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Finding the boundary of kindergarteners' subtraction understanding: prospective teachers' problem development and questioning

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Abstract

A challenge for prospective teachers (PTs) is to determine what students know about a topic through asking appropriate questions and being thoughtful about the wording of these questions so as to capture and reframe students' spontaneous mathematical thinking and eventually unriddle the fuzzy boundary of students' complex thinking. This study examined four PTs' efforts to elicit kindergarteners' subtraction strategies and make conclusions about their subtraction understanding. Drawing on PTs' plans for interviewing kindergarteners on subtraction problems, interview transcripts, and reflection papers, the results suggest that PTs provided effective scaffolds, adding context to numerical problems or explaining mathematical terms and symbols as needed. However, they avoided asking problems with subtrahends greater than five, numbers above ten, and missing starts or missing subtrahends, limiting their ability to draw targeted conclusions about the students' strengths and needs. Using effective questions allowed one PT who posed a limited variety of problems to make stronger conclusions about her student's subtraction understanding, while asking a broader variety of problems helped another PT who used limited questions make conclusions about her student's subtraction understanding. Based on these results, mathematics teacher educators could leverage their PTs' strengths to either encourage multiple interviews, each targeting different problem types, or one interview with a broader variety of problem types. The results of this study further highlight the need for PTs to move beyond just asking students to explain their strategies and have them justify or represent their strategies as well.

Keywords Prospective teachers · Assessment · Subtraction · Questioning prompts · Problem-solving strategies

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Introduction

Research about teaching practices focused on assisting prospective teachers (PTs) in exploring and advancing student thinking has provided valuable perspectives for PTs and teacher educators (Grossman 2018). Among those practices, PTs must hone two important practices in the elementary mathematics classroom: choosing appropriate problems for their students and using questions to probe their students' understanding (Doyle 1988; van Zee & Minstrell 1997; Wallach & Even 2005). For example, having PTs pose mathematics problems in a series of pen pal letter exchanges with students can help them develop an understanding of effective questions for students at a particular level (Crespo 2003); however, the opportunity for probing their in-the-moment understanding is limited. To develop and advance PTs' abilities of making in-the-moment choices (Jacobs & Empson 2016), interviewing students is a productive way for PTs to practice asking students questions about their strategies and to follow students' mathematical thinking by dynamically changing their questions (Huinker 1993; National Council of Teachers of Mathematics 1991, 1995). To support PTs in this process, mathematical teacher educators have had PTs use existing instruments or have given PTs instructor-chosen instruments to use for these diagnostic[1] mathematical interviews (e.g., Weiland, Hudson, & Amador 2014; Wilson, Mojica, & Confrey 2013), more generally referred to as clinical interviews.

Results of these studies, in the form of several categorizations of PTs' questioning, highlight the difficulties PTs experience; namely, many PTs ask leading questions (or no follow-up questions) and relatively few probing questions in a meaningful way. Rather than giving PTs problems to pose, having PTs use *self-designed* mathematics problems could provide mathematics teacher educators with greater insight into questions PTs think will help them better understand their students' thinking on a topic. Further, by interviewing students on self-designed or chosen problems, the PTs may use a different pattern of questioning in the interviews, providing a closer approximation of their future practice. Our study, therefore, casts deeper insight into the combined practice of PTs developing mathematics problems to pose *and* questioning students as they solve the problems in order to determine the landscape and boundaries of students' understanding.

Theoretical framework

From a conceptual change perspective, beyond helping students enrich their understanding of a topic, educators need to provide students with experiences that will challenge their thinking and support them in the process of restructuring their conceptions (e.g., Vosniadou 2003, 2007). To help PTs plan for supporting conceptual change (a lengthy process!), teacher educators first need to help PTs identify students' conceptions of a topic, including what they have mastered and what the current limits of their conceptions might be. Prior research in science education indicates that learning about students' conceptions is a complex process and often involves asking a variety of questions. For example, when trying to understand students' conceptions of the shape of the earth, Vosniadou and Brewer (1992) asked elementary students to identify the shape of the earth, draw the earth and indicate

[1] "Diagnostic interview" is used in some literature. We noticed the term "clinical interview" is used more generally, so we use this term throughout.

where people live on the earth, and had students explain what would happen if someone kept walking in a particular direction on the earth. Further, they used a set of follow-up probes depending on students' responses. By looking across students' response to the set of factual and generative items, they could construct a model of the students' conceptions ranging from initial to formal (Vosniadou & Brewer 1992).

Similarly, PTs need experiences identifying students' conceptions of mathematics topics through developing and asking a range of problems and questioning students' responses. Many previous studies aimed at helping PTs learn about students' conceptions either involved PTs creating questions and posing them through letter exchanges (e.g., Crespo 2003) or involved more dynamic, clinical interviews but with pre-developed questions for PTs to ask (e.g., Dunphy 2010; McDonough et al. 2002; Wilson et al. 2013). Yet, to approximate real teaching practice, PTs also need experience designing their own interview problems and revising probing questions in-the-moment to respond to students' mathematical thinking. We chose the topic of subtraction because it spans the elementary grades and number types (e.g., whole numbers, fractions, decimals). To build on the prior work, we present data from our study where PTs created their own protocol for a clinical interview, in which they were encouraged to probe students' thinking so that they could reflect on students' subtraction understanding.

Using mathematics problems to determine students' prior knowledge

Conducting clinical interviews (see Ginsburg 2009) can help PTs learn about students' prior knowledge, just as it does for researchers and teachers (McDonough et al. 2002; Wilson et al. 2013). When comparing novice teachers' (including PTs) with expert teachers' understanding of the role of students' prior knowledge in instruction, Meyer (2004) found that novice teachers used students' prior knowledge as an engagement device before introducing new knowledge; whereas, expert teachers treated prior knowledge as a logic bridge for connecting to new knowledge. Further, novice teachers were likely to regard students' naive conceptions as discrete information that was replaceable through teaching; conversely, expert teachers thought of these conceptions as part of the learning process from informal and fallible to formal and scientific knowledge (Meyer 2004). While learning to teach, PTs can possibly hold a limited understanding of students' prior knowledge and rigidly apply what they have learned in books to interactions with students. Through clinical interviews, PTs could benefit from students' responsive feedback and gradually alter their understanding of students' prior knowledge.

Conducting a series of clinical interviews, each around a target question, can help PTs shift from describing students' thinking in general terms to more mathematically focused interpretations, help them identify students' partial understanding and compare students' responses to known ways students think, and support PTs' prediction-making about students' future responses to similar tasks (Wilson et al. 2013). In order to be helpful to teachers, the content of clinical interviews needs to be considered carefully. Teachers need to understand the mathematics that is involved in the clinical interview, how children progress in understanding the topic, and how the mathematical ideas involved progress and relate to each other (Ginsburg 2009).

One benefit of pre-determined interview protocols is that they are often based on learning trajectories (e.g., Wilson et al. 2013) or knowledge of how children learn a concept (e.g., McDonough et al. 2002); however, interview protocols designed by PTs can provide different insight. For example, when PTs posed their own mathematics questions to children through letters, Crespo (2003) found that PTs posed problems that they thought would be easy for students. Over time, some PTs posed more difficult problems and found ways to encourage deeper thinking. Interview protocols designed by PTs (where they select or create tasks) could also provide insight in their conceptions of what types of problems are appropriate for students and what types of problems (in terms of presentation, numbers used, etc.) are sufficient to determine the boundary of a student's understanding of a topic.

Although there is little research about how PTs design one-on-one mathematics interviews, there are several factors to consider when creating or choosing problems for the interviews: amount of time available for the interviews, students' level of engagement in the tasks and how realistic (to the students) the contexts are, the use of manipulatives, the flexible use of problems (i.e., problems can be modified during administration), the grounding of the problems in educational research (e.g., have appropriate cognitive demand), and the relation of the problems to standards (Hunting 1997).

Questioning students

Beyond having teachers pose problems to students through interviews, the National Council of Teachers of Mathematics (2007) encourages teachers to use "questions and tasks that elicit, engage, and challenge each student's thinking" and ask "students to clarify and justify their ideas orally and in writing" (p. 45). During interviews with students, PTs can elicit such clarifications from students through the use of questioning. Questioning is a foundational practice that promotes teachers' understanding of what students know and helps them unpack students' difficulties and incomplete conceptions on a mathematics topic (Sahin & Kulm 2008; Weiland et al. 2014). Given the importance of teacher questioning in promoting teachers' knowledge of students and the ensuing implications for classroom instruction, many studies have contributed to the investigation and categorization of teacher questioning (e.g., Cotton 1989; Franke et al. 2009; Hancock 1995; Hiebert & Wearne 1993; Ilaria 2002; Moyer & Milewicz 2002; Sahin & Kulm 2008; Vacc 1993; Weiland, et al. 2014). A larger goal of some of these studies, especially the classroom studies, was to understand the relation between teachers' questioning and students' learning from a teacher perspective (e.g., Moyer & Milewicz 2002; Sahin & Kulm 2008). As our focus in this study is on PTs' understanding of students' thinking, we further describe the classifications from those studies that examined teachers' questioning within clinical interview settings (i.e., Moyer & Milewicz 2002; Weiland et al. 2014).

PTs in Moyer and Milewicz's (2002) study spent a class session of their mathematics methods course watching and analyzing a video of a clinical interview, focusing on questions the interviewer used to probe students' thinking. With the interview protocol provided by the researchers (i.e., the method course instructor) and a list of questions discussed in class, PTs conducted individual interviews with an elementary child. In the researchers' analysis of PTs' questioning, however, Moyer and Milewicz identified other categories of questions in addition to probes. Specifically, they organized PTs' questioning into three main categories: checklisting (completing the protocol questions with little follow-up), instructing rather than assessing (providing instruction or using leading questions, suggesting the PTs thought they already knew students' reasoning), and probing and follow-up questions (which included instances of probing questions that were vague or only focused on incorrect answers). Moyer and Milewicz found that one fourth of PTs used instructing questions; many PTs tried to use competent probing and follow-up questions but either used non-specific questions or did not attend to students' mathematical thinking in questioning.

Years later, Weiland et al. (2014) worked with two PTs who interviewed students 10 times over a series of weeks. In their analysis of PTs' questioning, they built upon Moyer and Milewicz's (2002) classification and also explored how two PTs'—who were working with first graders—questioning changed over time. Because they had PTs' original protocols available, Weiland et al. also compared the planned questions with actual questions and came up with the first code—*problem-posing* entailing four sub codes: protocol [questions directly read from the protocol, similar to checklisting (Moyer & Milewicz 2002)], framing (questions not in the protocol but composed to introduce protocol questions), new (questions not in the protocol but added impromptu), and repeat (questions already asked but repeated to reinforce students' understanding of the initial questions). The second set of codes—*instructing: teaching and telling* and *instructing: leading question*—dealt, respectively, with questions aimed at teaching students something and questions focusing students' attention on specifics, similar to Moyer and Milewicz's instructing rather than assessing category. Weiland et al.'s third code, *follow-up questions*, had subcategories similar to Moyer and Milewicz's, focusing on questions in response to incorrect student explanations, and nonspecific versus competent questions that open up opportunities for more student reasoning. Also similar to Moyer and Milewicz, Weiland et al. found that many PTs' questions were instructing rather than assessing, and several (one fifth) of the PTs' questions were non-specific questions. Tracking the changes of PTs' questions over those 10 interviews, Weiland et al. claimed that PTs showed great improvement with decreasing non-specific questions and increasing competent follow-up questions.

Reflective analysis of students' mathematical thinking

The Association of Mathematics Teacher Educators (2017) endorses assessment as an integral part of informing and improving teaching. Constant reflection on the effects of teaching on students' mathematical performance (Hiebert et al. 2007) contributes to developing a positive teaching cycle: plan, teach, reflect, then plan, teach, and reflect (McDuffie 2004). Compared with in-service teachers, PTs need more support in reflection on teaching (e.g., Hiebert et al. 2003; Melhuish et al. 2019).

PTs manifest some common practices when doing reflective analyses of student mathematical thinking. Morris et al. (2009) found that K-8 PTs focused primarily on students' explicit and visible actions instead of the concepts underlying those actions and that the correctness of students' responses also positively influenced PTs' analyses of students' mathematical understanding. In other words, PTs overestimated students' mathematical understanding when students correctly used procedures (e.g., the area formula) or gave correct answers; they deemed procedural understanding as evidence of conceptual understanding (Morris 2006; Spitzer et al. 2011).

Present study

The present study builds on the prior studies by identifying the types of problems PTs posed and questions PTs asked kindergarteners during a clinical interview, the goal of which was for PTs to identify the landscape and boundary of students' understanding of subtraction. Although there are benefits to interviewing students over time to hone the questioning practice and learn more about students, PTs also need to learn how to get a broad sense of students' understanding of a topic in order to plan classroom instruction. With this goal in mind, we address the following questions:

1. How do PTs define the subtraction landscape for kindergarteners through their development of interview assessment subtraction problems?
2. How and when do PTs question kindergarteners in the moment in order to elicit their subtraction understanding?
3. When PTs synthesize their data in relation to course readings, what conclusions do they make about the landscape and boundaries of their kindergarteners' subtraction understanding?

Methods

Participants and setting

Participants in the study were four female PTs (pseudonyms: Betsy, Jennifer, Jessica, and Sarah) taking the same section of an elementary mathematics methods course at a large midwestern university in the USA during the 2015 fall semester. This course is typically taken the semester before student teaching. As part of their course assignments, the PTs had to design an individual interview for a kindergarten student at an elementary school and use it to assess the student's subtraction understanding with the goal of finding the boundary of their understanding. In the USA, elementary schools routinely include kindergarten through fifth grade students, and this was the case for the schools with which we worked. For their fieldwork, the PTs were placed in kindergarten classrooms, composed of five- and six-year-olds. The kindergarteners were in the first half of the school year when the interviews took place and had not yet received formal instruction on subtraction, although some students could have learned about subtraction at home. The PTs had to be careful about which questions they asked given the kindergarteners' limited experiences with subtraction.

Procedures and data collection

This study involves a retrospective analysis of an interview assignment PTs completed during their mathematics methods course. Prior to crafting their subtraction interview questions, PTs read and took quizzes on the chapters from Wright et al. (2006) *Teaching Number in the Classroom with 4–8 Year-Olds*, which provided approaches to instruction on early numeracy (e.g., using dot patterns, ten frames, and number lines) and information about children's counting types and addition and subtraction strategies; they also consulted their grade-level standards to prepare appropriate interview problems and scaffolds for their kindergarten students. Further, the methods instructor incorporated into the class videos and activities on the counting types (i.e., figurative, perceptual, and figurative counters), counting strategies (e.g., count-all, count-down), and word problems and their relative difficulty. Therefore, the content of these activities was particularly relevant to the PTs placed in early elementary classrooms, as were the four PTs we studied.

Fig. 1 Base-8 chart

A	B	C	D	E	F	G	AO
AA	AB	AC	AD	AE	AF	AG	BO
BA	BB	BC	BD	BE	BF	BG	CO
CA	CB	CC	CD	CE	CF	CG	DO
DA	DB	DC	DD	DE	DF	DG	EO
EA	EB	EC	ED	EE	EF	EG	FO
FA	FB	FC	FD	FE	FF	FG	GO
GA	GB	GC	GD	GE	GF	GG	AOO

Base-8 word problem activity

One major activity in the course that the methods instructor used to help PTs make sense of their reading and other factors to consider when designing and implementing word and arithmetic problems was a base-8 word problem activity. This activity provided an opportunity for PTs to experience a different number system and understand the importance of and distinction between perceptual and figurative counting, as well as the relative difficulty of different problem types (e.g., word versus arithmetic problems and missing subtrahend versus missing result problems). In comparison with a hundred's chart, PTs explored a base-8 number chart using a letter system (see Fig. 1) and discussed the patterns they noticed. They practiced counting using the chart (e.g., A, B, C… to count by A's or AO, BO, CO… to count by AO's) and briefly thought about what addition and subtraction problems might be easy to solve. Following this discussion, PTs received a series of word problems for AO − C = ___; ___ − E = AO; and BB − ___ = AC. The class discussed their counting strategies for solving the problems and the relative difficulty of the problems.

Practice analyzing interviews activity

Before conducting their own interviews, PTs participated in a classroom activity where they listened to a student solving a repeated addition (multiplication) problem while being interviewed. For this activity, they first solved the problem in multiple ways, anticipated how students might solve the problem, took notes as they listened to the interview, and analyzed and discussed how the student solved the problem. Finally, the PTs discussed the interviewer's technique and use of questioning prompts and provided suggestions for improving the interview. The methods instructor encouraged PTs to consider questions they could ask students to determine their strategies for finding their answers (e.g., "How did you solve that?") and whether they could find their answers in more than one way (e.g., "Can you show me with these counters? Is there another way you could solve it?).

Table 1 Questions for PTs to Consider about the Boundary of Students' Understanding

Question	Example
Number choice	Do students have difficulty working with double-digit numbers? Do students make errors when the distance between two numbers being subtracted is small? Do students make errors when the problem would result in a negative number?
Problem format	Do students make errors when they have to solve a word problem, even if they can solve a similar number problem without context?
Use of manipulatives	Do students make errors only when they no longer have manipulatives to work with, regardless of the numbers involved?

Crafting interview problems

After the in-class practice interview activity, the methods instructor invited PTs to work in groups to craft interview questions for their students. PTs received a document with example questions (word problems and number problems), and the instructor encouraged PTs to create a set of varied mathematics problems so that they could better determine the boundary of students' subtraction understanding. In particular, the instructor encouraged them to use the Common Core State Standards for Mathematics (CCSSM; National Governors Association Center for Best Practices & Council of Chief State School Officers [CCSSO], 2011) as a guide and incorporate problems using a range of numbers, word problems versus number problems, and problems with and without manipulatives. The instructor also provided some guiding questions (see Table 1) to reflect upon as PTs crafted problems that could help them discover the boundary of students' mathematical understanding.

Number choice. The CCSSM ([CCSSO], 2011) require kindergartners to recognize whole numbers up to 20 and represent objects with written numerals up to 20. When specifying the number range in subtraction activities, however, these standards confine it to within 10 (CCSSO, 2011). Knowing where students are going in their mathematical learning can inform teachers' planning and help set goals for their students. Regarding subtraction, the CCSSM set the expectation that first graders should do subtraction with numbers up to 20; therefore, providing kindergartners with problems involving numbers between 11 and 20 could provide insight for the PTs. Also, although the CCSSM only require kindergarteners to work with whole numbers, multiple studies (e.g., Aze 1989; Behrend & Mohs 2006; National Research Council, 2009) have offered evidence that elementary students can reason about negative integers earlier than expected. The methods instructor required that students use at least one problem with negative integers (the example provided to be used or modified was *The temperature near the water was 5 degrees. That night it got colder and the temperature went down another 8 degrees. What was the new temperature?*) and encouraged them to use a variety of numbers.

Problem format. The primary delivery of subtraction problems in kindergarten is through word problems, although students are expected to represent word problems in equation form and then solve equations in first grade (CCSSO 2011). Despite the variety of number sizes and types of numbers involved in subtraction problems, "the basic structure involving actions and relationships remains the same" (Carpenter et al. 1999, p. 7). Four problem types involve subtraction, depending on which quantity is unknown (Carpenter et al. 1999; Clements & Sarama 2009): a) Join problems (e.g., Helen had some apples, later he got three more, and now he has nine apples. How many apples did Helen have at the beginning?) b) Separate problems (e.g., Helen had nine apples. He gave four apples to his brother. How many apples does he have now?), c) Part–part–whole problems (e.g., Helen has nine balloons, five are red and the rest are blue. How many balloons are blue?), and d) Compare problems (e.g., Helen has eleven toys. His brother has seven toys. How many more toys does Helen have?).

Although solving word problems may be more difficult than just computing an answer for number problems (Clements & Sarama 2009), without an understanding of the symbols in written numeral problems, a context can offer scaffolding for young children before they know how to write corresponding numerical equations (Riley et al. 1983). Especially for low-performing students, word problems in the form of stories facilitate the elicitation of "effective solution strategies" (Koedinger & Nathan 2004, p. 135). Regardless, students need to become familiar with the situations in word problems, and decode and convert the situations into numerical symbols and corresponding number sentences (Caldwell et al. 2011). The methods instructor encouraged PTs to include both word problems and numerical problems.

Use of manipulatives. Children's early computing knowledge is quite "context dependent, often depending on the presence of objects or fingers to represent sets" (National Research Council, 2009, p. 71). Students who are fluent in using manipulatives are more likely than those who do not to later adopt complex counting strategies (Siegler 1993). In particular, "the use of finger patterns in association with activities such as counting, adding, and subtraction is very widespread, being used in many cultures and by adults as well as children" (Wright et al. 2006, p. 65). Other manipulatives that provide space for children to solve subtraction problems in various ways include counters, sticks, cubes, drawings, and other concrete objects like coins (Wright et al. 2006). Once again, the methods instructor encouraged PTs to provide manipulatives for at least some of the problems. PTs then took the ideas generated from their groups and each completed their own final list of subtraction problems and prompts for the interview. When the PTs turned in their draft of problems and questioning prompts, the instructor provided them with feedback on their initial ideas.

Conducting interviews

Each PT chose one kindergarten student for interviews based on the recommendation of their placement teachers and interviewed the student for 20–30 minutes to elicit information about the student's subtraction understanding and use of strategies. Since this study uses archived, de-identified written data to analyze how PTs analyzed kindergarteners' subtraction understanding, we were not required to obtain PTs' or kindergarten students' consent; however, the authors did obtain University Institutional Review Board permission to use the data. PTs conducted their interviews in a hallway outside the students' classrooms, and the interviews were only audio recorded by the PTs. PTs were told they could change the questions if needed during the interview (see Appendix for their planned and actual interview questions). After their interviews, the PTs transcribed their interviews to facilitate further analysis.

Writing analysis papers

Based on their transcribed interviews, PTs each wrote a paper about the boundary of their student's subtraction understanding using evidence from the interview and student work. PTs were asked to reflect on the following questions: "What are the student's strengths and needs in terms of the mathematics concepts, language, etc.? How did the students approach the problems? Were certain ones easier or difficult? Why or why not? Did they use different approaches? Why do you think this is?" PTs were expected to refer to ideas about counting types and arithmetic strategies from the Wright et al. (2006) book or other readings to justify their analysis.

Analysis

To analyze how PTs defined the subtraction landscape for kindergarteners through their choice of interview assessment subtraction problems, we first categorized PTs' planned and posed problems according to their number choices, problem formats, and use of manipulatives. For number choice, we identified the number of problems each PT planned and posed involving numbers 0 to 5, 6 to 10, above 10, and we checked that each included a problem with negative numbers as requested by the methods instructor. For problem format, we determined the number of planned and posed problems that used a word-problem format versus a numerals-only format. Also, we recorded whether the missing number in each was the minuend, subtrahend, or result and the word-problem types used. Finally, we recorded the types of manipulatives or materials the PTs made available to the students and whether they were restricted to certain problems or not.

To analyze how PTs questioned kindergarteners in the moment in order to elicit their subtraction understanding, we used Weiland et al.'s (2014) revision of Moyer and Milewicz's (2002) questioning framework to code PTs' questions. As we coded, we identified a few areas where our interpretation differed slightly from their framework, in particular around the follow-up subcategories and with a new *interactive* category. Weiland et al.'s follow-up subcategories include *nonspecific*, *competent,* and *incorrect*, referring to times when questions lack specificity, where questions build on students' thinking, and when questions indicate that a student is incorrect, respectively. We had difficulty consistently applying their codes to our data and found it easier to classify the follow-ups in terms of their purpose. Therefore, we made four subcategories, depending on whether the follow-up focused on a *strategy*, *representation*, *justification*, or *verification*. Because PTs might use strategies such as checking a student's answer whether it was incorrect or correct, we chose to reclassify Weiland et al.'s *incorrect* category to a verification category. Finally, we added the *interactive* category to cover those times when PTs used questions as a support for students (see Table 2 for descriptions and examples of codes). To analyze when they used different types of questions, we organized their questions in the order they were asked and identified patterns in how their questioning shifted over the interview. Further, the order and patterns of PTs' questions helped clarify how PTs questioned kindergartners in response to their in-the-moment responses (e.g., a PT used multiple *problem-posing: framing* questions because her student knew little about subtraction).

To analyze how PTs used their data, the readings, and discussions related to designing subtraction problems to make conclusions about the landscape and boundaries of their kindergarteners' subtraction understanding, we identified instances in their reflections of addressing number choice, problem format, manipulatives or visuals, and textbooks ideas (i.e., counting types and subtraction strategies). We further identified when each PT discussed students' strengths and areas of need.

Table 2 Categorizations of Teacher Questions

Code	Definition	Example
Problem-posing		
Planned	Interview questions directly read as planned or slightly modified with same key information	Jennifer's planned interview question: I have 10 pennies and I use 5 of the pennies to buy a piece of gum. How many pennies do I have left? Jennifer's actual interview question: *So I have ten pennies. I need to use five of those pennies to buy a piece of gum. How many pennies will I have left?*
Framing	Questions not in the plan but used to introduce or help contextualize intended questions	Betsy's actual interview question: 8 – 2 = ___ Betsy's question before asking *8 – 2* = ___: *Do you know what this number is?* Student: 8
New	Actual interview questions different from planned interview questions	Jessica's actual interview question: *9 – 9* = ___ (no such a question on her planned interview question list)
Repeat	Interview questions already asked initially but repeated to focus students' attention on specific/key information or in response to students' previous responses	Sarah: So I have another question, if I have six markers and I share two of them with my friends to color a picture, how many pictures will I have? S: 8 Sarah: *Okay, so I have six markers… and I am going to share two of them with my friends. So I am going to give two of them away. How many markers am I going to have to color my picture?*
Follow-up		
Strategy	Questions in response to students and intended to learn more about their solution strategy	Betsy: *Can you tell me how you got 10 – 3?* Jennifer: *How did you get 5?* Jessica: *How did you get 11?* Sarah: *Can you show me how you counted that?*
Representation	Questions used to encourage students to show their work in a representation or explain a representation	Jessica: *Can you draw me a picture to help me understand?* Sarah: *How could you draw it to help someone else find the answer?*
Justification	Questions in response to students and intended to learn how they justify their answers	Jennifer: *How do you know that?* Sarah: *So why did you make this a two at the end here?*
Verification	Questions used to repeat students' explanations/ strategies to allow students to reconsider what they have said	Betsy: *So you are going to add 8 more, is that what you're saying?* Jennifer: *So nothing comes before zero?* Sarah: *So you counted with your fingers?*

Table 2 (continued)

Code	Definition	Example
Instructing		
Teaching and telling	Questions directing /encouraging students to do something procedural as the teacher expects	Betsy: *Do you think that you could write that down for me?* Sarah: *and I am going to give two of them away, can you circle them?*
Leading question	Questions with predicted answers and funneling students to think and answer as the teacher expects	The problem: Three of my apples are red. The rest of my apples are green. How many of my apples are green? (a visual of 5 apples was on the worksheet) After the student gave a correct answer, Jennifer: *How many red apples did I have again?*
Interactive	Questions irrelevant to problem-solving but acting as emotional communication with students	Betsy: *Do you want me to show it on my fingers again?* Jessica: *Are you ready for the first one?* Sarah: *Do you need another piece of paper?*

Results

We organize our results around the three research questions, starting with the problems PTs' planned and posed in order to try and determine the landscape and boundary of students' subtraction understanding.

PTs' subtraction problems

Table 3 presents a summary of PTs' planned and posed problems, separated out by problem type and the types of numbers involved in the problems.

Betsy

Betsy's subtraction landscape was the least varied of the PTs, including only separate, missing result problems with an even number of problems using numbers five and under versus six to ten. Further, she ignored suggestions from the instructor to modify her questions. She mostly asked numerical problems and asked them all before her word problems, even though the instructor suggested she start with a word problem. Consequently, when presenting the first problem 5–3, she had to re-present the problem as a word problem before rereading it as a numerical problem and then helped her student solve it using fingers. She reflected that being comfortable with visuals could help students solve larger-number subtraction problems, even though her student did not use visuals to solve the problems and ultimately added for all problems except two of them. Interestingly, Betsy's student succeeded in solving a subtraction word problem but added the numbers together in its corresponding numerical problem 3–2. In response to the student's recurring "I just know it," Betsy did not ask further questions and moved on.

Sarah

Sarah, like Betsy, posed the same number of problems using numbers five and under versus six to ten and originally planned to only include separate missing result problems; however, unlike Betsy, Sarah incorporated both suggestions made by the instructor. The first was to put the negative number problem last, and the second was to consider another problem type. She included two additional problems asking students to determine a number more or less than a given number. Sarah asked all of her word problems first, so she did not need to revoice her one numerical problem as a word problem. When reading her last word problem, Sarah indicated that using two hair ties would be taking away; therefore, when she read 5–3 as "five and take away three" the student may have been better able to make sense of what to do. Sarah encouraged her student to draw a visual for one problem. She reflected, "When asked about the subtraction word problems, [the] student had some complications and needed some more perceptual items for his comprehension and ability to solve these types of questions accurately." Subsequently, her student used fingers to solve problems and correctly answered

Table 3 PTs' Planned and Posed Subtraction Problem Types

		Separate, unknown			Part–part–whole, unknown		Other[1]	Material(s) PTs provided
		Start	Change	Result	Part	Whole		
Betsy	0 to 5	—	—	1W[2] + 2N[3]	—	—	—	Paper
	6 to 10	—	—	3 N	—	—	—	
	Over 10	—	—	—	—	—	—	
	Negative	—	—	1 W	—	—	—	
Sarah	0 to 5	—	—	1 W + 1H[4] → 1 W + 1 N	—	—	— → 1	Paper, whiteboard with markers, visuals
	6 to 10	—	—	1 W + 1H → 2 W	—	—	— → 1	—
	Over 10	—	—	—	—	—	—	
	Negative	—	—	1 W	—	—	—	
Jennifer	0 to 5	—	—	1 W + 1 N	1 W	—	—	Paper with two visuals; provide markers, cubes, & pennies
	6 to 10	—	—	2 W	—	—	1 → —	
	Over 10	—	—	—	—	—	—	
	Negative	—	—	—	—	—	1	
Jessica	0 to 5	—	1 W → 1 N	—	—	—	—	Paper, whiteboard with markers
	6 to 10	—	2 W → 1 N	5 W → 2 W + 3 N	1 W → —	—	—	
	Over 10	—	—	1 W → —	—	—	—	
	Negative	—	—	— → 1 W	—	—	—	

Each cell shows which problems PTs planned to ask. If there is no arrow in a cell, the PT posed the same problems they planned. Problems listed after an arrow show what PTs posed, if different from what they planned.

[1]Refers to questions like "what number comes after/before 7?" [2]"N" means numerical problem. [3]"W" means word problem. [4]"H" means hybrid problem, in which students would not need to interpret the symbols as in a numerical problem, but no context was provided either, for example, "What is five take away three?"

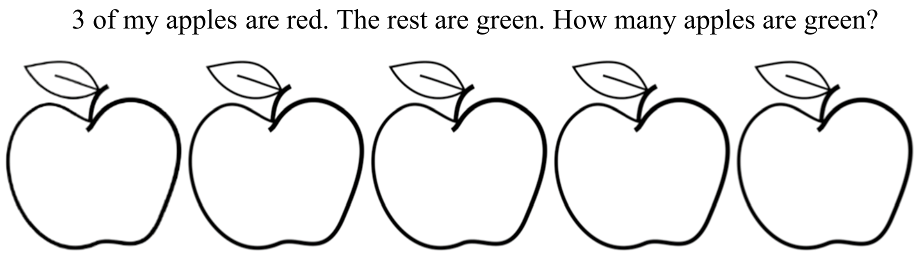

Fig. 2 Jennifer's part–part–whole problem showing five white apples

the problems. In her reflection, Sarah indicated that a good next step for this student and his class would be to focus on *counting on* in an effort to move them toward figurative counting.

Jennifer

Jennifer was unique in that, although she had a near balance of problems involving numbers five and under versus six to ten, she included a part–part–whole, missing part problem (see Fig. 2) and also asked, "What number comes before zero?" instead of a temperature problem to assess her student's knowledge of negative numbers. Similar to Sarah, she only asked one numerical problem at the end. As planned, Jennifer used visuals, cubes, and pennies to introduce the problems to help her student. She said that the student did not use the materials except for on the final numerical problem, although the visuals likely helped the student subitize the answers (or the student already knew the facts) because he answered quickly and took more time on the problem when a visual was not provided.

Jessica

Across the four PTs, Jessica had the broadest subtraction landscape based on her planning, although it narrowed a bit in practice. She is the only one who included separate, change unknown problems. She also originally included a problem with numbers over ten and a part–part–whole, missing part problem. Unlike the others, Jessica had originally planned only word problems; in practice, she was also unique because she included a majority of problems with numbers six to ten and intermixed word problems and numerical problems. Similar to Sarah, Jessica was fairly open to the suggestions made by the instructor, such as to include a negative-number problem and numerical problems. Interestingly, the two word problems she changed to numerical problems were the separate, change unknown ones. With regard to the manipulatives and visuals, Jessica did not originally plan to use manipulatives, although when encouraged by the instructor, she indicated she would use base ten blocks or counters. In practice, she had the student draw a picture after incorrectly answering a problem, a strategy the student later used without prompting. She also illustrated a separate, missing change problem with her fingers; the student did well when following her lead but otherwise counted up or down to numbers in the subsequent problems instead of subtracting.

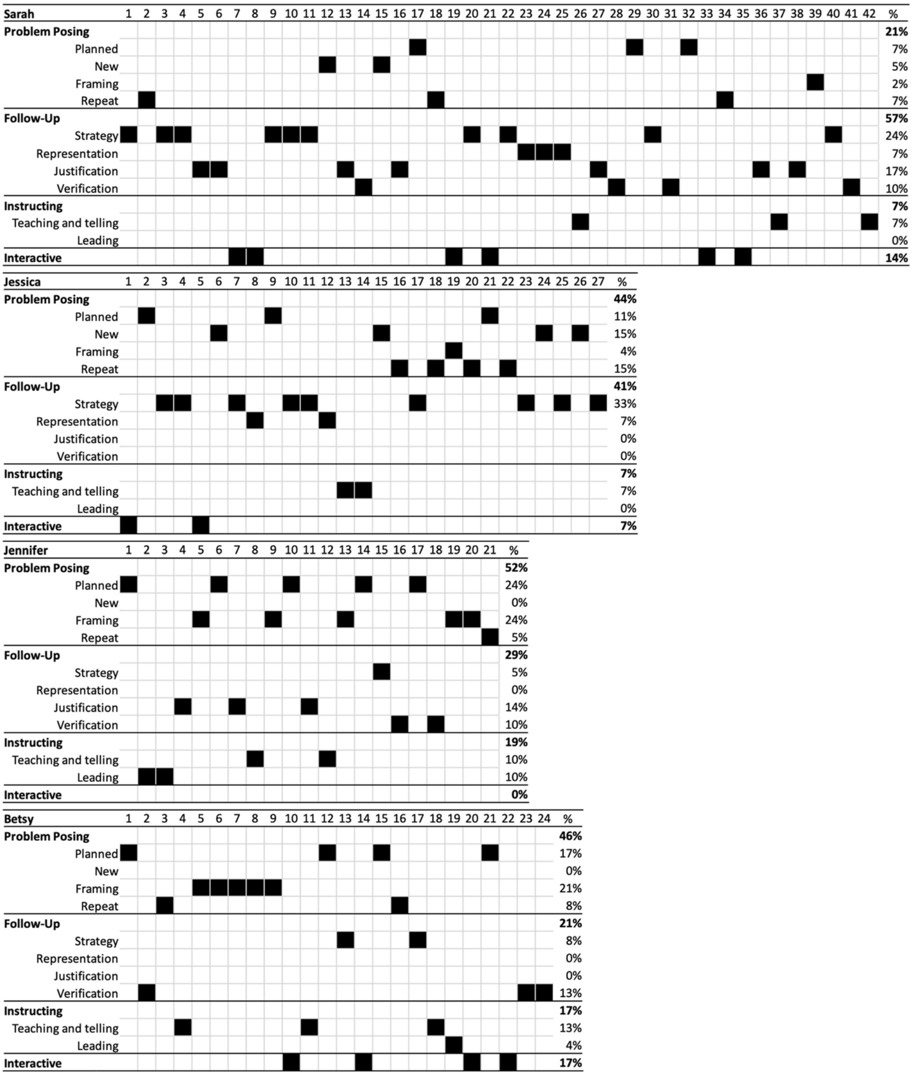

Fig. 3 Order and Percentages of Each Question Type among the Total Number of Teacher Questions (%) *Note*. Percentages may not add up to overall totals due to rounding. The last number in the top row represents the total number of teacher questions, for example, Sarah used 42 questions in total

Prospective teachers' questioning

On average, the PTs were most likely to ask *follow-up* (40% of questions) and *problem-posing* questions (38% on average) (see Fig. 3). The high *follow-up* question percentages suggest that PTs demonstrated some persistence in learning about students' solutions and strategies. Even with just four PTs, there were three different questioning patterns: Betsy and Jennifer had a high proportion of *problem-posing* questions, followed by *follow-up* questions; Sarah had a high proportion of *follow-up* questions, followed by *problem-posing* questions, and Jessica had a roughly equal proportion of *problem-posing* and *follow-up* questions. On a positive note, there was not much focus on using questions to lead students to an answer, and PTs asked *instructing* questions in cases of both correct and incorrect student answers.

Sarah

Sarah stands out in her use of questioning because she not only asked the most questions, but the majority of her questions focused on following up on the student's answers. She routinely used a *follow-up: strategy* question and asked the student how he got the answer or how he counted, although sometimes she did not get specific information from the student. Beyond asking for more information about the student's strategies, she also asked him to show her the strategy using a *representation* and *justification*, for example, asking, "How did you know that?" or in a more specific instance, "Why did you make this two at the end here?" These questions typically elicited adequate information about his strategy use. The student provided justifications indicating that he used his fingers to solve the problems. Even when Sarah used *instructing: teaching and telling* questions, she left space for the student to show his understanding, which in the following case led him to correct his answer. The student originally claimed that if there were six markers and two were shared with a friend, there would be eight left, which indicated a focus on addition. Sarah helped explain the potentially ambiguous word *sharing* as *giving away* by using a *problem-posing: repeat* question, "Okay, so I have six markers, and I am going to share two of them with my friends, so I am going to give two of them away. How many markers am I going to have to color my picture?".

Sarah's *follow-up: verification* questions captured the essence of what her student was explaining mathematically but provided an alternate way to describe it, such as when she described her student's counting by saying, "So you counted with your fingers?" In another instance, when her student explained that he determined four was before five because he "counted to," she provided a thorough way of explaining the student's potential mathematical reasoning but cut off the student's response in doing so, saying, "You counted to five and then thought what number comes before this?"

Jessica

Like Sarah, Jessica also made subtle changes to the language of the questions with her *problem-posing: repeat* questions. For example, Jessica originally posed, "Five minus what equals three?" and then immediately followed this by explaining, "So if you start out with 5 of something, what do you have to do to take away to get the answer 3?" When the student solved the problem by taking away the 5 and getting an incorrect answer of 3, Jessica tried repeating the problem once more with a change of language stating, "You start out with 5 and you don't know how much you're taking away yet. But you'll end up with 3. So how many fingers will you have to put down to get 3?" This explanation addressed the student's mathematical struggle and provided enough scaffolding for the student to successfully answer 2. Another type of rewording arose when Jessica posed the nonstandard problem 10–__ = 4. Seemingly aware of the student's struggle with subtrahend-missing problems, she first framed the problem by explaining, "So if you start out with 10, how many do you have to subtract to get to 4?" When the student counted *to 4*, "1, 2, 3, 4," Jessica repeated, "You want to get to 4. So how many do you subtract from 10?" This rewording emphasized that the student had to subtract *from 10*. However, the student then counted down *from four*, suggesting perhaps that the student now focused on the word "from" instead of "to" but ignored the 10. Jessica just moved on to the next problem.

Based on the student's answers, Jessica primarily focused on asking students *follow-up: strategy* questions of the form, "How did you *get* [answer]?" However, as shown in Excerpt 1, Jessica was often unsatisfied by the student's responses and probed further using *follow-up: representation* questions, encouraging the student to be more detailed.

Excerpt 1 (Jessica):
Jessica: Okay, the first one is just the number problem. It reads, 10 minus 3 equals?
Student: Okay. Equals 11.
Jessica: Okay, how did you get 11? [*follow-up: strategy*].
Student: Because I counted.
Jessica: Can you show me how you counted using the whiteboard? I want to know how you're thinking.[*follow-up: representation*].
Jessica: Can you show me how you counted backwards? [*follow-up: strategy*].
Student: I started with 6 and counted back to zero and got zero for my answer.
Jessica: Can you draw me a picture to help me understand? [*follow-up: representation*].
Student: Okay I have 6 doughnuts (she draws 6 circles), and I ate so take away four. (She crosses off 4 of the circles) and you have 0. I mean 2 left. Can I change my answer?

During the second part of the exchange, when Jessica asked the student to show her strategy using a picture, the student correctly changed her answer. However, in the first part of the exchange, she just demonstrated how she counted up from one to eleven. Unlike Sarah, Jessica did not ask the student to justify her answer or strategy here or for any problems, which could have provided better insight into the student's choices.

Jennifer

Jennifer asked the fewest questions, focusing on *problem-posing: planned* and *problem-posing: framing* questions followed by *follow-up: justification* questions. Unlike Sarah and Jessica who focused on the students' strategies first, Jennifer immediately asked justification questions in the form of "How did you *know* that?" in response to her student's answers as shown in Excerpt 2:

Excerpt 2 (Jennifer):
Jennifer: 3 of my apples are red. The rest of my apples are green. How many of my apples are green?…Okay and how many green apples did I have?
Student: 2.
Jennifer: How do you know that is right? [*follow-up: justification*].
Student: Because it says.
Jennifer: Let's do another problem. I have four cats. Can you count my four cats?
Student: One, two, three, four.
Jennifer: Okay, so one of my cats runs away! How many cats do I have now?
Student: Three (answered without counting cats).
Jennifer: How do you know that? [*follow-up: justification*].
Student: (Points to cat on the end.) Because that one ran away.
Jennifer: So I have eight cubes. I am going to take four of them away. How many do I have left?
Student: Four.
Jennifer: Four? How do you know that? [*follow-up: justification*].
Student: Cause there's eight. And eight is four plus four.

Although Jennifer did not probe the student's justifications further, they got increasingly more specific over the interview; toward the end, the student articulated number facts to justify answers. The students' ease in answering suggested that the student often knew the answer. Jennifer's framing questions, which focused less on helping students understand the language and more on getting the student to engage with the quantities, may have also helped the student solve the problems. For example, when she read, "I have eight cubes," before telling the student, "I take away four cubes," she first asked the student, "Can you count the total number of cubes?"

Betsy

Betsy had originally planned to start with a series of numerical problems, and the methods instructor had suggested that Betsy ask a word problem first or be prepared to change the problems into word problems if the student struggled. Although Betsy did not change the problems or order of the problems before the interview, she did end up providing language support when asking the first numerical problem because of her student's response.

Excerpt 3 (Betsy):
Betsy: We have 5–3.
Student: Uh, I don't even know all of these.
Betsy: You don't know any of these?
Student: Yeah.
Betsy: Ok, I will help you. Ok, let's say we have 5 apples and this sign means take away, like you are going to take away something. So say I have 5 and I take away 3, how many do I have left? [*problem-posing: repeat*].

When repeating another question, Betsy also immediately rephrased a word problem from "*I* had 3 candy bars" to "*You* have 3 candy bars." The shift in subject of the word problem put the student as a central figure of the problem.

A high percentage of Betsy's questions focused on *problem-posing: framing*, just like Jennifer's; however, Betsy used the questions to probe the student's recognition of the problem itself after her student said he knew nothing about the first problem.

Excerpt 4 (Betsy):
Betsy: Great! Ok let's look at the next one. Do you know what this number is? *[problem-posing: framing].*
Student: 8.
Betsy: Good, and do you know what this sign is? *[problem-posing: framing].*
Student: Minus.
Betsy: Great, minus what? What is this number? *[problem-posing: framing].*
Student: 2.

Rather than asking her student to elaborate on his answers, Betsy's used *follow-up: verification* questions, repeating what her student said in a way that provided encouragement for the student to keep talking, much like with a revoicing talk move. For example, her student said, "I don't even know all of these," and she responded, "You don't know any of these?" Unfortunately, these questions did not have the intended effect, and the student mostly added instead of subtracted. Rather than building on her success with the *problem-posing: repeat* questions or asking him to justify, she often just used an *instructing:*

teaching and telling question to ask him to write down his answer, so she did not get many details on how her student solved the problems.

PTs' assessment of their student's subtraction understanding

A key focus in the Wright et al. (2006) course reading was on types of counters and counting strategies. The PTs' reflections also followed these foci, except in the case of Betsy.

Betsy

Rather than focusing on any key themes from the interview or course readings, Betsy mainly summarized what happened in the interview. Namely, she identified that her student could solve the first numerical problem (i.e., 3–2) when she modeled the problem on her fingers, but the student added for the remaining numerical problems. When the problems shifted back to a word problem (i.e., 3 candy bars take away 2), her student once again quickly got the answer. Her follow-up questions were mainly oriented toward strategies; however, she rarely asked further questions of how her student solved the problems after getting repeated "I just knew it" responses. Although it would have been difficult for Betsy to discuss her students' strategies based on those responses, she could have looked deeper at the numbers involved and her scaffolding of the problems.

Jennifer

Although Jennifer had a similar questioning pattern as Betsy, she did ask a few justification questions and delved deeper in her reflection than Betsy by focusing heavily on her student's performance in relation to the numbers involved in the problems, and drawing successfully on the course reading to justify why it made sense for her student to know the answers to problems that involved doubles. In fact, Jennifer characterized her problems as those involving doubles versus those that did not. She concluded that when the problems did not involve doubles, her student "would count out the total number of manipulatives, take away a certain amount, and then count the remaining manipulatives." Jennifer's conclusion was not completely accurate. The first two problems she presented had pictures already drawn. For the first one, the student stated the answer before Jennifer had him count out the initial number and the number taken away (*problem-posing: framing*). Upon being asked how he got the answer (*problem-posing: strategy*), the student argued that the picture showed him. Similarly, for the second problem, Jennifer first had him count out four cats (*problem-posing: framing*). After she told him one ran away, he indicated there would be three left (without counting) and when asked to justify it *(follow-up: justification)* then pointed to the one cat that ran away, still not counting the remaining. Both of his strategies closely related to the pictures provided and suggest he either knew the answer or subitized rather than counted. For the last small-number problem, 5–2, the boy counted five pennies, removed two, and subitized the final three.

Although the instructor discussed subitizing in the class (and the PTs participated in dot pattern number talks), the course reading only included one paragraph on it. Regardless, Jennifer did indicate that these small numbers constituted an important class of problems, suggesting she made some connection between the size of the numbers and her student's strategies. Jennifer also reflected, "For questions that had small numbers to work with [the] student was able to answer without using the provided visuals or manipulatives," although this conclusion contradicted her earlier statement that her student had to count. It is possible that she was thinking of the doubles problems and overgeneralized in her statement.

Sarah

Compared to Jennifer, Sarah's reflection on the student's use of visuals and manipulatives led to more explicit reflections on what type of counter her student was, which partially resulted from her frequent use of the *follow-up: strategy* and *follow-up: representation* questions. Sarah accurately identified that her student needed perceptual items when solving the word problems; however, she also claimed that he was a figurative counter because he knew which number was before or after another, confusing knowledge of the number sequence with knowledge of number composition. Interestingly, Sarah also pointed out that her student did have some sense of number composition because he saw that six pies could be split into four that were eaten and two left over.

Jessica

Jessica, in particular, organized the description of her student's understanding by describing how the student could count, an explicit action: "The strategies that the student used were counting up from a number and counting back to a number." She explained that the student had an easier time counting down compared to counting back to, such as on the separate, missing change problems. However, her student initially counted from numbers not in the problems (e.g., solving 10–3 by counting up from 1), which she did not address in her reflection. Although Spitzer et al. (2011) found that PTs overestimated children's mathematical thinking by using unrelated evidence, our results suggest they also make these conclusions by omitting relevant evidence. While working on two subtrahend missing subtraction problems ($5 - \_\_ = 3$ & $10 - \_\_ = 4$), Jessica changed her pattern of first *problem-posing* and second *follow-up: strategy* and used four *problem-posing: repeat* questions to handle her student's misconception that getting to 3 and 4 mean the answer is 3 and 4 for the subtraction problems. These facts that Jessica mentioned irrelevant counting-back information and ignored her student's difficulty in conceptually understanding subtrahend missing problems imply that Jessica was more attracted to explicit student actions than to implicit student struggles (Morris et al. 2009). Further, Jessica identified that the student could solve problems when she scaffolded them with her fingers, using this as a justification that the student was a perceptual counter. Jessica also focused on the student's use of subtraction language (i.e., minus and take away) as an indication that the student had some knowledge of subtraction.

Discussion

Limitations of the Study

The results of this study provide interesting insights into how PTs may conceptualize the subtraction landscape and use problems and questions to better understand student thinking. However, we must interpret the results within the narrow scope of this particular study. In particular, the data we draw from involve only four PTs, placed at the same grade level at one elementary school, in one country, and involve one topic: subtraction. We acknowledge that part of the PTs' questioning patterns may differ simply because they each had a

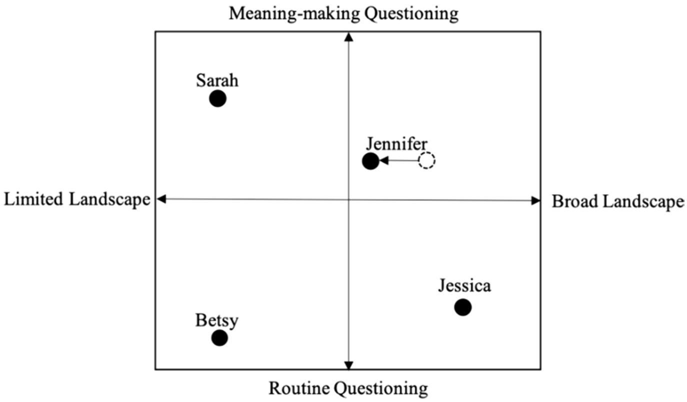

Fig. 4 Problem Landscape and Questioning Plane, Shown with the Four PTs' Placement

different focal student; however, the ways PTs responded in light of these differences are still relevant and important to explore. Because we relied on the PTs to transcribe audio from the interviews, we did not have access to gestures they (or the students) made or their intonation when asking questions. Also, we do not have explicit information on PTs' intentions for choosing their problems to ask or whether their choices would change if they interviewed a second child. Yet, by looking at the practice of four PTs, we were able to show some additional patterns of questioning to add to the work of Weiland et al. (2014). Although our sample was small and the PTs presented here had roughly similar percentages of *Instructing* codes as found by Weiland et al., our PTs' *instructing* questions left space for students to explain and explore their mathematical thinking. Moreover, these four PTs varied in whether the majority of their questions were on framing and reframing questions (through repeating questions in a slightly new way or helping students frame the questions), on *follow-up* questions to better understand students' strategies, or on both. We should continue to build on these results by widening the range of contexts in which we conduct similar investigations.

Cross-case synthesis

Characterizing the results across the research questions reveals four different types of PTs who could be placed along two continua: one indicating the breadth of the subtraction landscape they defined through their problem-type choices (from limited to broad) and the other indicating their use of questions (from routine to meaning-making); these continua form a problem landscape and questioning plane (see Fig. 4).

We define routine questions as those that involve asking questions without learning new information (e.g., asking students to describe how they solved a problem without following up unclear or simple responses); likewise, meaning-making questions involve those that encourage students to justify their answers or further illustrate their strategies or help students make connections.

Betsy defined a limited subtraction landscape with routine use of questions. Although when she chose to repeat questions, it helped the student, she did not probe when she knew the student was answering incorrectly. Therefore, her reflection on the student's understanding read as a summary of what the student did. Sarah also defined a limited subtraction landscape; however, she had meaning-making use of questions. She spent effort to fully understand the student's responses to each question and was able to identify some nuances in her student's understanding. Jennifer defined a broader landscape but then limited it through her overuse of visuals to show the quantities (see shifted dot in Fig. 4). Her visual problems were sufficiently easy for her student, so although she also used meaning-making questions, she underestimated her student's abilities because she made conclusions based on the counting she asked him to do rather than the student's original strategy. Finally, Jessica defined a broad subtraction landscape with routine use of questions. Even though she did not always gain insight into why her student chose particular strategies, because she used a variety of questions, she was able to notice differences in the student's strategies based on the problem type.

Overall, the PTs in this study posed a balanced variety of mathematics problems in terms of word problems and numerical problems with some attempt to include missing subtrahend problems. Yet, they were resistant to providing problems that might be difficult for the kindergarteners as stated in Crespo (2003). None of the PTs included (or added in) problems with numbers over 10, even Jennifer, whose student knew $4+4=8$ and $5+5=10$ or Jessica, whose student who successfully used visuals to solve $6-4$ and $7-2$ in word-problem forms. In fact, the PTs collectively posed only one problem that involved taking away more than five (i.e., $9-9$) and only one problem that involved taking away five (i.e., $10-5$ in the word-problem form). On the one hand, the variation in problem types helped Jessica compensate for some of her routine questioning when making conclusions about her student's strengths and needs. On the other hand, Sarah identified important aspects of her student's understanding through narrowing the boundary defined by the problems she posed and using multiple meaning-making questions per problem.

Implications

Posing mathematics problems

One goal of teacher education, and in particular this mathematics method course, is to prepare teachers who are equipped to pose relevant mathematics problems, interpret student thinking, and make instructional decisions to help students build on their thinking. Interpreting our results in terms of these goals illuminates some potential ways to differentiate the assignment. Building off of Jessica's case, we could provide some PTs with more guidance on broadening the subtraction landscape. Requiring them to extend their problems to include some larger numbers may have provided PTs similar to Betsy with a more complete picture of their students' changing strategies in relation problem types and use of visuals. Additional explorations of different problem types and requiring they pose non-standard problems might help PTs see the benefits of diversifying the types of problems they pose in terms of learning what students understand. Further, building off of Sarah's case and the strengths of letter writing exchanges found by Crespo (2003), we could have other PTs conduct a series of task-based interviews. These interviews could be with the same child or multiple children (with some in common among PTs) to allow them to focus on different areas of the landscape and iteratively test their conclusions about a single student's or groups of students' understanding. At the same time, instructors can help all PTs think about how to use problems as scaffolds for each other so that answering easier problems could help students on later items within an interview.

Questioning

PTs who used *problem-posing: repeat* and *follow-up: justification* and *representation* questions had more success learning about and then interpreting their students' thinking. In fact, PTs demonstrated that using *problem-posing: repeat* questions could help scaffold students' understanding of the language and symbols in the problems (e.g., explaining that *sharing* is *giving away*) and help them make sense of what they were being asked to solve (Wallach & Even 2005). Teacher educators could capitalize on this in instruction with PTs, using language examples as illustrations of ways to support language learners or students who may not be familiar with the everyday language or the mathematical language in the word problems. PTs could also use repeat questioning as an instructional tool, for example, when Betsy presented a numerical problem, rephrased it in terms of a word problem, and then once more posed the original numerical problem. She expected the student to solve the original problem but provided a contextual scaffold as well. Such strategies could help the student see the two questions as one, rather than as two separate questions.

Another way PTs provided scaffolding was through modeling problems with their fingers. Finger strategies are particularly helpful for students still learning about quantities and operations (Jordan et al. 2008), but the PTs should identify ways they could have students use their own fingers rather than just observing the PTs. Considering how to help students show and explain their strategies is an important challenge; as found by Weiland et al. (2014); PTs in our study sometimes missed opportunities to continue pressing students for more information after they provided incorrect or insufficient explanations of their strategies.

PTs did not adjust their problems mid-interview and only rarely did they successfully determine how students understood the problems when they answered incorrectly. Together, these results highlight a potential need to help PTs plan if–then interviews, developing a plan for what question to ask or what new problem to pose if a student provides an incorrect answer or if a student answers all of the problems with ease. Developing a bank of situations and corresponding interview steps could help better prepare PTs.

Importantly, the process of tracking PTs' changes in questioning illuminates patterns that PTs may rely on and need help to alter to become more effective in their questioning. Teacher educators could have PTs compare situations where they used different patterns of questioning to determine the relative advantages of each and then have them practice these alternative patterns of questioning.

Finding the boundary and beyond

The results of this study demonstrate that the PTs were able to develop interview problems for kindergartners with much variety in terms of number sizes, word and number problems, and manipulatives. They were also able to adjust their in-the-moment questions according to kindergartners' responses, which contributes to expanding previous research focused on PTs' use of teacher telling questions. However, the PTs revealed limitations in synthesizing their reflective analyses of students' thinking with some focusing on students' explicit actions or overestimating students' mathematical abilities. This suggests that while PTs could identify kindergarteners' subtraction boundaries through their use of interview problems and questioning skills, they need further support in raising awareness of the strengths and constraints of their problems and questioning in their reflection, and particularly paying increasing attention to the connections of mathematical concepts on the landscape. With deep reflection going along with problem development and questioning, PTs will eventually develop a positive teaching cycle.

Overall, the framework arising from this study, focused on placing PTs on the landscape and questioning plane, can help teacher educators identify areas to focus on in methods instruction. Teacher educators could use the framework to identify where their PTs fall and then strategically pair PTs for peer feedback or partner work to balance out their strengths. This study also shows the benefits of tracking the changes in PTs' self-designed interview problems and their questioning skills and advances current literature by removing the interview scaffolds proposed by Dunphy (2010) and Wilson et al. (2013). The overall practices of finding the boundary of student thinking provide PTs with an opportunity to synthesize their pedagogical and content knowledge of student thinking.

Appendix

PTs' Planned Interview Questions vs. Posed Interview Questions

See Tables 4, 5, 6, 7

Table 4 Betsy's planned interview questions vs. posed interview questions

Planned interview questions	Posed interview questions
5–3 = ___	5–3 = ___
8–2 = ___	8–2 = ___
3–2 = ___	3–2 = ___
9–4 = ___	9–4 = ___
10–3 = ___	10–3 = ___
I had 3 candy bars and my mom took 2 away from me. How many candy bars do I have now?	I had 3 candy bars and my mom took 2 away from me. How many candy bars do I have now?
It is 3 degrees out in the morning; by lunchtime it has gotten 8 degrees colder. How cold is it now?	It is 3 degrees out in the morning; by lunchtime it has gotten 8 degrees colder. How cold is it now?

Table 5 Jennifer's planned interview questions vs. posed interview questions

Planned interview questions	Posed interview questions
1. What number comes before 7?	1. 3 of my apples are red. The rest of my apples are green. How many of my apples are green?
2. 3 of my apples are red. The rest are green. How many apples are green?	2. I have four cats. Can you count my four cats? Okay so one of my cats runs away! How many cats do I have now?
3. I have 4 cats. 1 of my cats runs away. How many cats do I have now?	3. I have eight cubes. I am going to take four of them away. How many do I have left?
4. I have 8 cubes. I take away 4 cubes. How many cubes are left?	4. I have ten pennies. I need to use five of those pennies to buy a piece of gum. How many pennies will I have left?
5. I have 10 pennies and I use 5 of the pennies to buy a piece of gum. How many pennies do I have left?	5. Do you know what number comes before zero?
6. What number comes before 0?	6. 5–2 = ___
7. 5–2 = ___	

Table 6 Jessica's planned interview questions vs. posed interview questions

Planned interview questions	Posed interview questions
1. 10 birds were sitting on a cliff. Some birds flew away leaving only 3 birds left on the cliff. How many birds flew away?	1. It's 3 degrees outside in the morning. At lunchtime it got 8 degrees colder. What temperature is it now?
2. There are 6 doughnuts in the box. If you ate 4 of them for breakfast, how many do you have left?	2. 10–3 = ___
3. There are 8 cars in the parking garage. If 2 of them leave, how many are left in the garage now?	3. There are 6 doughnuts in a box. If you ate 4 of them for breakfast, how many do you have left over?
4. There were 7 mice. If a cat ate 2 of them, how many mice are there now?	4. 5– ___ = 3
5. There are 9 crayons. If three of them are in once piece, how many crayons broke in half?	5. 10– ___ = 4
6. 3 out of 12 eggs are used for my breakfast omelet. How many eggs are left?	6. There are 7 mice. If a cat ate 2 of them, how many mice are there now?
7. There are 9 spiders. I stepped on 2 of them, how many are alive?	7. 8–4 = ___
8. There are 5 baseballs. I hit some to the outfield and now there are 3 left. How many did I hit to the outfield?	8. 9–9 = ___
9. I have 10 fruit snacks. If I have 4 left, how many have I eaten so far?	
10. There are 8 water bottles. If the baseball team drank 4 of them, how many bottles are left?	

Table 7 Sarah's planned interview questions vs. posed interview questions

Planned Interview Questions	Posed Interview Questions
1. It is 3 degrees outside in the morning; by lunch it was 8 degrees colder, how cold is it at lunch time?	1. If I have six apples and used four of them to bake an apple pie, how many apples will I have left?
2. If I have six markers and I share two of them with my friend, how many markers will I have to use?	2. Could you tell me what number comes before five?
3. I have three hair ties, I use two of them to put pigtails in my hair, how many hair ties do I have left?	3. Can you tell me what number comes after seven?
4. What is five take away three?	4. If I have six markers and I share two of them with my friends to color a picture, how many pictures will I have?
5. What is seven minus four?	5. I have three hair ties, I use two of them to put pigtails in my hair, how many hair ties do I have left?
	6. 5 – 3 = ___
	7. It is 3 degrees outside in the morning; by lunch it was 8 degrees colder, how cold is it at lunch time?

Questions underlined are new questions

References

Association of Mathematics Teacher Educators. (2017). *Standards for preparing teachers of mathematics*. Available online at amte.net/standards.

Aze, I. (1989). Negatives for Little Ones? *Mathematics in School, 18*(2), 16–17.

Behrend, J. L., & Mohs, L. C. (2006). From simple questions to powerful connections: A two-year conversation about negative numbers. *Teaching Children Mathematics, 12*(5), 260–264.

Caldwell, J. H., Karp, K., Bay-Williams, J., Rathnell, E., & Zbiek, R. M. (2011). *Developing essential understanding of addition and subtraction for teaching mathematics in Pre-K-Grade 2*. Reston, VA: National Council of Teachers of Mathematics.

Carpenter, T. P., Fennema, E., Franke, M. L., Levi, L., & Empson, S. B. (1999). *Children's mathematics: Cognitively guided instruction*. Portsmouth, NH: Heinemann.

Clements, D. H., & Sarama, J. (2009). *Learning and teaching early math: The learning trajectories approach*. New York, NY: Routledge.

Cotton, K. (1989). Classroom questioning. *School improvement research series* (SIRS), Series 3 (close-up 5). Retrieved from https://educationnorthwest.org/sites/default/files/ClassroomQuestioning.pdf.

Crespo, S. (2003). Learning to pose mathematical problems: Exploring changes in preservice teachers' practices. *Educational Studies in Mathematics, 52*(3), 243–270.

Doyle, W. (1988). Work in mathematics classes: The context of students' thinking during instruction. *Educational psychologist, 23*(2), 167–180.

Dunphy, E. (2010). Exploring young children's (mathematical) thinking: Preservice teachers reflect on the use of the one-to-one interview. *International Journal of Early Years Education, 18*(4), 331–347.

Franke, M. L., Webb, N. M., Chan, A. G., Ing, M., Freund, D., & Battey, D. (2009). Teacher questioning to elicit students' mathematical thinking in elementary school classrooms. *Journal of Teacher Education, 60*(4), 380–392.

Ginsburg, H. P. (2009). The challenge of formative assessment in mathematics education: Children's minds, teachers' minds. *Human Development, 52*(2), 109–128. https://doi.org/10.1159/000202729

Grossman, P. (2018). *Teaching core practices in teacher education*. Cambridge, MA: Harvard Education Press.

Hancock, C. L. (1995). Enhancing mathematics learning with open-ended questions. *The Mathematics Teacher, 88*(6), 496–499.

Hiebert, J., Morris, A. K., Berk, D., & Jansen, A. (2007). Preparing teachers to learn from teaching. *Journal of teacher education, 58*(1), 47–61.

Hiebert, J., Morris, A. K., & Glass, B. (2003). Learning to learn to teach: An"experiment"model for teaching and teacher preparation in mathematics. *Journal of mathematics teacher education, 6*(3), 201–222.

Hiebert, J., & Wearne, D. (1993). Instructional tasks, classroom discourse, and students' learning in second-grade arithmetic. *American Educational Research Journal, 30,* 393–425.

Huinker, D. M. (1993). Interview: A window to students' conceptual knowledge of the operations. In N. L. Webb (Ed.), *Assessment in the mathematics classroom* (pp. 80–86). Reston, VA: NCTM.

Hunting, R. P. (1997). Clinical interview methods in mathematics education research and practice. *The Journal of Mathematical Behavior, 16*(2), 145–165.

Ilaria, D. R. (2002). *Questions that engage students in mathematical thinking*. Paper presented at the annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education (24th), Athens, GA.

Jacobs, V. R., & Empson, S. B. (2016). Responding to children's mathematical thinking in the moment: An emerging framework of teaching moves. *ZDM Mathematics Education, 48*(1–2), 185–197.

Jordan, N. C., Kaplan, D., Ramineni, C., & Locuniak, M. N. (2008). Development of number combination skill in the early school years: When do fingers help? *Developmental Science, 11*(5), 662–668. https://doi.org/10.1111/j.1467-7687.2008.00715.x

Koedinger, K. R., & Nathan, M. J. (2004). The real story behind story problems: Effects of representations on quantitative reasoning. *The Journal of the Learning Sciences, 13,* 129–164.

McDonough, A., Clarke, B., & Clarke, D. M. (2002). Understanding, assessing and developing children's mathematical thinking: The power of a one-to-one interview for preservice teachers in providing insights into appropriate pedagogical practices. *International Journal of Educational Research, 37*(2), 211–226.

McDuffie, A. R. (2004). Mathematics teaching as a deliberate practice: An investigation of elementary pre-service teachers' reflective thinking during student teaching. *Journal of Mathematics Teacher Education, 7*(1), 33–61.

Melhuish, K., Thanheiser, E., & Fagan, J. (2019). The student discourse observation tool: Supporting teachers in noticing justifying and generalizing. *Mathematics Teacher Educator, 7*(2), 57–74.

Meyer, H. (2004). Novice and expert teachers' conceptions of learners' prior knowledge. *Science Education, 88*(6), 970–983.

Morris, A. K. (2006). Assessing pre-service teachers' skills for analyzing teaching. *Journal of Mathematics Teacher Education, 9*(5), 471–505.

Morris, A. K., Hiebert, J., & Spitzer, S. M. (2009). Mathematical knowledge for teaching in planning and evaluating instruction: What can preservice teachers learn? *Journal for Research in Mathematics Education, 40*(5), 491–529.

Moyer, S., & Milewicz, E. (2002). Learning to questions: Categories of questioning used by preservice teachers during diagnostic mathematics interviews. *Journal of Mathematics Teacher Education, 5,* 293–315.
National Council of Teacher of Mathematics. (1991). *Professional standards for teaching mathematics*. Reston, VA: Author.
National Council of Teachers of Mathematics. (1995). *Assessment standards for school mathematics*. Reston, VA: Author.
National Council of Teachers of Mathematics. (2007). *Mathematics teaching today*. Reston, VA: Author.
National Governors Association Center for Best Practices & Council of Chief State School Officers. (2011). *Common Core State Standards for mathematics*. Washington, DC: Authors.
National Research Council. (2009). Mathematics learning in early childhood: Paths toward excellence and equity Committee on Early Childhood Mathematics. In T. C. Christopher, A. W. Taniesha, & S. Heidi (Eds.), *Center for Education, Division of Behavioral and Social Sciences and Education*. Washington, DC: The National Academies Press.
Riley, M. S., Greeno, J. G., & Heller, J. I. (1983). Development of children's problem solving ability in arithmetic. In H. P. Ginsburg (Ed.), *The development of mathematical thinking* (pp. 153–200). New York: Academia Press.
Sahin, A., & Kulm, G. (2008). Sixth grade mathematics teachers' intentions and use of probing, guiding, and factual questions. *Journal of Mathematics Teacher Education, 11,* 221–241.
Siegler, R. S. (1993). Adaptive and non-adaptive characteristics of low income children's strategy use. In L. A. Penner, G. M. Batsche, H. M. Knoff, & D. L. Nelson (Eds.), *Contributions of psychology to science and mathematics education* (pp. 341–366). Washington, DC: American Psychological Association.
Spitzer, S. M., Phelps, C. M., Beyers, J. E., Johnson, D. Y., & Sieminski, E. M. (2011). Developing prospective elementary teachers' abilities to identify evidence of student mathematical achievement. *Journal of Mathematics Teacher Education, 14*(1), 67–87.
Vacc, N. N. (1993). Questioning in the mathematics classroom. *Arithmetic Teacher, 41,* 88–91.
van Zee, E., & Minstrell, J. (1997). Using questioning to guide student thinking. *The Journal of the Learning Sciences, 6,* 227–269.
Vosniadou, S. (2003). Exploring the relationships between conceptual change and intentional learning. In G. M. Sinatra & P. R. Pintrich (Eds.), *Intentional Conceptual Change* (pp. 377–406). Mahwah, NJ: Lawrence Erlbaum Associates.
Vosniadou, S. (2007). The cognitive-situative divide and the problem of conceptual change. *Educational Psychologist, 42*(1), 55–66.
Vosniadou, S., & Brewer, W. F. (1992). Mental models of the earth: A study of conceptual change in childhood. *Cognitive psychology, 24*(4), 535–585.
Wallach, T., & Even, R. (2005). Hearing students: The complexity of understanding what they are saying, showing, and doing. *Journal of Mathematics Teacher Education, 8*(5), 393–417.
Weiland, I. S., Hudson, R. A., & Amador, J. M. (2014). Preservice formative assessment interviews: The development of competent questioning. *International Journal of Science and Mathematics Education, 12*(2), 329–352.
Wilson, P. H., Mojica, G. F., & Confrey, J. (2013). Learning trajectories in teacher education: Supporting teachers' understandings of students' mathematical thinking. *The Journal of Mathematical Behavior, 32*(2), 103–121.
Wright, R. J., Stranger, G., Stafford, A. K., & Martland, J. (2006). *Teaching number in the classroom with 4–8 year-olds*. London: Thousand Oaks.