Journal of Mathematics Teacher Education (2020) 23:341–361
https://doi.org/10.1007/s10857-018-09425-w

Posing mathematically worthwhile problems: developing the problem-posing skills of prospective teachers

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Published online: 28 January 2019

Abstract

Problem solving is a key priority in school mathematics. Central to the valuable role played by problem solving is the quality of the problems posed. While we recognize the features of good problems and how to support learners in solving problems, less is known about the ways in which prospective teachers' (PTs) conceptions of what constitutes a 'good' problem develop within the confines of an Initial Teacher Education program. This study explored the effect of engagement in a mathematics education course on the problem-posing skills of 415 prospective primary teachers. A 3-week instructional unit consisting of a series of lectures and tutorials on problem solving and problem posing was implemented. A questionnaire examining participants' understandings of and ability to pose problems was administered prior to and following instruction. Results reveal that participation brought improvements in conceptions of what constituted a good problem and in the ability to pose good problems (targeted at grades 1–4). Initial problems generally were arithmetic, required one step to solve and had only one correct solution. Following the instructional unit, attention was paid to designing problems that had the potential of multiple strategy use, multiple possible correct solutions, multiple modes of representation and the incorporation of extraneous information. Despite these improvements, the complexities of problem posing and the challenges that persist for PTs in posing good problems are evidenced. Recommendations are made for the enhancement of problem-posing experiences, most notably developing skills in identifying mathematically worthwhile problems from a selection of problems or in reformulating given problems to make them better, that support PTs in developing the knowledge and understandings required to pose mathematically worthwhile problems.

Keywords Problem solving · Problem posing · Prospective teachers · Mathematics education · Teacher education

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Introduction

Over the past century, perspectives on mathematics education have expanded to emphasize conceptual understanding, higher-level problem-solving processes and children's internal constructions of mathematical meanings in place of, or in addition to, procedural and algorithmic learning. Alongside this broadening perspective is acknowledgment of the critical role of both problem posing and problem solving in school mathematics (Ellerton 2013). There is growing acknowledgment that better teacher preparation will lead to better problem posing and solving in schools due to the belief that the quality of pupils' exposure depends on the teacher's own approach to this very important topic (Schoenfeld 1989). Having teachers pose more mathematically worthwhile problems will, it is hoped, counter what is termed 'the suspension of sense making' (Schoenfeld 1991) arising from the practice of solving traditional word problems in school settings.

What teachers understand or believe mathematical problems to be is crucial in the context of what is taught and how it gets taught in schools. Indeed, Ernest (1989), Thompson (1984) and Chapman (1999) all assert that the conceptions, personal ideologies, worldviews and values that shape practice and orient knowledge of a teacher impact hugely on their classroom approach. In order to support and shape prospective teachers' (PTs) understandings and views on problem posing, it is important that we develop an awareness of their initial conceptions of problem posing. These initial conceptions impact the ways in which PTs interpret, respond to and engage with the experiences provided to them in Initial Teacher Education (ITE).

What is a mathematical problem?

There are many conceptions of what constitutes a mathematical problem. While some people construe problems as routine exercises for the consolidation of newly learned mathematical techniques, others view them as tasks whose complexity makes them problematic or non-routine (Schoenfeld 1992). It is this latter conception of a problem that is the focus of contemporary research. In the early 1960s, Polya (1961) asserted that solving a problem is finding a way out of a difficulty, a way around an obstacle or attaining an aim which was not immediately attainable. Different conceptualizations of problem exist. Polya outlines two types of problems: problems to find in which we are asked to construct, to obtain, to identify, what is the unknown?, i.e., What did he say? Problems to prove in which we are asked is this true or false, what is the conclusion. i.e., Did he say that? These two types of problems require different approaches from the problem solver. Vacc (1993) further categorized all problems as factual, reasoning or open. Factual problems are those which provide little information as to whether the pupils understand the concept or not. Reasoning problems are not immediately solvable and require higher order thinking and figuring out. Open problems have a wide range of acceptable answers. Another way to conceptualize problems is those that are purely mathematical or those that are applied. Blum and Niss (1991) and Xenofontos (2014), among others, identify this delineation as being present in contemporary international assessments of mathematics.

What is problem posing? Why is problem posing important?

Problem posing is critical to the problem-solving process and focuses on the problem itself rather than on the solution to the problem (Polya 1954; Brown and Walter 1983; Silver

1994). Despite the primacy of problem posing within the process of problem solving, Ellerton (2013) remarks that problem posing has received relatively scant research attention as compared to problem solving. Nonetheless, the emergence of research on problem-posing (Brown and Walter 1983; Ellerton 1986; Kilpatrick 1987; Silver and Cai 1996) has resulted in problem-posing gaining a more prominent role in the teaching and learning of mathematics and occupying a component of international mathematics curriculum and pedagogy (Silver 2013).

Problem posing is deemed important for a number of reasons. Problem posing is valuable as a goal in itself but also constitutes a means to accomplish a myriad of other mathematical goals such as developing confidence in mathematics, deepening mathematical understanding, advancing mathematical problem-solving skills and developing mathematical aptitude and learning autonomy (English 1998; Kilpatrick 1987; Silver 1994). Problem-posing activities are seen as cognitively demanding tasks (Cai and Hwang 2002) and thereby 'can provide intellectual contexts for students' rich mathematical development' (Cai et al. 2013, p. 60). Problem-posing activities have also been used to promote and evaluate creative thinking (Leung and Silver 1997; Silver 1997; Sriraman 2009) due to problem finding being viewed as the first process in creative problem solving (Getzels and Csikszentmihalyi 1976). There is also evidence of the value of problem posing in supporting mathematical learning and in assisting learners in becoming better problem solvers (Cai 1998; Ellerton 1986; English 1997, 1998; Silver and Cai 1996). While the aforementioned study interventions provide valuable insights into approaches to incorporate problem posing within classrooms, Silver (2013) cautions that 'progress has been stymied by the lack of an explicit, theoretically based explanation of the relationship between problem posing and problem solving' (p. 160).

Research on prospective and practicing teachers' problem-posing abilities

While the importance of posing good problems is widely accepted, Ellerton (2013) argues that an enculturation process exists in our schools and ITE institutions where learners 'have little or no opportunity to be involved in any problem-formulation processes. Thus begins the enculturation process of accepting problems that others create as those which need to be solved' (p. 87). The concomitant reliance on the use of ready-made mathematics problems from books and the Internet needs to be replaced by providing opportunities for PTs to learn to pose mathematics problems. Such problem-posing opportunities can come about from the formulation of a new problem or the reformulation of a given problem into a novel problem (Duncker 1945; Silver 1994); The majority of research focuses on the former. Silver et al. (1990) and English (1997) consider generating new problems from given mathematics situations to be the main activity of posing problems. Indeed, Cai et al. (2013) report that many studies examining the types of problems that teachers and students pose reveal the abilities of both groups to pose interesting and important mathematical problems.

However, the task of posing worthwhile mathematical problems is not trivial; learning to pose problems is one of the challenges of learning to teach mathematics. Knapp et al. (1995) and Stigler and Hiebert (1999) found that teachers at all levels need to understand the important role of posing problems. Studies on problem posing are divided between those that focus on the problem-posing abilities of children and those that focus on teachers. Studies focusing on the latter group, teachers, either draw on the diagnostic potential of problem-posing activities such as those that explore PTs rational number understanding

(Tichá and Hošpesová 2015; Xie and Masingila 2017) or have a broader non-content examination of the features and qualities of the problems posed (Sarrazy 2002). It is the latter type of research that is the focus of this study. There is consensus that if prospective and practicing teachers are to provide new and different sorts of problem-solving experiences for their pupils, it is important that they have such experiences themselves, as learners of mathematics, in posing and solving problems (Singer et al. 2013). Studies that have provided such experiences have found that teachers who are comfortable with their own problem posing will introduce their pupils to this skill (Brown and Walter 1983; Crespo and Sinclair 2008; English 1998; O'Shea and Leavy 2013). Arising from this, problem-posing research is fast becoming 'an emerging force in mathematics education research' (Singer et al. 2013, p. 3) with much of the focus being placed on exploring ways to support PTs in developing the skills necessary to pose mathematically worthwhile problems.

A study by Ellerton (2013) exploring the integration of problem-posing and problem-solving opportunities into a middle school teacher education program revealed both challenges and opportunities arising from the experience. While PTs found creating problems more challenging than solving problems, they acknowledged the importance of developing problem-posing skills. Importantly, they reported that the act of posing problems helped improve their understanding of the structure of problems. This study by Ellerton (2013) and other studies (Chapman 2005, 2012; Crespo 2003; Crespo and Sinclair 2008) reveal that even though PTs are capable of posing problems, they demonstrate shortcomings in their initial efforts to do so. These shortcomings arise from limited understandings of problem posing and solving; such understandings appear closely aligned to teachers' own classroom experiences, as learners themselves, of problem solving and posing. For example, a study by Crespo (2003) revealed that when asked to extend a mathematics problem, prospective and practicing teachers did so in undemanding, ill-formulated and sometimes unsolvable ways. Even when they had access to potentially rich and worthwhile problems, they lowered their cognitive demand. Similarly, a later study by Crespo and Sinclair (2008) found that PTs often pose trivial, non-mathematical or poor problems due to a lack of opportunity to engage in and explore a problem situation before and during the posing process.

While the previous studies suggest the tendency of initial problems to be ill-formulated and lack cognitive demand, they also provide valuable information. Most notably, these studies reveal gains in PTs understandings of and approaches to problem posing as a result of experiences provided during ITE. The study by Crespo (2003), for example, which took the form of letter-writing exchanges between the PTs and a fourth-grade class showed that over the course of the study, PTs' problems were less typical in their structure and became more adventurous, more puzzle like and open ended, encouraged exploration, extended beyond the arithmetic and became more cognitively complex. A similar letter-writing exchange study by Norton and Kastberg (2012) also found that such opportunities can bring about improvement in PTs' ability to pose more cognitively complex problems. Other studies provide evidence of instructional practices bringing about improvements in how PTs view and discern the mathematical features of word problems (Chapman 2004) and how they analyze and select mathematically worthwhile and cognitively challenging problems (Arbaugh and Brown 2004). These studies reveal that changes can occur in beliefs and understandings relating to problem posing as a result of allowing PTs to be reflective problem posers and solvers themselves. They point to the value of introducing PTs to rich and varied problem-solving experiences, providing opportunities to reflect on their problem solving and to work collaboratively on posing problems.

This study responds to recommendations calling for the incorporation of problem-posing experiences in mathematics education courses in ITE (Ellerton 2013; Lavy and

Bershadsky 2003). It examines initial conceptions of PTs around problem posing and solving and explores changes in these conceptions as PTs engage in a series of experiences designed to develop rich understandings of what constitutes a worthwhile mathematical problem. The research questions are:

> Question 1: Prior to instruction, what are prospective primary teachers beliefs about what constitutes a primary-school mathematical problem? Are these beliefs subject to change?
>
> Question 2: In what ways does engagement in an instructional unit on problem posing and solving influence the types of problems designed for use with children in grades 1–4?

Methodology

Participants

Participants were 415 first-year prospective primary teachers enrolled in a 4-year (eight-semester) undergraduate ITE program [Bachelor of Education (B.Ed.)] in Ireland. In the first three semesters, mathematics education received two contact hours per week in the form of one lecture (groups of 50) and one tutorial (groups of 25). In the subsequent two semesters, mathematics education had one contact hour weekly (in groups of 50) (see Table 1). Participants were enrolled in the second semester of the program and had completed one mathematics education course in the previous semester that focused on number (see Table 2). The focus of their current course was number, problem posing and problem solving and algebra (see Table 2). Details of the foci and outcomes arising from other courses and program participation can be accessed from other sources (Hourigan et al. 2016; Hourigan and Leavy 2017a; b; Leavy and Hourigan 2018; Leavy et al. 2017). Participants had completed one school placement during the first semester and were involved in their second school placement while taking part in this study. Of participants, 76% were female, which reflects the gender breakdown in the teaching profession in general

Table 1 Overview of mathematics education program

Year	Semester [Number] Name	Contact hours [Number of contact hours per week] Nature (group sizes)	Focus of module
1	[1] Autumn	[2] 1 lecture (50); 1 tutorial (25)	Number
1	[2] Spring	[2] 1 lecture (50); 1 tutorial (25)	Number, problem posing and problem solving, algebra
2	[3] Autumn	[2] 1 lecture (50); 1 tutorial (25)	Measures
2	[4] Spring	[1] 1 lecture (50)	Geometry
3	[5] Autumn	[1] 1 lecture (50)	Statistics and probability

Table 2 Summary of the mathematics education program content

Semester	Focus of module	Content Summary
[1] Autumn	Number	Looking back: reflecting on personal experiences of mathematics education, looking forward: reform approaches to teaching mathematics, teaching pre-number concepts, early number concepts, place value concepts, number operations (addition, subtraction, multiplication, division), developing understandings of rational number concepts (fractions, decimals, percentages), effective pedagogies for teaching number
[2] Spring	Number, Problem posing and solving, algebra	Personal experiences of problem Solving, task development, problem posing, problem solving, developing estimation strategies, effective pedagogies for teaching problem solving, understanding number theory, understanding operations on rational numbers, developing understandings of and approaches to teaching algebraic concepts (pattern, equality, functions, conjectures, variables, directed numbers), effective pedagogies for teaching algebra

in Ireland. Participation in the study was on a voluntary basis. Throughout the study, the necessary ethical obligations were adhered to and the study was granted College Research Ethics Committee approval.

Research design

The research is an evaluation of a problem-posing and problem-solving unit of instruction, taught as part of a mathematics education module (Table 2). It involved the collection and analysis of pre- and posttest data pertaining to the instruction. The pre- and posttests consisted of a questionnaire to evaluate participants' understanding of what constitutes a mathematical problem and their ability to construct a mathematical problem for primary students in grades 1–4. The questionnaire was developed and modified from that used by Chapman (2008) in her study of PTs' abilities in relation to the pedagogical and mathematical aspects of problem solving. It consisted of the following five items:

1. What is a problem?
2. Choose a class level (first to fourth grade). Make a maths problem that would be a problem for those children.
3. What did you think of to make the problem?
4. Why is it a problem?
5. Is it a "good" problem? Why?

Given Schoenfeld's (1985, p. 74) assertion that '…being a "problem" is not a property inherent in a mathematical task. Rather, it is a particular relationship between the individual and the task that makes the task a problem for that person,' the authors acknowledge that a limitation of this study is the lack of opportunity for participants to trial their created problems with the intended audience (i.e., Grade 1–4 pupils). While such insights would further inform participants' beliefs regarding what constitutes a problem and provides insights into the quality of their problems, it did not prove possible to coordinate with schools to facilitate such a process given the large number of participants ($n=415$).

Instructional unit

The 3-week instructional unit, taught as part of a 12-week mathematics education module, consisted of a series of three lectures and three workshops on problem posing and solving. Lectures, in groups of 50, focused on the theories and research underpinning problem posing and solving. Workshops, in groups of 25, were closely aligned to the lecture and provided opportunities to engage in problem-posing and problem-solving activities. Thus, there were eight lectures and 16 tutorials each week for the 3-week period provided to the cohort of 400+ PTs. Lectures and workshops were taught by a team of four mathematics educators with a wealth of experience in classroom and college-level mathematics teaching. Two of the mathematics educators, the authors of this paper, designed the lectures and tutorials. They also each taught four lectures and two tutorials every week. The other two mathematics educators provided feedback on the lectures and tutorials prior to their implementation. To ensure continuity between lectures and tutorials, each week they observed one lecture and taught the remaining six tutorials.

Throughout the study, participants were provided opportunities to explore and design non-traditional problems, to solve problems themselves and examine problem-solving heuristics and strategies. All these components were designed to develop and nurture participants' understandings of and skills in posing and selecting problems for their own practice as future primary teachers. The instructional unit consisted of six elements and foci.

Element 1: the value of problem solving

The viewpoint that problem solving is a mathematical skill which permeates across the mathematics curriculum underpinned this element. To emphasize this, problems presented were taken from various strands of the Irish Primary School Mathematics Curriculum (NCCA 1999): data and chance, number, geometry, algebra and measures. The notion of 'the problem-solving classroom' was introduced, and it was emphasized that teaching through problem solving focuses pupils' attention on ideas and on making sense and helps develop confidence in doing and understanding mathematics. The potential that problem solving provides for opening up the opportunity for discussion, mathematical argumentation and collaborative work in the classroom was also explored. Moreover, the role of problem solving in developing higher order thinking skills and in promoting curiosity through trial and error and risk-taking was stressed.

Element 2: what is a problem?

Participants explored what it means for a mathematical task to be a problem. They were introduced to research that draws on the perspective that a problem is something for which we do not know the answer and for which a strategy is not immediately obvious (cf. Polya 1954, 1961; Van de Walle 2013). The implications of this perspective on the experiences of classroom pupils were also explored. For example, pupils would be presented with problems for which they are uncertain of the solution or the solution path. Moreover, pupils may have no learned algorithm to draw from; hence, they must figure out a strategy to arrive at the answer. Links to the Irish primary school mathematics curriculum and constructivist theories were made.

Element 3: types of mathematics problems

Many different problem types were explored using the Irish Primary Mathematics Curriculum Teacher Guidelines as a reference (NCCA 1999, p. 35). The curriculum states that there are seven different types of problems: word problems, practical tasks, open-ended investigations, puzzles, games, projects and mathematics trails. These seven categories were discussed, and examples of different problem types, with the exception of projects and mathematical trails (which are the focus of a different module in the program), were explored during the instructional unit. The representation of problems in common mathematics textbooks was also discussed, i.e., the closed, routine one-step, one-right-answer-type problems that present exactly sufficient information. During the workshop sessions, participants were asked to reflect on their experiences of problem solving and examples of problems they had encountered in primary school.

Element 4: exploring problem structures

The Vacc (1993) framework for categorizing problems was introduced: factual, reasoning or open. As an example, one of the workshop activities focused on the posing of problems using a set of tangrams. Participants were asked to work in pairs to write 3–4 problems based on the set of tangrams. They were then asked to categorize the problems according to the Vacc framework. Most of the problems written were factual, i.e., how many triangles can you find? The participants were asked to look at their problems through the Vacc lens and rework them so that there was a variety representing factual, reasoning and open structures. Another perspective on problem structure developed by Crespo and Sinclair (2008) was presented and explored. Using a food metaphor, this perspective categorizes all problems as either nutritious or tasty. Nutritious problems are factual in nature and provide opportunities to practice operations or procedures with which one is familiar. Tasty problems are those which have aesthetic criteria such as surprise, novelty or fruitfulness. Surprise arises when a problem throws up things which were unexpected such as a pattern emerging, novelty can relate to the way in which the problem is stated, while fruitfulness is when a problem leads on to more problems and maybe answers other questions. For instance, in the tangram problem-posing session, a nutritious problem would be 'how many triangles can you see?' A tasty tangram problem would be 'can you make a shape using all the tangram pieces?'

Element 5: problem-solving strategies

During the workshops, participants were presented with a variety of non-traditional problems to solve. They were reminded of the problem-solving strategies discussed in the previous lecture and were encouraged to consider and use strategies they found most useful.

For example, one approach explored was Polya's four-step problem-solving process (Polya 1954, 1961) consisting of understanding the problem, devising a plan, carrying out the plan and looking back. We also reviewed recommended strategies suggested by Van de Walle (2013), the National Council for Curriculum and Assessment (1999) and Posamentier and Krulik (1998). These included strategies such as Guess and Test (guess and check, guess and improve), Act It Out (use equipment), Draw (pictures and diagrams), Organize data (make a list or table), Find a Pattern, Solve a Simpler Analogous Problem and Work Backwards. During these sessions, instructors focused on modeling ways to support discourse while problem solving and on sharing various strategies and paths used to arrive at these solutions. Emphasis was also placed on the actions of the teacher to support pupils before, during and after problem solving. These *before, during and after activities* were adapted from Van de Walle (2013).

Element 6: considerations when designing and selecting problems

In the workshops, participants were alerted to particular design features or considerations when designing or selecting problems. While not a comprehensive list, the main features examined were: problem type (word, computational, exploratory, puzzle, etc.), curriculum strand, entry points (single vs. multiple), single or multiple steps, single or multiple solutions, understanding required to solve the problem (procedure or conceptual) and level of cognitive demand (low, medium, high). Participants were presented with a variety of traditional and non-traditional problems and required to categorize, analyze, solve and modify these problems.

Data collection and analysis

Our approach to this qualitative inquiry reflects a grounded theory approach (Glaser 1978; Strauss and Corbin 1998). We took an inductive approach to the treatment of the data thus allowing '…the theory to emerge from the data' (Strauss and Corbin 1998: p. 12). The qualitative data from the pre- and post-instruction questionnaires were hand-coded and analyzed to ascertain the results from the collected data. Themes that appeared regularly were identified, and within these themes, common beliefs or ideas were uncovered.

The pre-instruction data were analyzed first on a question-by-question basis. For example, the responses to questionnaire item 1 were examined for each of the 415 participants. This involved taking a piece of data, applying a code to it and providing a description for that code. The data were then clustered into categories in an effort to identify themes or patterns. For example, in relation to item 1, 26 codes were established and seven categories were constructed from these codes. The fit between the data and categories was a process of continual refinement. One researcher made an initial 'first pass' on the pre-instruction data to establish codes, clustered these codes into categories and identified emerging themes. As the findings emerging from qualitative analysis may be influenced by the researcher's personal biases (Suter 2012), the researcher then presented the original data and codes to a second researcher (who was involved in the study and understood the context and data). This researcher completed a 'second pass' through the established codes and examined and critiqued the categories and emergent themes established by the first researcher. Both researchers discussed these codes, categories and emerging themes, and where necessary, the initial data were revisited to assess the evidence for the claims. This process was carried out for each of the five items posed on the questionnaire. At the completion of this process, dominant themes were identified (Merriam 2009).

A similar strategy of establishing codes, categories and themes for the analysis was followed for analysis of the post-instruction data. However, the post-instruction responses and hence data were more complex resulting in the establishment of additional new codes and hence categories and themes. For example, in relation to questionnaire item 2, 23 codes were established and five categories were constructed from these codes from the post-instruction data. This compares to the 14 codes and three categories established from examination of the pre-instruction data for the same item.

Findings and discussion

Research question 1: Prior to instruction, what are prospective primary teachers' beliefs about what constitutes a primary school mathematical problem? Are these beliefs subject to change?

The answers to items 1 and 3 from the questionnaire were used to inform the answers to this research question. In response to the question 'what is a problem?', each participant listed several features of problems. Thus, several codes were assigned to each response resulting in 1101 pretest and 1241 posttest responses. Codes were then collapsed into categories. The most common categories are presented in the first column of Table 3. Table 3 shows the percentage of participants who mentioned problem features or characteristics. Examination of Table 3 indicates that across both pre- and post-instruction, participants referred to the structure and layout of problems (i.e., a problem contains words and numbers, is a question), the activity of the problem solver (the problem solver has to use strategies, find the solution, use

Table 3 Pre- and post-instruction responses to the question 'what is a problem?' (questionnaire item 1, $n=415$)

Most common category of responses	Pre-instruction		Post-instruction	
	% Participants [Total number of mentions]			
Contains words, numbers, is a story	24%	[100]	16%	[66]
Is a question	33%	[136]	33%	[137]
You have to use various strategies	29%	[122]	46%	[190]
Can have more than one correct solution	4%	[16]	31%	[129]
You have to find the answer or solution	19%	[77]	<1%	[3]
Needs higher order or critical thinking	34%	[140]	23%	[94]
The solution strategy is unclear, is not straightforward or obvious	14%	[58]	46%	[190]
You must use prior math knowledge	13%	[53]	8%	[34]
It must be solved, worked on or figured out	40%	[149]	15%	[63]
Is difficult, is a challenge, is a struggle	21%	[86]	14%	[56]
Can be a puzzle-type question	0%	[0]	9%	[39]
Contains relevant and irrelevant information	0%	[[0]	7%	[30]

prior knowledge) and common problem features (unclear strategy, various solutions, must be solved, is a challenge).

The largest decreases in responses from pre-instruction to post-instruction were in relation to problems needing to 'be solved, worked on or figured out' (25% decrease), as involving the problem solver in having to 'find the answer or solution' (18% decrease) and in 'requiring higher order or critical thinking' (decrease 11%). In the case of the first two responses, which refer to effort in arriving at a solution, the decrease may indicate a growing awareness that the process of problem solving is more important than the product. Support for this conjecture may be found in the increase in responses that referred to the process of problem solving most notably in responses mentioning that the solution strategies were 'unclear, not straightforward or obvious' (mentioned by 32% more PTs). The increase in this response type may indicate the growing realization that the purpose of problems extends beyond providing practice in algorithmic procedures as is common practice in textbooks (i.e., arriving at 'the solution' through enacting a known procedure) toward focusing on finding a strategy to arrive at a solution (i.e., no known procedure at the outset). Other noticeable increases as a result of the instruction indicate a greater awareness of the openness of the problem-solving situation both in terms of the 'use of various strategies' (increase 17%) and those responses that refer to 'more than one correct solution' (increase 27%). Two new responses emerged in the post-instruction that were not present in the pre-instruction. Following instruction, 9% of PTs referred to the possibility that problems can be 'puzzle type questions' and 7% mentioned that problems can 'contain relevant and irrelevant information.'

Changes in participants' understandings of what constitutes a problem were also ascertained by examining the answers to questionnaire item 3. In response to 'What did you think of to make the problem?', it is evident that even prior to instruction, PTs attended to a number of relevant and important factors such as the importance of context in posing problems, the ability and prior knowledge of children and the challenge set by the problem (see Table 4). There was also attention to strategy use and the type of language used in the problem. As expected, following the instruction, PTs were more alert to other problem features that were not mentioned in the pre-instruction data. This included attending to features such as the incorporation of extraneous information and the possibility that problems have multiple steps, multiple strategies and multiple solutions. References to the use of drawings/diagrams and concrete materials indicate that participants are now aware of the use of multiple representations and strategies to use when problem solving.

In summary, the pattern in responses as a result of the instruction is generally as would be expected given that participants' experiences of problem solving prior to the instruction had been limited to traditional one-step, one-solution textbook word problems. The instruction exposed them to experiences of both posing and solving a wide variety of non-traditional problem types. The new categories of responses that emerged indicate that participants' understandings of what constitutes a good problem have now been extended. Moreover, as will be seen in the next section, many of these new categories that emerged were also incorporated as elements in the problems they constructed. This suggests that the new knowledge and understandings are not inert; participants not only have awareness of these features but can also translate this knowledge into practice.

Table 4 Pre- and post-instruction responses to the question 'what did you think of to make the problem?' (questionnaire item 3, $n=415$)

	Pre-instruction		Post-instruction	
	% Participants [Total number of mentions]			
Interesting context	44%	[182]	35%	[143]
Children's ability	44%	[184]	36%	[150]
Strand unit	31%	[128]	19%	[77]
Children's prior knowledge	20%	[84]	19%	[77]
Strategy children would use	15%	[61]	8%	[35]
A challenge for children	15%	[56]	17%	[79]
Language used in the problem	15%	[60]	15%	[58]
Inclusion of extraneous information	0%	[0]	13%	[52]
Requires more than one step to solve	0%	[0]	6%	[26]
Potential for more than one strategy could be used to solve	0%	[0]	13%	[55]
More than one correct solution	0%	[0]	11%	[45]
Possible to use a diagram, drawing or concrete materials to support solution process	0%	[0]	12%	[48]

Research question 2: in what ways does engagement in an instructional unit on problem posing and solving influence the types of problems designed for use with children in grades 1–4?

In response to item 2 of the questionnaire, 'Choose a class level (1st to 4th grade). Make a maths problem that would be a problem for those children,' participants constructed a mathematical problem suited for first- to fourth-class children (ages 7–11). This grade bracket was chosen as participants' experiences of school placement were only at these grades; hence, they had greatest familiarity with this age-group. The problems written were analyzed according to the following design features explored in the instruction (see element 6): curriculum strand (number, measures, geometry, algebra, data and chance), number of steps taken to solve problem (one/multiple), number of correct solutions (one/multiple) and problem type (arithmetic/puzzle/other). Other features emerged in the comparison of problem types as a result of the instruction, most notably the inclusion of irrelevant/extraneous information when posing problems. Examination of Table 5 provides insights into the components of problems that participants attended to prior to and following the instruction.

The most evident characteristic of problems posed in pre- and post-instruction was attention to the use of contexts and situations involving food, fun scenarios (parties, concerts) and popular games and characters (e.g., Moshi Monsters and Match Attax Cards). It was almost without exception that problems posed were situated within meaningful contexts for children, thus indicating awareness of the importance of engaging children's interest in the problem-solving process. The vast majority of questions were situated within the number strand and to a lesser extent the measures strand. This was most likely due to a number of factors. First, participants had completed almost two mathematics education modules; both of which focused on number. In addition, in their school placement they were required to teach number. Second, participants' own experiences of problem solving

Table 5 Pre- and post-instruction problem features in response to the task 'Choose a class level (1st–4th grade). Make a maths problem that would be a problem for those children' (questionnaire item 2, $n=415$)

	Pre-instruction ($n=415$)	Post-instruction ($n=415$)
Curriculum strand		
Number	84% [349]	68% [284]
Measures	11% [44]	9% [36]
Geometry	0	2% [8]
Data and chance	0	0
Algebra	0	1% [4]
Number of steps		
One-step problems	68% [282]	40% [164]
Two- or more-step problems	29% [121]	29% [120]
Number of correct solutions		
One correct solution	99% [413]	87% [368]
More than one correct solution	<1% [2]	11% [47]
Problem type		
Arithmetic problems	96% [398]	74% [305]
Puzzle problems	3% [12]	24% [101]
Other	<1% [5]	2% [10]
Irrelevant/extraneous information	0% [0]	13% [53]

as students were related to traditional computational textbook problems which are situated within the number strand and provided practice in computation of whole numbers, decimals and fractions.

The majority of problems constructed prior to the instruction were fairly typical of the problems one would encounter in a mathematics textbook. They provided practice in arithmetic computation, generally involved a one-step solution process and had only one correct answer. Indicative examples are:

> George had 17 Moshi Monsters. He gave 9 of them to Joanne. How many did he have left? P2

> The Easter Bunny has 125 eggs. He gives 5 eggs to each child he visits. How many children will he be able to visit with 125 eggs? P61
> If John has 2/3 of a pizza and Jess has 1/6. How much is left for James? P39

The instruction brought about changes in the characteristics of the problems posed. Most notable are the decrease in one-step problems (28% decrease) and an increase in puzzle-type problems (21% increase) and those with more than one correct solution (11% increase). Indicative examples are:

> John has a farm. He drives a John Deere tractor. John has cows and hens on his farm. Overall there are 30 legs on his farm and there is at least 4 of each animal. Find out the possible number of cows and hens on the farm. P169

> The children in 1st Class are making Play-Doh caterpillars. They have four colours to choose from. How many different ways can they make a caterpillar that is four sections long? They can only use each colour once. P216

Increased awareness of the inclusion of extraneous information is evident in both analysis of the problem posed (see Table 5) and in the narratives provided (see Table 4) by participants. It is interesting to note that extraneous information was not addressed explicitly nor provided the status of a 'problem characteristic' during the instruction. Thus, the time or attention received by other problem characteristics such as problem type or multiple solutions (see element 6) was not afforded. Nonetheless, participants' attention to the incorporation of extraneous information in problems is evident in the problems posed following the instruction. Two examples of problems with extraneous information are:

> Jane is 9 years old and has blue eyes. On Saturday morning she goes cycling on her bike to the local park. When she gets there all of her friends are there too. They are all aged 9 or 10 and their younger brothers and sisters are with them riding their tricycles. Jane can count 18 wheels altogether. How many bicycles and tricycles are in the park? P285

> There are 7520 people at a One Direction concert. One quarter are parents who paid €45 for a ticket. The rest are children who paid €40 per ticket. How many children were at the concert? P74

Insights gleaned from examining problems alongside participant descriptions

Focusing on the problem characteristics and structures without concomitant attention to participant narratives surrounding the posed problems may lead to overgeneralizations regarding what the problems reveal about the effectiveness of the instruction. Consideration of the narrative participants construct around the problems they pose (as revealed in questionnaire items 1, 3–5) alongside analysis of the features of the posed problems provides additional contextual information regarding participants' developing understandings about problem posing. This has led to two caveats.

Caveat 1: not all that appears good is necessarily good

Similar to findings of Crespo (2003) and Crespo and Sinclair (2008), close investigation of the problems that incorporated desirable problem features, such as multiple solution paths or multiple points of entry, revealed that a proportion of these problems had no solution, an infinite number of possible solutions, or were too difficult for the intended audience. In these situations, it appears that effort was placed on the design features of the problems and not sufficient attention paid to the problem solution (see problems below). While P119 has multiple possible solution paths and has multiple entry points, there appears not to be a solution to the problem. P34 has similar strengths to P119 and supports the use of representation; however, there are an exceedingly large number of possible combinations thus requiring an inordinate amount of time for a third-grade student to solve. A problem also supporting the use of representations is that written by P35; however, unless the problem indicates the maximum number of slices into which the pizzas are divided, there are an infinite number of possible solutions. Other problems, such as those of P236, contain many desirable design features; however, while possible to solve, it is too complex for the intended students. Thus, some of the more desirable problems posed by participants contain errors and flaws within their mathematical structures.

> Mary has a certain number of sweets. The number is a multiple of 3, has a digit sum of 6 and is 2 less than a square number. How many sweets does Mary have? P119

> There are 72 people waiting for the train. The train has 6 carriages. How many different combinations of people will fit in each carriage? P34
> Mary has two pizzas. How many slices could both her friends John and Mary get if she had to share the pizza equally among the three of them? Display how you solved this using diagrams. P35
> I have 2 more sweets than Mary. Mary has 5 more sweets than John. We have 30 sweets altogether. How many sweets does John have? Who has the most amount of sweets? P236

Problem posing requires that the problem poser, in this case the PT, be able to solve their problem. The findings suggest little attention was given to ensuring the problem was solvable for its intended audience.

Two common misconceptions were revealed regarding the characteristics of mathematically worthwhile problems. What constitutes as extraneous information posed difficulties for some participants. It appeared that almost 20% ($n=10$) of the post-instruction problems that incorporated extraneous information confused the provision of context to a problem as analogous to extraneous information. This was manifested in analysis of participants' responses to questionnaire item 5 alongside their posed problems. In these cases, participants identified the incorporation of extraneous information as a desired design feature as 'it contains irrelevant information that is not necessary to work out the question' (P316). However, the information was not numerical and therefore was not extraneous to the mathematical features of the problem.

> Laura has been invited to her cousin's birthday party. The party will take place in Cork and there will be loads of food there. Her mother tells her that her cousin's age is divisible by 2 and next year her age will be a multiple of 5. What age is the cousin? P316

A second misconception was in relation to problem complexity. Many participants who constructed one-step problems in the pre-instruction moved to constructing two-step problems post-instruction. They reported these problems to be good problems as they were more complex than their initial one-step problem. These participants provided justifications such as 'this problem isn't as straightforward and requires a lot of thinking' (P217) and 'it not only tests addition but subtraction as well' (P281). However, rather than increasing the reasoning or higher order thinking required of the problem solver, these two-step problems merely involved application of one more operation or arithmetic procedure in order to reach a solution.

Caveat 2: not all that appears bad is irredeemable

A proportion of problems classified as one- or two-step problems required little more than recall of a number fact or arithmetic procedure. While these problems did not explicitly incorporate structures or components discussed during the instruction, consideration of participants' descriptions (questionnaire items 4, 5) indicated that the instruction did enhance their understanding of problem posing. This was evident in two ways: responses justifying the richness of an otherwise simple arithmetic problem and responses that acknowledge the limitations of the problem posed.

Arguing for the richness of the problems Some participants argued that their one-step problems were good problems. Common justifications referred to the potential for multiple solution paths, the potential for use of representations and the incorporation of extraneous information. P6 justified her problem as a good problem as 'it can be solved in many ways. A child could draw the balloons and cross out half of them. Or they could use counters and find half of them. Or they could use what they know from doubling to see double (it is 22). Or they could also use knowledge that half of 20 is 10 and half of 2 is 1 and add them together.' Similarly, the one-step problem requiring subtraction of two numbers developed by P15 may be considered to have low cognitive complexity. However, the participant explains 'this is a good problem because it has no clear-cut algorithm telling the children what to do but requires them to understand what is being asked, to sift the problem for relevant and irrelevant information and to create a strategy to arrive at the answer.' The inclusion of extraneous irrelevant information was also used to justify the quality of the problem posed by P223 in her statement that 'there is a lot going on in this problem. To get the correct solution a child needs to read carefully the information and extract what is relevant and what is not.'

> It's Mary's 8th birthday so her brother John decides to decorate the room. He blows up 22 balloons to hang up but half of them burst. John is 11 years old, how many balloons popped? P6

> Every day Katie Taylor runs for 12.38 km on the treadmill before training. However yesterday she was 15 minutes late for training and ran 6.53 km less that she usually does. How many km did Katie Taylor run on the treadmill yesterday? P15
> Bob went to the shop and bought 100 sweets and two bags of crisps. He gave 10 sweets to Mary, 40 sweets to Dan and a packet of crisps to Orla. What fraction of his sweets does Bob have left? P223

Acknowledging the limitations of the problem posed Other evidence to suggest that the instruction was effective comes from analysis of participants' commentary which revealed that many were aware of the limitations of their problems and provided suggestions for how to make their problem better. A number of participants (7%) who constructed simple arithmetic problems indicated that their problem could be improved. While these problems incorporated attention to interesting contexts and many incorporated extraneous information, many commented that there was only one correct answer and acknowledged this as a possible area for improvement. For example, P56 explains that while her problem 'is fairly good it could be better. It is a closed problem therefore it has only one correct answer.' P65 refers to the lack of cognitive complexity associated with her problem in her statement 'In fairness, it is not an exceptionally good problem. It is not particularly challenging because if the children have some idea of how to estimate, then they will be able to solve the problem with ease. But it is good to get them to estimate. I think this problem has both positives and negatives.' P231 states 'On the negative side, it is still relatively straightforward as the child only has to complete two addition sums to obtain the answer. Also, it does not require a lot of thinking outside the box. So there is room for improvement. On the positive side, it is not solved in just one step. The children are required to engage with the problem for two separate operations. They could also draw it out. It is relatable and interesting to them, therefore they will see maths relate to real life. The language is not too challenging.' This awareness shows that participants were able to apply a critical lens to their problems and draw from the experiences provided in the instruction when critiquing problems.

> Damien, Michael and Cathal went into a pizzeria. Damien wanted cheese, Michael wanted ham and Cathal wanted pineapple. If Damien took half the pizza, what fraction did Michael and Cathal get? P56

I have 67 bottles of shampoo and Pat has 16 bottles. Make an estimate of how many bottles we have altogether. P65

John has 12 moshi monsters. His best friend Alex has 8 more than him. If Alex's older brother Steven has 6 more than Alex, how many moshi monsters does Steven have? P231

Conclusions and recommendations

Prior to the instruction, participants held a variety of understandings about the nature of problem posing. They demonstrated awareness of the importance of incorporating meaningful contexts and the careful selection of mathematical language when posing problems. They were cognisant of the prior mathematical knowledge of the learner and possible strategies that they may use when solving the problems. All these considerations are critical when embarking on posing mathematical problems. Nonetheless, analysis of the problems posed prior to the instruction reveals the majority to be arithmetic in nature, requires one step to solve and has only one correct solution.

Similar to the findings of Crespo (2003), Chapman (2004) and Ellerton (2013), this study shows the potential of instruction in problem posing and solving to bring about improvements in the problem-posing skills of PTs. Post-instruction problems became more complex and extended beyond the predominantly arithmetic problems evidenced prior to the instruction. More puzzles were constructed, and greater attention was paid to opening up possibilities for multiple correct solutions and problem paths. Even in the cases of those participants who continued to construct traditional arithmetic style problems, there was evidence of growth. More specifically, these participants showed gains similar to those found by Chapman (2004) in the ability to critique and discern the mathematical features of word problems.

However, the shortcomings of initial problems posed considered alongside the limitations of a large proportion of the post-instruction problems support the findings of studies by Chapman (2005, 2012) and Crespo and Sinclair (2008) that problem posing is a multifaceted and complicated task. Furthermore, analysis of the problems posed throughout the study supports Ellerton's (2013) contention that the enculturation process is pervasive in ITE. Even following the instruction, where PTs were exposed to a wide variety of problems, the powerful influence of traditional school problems could be seen in the problems posed. Within the more traditional one and two-step problems, the situations posed in the main part were limited predominantly to sharing items (sweets and fruit) between friends. Less traditional problems with multiple possible solutions generally involved identifying different combinations of animals (by counting legs) or vehicles (by counting wheels) that could be constructed. Even at the most fundamental level of naming characters in the problems, post-instruction problems named 55 Mary's and 64 John's. This finding suggests strongly the need to expose PTs to a large and varied assortment of problems in order to broaden both their experiences and repertoire of problems, thus echoing a recommendation of Singer et al. (2013).

This study contributes to our understandings of the problem-posing abilities of PTs. It provides evidence that problem-posing initiatives can work even within the short confines of a 3-week instructional unit. Participants' broadening understanding of what constitutes a 'good' problem, considered alongside the improvements in the post-instruction problems posed, suggests that a large proportion of participants were able to transfer their new knowledge into practice. Moreover, for those participants who did not build the bridge between their newfound understandings and the problems they posed, their narratives revealed awareness of the shortcomings of their own problems. It is this latter group of participants that are of interest for future studies. It might be argued that we expect too much of prospective elementary teachers in having them construct entirely new problems. In a knowledge economy where there is growing access to resources, a recommendation of this study is that perhaps we should consider investing our energies in supporting teachers in developing skills in identifying more mathematically worthwhile problems from a large selection of problems or in reformulating given problems to make them better. This may provide support and serve as an initial step, or crutch, on route to posing entirely new problems. Another way that this study contributes to our understanding is by revealing the shortcomings of analyzing problems alone without accompanying participant descriptions or narratives. This may, in some cases, underestimate the effectiveness of the instruction in developing understandings of problem posing.

While this study focuses entirely on the characteristics of the problems posed, there may appear a tacit assumption that these problems will be received well in any classroom. This is overly simplistic as there are a multitude of other factors that will influence the interactions between a learner and a problem including the learner characteristics, the didactical contract in place in the classroom (cf. Brousseau's theory of didactic situation) and the teacher's ability to operate relevant variation in the conception of problems (Sarrazy 2002), to name but a few. Thus, a limitation of this study is that the problems posed are examined devoid of any consideration of the classroom situation. As mentioned, we have not observed participants as they implement problems in their classrooms. Studies by Stigler and Hiebert (1999) and Stein et al. (1996) suggest that teacher enactment of tasks in classrooms frequently reduces the opportunities to learn. Within this study, there is the possibility that more complex problems that appear to lend themselves to higher order reasoning may not be enacted in a classroom in a way that allows the potentialities to be reached. However, evaluation of participants' responses also suggests that the converse is also possible. Our analysis revealed that while the surface features of some problems may indicate that a one-step procedure is sufficient to come to a solution, the teacher intentions around supporting children when solving the problem (such as encouraging the use of multiple representations and focusing on multiple strategies) may provide opportunities to enhance the learning outcomes for children from engaging with what otherwise appear to be relatively simple problems. Another limitation is that we do not know whether the instruction effect lessens over time. The posttest happened immediately after the study conclusion, and we did not follow up again at a later date.

We recommend that studies of PTs implementing their problems in classrooms would be beneficial in terms of identifying the classroom factors that support mathematical problem solving. Furthermore, consideration and development of the letter-writing exchanges between PTs and classroom pupils in studies by Crespo (2003) and Norton and Kastberg (2012) could harness the benefits arising from linking with learners and classrooms at the problem design phase.

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