatening. For both of these reasons, she worked on her PD content goal CK-C1 by framing it as PCK-C1 and PCK-C2 (Moment 1). The PD activity itself was structured according to CK-C1 and PCK-C1 (see list above). The learning

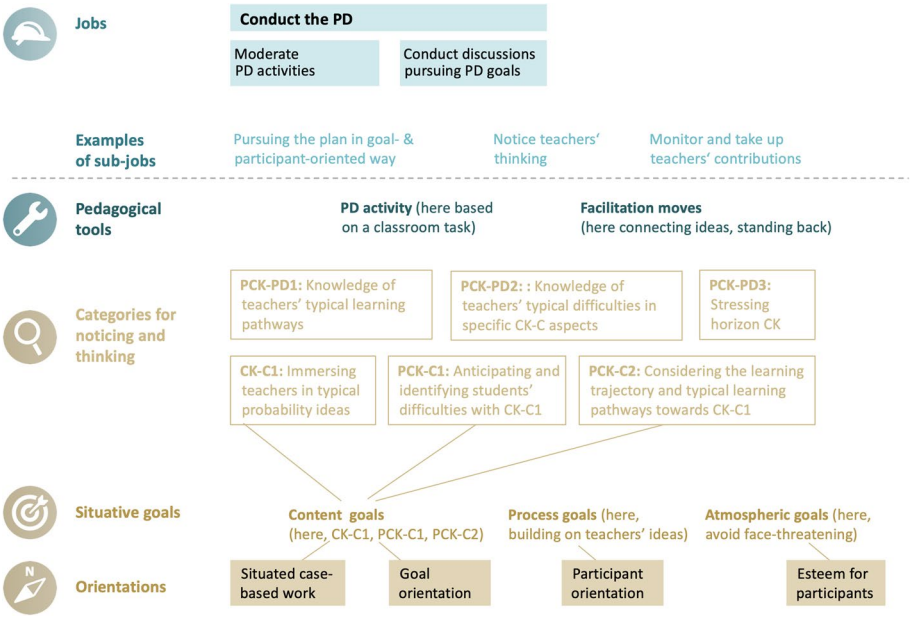


Fig. 6 Erin's enacted facilitator expertise with content-specific substantiation for the analyzed episode

trajectory was then explicitly reflected once the teachers had finished. Her plan was to follow an approach that works with the participants' ideas for reaching the PD goals CK-C1a-c in a < participant-oriented > way.

In Moment 3, when it came to the discussion of possible solutions, she was aware that not all participants used a structured list themselves (PCK-PD about CK-C1b). As she wanted to avoid face-threatening, she presented the structured list and disguised the discussion about its relevance as PCK-C1. However, she always kept her < goal orientation > of working toward the PD content goal CK-C1c.

The goal CK-C1c is implicitly addressed in the task, so that children in primary school can discuss qualitatively the probability of events. However, the facilitator pursued this goal not only disguised as a PCK goal, but also as a CK content goal for the teachers, to enable them to gain a higher viewpoint, which is necessary to respond to their student learning adequately. In this respect, the teacher question concerning "throwing one coin four times or four coins at once" created a relevant learning opportunity for the teachers, showing the relevance of adequately determining the sample space and its consequences for later determining the probability.

The facilitator explained in the post-PD reflection that according to her experience, the issue is always brought up by the teachers themselves. When the issue first occurred in Moment 2, she left the teachers without an answer in order to give them the opportunity to deeply reflect upon it. Here, the < participant orientation > led to a decision whose outcome was different from what the atmospheric goal of avoiding face-threatening would have caused. Moments 4–7 were dedicated to CK-C1c and PCK-C2, because she knew that this issue required intense discussion. She maintained her process goal of building upon teachers' ideas but made a strong statement to contradict the teacher in Moment 7.

Summing up, we can substantiate the framework for expertise in content-related ways by collecting what Erin enacted in this episode (see Fig. 6). Erin, an experienced facilitator, consequently worked toward her PD content goals although she sometimes decided to take deviations. In this deviating practice for reaching CK-C1c, she implicitly or explicitly drew upon the following *categories in her PCK-PD* about the target group:

- *PCK-PD1 Knowledge of teachers' typical learning pathways:* Intertwining CK-C and PCK-C goals strengthens acceptance, e.g., by indirectly addressing CK-C goals in the disguise of PCK-C goals (in the orientation of < esteem: for participants > aimed at reaching the atmospheric goal of avoiding face-threatening)
- *PCK-PD2 Knowledge of teachers' typical difficulties:* Less accessible CK-C aspects can be deepened by anticipated students' ideas (in the orientations < situated case-based work > and < participant orientation >). Thereby, they will become more immersed in probability practice, thus reaching the content-learning goals.
- *PCK-PD3 Knowledge about crucial horizon CK:* Erin stressed horizon CK by emphasizing CK-C aspects going beyond the classroom level, thus enhancing teachers' abilities to make well-founded decisions in the classroom (in the < CK goal orientation >, based on the facilitators' knowledge about teachers' tendency to avoid a CK-C aspect for students when they themselves struggle with this content).

This list of categories goes beyond the very generic orientations or generic categories in the facilitators PCK-PD. Instead, they must be substantiated by content-related knowledge about teachers' thinking and learning of different aspects of the PD content.

Illustrative case 2: facilitator expertise for the PD content language-responsive mathematics teaching

Background and sample of an episode from a PD session

Our illustrative Case 2 stems from a PD program with the overall goal of promoting language-responsive mathematics teaching (Prediger, 2019). The PD aims at increasing teachers' willingness to assume responsibility for language learning, their content knowledge about language (CK-C), and pedagogical content knowledge about teaching approaches for fostering language in mathematics classrooms (PCK-C).

The selected short episode stems from a first PD session with 17 secondary mathematics teachers from two schools. The two facilitators, Alice and Christin, provide the PD on language-responsive mathematics teaching for their first time. Whereas Alice is an experienced secondary mathematics teacher with some experience as a facilitator for other PD content, Christin has considerable experience as a researcher in the field of language-responsive mathematics teaching and as a teacher educator at a university, but limited experience in classrooms and PDs. As in Case 1, we investigate the jobs *introducing and moderating a PD activity* and *conducting discussions*.

The PD activity focused on a well-established principle for language-responsive mathematics teaching (see Fig. 7; Prediger & Wessel, 2013): Teachers' awareness of multiple representations (concrete, graphical or symbolic representations) is extended to multiple language varieties (technical language, academic language [i.e., school language] and everyday language). The productivity of relating them to each other

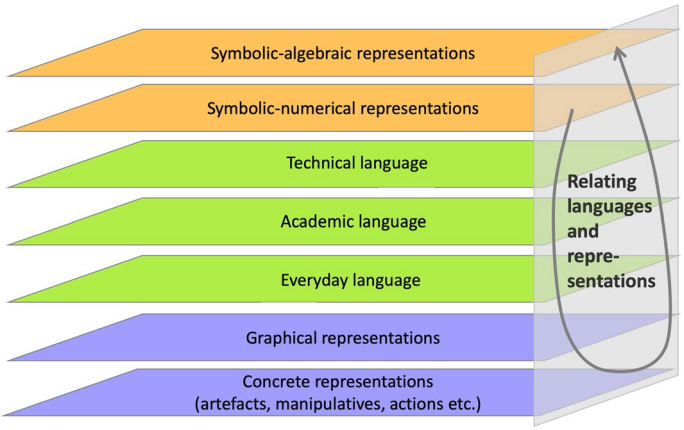


Fig. 7 PD content in view: Principle of relating language varieties and representations (Prediger & Wessel, 2013)

(indicated in Fig. 7 by the round arrow) is explained for mathematics learning, as well as for language learning. This principle was introduced by the facilitator, together with a list of typical classroom tasks for implementing it (and color codes for identifying the

PD activity for connecting language varieties and representations

On the right, you will find the list of possible types of tasks for connecting language varieties and representations, together with examples for the mathematical concept fractions.

(1) Build small groups and choose a mathematical concept you want to work on (e.g., conditioned probability or inflection points or multiplication of fractions).

(2) Design a similar set of tasks for your concept.

(3) Which task seems to be the most powerful for language learning in your context?
Which one is difficult to realize?

Multiple tasks for initiating the connection of language varieties and representations

Type of tasks	Examples for fractions
A. Translate from one language variety or representation into another (freely chosen or determined)	• Here is a given fraction in symbolic form: • Find a situation for it or draw a picture. • Here is a difficult journal text: Translate it into your own words.
B. Find fitting language variety or representation, also for consolidating vocabulary	• On these 15 cards, you find fractions, situations, and drawings. Group those that belong together. Add missing cards. • Mathematicians use these words to describe fractions. Connect them with the given example of a symbolic fraction, denominator, numerator, part, whole. What is their meaning in a situation of fair share?
C. Examine whether different language varieties fit or correct if not	• Tim has offered this picture for a sharing situation. Decide whether it is correct. • 5 kids explained the meaning of 3/5. Which explanations are correct? You can use pictures and own words, or technical language to support your conclusion.
D. Finding mathematical relations or structures by means of a certain language variety or representation	• Which of the fractions 3/4 or 3/10 is bigger? Explain with the help of fractions bars. • Find the equivalent fraction to 3/4 with the help of a picture or situation.
E. Systematic variations	• Systematically vary a representation and investigate the effects of the variations in other registers (examples in Figure 3 and 4).
F. Collect and reflect different means in a language or representation	• Collect different questions in everyday language and technical language that ask for 3/4. • Collect different pictures for 3/4 and 3/10. Which is better to compare the sizes of the fractions?

Fig. 8 PD activity for appropriating the principle of connecting language varieties and representations

language varieties; see Fig. 8).

The PD activity in view of the analyzed episode was to transfer the given list of classroom task types to another area of mathematical content (Fig. 8). In this way, the

facilitators pursued the goal that the teachers appropriate and deepen the following PD content goals:

CK-C Distinguishing technical, school academic and everyday language
PCK-C Situating the language varieties through connecting them to the construct of multiple representations

- *PCK-C1* All representations (concrete, graphical, verbal and symbolic) are required to promote conceptual understanding.
- *PCK-C2* Connecting is more than juxtaposing or switching, as it includes explaining the connections.
- *PCK-C2* All language varieties and representations should be *connected* in order to promote language learning.

Presenting the episode in a PD on language-responsive mathematics teaching

(1) The facilitator Alice starts the PD episode by introducing Fig. 7 and its ideas in a brief presentation. She emphasizes connecting the three language varieties among themselves and in relation to the familiar principle of multiple representations:

Alice I think you all know it: the principle of multiple representations. And we called it the principle of relating language varieties and representations now here […] What is new here is the verbal representation—the language aspect, which plays a crucial role in all areas [*other representations*], not only in one. You see here this interrelation, which is important. […] We differentiate between our everyday language, our school academic language, and of course our formal [*technical*] language [*The facilitator provides explanations on the three language varieties*] […] If we now conclude, what is important is that we see this arrow here. It is important to relate all language varieties and representations.

Finally, Alice presents different types of tasks with example tasks for fractions (from Fig. 8 on the right, with color codes for language varieties and representations in Fig. 7) by which the principle can be put into practice for supporting students' language in mathematics classrooms.

(2) Then, Alice introduces the PD activity, asking teachers to transfer the tasks (as shown in Fig. 8, on the right) to a self-chosen mathematical concept.

(3) After 20 min of group work, Alice invites the teachers to share their experiences with the PD activity on connecting language varieties and representations.

Alice We know that at this time, after two hours of our session, this activity was very exhausting. It provided first insights into this topic. And first of all, we would like to know: How did you feel while working on the activity? We liked that you were so deeply immersed in the topic, exchanged ideas and were very active. […]

(4) Only one teacher responds to Alice's prompt by emphasizing the good accessibility of the PD activity.

Teacher 7 Actually, we found examples for all types of tasks. Thematically, we have chosen linear functions, and there you can actually find tasks that can be connected.

(5) Subsequently, Christin intervenes by trying to determine whether the teachers perceived the activity of finding tasks for connecting language varieties as something new or something they already used in mathematics classrooms. Teacher 7 and Teacher 3 confirm the familiarity with changing between representations and it shows that they do not recognize the new aspects of the principle.

Teacher 7 I have always done this. For example, we have a game of dominoes [for connecting representations]. You have to relate something. You have to decide which symbolic expression, which graph and which function fit together. […]

Alice So how did it go for the other groups? This time you hopefully left the activity with a positive feeling, although it was quite challenging.

Teacher 3 I think, […] many textbooks emphasize the representations, and levels of representations. That the students change and compare them, that they recognize structures in order to identify the connections better. Thus, I think that this is done regularly and automatically. And for me, it is very relevant.

(6) After asking for a contribution from another group, which also emphasizes the good accessibility of the PD activity, Alice closes the episode by saying: "Good. Let's continue."

Analysis of the episode with respect to the facilitator expertise in the descriptive mode

Within the selected episode, the facilitators handle the *job* of providing presentations (in Moment 1), the *job* of introducing and moderating PD activities (in Moments 2 and 3), and the *job* of conducting discussions (in Moments 3–5). With regard to the aim of reaching the PD goals (CK-C and PCK-C1 to PCK-C3), the *sub-jobs* of pursuing the plan in a goal- and participant-oriented way, noticing teacher thinking and monitoring and taking up teachers' contributions become relevant.

Alice and Christin draw upon the *pedagogical tool* of a PD activity (in Moment 2) and perform different facilitation moves (in Moment 3) such as *launching* (posing generic prompts to elicit participant ideas; A: "How did you feel while working on the activity?"), *distributing participation* (A: "How did it go for the other groups?"), and *pressing* (prompting participants to explain their reasoning and/or elaborate on their ideas; C: "Is this something you always made anyway or could you take something with you with regard to this task?").

However, to understand facilitators' practices, it is also necessary to analyze their activated orientations and categories and their situative goals.

Alice's way of starting the discussion (in Moment 3: "And first of all, we would like to know: How did you feel while working on the activity? We liked that you were so deeply immersed in the topic, exchanged ideas, and were very active.") shows that she primarily pursues atmospheric goals in the beginning, as she asks the participant teachers about their feelings while working on the activity. She articulates these atmospheric goals in her pre-PD questionnaire and the post-PD interview. The utterance also reflects her strong orientation of <esteem for participants>, which overrides the content-related <participant orientation> during the discussion. For Alice, a pleasant PD atmosphere and the well-being and the activation of the participants seem to be of high importance. She articulated this orientation explicitly in the post-PD reflection session, when reflecting the PD together with one

of the researchers: "I understand myself as a moderator, who not only provides input but is part of this group, and is also there for keeping it alive." Christin articulated in the post-PD reflection session that both facilitators reached this process goal of activating the teachers in the PD session: "The PD activity on relating language varieties and representations at the end proceeded surprisingly. Suddenly, the room was full of life."

Although Alice addresses the PD content goals more or less in her brief presentation in Moment 1 (by marking PCK-C1 as assumedly known and PCK-C2 and PCK-C3 as new), Moment 3 reveals only a partial resonance: the teachers only refer to PCK-C1 and design juxtaposings of representations without reflecting how they can be related consciously (PCK-C2), and without taking into account the different language varieties (PCK-C3); these teacher contributions might also reflect a missing perception of the distinction between the language varieties (CK-C).

However, Alice does not provide a content-related comment on teachers' contributions, and in the post-PD reflection, she also does not reflect the different degrees of teachers' familiarity with PCK-C1 and PCK-C2, PCK-C3, and CK-C: "I had the feeling […] that everything we said was known." Due to not noticing the differences in teachers' appropriation (missing PCK-PD2), she does not react in any content-related way and can therefore not exploit the situation for steering more toward the PD goals.

The observation that the teachers do not recognize the new aspects PCK-C2 and PCK-C3 is further underpinned by their reactions to Christin's explicit question ("Is this something you always made anyway or could you take something with you with regard to this task?"). Christin then makes a slight shift to a goal-oriented perspective, thereby her presumptive pedagogical content knowledge (PCK-PD2) on typical teachers' learning pathways, that this activity often seems to be nothing new for teachers, could be confirmed. However, instead of taking teachers' contributions up, she leaves their comments untouched.

Summing up, Alice and Christin, the new facilitators for language-responsive mathematics teaching, pursue different goals in conducting the discussion after the teachers worked on the PD activity. Alice starts by giving a brief presentation on relevant PD content goals in Moments 1 and 2 (as shown in Fig. 9 on the left with situative goals and categories for noticing and thinking) but misses relating all of them in Moment 3 (as shown in Fig. 9 on the right with goals and categories for noticing and thinking). As in our illustrative Case 1, we can substantiate the framework for expertise in content-related ways by collecting what the facilitators did and did not enact (aspects marked in light gray in Fig. 9) in this episode:

- *PCK-PD1 Curricular knowledge on relevant knowledge aspects:* Alice activates all the knowledge aspects PCK-C1–PCK-C3 that are relevant for teachers as long as she gives her presentation. However, Alice seems not to draw upon this knowledge in Moment 3 and does not connect it to CK-C (see missing lines in Fig. 9 on the right).
- *PCK-PD2 Knowledge about teachers' typical difficulties:* Christin activates PCK-PD2 in understanding that teachers recognize some of the examples of tasks within the activity as familiar, and that typically they tend to perceive the goals PCK-C2 and PCK-C3 as being similar to PCK-C1. In contrast, Alice does not draw upon this knowledge and does not recognize the < situatedness of CK-C > (here, distinctions between language varieties can be reflected by a PCK activity focusing on tasks for initiating the connections).

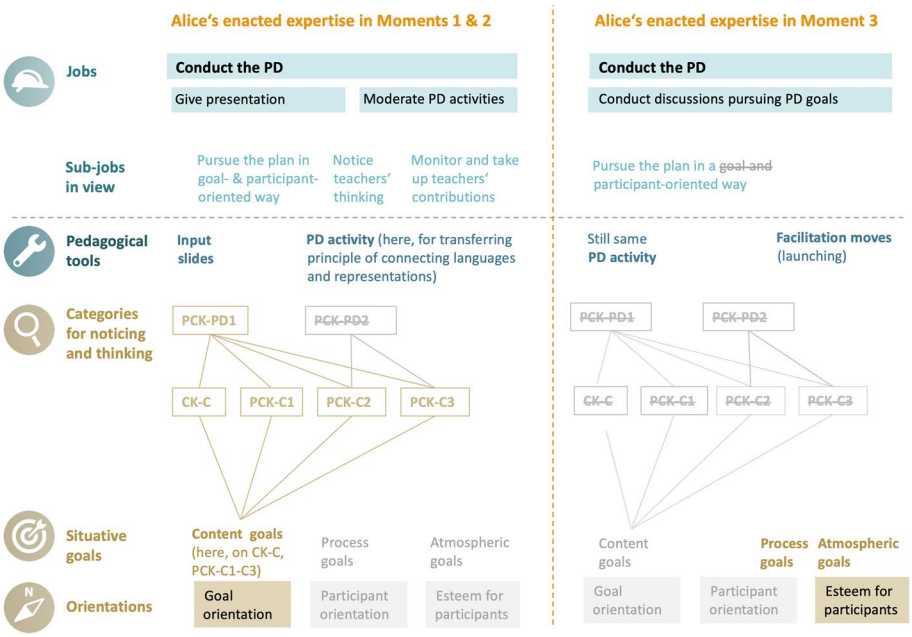


Fig. 9 Alice's enacted facilitator expertise with content-specific substantiation for Moments 1 and 2 and Moment 3 (aspects marked in light gray are not addressed in this Moment)

Rather than mobilizing PCK-PD2, Alice gives priority to the orientation < esteem for participants > instead of a content-related < participant orientation > in this situation (see marked orientation in Fig. 9, on the right): A pleasant PD atmosphere and the well-being and activation of the participants are more important to her than noticing and reacting to teachers' contributions in a content-related way. Christin intervenes after having used PCK-PD2 for noticing teachers' ideas and their limitations in pursuing < goal orientation > and < participant orientation > at the same time, but Alice closes the episode.

Discussion of the analyzed content-related facilitator expertise in a descriptive mode

Both episodes show that all components of the conceptual framework for facilitator expertise are highly relevant to describing the facilitators' practices and their background (similar to Karsenty et al., submitted). Even if we focus only on specific *jobs* (here, moderating PD activities and conducting discussions), with very expectable *pedagogical tools* such as PD activities and facilitation moves, the situations unrolled differently in the analyzed PD episodes. In principle, all three facilitators shared the same three *orientations*: < goal orientation >, < participant orientation >, and < esteem for the participants >. But they prioritized them very differently with respect to the situational demands occurring during the PD sessions, which had direct impacts on their *situative goals*, whereas < goal orientation > was expressed in pursuing PD content goals, a priority on < esteem for participants > can supersede the content goals by prioritizing atmospheric goals such as avoiding

face-threatening and creating a safe environment for participants (as in Alice's Moment 3). <Participant orientation> was mostly expressed in process goals such as building upon teachers' ideas, and this worked best when facilitators set their personal *sub-job* of simultaneously pursuing content and process goals. It was the experienced facilitator Erin who was most able to act in line with the three orientations simultaneously, whereas the two less experienced facilitators prioritized some at the expense of others.

These observations in our analysis are in line with general observations of previous research. Again, the importance of pursuing content goals became apparent (Borko et al., 2014; González et al., 2016; Lesseig et al., 2017), as well as the crucial role of prioritizing goals (Schoenfeld, 2010; Törner, 2010). Similarly, it was confirmed that balancing content and process goals is critical (Jackson et al., 2015; Karsenty et al., submitted; Schwarts, 2020), as well as facilitators' PCK-PD about possible teacher answers to and difficulties with tasks (Tekkumru-Kisa & Stein, 2017b). Thus, the relevance of combining a situated and a cognitive perspective could be emphasized.

The current study was able to replicate these general observations but also adds to the state of research by content-specifically unpacking the goals for reaching a deeper understanding of the episodes: All three facilitators seemed to have their PD content goals in mind (consisting of CK-Cs and PCK-Cs, as listed for Case 1 and Case 2). All three facilitators were able to launch them while providing prepared presentations or introducing prepared PD activities, but they succeeded differently in pursuing the PD content goals while conducting and monitoring the discussions after an activity (Tekkumru-Kisa & Stein, 2017b). The scrutinized analyses of the PD content goals that were addressed in different moments and how the facilitators navigated between them are summarized in Figs. 6 and 9. These graphical summaries allowed us to extract from the empirical analysis characterizations of the categories for noticing and thinking, in other words, the PCK-PD elements that were activated by the facilitators. Looking for communalities and differences results in the following relevant PCK-PD elements. The first three PCK-PD elements could be observed in both PD courses (one with a CK focus, the other with a PCK focus), and the fourth one only in the PD course on probability.

- *Curricular knowledge of PD content goals in CK-C and PCK-C* (activated by Alice in her brief presentation, but not yet in her way of conducting the discussion, this particularly concerns dealing with language varieties; activated by Erin through aligning CK-C with PCK-C goals to guide teachers toward the CK-C goals)
- *Knowledge about a suitable learning trajectory for teachers* (activated, for example, by Erin for deciding not to react too early to teachers' questions in order to allow deeper probability insights)
- *Knowledge about teachers' typical learning pathways with their starting points and possible obstacles* (activated, for example, by Erin and Christin to connect teachers' ideas to the pursued PD goal, a resource only discovered by scrutinizing content-related aspects)
- *Knowledge about a possible horizon CK that can strengthen teachers' CK-C* (as addressed by Erin to help teachers deeply understand the topic they teach in the classroom)

Regardless of the specific area of content, these forms of PCK-PD can be articulated here in generic ways and do not seem to be specific to mathematics, but can be assumed to be equally relevant for other subject matter such as physics or music education. Rather than being *subject specific* per se, they need to be substantiated in highly *content-specific* ways.

That is, these PCK-PD elements constitute a facilitator's expertise and reflect aspects of the CK-C and PCK-C in the specific PD content goals. Knowledge about teachers' typical content-specific learning pathways in particular seems to be a key to reacting in participant- and goal-oriented ways. When facilitators know about teachers' typical reactions for a particular aspect, they are able to notice these in the moment when conducting the PD, as Case 1 involving Erin shows. When facilitators only know generic pattern, they tend to easily consider the learning goals as achieved, as Case 2 involving the novice facilitator Alice indicates. For specifying the relevant PCK-PD elements that facilitators need to know for a particular content-related sub-job, the analysis of facilitators' practices for concrete subjobs by means of the framework for content-related facilitator expertise seems promising.

Of course, the analyzed episodes present only very small extracts of the material and have methodological limitations in that they can only partially be triangulated by facilitators' explicit reflections (Karsenty et al., submitted). Further analyses for many more episodes will be necessary to provide a more comprehensive view of the landscape of necessary PCK-PD elements, also for the other sub-jobs not in view of our illustrative cases. However, the current limited analysis has already led to substantial consequences for our facilitator preparation programs as the following section shows.

Consequences for preparing facilitators in the prescriptive mode

The two episodes allude to the complexity of facilitating PD toward supporting teacher learning. Facilitating PD in a goal- and participant-oriented way requires facilitators to manage many jobs for which they are often only partially prepared (see Fig. 4). Mastering these jobs requires not only generic orientations and standard pedagogical tools, but many knowledge resources to navigate between the different content goals, to notice what teachers can already bring in, and to build upon the fruitful starting points to push their learning pathways further.

However, most facilitator preparation programs (at least in Germany) mainly provide learning opportunities for the generic aspects of facilitator expertise (mainly from GPK-PD), the analysis based on the framework for content-related facilitator expertise revealed the high relevance of content-specific aspects forming the core of PCK-PD for different areas of PD content, and an analysis of further jobs will surely reveal more.

However, we could already draw consequences for the design of our facilitator preparation programs. In particular, we strive for more content-related preparation programs. In these programs, facilitators are offered content-related learning opportunities in situated activities, not only to widen their knowledge of PCK-PD, but to actively apply these aspects as PCK-PD categories for typical practices of preparing and conducting PD (Grossman et al., 1999). Thereby, we also want to foster facilitators' ability to follow up on content goals when situative or atmospheric goals become salient in a PD episode. Figure 10 sketches these kinds of situated facilitator PD activities, which can include:

- Unpacking the PD content goals and planning PD,
- Analyzing PD activities with respect to content goals,
- Noticing teacher thinking with respect to the content goals,
- Organizing a content trajectory through the PD,
- Analyzing typical obstacles in teachers' learning pathways and

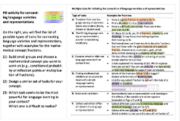
Cognitive perspective Knowing PCK-PD	**Example**
Facilitators **know** the CK-C and PCK-C themselves	Classroom level: The following categories for noticing and thinking are relevant when dealing with the multiple language varieties and representations **CK-C:** Distinguish technical, academic, and everyday language **PCK-C1:** Connect multiple representations to enhance conceptual understanding **PCK-C2:** Connecting is more than switching or juxtaposing, it includes explaining connections **PCK-C3:** Connect all languages and representations to promote language learning
Facilitators **know** about teacher learning • PCK-PD1: Relevant curricular aspects • PCK-PD2: Teacher' learning pathways, e.g., PCK-C1 is easier to achieve than PCK-C2 and PCK-C3	PD level: **PCK-PD1:** Curricular knowledge on relevant knowledge aspects **PCK-PD2:** Knowledge about teachers' typical difficulties (distinguishing the knowledge aspects) **CK-C:** Distinguish technical, academic, and everyday language **PCK-C1:** Connect multiple representations to enhance conceptual understanding **PCK-C2:** Connecting is more than switching or juxtaposing, it includes explaining connections **PCK-C3:** Connect all languages and representations to promote language learning
Situated perspective Facilitators use the categories for the following PD activities	**Example activities for learning to activate the PCK-PD elements**
Unpack the PD content goals and plan PD	• Which of the four aspects of the PD content goals (CK-C and PCK-C1 to PCK-C3) will you prioritize in the first hour of the PD? Why exactly these?
Analyze PD activities with respect to content goals	• Analyze the PD activity suggested in the PD curriculum material. Which part is most suitable for enhancing teachers' learning of the four PD content goal aspects (CK-C and PCK-C1 to PCK-C3)? 
Notice teacher thinking with respect to the content goals	• Here are five teacher utterances from a typical discussion after the PD activity. Analyze which utterance refers to or contradicts which PD content goal aspect.
Structure a content trajectory through the PD	• You have been informed that most teachers are acquainted with PCK-C1, but not with PCK-C2 and PCK-C3. How will you structure the PD session so that you can build upon teachers' resources and expand them successively?
Analyze typical obstacles in teachers' learning pathways	• Analyze this teacher utterance (e.g., in a video scene) in depth. Which content-related orientation or category seems to hinder the teacher's acceptance of a new strategy? • By which focus question can we sensitize the teacher for aspect *x*?
Find suitable **bridges** between teachers' current ideas and the PD content goals to be achieved	• In this video episode, you see an early idea a teacher developed. What would be your next move to expand the teacher's approach and develop it into an idea in line with the PD content goal? • What is the main argument needed to bring the initial idea forward?

Fig. 10 Sketch of activities for enhancing the use of PCK-PD categories in facilitator preparation programs

- Finding suitable bridges between teachers' current ideas and the PD content goals to be achieved.

For Case 2, we first provided an overview of the relevant categories for noticing and thinking, thus the decisive PCK-PD elements. Against this background, we then exemplarily listed learning activities for prospective facilitators to help them to activate these PCK-PD elements in different situations in order to manage different sub-jobs.

Situated activities such as those sketched in Fig. 10 can help facilitators to prepare for managing the real jobs during the PD, in a category-led way with the unpacked PD content goals being major categories.

In addition to discussing authentic protocols or teacher solutions from PDs, we are currently developing and testing video clubs that enable facilitators to analyze and compare episodes from authentic PDs (González et al., 2016). Focus questions concentrate the discussions in such video clubs on (1) teachers' learning pathways and (2) alternative facilitations moves (Schueler & Roesken-Winter, 2018). Our first experiences seem to reveal that these learning opportunities can be substantiated by a deliberated navigation through content-related landscapes of PD content goals to foreground the categories for noticing and thinking.

Even if these consequences are only sketched here and will be elaborated in later articles, they give a first idea how the suggested framework for content-related facilitator can serve as a powerful tool not only for PD researchers in the *descriptive mode*, but also for facilitator preparation designers in the *prescriptive mode*. In the long run, we hope it can also serve in an *evaluative mode* for capturing each individual facilitator's expertise in situated formative assessment purposes as suggested by Gasteiger et al. (2020).

Acknowledgements The presented framework has grown within the DZLM (Deutsches Zentrum für Lehrerbildug Mathematik; German Center for Mathematics Teacher Education), which is financially supported by the German Telekom Foundation. We thank our DZLM colleagues for inspiring discussions and feedback. Case 1 stems from the NoTe project "Fostering multipliers' noticing of teachers' mathematics learning by means of video examples" (funded by the Deutsche Forschungsgemeinschaft [DFG]; German Research Foundation) under Germany's Excellence Strategy—The Berlin Mathematics Research Center MATH+ (EXC-2046/1, Project ID: 390685689). Case 2 stems from the GIF project "Professionalization processes of facilitators in mathematics teachers' professional development" (funded by the German-Israeli Foundation for Scientific Research and Development, Grant No. I-1426-117.4/2017 to R. Karsenty, S. Prediger, and A. Arcavi).

Funding Open Access funding enabled and organized by Projekt DEAL.

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