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Teachers' knowledge of student mathematical thinking in written instructional products

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Abstract

The successful use of lesson plans as the primary vehicle for storing and sharing teachers' instructional knowledge in Japan has given impetus to calls by US researchers for the development of a system for sharing teachers' knowledge through instructional products to improve teachers' capacity to implement high-quality instruction and to build a knowledge base for instruction. These products would be created by, and for, teachers to use in guiding instruction, thus building and sharing teachers' instructional knowledge. In this study, we try to characterize one aspect of teacher knowledge that is central in building a knowledge base for instruction, knowledge of student mathematical thinking. We analyze ten written instructional products from the USA and Japan to better understand what knowledge of student mathematical thinking can be shared in such products. We also look at how knowledge of student mathematical thinking is used to guide and justify instructional decisions. One key finding is that the knowledge of student mathematical thinking shared in the top written instructional products is *specific* to a task or mathematical topic, *varied* with descriptions of multiple solutions or ways of reasoning, and sufficiently *detailed* to make the knowledge usable for teachers.

Keywords Student mathematical thinking · Lesson plans · Knowledge base for teaching · Japanese lesson study

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Introduction

Nearly two decades ago, based on the TIMSS 1999 Video Study (Hiebert et al. 2003a), scholars began calling for the creation in the USA of a way to accumulate the knowledge, experiences, and insights of teachers about the practice of teaching—what Hiebert et al. (2002) have characterized as building a *knowledge base of teaching*. Since that time, Hiebert and his colleagues have written extensively both on the characteristics of a knowledge base for teaching and on the critical role that it could play in improving the effectiveness of teaching at all levels (Hiebert et al. 2002, 2003b; Morris and Hiebert 2009; Hiebert and Morris 2009; Berk and Hiebert 2009).

Stigler and Heibert (1999) were motivated in part by the example of Japanese mathematics teachers, and it remains an important example of the vision that Hiebert and his colleagues have pursued (Hiebert et al. 2002; Morris and Hiebert 2011). Japanese teachers, particularly elementary and junior high school teachers, generate, test, and revise instructional knowledge through lesson study (Stigler and Heibert 1999; Lewis 2002), store instructional knowledge primarily in detailed lesson plans, and share that knowledge through participation in lesson study, distribution of lesson plans, and other means. Recently, long-term projects by Hiebert and his colleagues at Delaware and by Ball and her colleagues in Michigan have provided additional insights into how such knowledge bases can be built. As with Japanese lesson study, both projects employ lesson plans to capture knowledge for teaching. In a recent editorial, Cai et al. (2018) proposed a model of building knowledge for improving instructional practice that also utilized (among other aspects) detailed, context-sensitive lesson plans.

Lesson plans have three critical characteristics that allow both for improvement of instruction and accumulation of professional knowledge. First, they are testable. Lessons can be seen as experiments (Hiebert et al. 2003b; see also Berk and Hiebert 2009; Morris et al. 2009) and lesson plans as hypotheses about what teacher actions will lead to desired learning goals. Second, they are revisable. Results of actually teaching the lessons can be recorded, and based on those results the proposed activities, assumptions, and methods can be revised. Third, they are sharable. In the context of agreed-upon instructional goals, these lesson plans can allow for collaborative experimentation and refinement, leading to the accumulation of valuable professional knowledge over time and across instructional settings.

Our hope for this paper is to add to our understanding of how a knowledge base for teaching can be constructed by more fully understanding the knowledge of student thinking captured in high-quality written instructional products. We broaden the focus slightly from lesson plans to *written instructional products* (WIPs). Unlike in Japan, teachers in the USA and many other countries do not generally have access to published annotated lesson plans that are the results of careful lesson study or refinement of a lesson. Thus, we chose to also consider published articles from practitioner journals in an effort to include aspects of teaching cultures that may be more familiar to non-Japanese teachers and could play a similar role. At the same time, we narrow our attention to one particular aspect of instructional knowledge that we see as critical to the improvement of teaching in the USA: the attention paid by teachers to students' thinking during a lesson. In the next section, we justify this choice and review literature related to the role of students' thinking in instruction, lesson planning and WIPs, and instructional knowledge (*or knowledge for teaching*).

Literature review and background

Importance of attending to student thinking in teaching

As mentioned earlier, Japanese lesson study provided an exemplar for the building of a knowledge base for teaching. In part, interest in lesson study was motivated by the results of the TIMSS 1999 Video Study (Hiebert et al. 2003a) which argued for the advantages of the Japanese instructional model. This model, which has been referred to as "structured problem solving" (Takahashi 2006; Hino 2007; Doig and Groves 2011), makes careful use of tasks in lessons characterized by these four segments: presenting the problem for the day, students working individually or in groups, discussing the solution methods to deepen students mathematical understanding, and highlighting and summarizing the major points. Although this model is not widespread in the USA, it has been recognized for some time that good instruction should focus on choosing good mathematical tasks (e.g., National Council of Teachers of Mathematics 1991) that "leave behind important residue" (Hiebert et al. 1997, p. 22)—the student learning that results from their engagement in the task (cf. Davis 1992). Thus, attention to student thinking while engaged in such tasks provides valuable information to teachers about the mathematical knowledge available to students and helps them make decisions about their next instructional steps.

Thus, there has been a great deal of focus in the last three decades on the importance of teachers using their students' thinking to guide instruction. The National Research Council noted that, in addition to focusing on high-quality mathematical content, effective instruction always "takes sensitive account of students' current knowledge and ways of thinking as well as ways in which those develop" (National Research Council [NRC] 2001 p. 315). They go on to note that "such instruction is effective with a range of students and over time develops the knowledge, skills, abilities, and inclinations that we term mathematical proficiency" (p. 315).

Research has also shown the efficacy of such a focus. The Cognitively Guided Instruction project (Carpenter et al. 1988, 1996) was among the first to demonstrate that teachers could make instructional decisions based on knowledge of student thinking and that such decisions provided academic advantages for their students (Carpenter et al. 1989). This work was cited as an example of building a knowledge base for teaching by Hiebert et al. (2002) and today remains the basis for numerous teacher development projects throughout the United States (Franke et al. 1998, 2001).

More recently, other projects have focused on the importance of teachers' being attuned to student thinking. Work growing out of the QUASAR project (Silver and Stein 1996) has demonstrated the importance of teachers maintaining a high level of cognitive demand in the classroom tasks they pose (Stein et al. 2000). Others have written on professional noticing of children's mathematical thinking as a critical skill for mathematics teachers (Jacobs et al. 2010; Nickerson et al. 2017). Deborah Ball and her colleagues include aspects of student mathematical thinking as an important part of Mathematical Knowledge for Teaching (Ball et al. 2008).

Of course, attention to student thinking is also a vital component of Japanese instruction, and a major focus of lesson study. Corey et al. (2010) examined conversations between student teachers and cooperating teachers in Japan and found that an important focus of lesson planning was student thinking—times when students had to "use their heads." Lewis and Perry (2015) note that an important component of lesson study is analyzing students' thinking and students' work. Indeed, written materials for teachers in Japan focus on variations in students' thinking to a much greater extent than those in the USA (Lewis et al. 2011).

In summary, although attention to student thinking is by no means the only important component of knowledge for teaching, it does characterize both successful teaching and continuing efforts to improve instruction. For this reason, we chose to make it a focus of our research into WIPs.

Written instructional products and lesson plans

The building of a knowledge base for teaching necessitates the storing and sharing of instructional knowledge. Such knowledge can be stored in what Morris and Hiebert (2011) call *instructional product*s, by which they mean annotated lesson plans and associated assessments. As mentioned above, we expand this from lesson plans to include articles in practitioner journals. No literature seems available on the use of practitioner journal articles, and there is limited literature on lesson plans in our field.

Roche et al. (2014) point out that despite the importance of planning to teaching, few studies of effective mathematics teaching have looked at the role of planning. It is not surprising, then, that mathematics lesson plans have received very little attention recently from the USA mathematics education research community. Few studies have looked at lesson plans explicitly to make sense of what they are, what they could be, and their role in teaching and teacher learning. A review of handbooks and compendia of research show few mentions of *lesson plan* or *lesson planning* and no work focusing on lesson plans (Cai 2017; English 2002; Lerman 2014; Lester 2007). Outside of the USA, however, there is more discussion about lesson plans. Japanese teachers, for example, use annotated lesson plans (sometimes referred to as "lesson proposals" (Fujii 2015, p. 275)) as a key resource in, and product of, lesson study. Lesson plans (or proposals) and a variety of other instructional materials (textbooks, teaching books, practitioner journals, etc.) and human resources (colleagues, lesson study groups, teacher math circles, etc.) are key for the Japanese planning and development practice of *kyōzaikenkyū* (Melville 2017; Miyakawa and Winsløw 2019; Fujii 2015).

An analysis of the research literature on lesson planning in a variety of content areas reveals two main purposes for writing lesson plans. The first is use for the teachers' immediate responsibilities, such as for their own teaching, for substitutes, or for administrators/evaluators (McCutcheon 1980; Morine-Dershimer 1977; Neale et al. 1983). The second is to share knowledge with other teachers. In this latter case, there are two kinds of lesson plans. First, "plan[s] for instruction" (Berk and Hiebert 2009, p. 351), focus on "what to do" to reproduce a lesson in another teacher's classroom. Second, "annotated lesson plans" (Morris and Hiebert 2011, p. 9), for which the purpose goes beyond helping a teacher implement the *what* and focus on the *why's and how's* of a specific lesson. Such lesson plans constitute a case, in a particular context, that reveals teacher reasoning, judgment, and knowledge, making the work of teaching visible to the consumer of the lesson plan. Thus, a lesson plan of this type has a core purpose of sharing instructional knowledge.

There are two documented cases in the USA where annotated lesson plans have been used to build a knowledge base of teaching in specific local contexts. Both cases were efforts in teacher education, not K-12 teaching. The first is from the University of Delaware (Morris and Hiebert 2009); the second is from the University of Michigan (Ball et al. 2009). Both efforts focused on mathematics courses for elementary education majors, and their WIPs were lesson plans. Research on these efforts gives evidence that their lesson plans can move beyond the traditional role of laying out the plan for instruction to include "the kinds of thinking, reasoning, and communicating used in teaching" (Ball et al. 2009, p. 462). These additional elements created valuable opportunities-to-learn for teacher educators to support them in a particular lesson as well as making instructional principles available that could extend to other lessons.

Another research tradition that examines what we have called WIPs is the *documenational approach to didactics* (Gueudet et al. 2012; Gueudet and Trouche 2009). From this perspective, teachers or groups of teachers find and modify various resources to achieve their instructional goals. These interactions, which include cycles of both locating and modifying resources, produce a *document*, which is "a mixed entity integrating a material component (the resources gathered for a given teaching objective), a practice component (the usages of these resources) and a cognitive component (knowledge guiding these usages)" (Gueudet and Pepin 2019, p. 142). Ideally, this process results in a "shared resource system" (p. 148) which includes a component of teacher knowledge and could thus contribute to the building of a knowledge base for teaching.

There is much in this approach that is compatible with our goals. However, our purpose in this research is to look at the *results* of the documentational genesis as an entity, to see how teachers' knowledge of student thinking is stored in the written parts of the document produced. Thus, we focus mainly on the (written) content of the document produced and not on the practice-oriented aspects.

Knowledge for teaching

The building of a knowledge base for teaching presupposes a view of what is meant by knowledge for teaching. Attempts to define what knowledge is needed for teaching stretch back at least as far as attempts to correlate teachers' course-taking or content knowledge with their instructional success (Ashton and Crocker 1987; Ball et al. 2001). A more nuanced view came with Shulman's (1986) introduction of *pedagogical content knowledge* (PCK). This notion has been extended and expanded in mathematics education, perhaps most successfully by Ball and her colleagues (Hill et al. 2005; Ball et al. 2008), who introduced the term *mathematical knowledge for teaching* (MKT). A recent review by Depaepe et al. (2013) of how scholars in our field use PCK in published research reports suggests that most ground their understanding of knowledge needed for mathematics teaching either in the work of Shulman or the work of Ball and her colleagues.

Depaepe et al. (2013) found four common characteristics of PCK assumed by most scholars: It is a merging of content and pedagogical knowledge; it is a form of practical knowledge, aimed at successful teaching; it is subject matter specific; and it is built on content knowledge. Because it is aimed at the practice of teaching, we feel that written instructional products, embedded as they are in that practice, hold the possibility of capturing this kind of knowledge.

Depaepe et al. (2013) also suggest that "…most authors agree on the core components that constitute PCK, specifically knowledge of students' (mis)conceptions and knowledge of instructional strategies and representations" (p. 22). Thus, knowledge of students' thinking forms one of the pillars of knowledge for teaching. As we have argued above, students' thinking (and therefore teachers' knowledge of student thinking) is a critical aspect of successful teaching in mathematics classrooms. For this reason, we choose to narrow our focus to the student thinking that is manifest in written instructional products.

Identifying student thinking in lesson plans

It is clear that Japanese lesson study takes students' mathematical thinking as a critical component of lesson planning. The written lesson plans from the Michigan and Delaware projects also contained substantial amounts of student mathematical thinking. Researchers at Delaware (Hiebert and Morris 2009), for example, modified a framework proposed by Grossman (1990) that includes four kinds of knowledge for teaching that guided their work. One category explicitly focused on knowledge of students' thinking; the others focused on knowledge of the lesson's purpose, knowledge of the curriculum, and knowledge of strategies and representations for teaching toward particular learning goals.

Our goal is to better understand how teachers' knowledge of student mathematical thinking is manifested in WIPs. We chose to focus on student thinking in our analysis, but we recognize that student thinking is also manifest in its relations to curriculum, goals, and instructional strategies. One intuitive and widely accepted way of looking at these relations is the instructional triangle (Cohen et al. 2002) displayed in Fig. 1. The instructional triangle consists of interactions of four main components. The first three are the vertices of the triangle: the teacher, the content, and the students. These interactions happen in a particular context or environment, which constitutes the fourth component. These interactions are complex, with the teacher mediating the effects of other possible influences in the classroom.

We find this framework useful because WIPs, such as lesson plans, can be viewed as proposed or hypothesized interactions between the teacher and the student around specific content. Moreover, we assert that many of the teacher decisions and actions depicted in WIPs deal with the interactions described by the instructional triangle.

Although we recognize the complexity of both instruction and of the instructional triangle as a model, our focus on student mathematical thinking suggests we focus on two arrows where knowledge of student mathematical thinking would be most evident: the horizontal arrow representing the student-content interaction and the vertical arrow representing the mediating effect a teacher has on the student-content interaction. We call the first *Student-Content Interaction* and the second *Teacher Mediation*. These two features of the instructional triangle are intimately connected. Knowledge about student mathematical

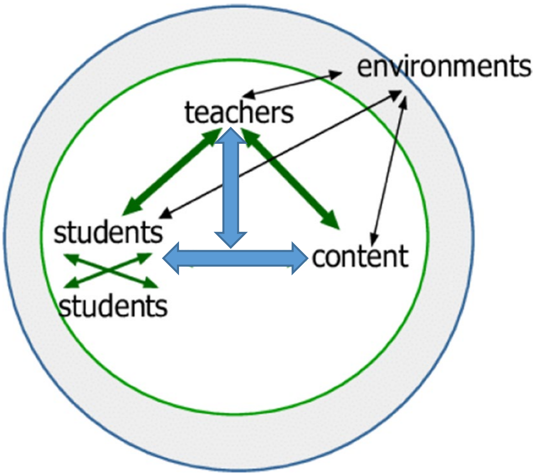


Fig. 1 The Instructional Triangle (Cohen et al. 2002) with two arrows bolded for emphasis

thinking is important because it allows teachers to better mediate the student-content interaction to achieve specific goals. We began coding and subsequently organized our subcodes beginning with these two categories. The details of this analysis are in the methods section.

Research questions

We sought to answer the following research questions. First, what is the nature of the knowledge of students' mathematical thinking captured by high-quality WIPs? For this study, we define high-quality WIPs to be those that more fully address the questions related to student thinking in the Thinking Through the Lesson Protocol (Smith et al. 2008), discussed more fully in the methods section. Second, how is that knowledge used to justify and inform instructional decisions in WIPs? Since we are trying to contribute to understanding the development of a knowledge base for teaching, we feel compelled to connect the knowledge of student mathematical thinking explicitly to the instructional decisions of teachers that constitute that teaching.

Methods

Sample

To answer our research questions, we gathered sets of WIPs from various sources. Our primary data set comprises Japanese lesson study lesson plans because prior research has documented such lessons as rich in student mathematical thinking, as we argued in the introduction. Our sample of Japanese lesson study lesson plans come from two prefectures across Japan. We gathered 9 lessons from Saitama prefecture and 3 from Osaka. All lesson plans were from elementary school teachers in public elementary schools.

We gathered other sets of WIPs to include in our study for two reasons. First, we wanted to see if there were other available lesson plans or written instructional materials that capture knowledge of student mathematical thinking besides the Japanese lesson study lesson plans. Second, comparing and contrasting a variety of WIP sets could help us understand variations in capturing knowledge of student mathematical thinking that would be hard to notice otherwise. All of the WIP sets are summarized in Table 1. (More details on the selection of the WIP sets are found in "Appendix A"). The purpose of using multiple data sets was not to see which set is "best." We thought that WIPs that capture and use student mathematical thinking might be found in more than one of these lesson plan sets. We were trying to learn about the characteristics of WIPs that can serve as a basis for a knowledge base for teaching, independent of the source of the WIPs.

Analysis

We performed two different analyses on the WIPs in our data set. The first analysis is primarily to find a set of WIPs that contain a substantial amount of knowledge of student mathematical thinking. The second analysis uses the set that resulted from the first analysis to answer the research questions of this article: to understand the teacher knowledge of student mathematical thinking that is evident in WIPs, and how that knowledge is involved in, and used to justify, instructional decisions.

Table 1 Descriptions of the seven lesson plan data sets

Name	Description	Grade levels	Number of lesson plans/articles	Public availability
Japanese Lesson Study (JLS)	Common lesson study lesson plans from groups of teachers in 9 schools in Saitama Prefecture and 3 in Osaka	1–6	12	No
US PD Lesson Study (from a local context) (US LS)	Inservice elementary school teachers earning an elementary mathematics endorsement engaged in a few lesson study cycles and produced lesson plans for an 'inquiry' lesson that followed a launch/explore/summarize structure	1–6	12	No
Chicago Lesson Study Group (CLSG)[a]	The lessons of a group conducting lesson study in Chicago (lessonstudygroup.net)	1–9	11	Yes
US Student Teachers (from a local context) (US ST)	Lessons taken from preservice teachers at a large public university	7–9	6	No
Betterlesson.com (BL)[a]	Free resources for teachers with lesson plans made by other practicing teachers	1–8	16	Yes
Japanese Student Teachers (JST)	Lesson plans from three Japanese student teachers in southern Japan	7–9	8	No
NCTM lesson or activity articles from MTMS and TCM (NCTM)[a]	Articles found on nctm.org in the journals *Teaching Children Mathematics* and *Teaching Mathematics in the Middle School*	1–9	13	Yes

[a]See Appendix A for the selection process for the publicly available lessons

Selecting WIPs with high levels of student thinking

We chose to study 10 WIPs that have a substantial amount of knowledge of student mathematical thinking. We scored each lesson plan using the questions from the *Thinking Through the Lesson Protocol* (TTLP) (Smith et al. 2008). The TTLP has many questions about student mathematical thinking, so it was a good fit to help us find WIPs with a high amount of student mathematical thinking. We selected the 27 questions from the TTLP that were directly or somewhat related to either the student-content arrow or teacher mediation arrow in the instructional triangle. These questions are found in "Appendix B".

The TTLP analysis used the WIP as the unit of analysis. WIPs that answered more of the selected questions from the TTLP were considered to have more knowledge of student mathematical thinking. For this analysis, researchers read through lessons to see if they answered the selected questions on the TTLP. Each question was coded as a 0, 1, or 2 based on the completeness of the answer. The code 0 signifies that the WIP did not answer the question. A code of 1 signifies that there was a partial or incomplete answer to the question. Finally, a code of 2 signifies that the WIP answered the question completely. The standard for 2 was based on presence and completeness, not necessarily quality, so there is a possibility of variation in the quality of responses that were coded as 2. To illustrate the differences between WIPs that received a code of 1 or code of 2, we have included a few examples in Appendix C.

Although an analysis of the quality of answers to individual questions was not the focus of this study, the sum of the scores for an individual WIP did seem to have a good correspondence with the richness of the description of student mathematical thinking, not just the amount. When the researcher overseeing the coding and each of the two coders independently each selected the two WIPs that they felt captured the richest descriptions of student mathematical thinking (6 total), 5 were in the top 11 (including the highest ranked lesson plan), and the other was number 19 based on the ranking method described below. Although theoretically one could devise a WIP that could score high on the TTLP and lack rich descriptions of student mathematical thinking, we could not find any examples in our data set. The higher scoring lesson plans, especially the top 20, all had rich descriptions of student mathematical thinking.

After receiving instruction and reviewing of initial coding by a senior researcher, TTLP coding was conducted by two research assistants. The research assistants coded independently and compared their coding for agreement. Inter-coder reliability was above 90%, and all discrepancies were then discussed until they were satisfactorily resolved.

An individual WIP score was calculated by calculating the arithmetic mean across all selected questions. Final ranking of the WIPs was accomplished by sorting the lessons from the highest score to the lowest score and selecting the 10 with the highest score. The top 10 individual WIPs consisted of five Japanese lesson study lessons (JLS), three CLSG lessons, and two NCTM articles. Thus, not all of the top WIPs were from the same set. Twenty-eight of the top 30 individual WIP scores were from these three groups, which indicates that these three data sets may capture more knowledge about student mathematical thinking than other sets. The average scores for these three groups were JPLS: 0.88, CLSG: 0.77, and NCTM: 0.76.

A set of lower scoring WIPs was also selected for analysis to help the coders notice aspects that might be important in depicting knowledge of student mathematical thinking (e.g., present in the high-scoring WIPs and largely absent in the lower scoring WIPs, or different in nature between the two groups). The lower scoring set did not comprise the 10 lowest WIPs. We did not want the majority of the 10 lowest scoring lesson plans from a single data set, so we only included at most 3 lesson plans from any single set in the lower 10 group. The lower 10 individual lessons consisted of one lesson from JP ST, and three each from US ST, Betterlesson.com, and US LS.

Student mathematical thinking analysis

The WIPs were coded to discover the kinds of knowledge of student thinking manifested in WIPs. Because we did not know which grain size might be most useful for analyzing features of WIPs, we performed analyses at two different levels: statement and topic.

Statement-level analysis The smallest unit of analysis for the open coding was a statement, which was a full sentence for expository text, or segments smaller than a sentence in non-expository texts (for example, mathematical statements). Lesson plans are not all expository text; therefore, we could not strictly use a sentence as a unit of analysis. Lesson plans include such elements as:

- Problem statements, for example: “1.36/0.4 = ___”
- Student responses, for example: “One and one-third” or “Student 1: 37 × 15 = 555”
- Labels for student or teacher actions: “estimate 10 equal parts” or “vocabulary: tick marks”

Not all statements in a lesson plan relate to instruction, and so some statements were not coded. A statement was codable if it related to at least one of the vertices of the instructional triangle (teacher, student, mathematics). Headings were not coded. Examples of non-codable phrases include: “This is one of my favorite lessons” or “In the next section we describe the children’s book we used in our lesson.”

The coding was accomplished using an open coding methodology (Strauss and Corbin 1998), with the initial codes based on our view of instructional interactions manifested in the instructional triangle. Initially, all codable sentences and phrases were coded into one of the three initial categories: mathematics, students, or teachers. A sentence or phrase could have more than one code assigned, and many had multiple codes. For this study, we only focused our subsequent analysis on units that were coded either as relating to both content and students (the units related to the student-content interaction arrow of the instructional triangle) or as units that were connected to all three vertices of the instructional triangle (units that might relate to the teacher mediation arrow of the instructional triangle).

After initial coding, the statement-level codes were refined and sub-codes created as researchers sought to categorize units into meaningful groups and categories. The coding required several iterations. As researchers coded statements, they looked for significant features of the lesson plans or for potentially important variations within each code. For example, we found that there was a difference among statements of student mathematical thinking relating to when they occurred relative to the lesson: *before the lesson*, *during the lesson*, and *after the lesson*. We also noticed that the top WIPs tended to have different kinds of descriptions of student mathematical thinking than the lower WIPs. This difference gave rise to a distinction that we referred to as descriptions of *students’ mathematical* *reasoning* on the one hand and statements of *students' mathematical knowledge* on the other. (These categories are explained more fully in the Results section).

During the refining process, WIPs were coded by both a research assistant and a senior researcher. Discussions after coding passes helped us to make sub-codes or refine codes. After at least five revisions, we coded WIPs for which there were only very minor changes to the coding protocol, and the mean interrater reliability was 85% across all categories and data sets combined. We considered this sufficient reliability and refinement for this study. The remaining WIPs were coded by the research assistant only.

Topic level analysis We also analyzed the statements related to student mathematical thinking by topic to better understand how individual statements were related. A topic consisted of related statements, usually 2–10 in number. Often related statements constituted a paragraph, but the most common format was a teacher question and a set of student responses. A typical example among the top 10 WIPs is shown in Fig. 5 later in the paper, where a task is given and various student responses are described. The analysis of the topic started by drawing on the codes of the statements within each topic. Patterns of combinations of codes were noted, and the most common patterns were then examined more closely to understand the meaning of the patterns. One example of a code that emerged from the topic level analysis was that of *variation in student thinking*. Some topics included multiple ways students could understand (or misunderstand) or multiple ways students could reason about a particular problem or idea.

Results

From the above analyses, we summarized the coding to capture the most salient and central patterns into a characterization of knowledge of student thinking. We organize our results in two categories: student-content interaction and teacher mediation. These are the two general categories of codes that stemmed from our framework and remained throughout our coding. Content of the WIPs that were classified as student-content interaction felt like important background knowledge, enabling the teacher to make reasoned instructional decisions. The teacher mediation category had knowledge about instructional decisions and how they follow, or are based on, the knowledge of student mathematical thinking in the previous category.

As we illustrate our findings, we do so with examples from three particular WIPs: a lesson on ratios for fifth grade students in Japan (JPLS lesson 1), a third grade CLSG lesson on decimals (CLSG lesson 5) (Carter et al. 2009), and an NCTM article (Lewis et al. 2015) from *Teaching Children Mathematics* on the topic of equal-sharing problems (NCTM article 7). We hope our sharing examples from three WIPs, each from one of the top sets, will allow the reader to see the variation across the top WIPs, but also to see some coherence and gain a more holistic picture as multiple examples are shared from the same WIP. Two of these lessons are published with the CLSG lesson on the web and the NCTM article in a journal, making them available to the reader for further investigation if desired.

We purposely avoided setting up this study as a comparison of Japanese vs US instructional products, but instead wanted to understand how high-quality WIPs captured knowledge of student thinking independent of the source of the WIP. Since there are no JLS WIPs from our sample that are publicly available for the reader to examine, we would like to mention that the CLSG WIPs are similar to JLS WIPs in style, structure, and information.

Readers can get a very good idea of the nature of Japanese lesson study lesson plans by investigating the publicly available lesson plans at the CLSG website.

Overall characteristics of knowledge of student thinking

There were three overarching characteristics that describe the student mathematical thinking in the top WIPs. The student mathematical thinking was *specific*, *varied*, and *detailed*. The student mathematical thinking evidenced in the WIPs was *specific* to the mathematical topic of the lesson, even specific to a particular task or problem. The WIPs showed *variety* in how students think about and solve problems in the lesson. Finally, the student mathematical thinking was described in such *detail* that teachers could recognize it in their own students and therefore could reason about how to use it in instruction. These three features will be evident in the examples that we share to illustrate the kind of mathematical thinking evident in the top WIPs.

One way in which the detail of student mathematical thinking was illustrated in our results is in the difference between statements about what mathematics students know or do not know (*student mathematical knowledge or SMK*) and statements that described how students reason through problems or think about mathematical phenomena (*student mathematical reasoning or SMR*). We refer the reader to Table 2 for deeper descriptions of these categories. These two sub-codes of student mathematical thinking yielded statements that highlight a key difference between the top 10 WIPs and the lower 10 WIPs. The former had about 40% of their statements connected in some way to student mathematical thinking. It was 9% for the lower 10 WIPs. A more dramatic difference is seen in the number of statements that fell into the student mathematical reasoning (SMR) category. The top WIPs had 16% of their statements coded as student mathematical reasoning (SMR), while only 1% of the statements in the lower WIPs fell in this category. The remainder of the statements connected to student mathematical thinking (about 24% for top 10 and 8% in the lower 10) were about student mathematical knowledge (SMK). Teacher knowledge of both kinds of student mathematical thinking are important, and our results show that the top 10 WIPs had much more of both.

Table 2 Differences between the student mathematical reasoning and student mathematical knowledge codes

Code	Definition	Example
Student Mathematical Reasoning	Statements containing any information about how a student reasons about mathematics or about specific mathematical idea. This includes any reasoning by students, anticipated student responses, solution methods, example of student work, etc.	(Student says) It is easy to compare which room is more congested in the first example because the areas of the rooms are the same, so whichever room has the most people, will be the most congested…. the denominator of the ratio is the same so the larger numerator gives a larger ratio (JPLS lesson 1)
Student Mathematical Knowledge	Statements of what students know/do not know; or understand/do not understand. It includes past, present, and future tense, so mathematical learning goals fall into this category	(CLSG lesson 5) Students need to understand how decimals fit into the number line and how decimals can be used in measurement (Carter et al. 2009)

Student-content interaction

We have illustrated the main codes and relationships within this category in Fig. 2. The structure is largely temporal, with the student mathematical thinking represented when it occurs: prior to the lesson, during the lesson, or at the end of the lesson. We note that the WIPs themselves do not necessarily follow this order, but tend to spend time at the beginning discussing the mathematical thinking students will be bringing to the lesson (prior) and the desired student mathematical thinking (end). After understanding these two features, teachers can carefully reason about the advantages and disadvantages of particular instructional choices. Student mathematical thinking during the lesson largely came at the middle or end of the WIPs as decisions were made about what activities/tasks/questions to use and how to implement them in instruction. We present our results in the order they tended to occur in the WIPs.

Specific student thinking prior to the lesson

In this section, we discuss two pairs of categories of student mathematical thinking prior to the lesson: the instructional advantages and disadvantages; and the source—in school or out of school—of the student thinking. We begin with the former.

Instructional advantages and disadvantages The two most prominent features we found in this category are understanding current student mathematical thinking which might (1) cause difficulties in helping students understand the topic of the lesson, and why (*disadvantages*); and (2) be profitable to build upon to help students understand the topic of the lesson, and why (*advantages*). One way to view these categories is as the costs on one hand and the benefits on the other which students' current mathematical thinking contribute to the resources the teacher has to work with in developing and facilitating the lesson.

One example illustrating that student mathematical thinking may be either an advantage or disadvantage comes from a second lesson on decimals (CLSG lesson 5). The lesson plan discusses many contexts in which students may have already encountered decimal numbers (money, gas pumps, FM radio stations, digital thermometer, digital stopwatch, electronic scale, etc.) (Carter et al. 2009). This information allowed the teachers to understand what students might already know about decimal numbers. However, students would have various levels of exposure to these contexts, so there may not be one that all students were

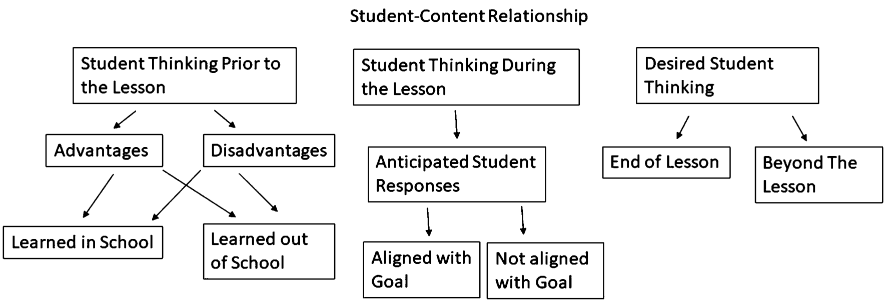


Fig. 2 Main codes in the student-content relationship category

comfortable with and was a good context for building the kind of thinking about decimals needed in later grades.

The context with which students would have the most wide spread exposure would be money, with its dollars and cents decimal representation. However, the authors explained some disadvantages of using this context based on the way students think about and talk about money.

> For instance, $1.23 is read “one dollar, 23 cents” rather than “one point two three dollars.” This leads students to view the decimal point as a separator between dollars and cents and to make the predictable error of interpreting a number like 0.4 as 4 cents (we observed this with some 7th graders recently). (Carter et al. 2009, p. 3).

Another example of advantages and disadvantages of student thinking comes from the fifth-grade Japanese lesson on ratios (JPLS lesson 1). The detailed lesson plan included a problem from a district exam that tested sixth graders’ ability to compare two situations requiring multiplicative comparison (comparing mixtures). Only 54% of the sixth graders were able to complete the problem correctly. Authors’ analysis of the problem concluded that students have had extensive experience with additive comparisons (length, area, volume, time, and angles), but very little experience in contexts that require multiplicative comparisons. Students’ tendency to compare situations with concepts of additivity could be a disadvantage for this lesson.

However, students do have knowledge and experience that could be an advantage for this ratio lesson. For example, the lesson plan explains that students have experience with crowdedness and congestion, a concept that requires rational thinking to make formal. This could be a phenomenon teachers could use to help students understand that different mathematical thinking is needed from just comparing the number of people in two rooms if the rooms are of different sizes. Furthermore, the lesson plan describes two closely related ideas that students understand from previous grades that can be used to make sense of comparisons requiring multiplicative thinking: equal sharing and average. Through equal sharing of one quantity with respect to another (people per square meter or square meters per person), students can create a situation that can then be used to compare the congestion of rooms of different sizes. Averaging one quantity with respect to another leads students to a similar strategy.

Our final example comes from an elementary school lesson (NCTM article 7) on understanding fractions by solving equal-sharing word problems. The lesson plan authors emphasized that students may begin to give answers in terms of number of pieces, even though the pieces may not be the same size. Young students have vast amounts of experience, in school as well as out of school, without having to consider the size of pieces, but only the number, either because the size does not change from piece to piece or size is irrelevant (when using people or vehicles, for example, as the units). Not attending to the size of the pieces could lead students to struggle to understand fundamental fraction concepts, creating a serious obstacle for some students.

Source of student thinking What students know and understand about mathematics does not come only from what they learn and experience in school, but also from what they experience outside of school. Teachers must be aware of ways of thinking that might cause difficulties or give teachers advantages, whether that way of thinking is something that students learn in school or pick up outside of school. Nine of the top 10 WIPs drew on student mathematical thinking from both sources, with the 9th grade CLSG lesson on reciprocals of quadratic functions being the one lesson that did not draw on student thinking outside of school.

Notice that two of the examples in the previous section, the lesson on decimals (CLSG lesson 5) and the lesson on ratios (JPLS lesson 1), drew heavily on students' exposure and thinking related to the mathematical phenomenon outside of school. In the former, it was exposure to decimals outside of school. In the later, it was the idea of congestion, which is very salient to Japanese metropolitan life.

A principle here seems to be that students' experiences outside of school can influence the way they interact with the mathematics in school. The top WIPs showed how outside experiences could be confusing withough adequate clarification, such as the case with money as a context to learn decimals. Conversely, common outside experiences can serve as an advantageous context as in the case of congestion in the Japanese lesson.

Specific desired student thinking

All of the top WIPs included background information for the readers about the kind of student thinking that was the goal of the lesson. Interestingly, all of the lower WIPs had goal statements for the lessons, but none of them had statements that were coded in the student mathematical reasoning category that illustrated the kind of reasoning and understanding students should develop by the end of the lesson. Desired thinking in the top WIPs at times included the kind of student mathematical thinking that was desired beyond the lesson, perhaps later in the unit, or in subsequent grades (what may be referred to as *horizon knowledge*, see Ball et al. 2008). We organize our examples between desired mathematical thinking that is the goal at the end of the lesson and that which is beyond the end of the lesson.

End of lesson Because both the top WIPs and the lower WIPs had goal statements, we focus our examples on the kind of descriptions from the WIPs that were absent in the lower WIPs. Examples from the lesson on decimals (CLSG lesson 5) provided detailed descriptions of the way that they wanted students to understand decimals by the end of the lesson:

> We want students to see that decimal numbers are a natural extension of the whole number system in which each place moving to the right is one tenth of the previous place. But typically, most early work with place value goes the other way, i.e. viewing each place moving to the left as ten times the previous place.… that just as there are ten intervals of 10 from 100 to 200, and ten intervals of 1 from 10 to 20, there are ten intervals from 1 to 2, and those intervals can be represented with a decimal point and another digit.

Even more detail was given in the Japanese lesson on ratio (JPLS lesson 1). The authors explain that students can make sense of a situation comparing two or more congested spaces formally by using "population per 1 m^2." Students will understand that this strategy works because it is one way of applying the principle that two ratios can be compared if we can find equivalent ratios with one of the values (population or area for this context) equal in both ratios. Although there are different ways of finding equivalent ratios (using least-common multiples, for example), there are many benefits for students being able to think in "quantity per unit." The general form of quantity per unit, not just population per area, is explicit since the authors want students to generalize their knowledge of ratios to other situations: "For example, population density on a bullet train can be determined as 'people on the train/number of seats;' in traffic it can be 'area of traffic/speed of cars.' "

A different technique was used in the NCTM lesson on equal-sharing problems. The authors of the article included a table explaining different strategies students could use to solve equal-sharing problems. The authors adapted results of formal research about students' strategies and presented five strategies, from least-sophisticated to most-sophisticated. The strategies and explanations are included in Fig. 3. In the article, teachers read about how to help their students move from their current strategy to the one that is up one level in sophistication, giving them a clear goal on how students could effectively think through equal-sharing problems.

The pattern in these top WIPs is that it is not just important to know what you want your students to know, but how you want them to know it: That is, specifically how do you want students to be able to reason mathematically about the mathematical ideas and related problems. This pattern is an example of how the knowledge of

TABLE 3

These children's strategies for solving equal-sharing problems are adapted from Empson and Levi 2011, p. 25.

Problem: 6 children are sharing 8 small sandwiches. They are sharing so that each child gets the same amount. How many sandwiches will 1 child get?

Strategy name	Strategy description
Nonanticipatory sharing	Child does not think in advance of both number of sharers and amount to be shared. For example, child splits each sandwich into halves, because halves are easy to make, and gives each person one-half. Child may or may not decide to split the last two sandwiches into sixths. Each person gets one-half sandwich and "two little pieces," if the last two sandwiches are split. Example: See Muna's work in **figure 2a**.
Additive coordination: sharing one item at a time	Child represents each sandwich, splitting the first sandwich into sixths because that is the number of sharers. Each person gets one-sixth piece. Child repeats the process until all eight sandwiches are shared. Each person gets eight-sixths sandwiches altogether. Example: See Sahra's work in **figure 2b**.
Additive coordination: sharing groups of items	Child represents each sandwich and realizes that six pieces can be created by splitting two sandwiches each into thirds. Each person gets one-third. Child moves on to another group of items and continues similarly until all the sandwiches are used up. Each person gets four-thirds sandwiches. OR Child represents each sandwich. Realizing that there are more sandwiches than people, child gives each person a whole sandwich. Child moves on to remaining two sandwiches and divides them into sixths or thirds. Each person gets one and two-sixths or one and one-third sandwiches. Example: See Abdi's work in **figure 1**.
Ratio (repeated halving, factors)	Child may or may not represent all the sandwiches and people. Uses knowledge of repeated halving or multiplication factors to transform the problem into a simpler problem: three children sharing four sandwiches. Solves the simpler problem. Each child gets four-thirds sandwiches.
Multiplicative coordination	Child does not need to represent each sandwich. Child understands that a thing shared by b people is a/b, so eight sandwiches shared by six people means each person gets eight-sixths sandwiches.

Fig. 3 Leveled student solution strategies for equal-sharing problems (Lewis et al. 2015)

student mathematical thinking shared in the top WIPs was specific (the thinking is specifically about solving equal-sharing problems), varied (multiple and common solutions are explained), and detailed. (The solutions are described in enough detail that teachers could classify their own student work based on the description.)

Beyond the lesson Understanding the larger curricular context or learning trajectory in which the instruction is happening allows teachers to better make instructional decisions. To describe this context, many of the top WIPs shared a vision or information about what students would learn related to the topic at hand after the lesson(s), which was the focus of the WIP. This practice was especially common among JPLS and CLSG lessons. The introductory lessons on decimals (CLSG lesson 5) displayed the prior and subsequent lesson in the decimal unit. The unit plan is displayed in Fig. 4.

The NCTM article (NCTM article 7) took a different approach to help teachers understand the end to which they were teaching. As explained earlier, the article contained five strategies for solving equal-sharing problems (see Fig. 3). The last two strategies are advanced strategies, which few students will use at the grade when they are first exposed to equal-sharing problems, yet they allow teachers to see where students are heading in the future and can support that kind of thinking in their instruction.

4. Unit Plan

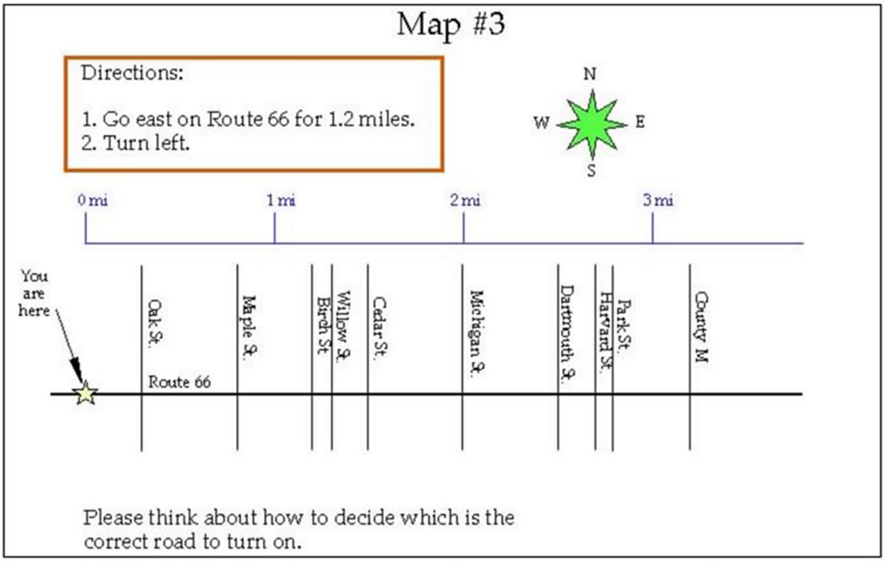
Lesson	No. of days	Description
1	2	Students explore how decimals are used in real life and begin to see how they behave. Contexts include: weights on an electronic delicatessen scale, body temperature, a gas pump displaying cost and quantity, time measured by a stop watch, and driving directions from an internet site. As part of the discussion, students are introduced to the terms *decimal number* and *decimal point*.
2	1	(RESEARCH LESSON) Students discover how decimal values can be used to express distances in between whole numbers of miles. They learn the term *whole number* as contrasted to *decimal number*.
3	1	Students place decimals on a number line. They also learn to use "first decimal place" to refer to the tenth place.
4	1	Students learn how decimals can be used for linear measurement (in cm).
5	1	Students learn how decimals can be used for liquid measurement.
6	1	Students investigate the number 1.8 and consider it from different perspectives. In particular, they learn that 1.8 can be decomposed either into 1 and 0.8, or into eighteen 0.1s. They also compare decimals (using < and >) and compare decimals to whole numbers.
7	3	Addition of decimals: (a) with sum less than one; (b) with sum equal to one; (c) with sum greater than one (e.g. 0.8+0.4).
8	3	Subtraction of decimals: (a) less than one (0.6 – 0.2); (b) from one (1 – 0.3); (c) across one (1.4 – 0.6).
9	2	Practice.
10	1	Assessment.

Fig. 4 Unit plan from CLSG lesson 5

Specific student thinking during the lesson

The previous two categories about student thinking (prior to the lesson and desired student thinking) were often discussed as background information to the lesson(s). It set the context for what instructional decisions might be reasonable, taking the information from the previous categories as givens. (We discuss instructional decisions in more detail in the Teacher Mediation category below.) The student thinking that happens during the lesson helps the readers understand how students may respond to the activities and questions in the lesson. Knowledge of how students will solve particular problems has been shown to help US teachers create better lessons (Lewis and Perry 2015) and improve student achievement (Lewis and Perry 2015; Carpenter et al. 1996). Having a knowledge beforehand allows teachers to anticipate how they can help students and use different solutions and responses as discussion points to deepen students' mathematical understanding. All of the top WIPs illustrated how students did (or might) respond during the lesson.

One of the tasks that students were given in the decimal lesson (CLSG lesson 5) asked students to find which road a car should turn on given a map, a scale, and directions to turn 1.2 miles down the road (see Fig. 5). The WIP makes teachers aware of seven different possible responses. These responses are also in Fig. 5.



Anticipated Student Responses S1: Use estimation to choose Birch or Willow. *S2: Make tick marks for 1.1 and 1.2 and stop. *S3: Make too many/few tick marks between 1 and 2. S4: Count two streets after 1 mi and chooses Willow St. *S5: Divide the interval into two parts; choose Cedar St. S6: Misread 1.2 as twelve. "It's not on the map." S7: Make 9 tick marks between 1 and 2 and choose Birch St.	S1: "Remember, we need to be sure. How can you be sure?" S2, S3, S5 will be discussed.	Do students make a connection to their experiences with decimals the previous day? Do they relate this problem to the previous ones?

Fig. 5 A task from CLSG lesson 5 with accompanying anticipated student responses

The authors have indicated how students might respond and which responses might be further discussed as a class to deepen the students understanding of decimal numbers and place value. Notice that the task was engineered so common but unproductive thinking would emerge and could become a point of discussion to help students develop a better understanding of decimals. For example, students that follow the solution path represented by S5 might think the numeral 2 in 1.2 means two equal parts, not two parts of size one-tenth.

Some WIPs used student work to illustrate student responses. This was the strategy used in NCTM article 7 on equal-sharing problems. The article shared effective as well as ineffective strategies students used to solve these type of problems. An example is illustrated in Fig. 6. Showing a variety of student work was one way that WIPs illustrated the kind of thinking that might happen in response to particular questions or problems.

Summary

We have shown that there was a substantial amount of student mathematical thinking in these top WIPs. The student mathematical thinking was specific to the mathematical topic, was shown to vary across students, and was detailed enough that teachers could recognize it in their own students. The WIP provided enough information about the student mathematical thinking in the context of the lesson that teachers might have a better

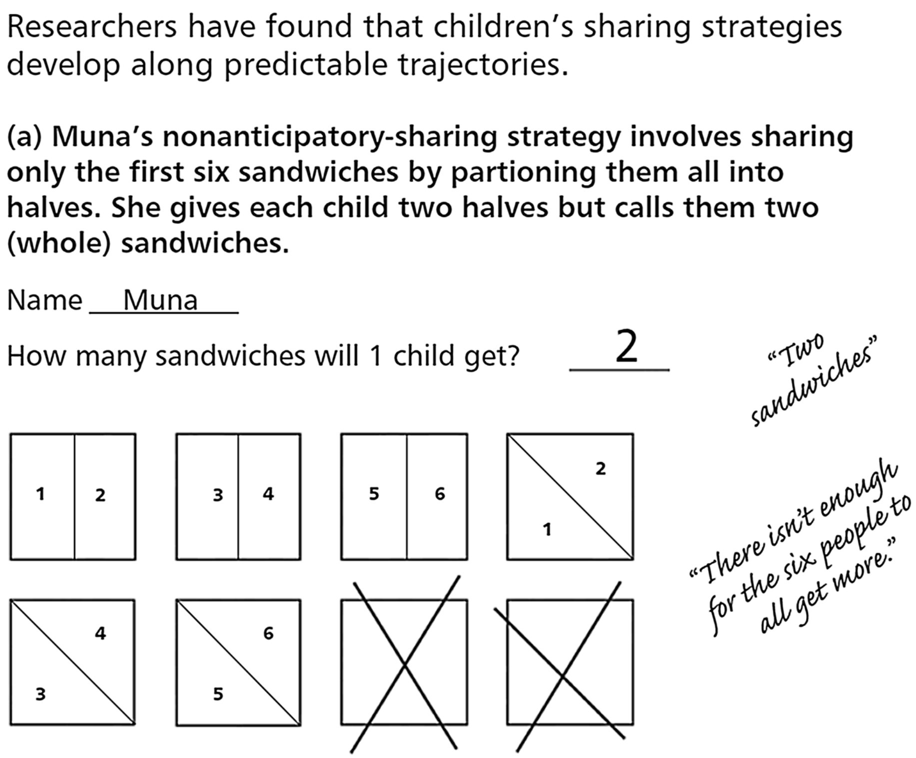


Fig. 6 A recreation of a student's solution to an equal-sharing problem (Lewis et al. 2015)

understanding of the kind of student mathematical thinking students might bring into the lesson, the desired student thinking the lesson was designed to develop, and the particular student thinking connected to the specific tasks or questions used in the lesson. Enough background or analysis was given about the student mathematical thinking of the lesson that teachers might better recognize the significance of an instance of student mathematical thinking. How to respond or facilitate a discussion based on particular responses is largely the focus of the next category of codes.

Teacher mediation

As illustrated by the instructional triangle, teachers mediate the student-content relationship. They do this in a variety of ways: selecting activities, asking questions, giving feedback, explaining concepts, working problems, etc. About 36% of the statements in the top-rated WIPs were teacher decisions or actions. These decisions related to various aspects of the lesson: content, organization, questions for students, boardwork, etc. Many of these decisions were specific instructional moves during the lesson, while others were more general. About 88% of the statements in the lower-rated lesson plans were teacher decisions. These lesson plans tended to be long to-do lists for teachers, along with a few goal statements but little else. The percentages could be deceiving. The top-rated lesson plans tended to be longer lesson plans, so they actually included a comparable number of (actually slightly more) teacher decision statements than the lower-rated lesson plans included. Moreover, the instructional decisions in the top-rated lesson plans were much more likely to be justified or connected to student mathematical thinking.

The top-ranked WIPs had a clear pattern of establishing connections between student mathematical thinking and teacher decisions. Two common patterns emerged to make these connections: justifying a teacher action or instructional decision based on student mathematical thinking and illustrating the development of student mathematical thinking within a lesson. The development of student thinking was described from two different perspectives: retrospective and prospective. We have displayed the structure of our central results in the teacher mediation category in Fig. 7.

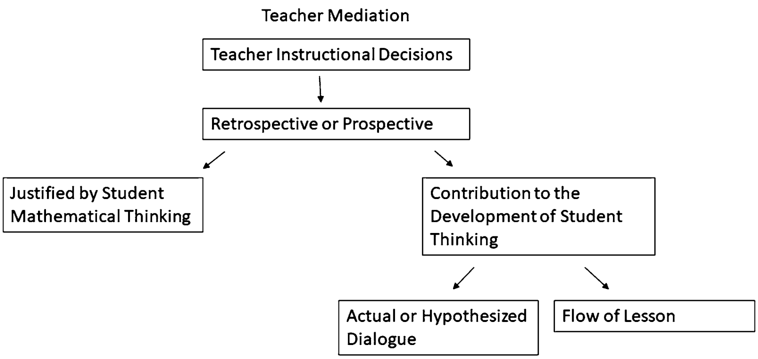


Fig. 7 Main codes in the teacher mediation category

Instructional decisions justified by student mathematical thinking

The specific information provided by the top lesson plans about what students knew and how they might think about the mathematical topic was often then used to help teachers reason about the actions needed to impact student thinking. For example, several of the instructional choices in the Japanese lesson on ratios (JPLS lesson 1) followed directly from the way the authors of the lesson plan anticipated students thinking (described earlier). For example, they anticipated that some students would initially want to find which room was most congested by simply counting which room had more people or finding which room had the smallest area. The lesson plan authors carefully chose numbers so that this strategy would not work, and other students could argue that another strategy is needed.

Some of the teacher decisions that were explicitly justified by student mathematical thinking were quite specific, enough so that a teacher had sufficient information to perform the instruction in class (the teacher would know what to do, write, present, or say). One such example comes from the lesson on equal-sharing problems (NCTM lesson 7). The authors explained choices in the numbers they selected for their problems: (Decision) "*We select quantities so that the number of items (sandwiches) is greater than the number of people sharing*." The statement of this teacher move is followed by a justification based on uncovering something specific about students' solution strategies: (Justification) "*so we can see if students distribute whole items before partitioning as well as how students deal with the remaining wholes*." The authors follow up with another justified specific move based on what students know how to do: (Decision) "*We select the number of people so that each person's share involves fractions other than halves (e.g., thirds, sixths,* etc.*); we avoid such numbers as five or seven shares"* (Justification) "*because partitioning shapes into these fractional pieces is quite difficult, even for adults"* (Lewis et al. 2015, p. 160, italics added).

Contribution to the development of student mathematical thinking

We use this phrase to capture the back and forth between instructional moves and student mathematical thinking across time. Sometimes the development is an overview at a general level, what some have called the flow of the lesson (Schmidt 1996). Sometimes it is very detailed, with dialogue (either hypothesized or actual) between student(s) and teacher.

Flow of the lesson In the Japanese lesson on ratios (JLS lesson 1), the development of the lesson is partially depicted in a section of the lesson plan titled *Instructional Perspective.* This section explains that the goal is for students to make comparisons of different situations by using ratios, particularly to see the advantage of comparing situations through unit ratios. Then the lesson plan proceeds to describe the logical flow of the lesson. First, students will be asked to decide which room is most crowded, given rooms of different sizes and containing different numbers of people. Some students will provide answers based on just the number or size of the room without considering them together. Other students will recognize that this strategy does not work and that both quantities need to be taken into account. Some students will use knowledge from past classes and be able to compare how to answer this question if the area of the rooms is the same and will probably use a common multiples approach to solve the problem by creating two larger rooms with equal areas and then comparing the number of people. This solution is not the most straightforward, so by comparing the common multiples solution to the unit rate solution, students will hopefully see the advantages of the latter.

We have summarized the Instructional Perspective section from the lesson plan into just a few sentences. The full section more fully captures the flow of the lesson to help teachers understand the general plan of the lesson. Understanding the logic of the lesson can still guide the teacher in the moment of teaching if, and when, students might respond differently than anticipated.

Actual or hypothesized dialogue Examples of detailed development of mathematical ideas can be found in every top lesson. Nine of the 10 top-ranked WIPS used actual or

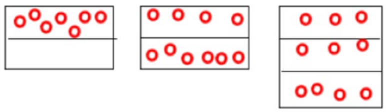
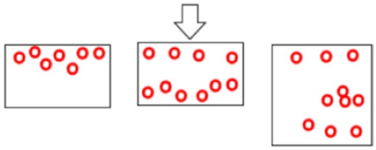
Primary Lesson Activity and Expected Student Response	Points to Consider During Instruction
Understand the problem and think about congestion (as a class) T: A group of people are changing in the changing room. Which room is busier? Room A Room B Room C  C: We don't know how many people are at the bottom half of the room, so we can't tell yet which room is the busiest. T: So we can't tell the busiest room by what's shown right now? C: Yes, since the bottom half of the room may be occupied  as well. . . 7 People 10 People 10 People C: Between rooms A and B, they have the same area, yet B is occupied by more people; therefore, B is busier. C: Between rooms B and C, the number of people is the same but B is smaller, so B is busier." T: How can we move the people so that each person share equal space? T: In two rooms with the same area, when people are equally dispersed, the room with the larger population is busier. When there is the same number of people, the busier room is the smaller one.	• State that a circle (O) represents a person. • By showing the room slowly, have students realize that population is dispersed. • **By asking the students which person they would like to be, confirm that, in this figure, one person's space is not evenly distributed**. • Have students imagine moving the people around so space is distributed evenly. • Cover "average disbursement," "the room with more people and same area is considered busier," and "the smaller space per person is busier." • Have students realize that the degree of busy-ness depends on the number of people per area. • Have students predict, and begin on problem-solving by checking their predictions.

Fig. 8 A portion of a lesson plan showing hypothesized dialogue between teacher and student, along with supporting statements for teachers

hypothesized student–teacher dialogue to describe the development across the lesson. In the Japanese lesson study and CLSG lessons there were multiple pages, at or near the end of the WIP that laid out hypothesized dialogue between the student and the teacher to illustrate the development of mathematical ideas across the entire lesson. A portion of the Japanese lesson on ratio shows this strategy in Fig. 8. Although the scripts have a potential dialogue, it is clear that it does not contain everything the teacher or students will say. The hypothesized dialogue seems to serve as guideposts for the nature of the dialogue during the lesson. The interaction between teacher and students in the lesson plan illustrates how teachers' questions/comments are designed to elicit student thinking and help them come to understand mathematical ideas. Another column was used to help instruct teachers on aspects of the lesson that are not clear from the dialogue itself. NCTM articles either had short dialogues to show the development of an idea for a small piece of a lesson, or intermixed actual dialogue with commentary and summary to show the development of ideas across the lesson.

Perspective: retrospective or prospective

The development in the top lesson plans had two different perspectives: retrospective and prospective. Retrospective lesson plans included artifacts and records from a lesson (or set of lessons) that had already been implemented. Actual lessons had been recorded, and the dialogue and strategies that students used as well as images of student work were shared. Such lessons plans were typical of the NCTM articles included in this study. The hypothesized dialogue of the Japanese lesson study lesson plans and the CLSG were prospective. The authors presented the WIP as if they had not taught the lesson before. Students' anticipated strategies might come from documented cases in textbooks or other resources as well as teacher knowledge of student thinking acquired from teaching similar tasks.

Discussion and conclusion

In this study, we analyzed a sample of WIPs from the USA and Japan to understand what knowledge of students' mathematical thinking is evident in WIPs and how that knowledge is used to justify instructional decisions. The content of and discussion surrounding Figs. 2 and 7 provide answers to those questions. Beyond these categories of knowledge and forms of justification, we emphasize another key finding of our analysis. The student mathematical thinking in the top WIPs was specific, varied, and detailed: specific to a lesson-sized mathematics topic or smaller; varied across the type (prior student thinking, desired student thinking, anticipated in-lesson student thinking), source (in school, out of school), and the students (anticipated student responses); and it was detailed enough to allow teachers to recognize and respond to particular instances of student reasoning in their classroom.

These three features connect strongly to findings of previous research on teacher knowledge of student mathematical thinking. For example, Lewis et al. (2011) found that Japanese teacher manuals are rich with detailed explanations of student mathematical thinking that is specific to particular tasks or problems and illustrates the potential variation in student solutions. Moreover, the manuals include analyses that help the teacher know how to use the various solutions or responses to deepen students' mathematical knowledge. These features are largely lacking in US teacher manuals. Moreover, teachers in the USA, and perhaps other countries, do not have access to instructional resources rich in student mathematical thinking characterized by these three features: specific, detailed, and varied. Hiebert and Morris (2009) called for lesson plans to share instructional knowledge where student mathematical thinking was central to the lesson plan. Additionally, the researchers' description of student mathematical thinking seems to fit these three characteristics. Much of the discourse about instruction focuses on general principles or characteristics (for example, Principles to Actions (NCTM 2014) or the teaching standards of the Common Core State Standards (NGA 2014), and does not focus on sharing resources detailing the variety of student thinking for specific lessons or tasks.

Although 5 of the top 10 WIPs were from the USA (and interestingly the top 15 had 5 each from the JPLS, NCTM, and CLSG groups), they were not from work by typical US teachers. Few teachers seek to publish in the NCTM teacher journals, and presumably even fewer are engaged in the small Chicago Lesson Study Group. Our sample of the highest user-rated lesson plans from betterlesson.com showed very little focus on the knowledge of student mathematical thinking (but, of course, they might have other strengths). We anticipate, though future research is needed, that open sharing systems such as betterlesson.org or teacherspayteachers.com will not have the focus on knowledge categories key to building a knowledge base for teaching (see Hiebert and Morris 2009) and student mathematical thinking in particular. We are aware of shared digital resources or learning platforms such as MinUddannelse in Denmark (Tamborg 2017) and the Digital Educational Resources Bank (DERB) in France (Gueudet and Pepin 2019) that are possible models for dissemination of WIPs and merit further study.

Our findings point out for teachers and teacher educators the kind of knowledge of student mathematical thinking that should be captured or shared in efforts to build instructional knowledge. Our findings are not exhaustive, but can serve as a guide. We wonder about the extent that these are the focus of professional development activities or preservice courses, and the extent they are included in resource materials for teachers. We know that written materials for teachers in Japan focus on variations in students' thinking to a much greater extent than those in the USA (Lewis et al. 2011). Our experience with professional development in the USA is that the focus is on general ideas and principles and does not capture the variety of student thinking in great detail specific to a lesson or task.

Because Japan has a structured, common, well-taught curriculum, the teachers there can predict with fair reliability the kind of thinking students will exhibit in response to particular tasks/questions. The USA and many other countries do not have such a resource, so it may be a better fit for such countries to discuss and share student mathematical thinking in a retrospective manner by documenting actual student responses to specific tasks/questions in a particular context.

We are encouraged that others (Cai et al. 2018) are not only calling for more work to be done, but offering potential models and frameworks for a robust system that will generate, store, and share instructional knowledge in a manner that will fit the contexts, resources, and culture of the US education system. Knowledge of student mathematical thinking is a key component to share as part of this effort. Characterizations of knowledge categories, such as ours for student mathematical thinking, will be valuable supports in this work because they point to the kind and breadth of knowledge that seems to be most valuable to document and share.

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