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Developing mathematical knowledge for teaching teachers: potentials of history of mathematics in teacher educator training

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Abstract
What mathematical knowledge is required for teaching has been researched by many, with Ball et al. (J Teach Educ 59(5):389–407, 2008) and their practice-based theory of mathematical knowledge for teaching (MKT) primary among them. However, what is required in terms of mathematical knowledge for teacher educator training has been researched much less. The present study builds on an emerging framework on mathematical knowledge for teaching teachers (MKTT) by Zopf (Mathematical knowledge for teaching teachers: the mathematical work of and knowledge entailed by teacher education, University of Michigan, Ann Arbor, 2010) augmented with more recent developments in the field. The empirical basis for the study stems from an implementation of a short course on history in mathematics education at the Danish School of Education, and in particular on two case studies involving three teacher educator students participating in that course. Based on analyses of their developments in terms of MKTT, we point to the potential of using history in both teacher and teacher educator training, and in particular historical source material. As part of the analyses, we also consider the teacher educator students' development of disciplinary knowledge of mathematics, especially in relation to developing and making use of knowledge of the epistemology of mathematics and mathematical work. We assert that the cases we present here address Zopf's call for case studies from different teacher education contexts as well as investigations of novice teacher educators, and further contribute to the developing theory of MKTT.

Keywords Mathematical knowledge for teaching · Mathematical knowledge for teaching teachers · Teacher educator students · History of mathematics · Primary historical sources

Introduction

Numerous studies have been conducted on the mathematical knowledge needed for teaching mathematics (for an overview, see Hoover et al. 2016), but there has been much less interest in the knowledge needed for *teaching* MKT and how to facilitate the development

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of such knowledge. In her study of two expert teacher educators, Zopf (2010) made some initial efforts to develop a theoretical framework for mathematical knowledge for teaching teachers, labeled MKTT. Toward the end of her study, Zopf called for further (empirical) case studies from different teacher education contexts as well as investigations of novice teacher educators. The present study is a response to her call in both respects. Kim (2013) suggested a further development of the MKTT framework to emphasize two interrelated entities: mathematical work of teaching and knowledge *about* mathematics. The latter concerns knowledge about mathematics as a discipline, where it originates from, how it comes into being, changes over time, establishes truth, etc. From our perspective, the history of mathematics lends itself here. Within the research literature on history *in* mathematics education—the so-called HPM literature[1]—it is well known that history has a role to play in relation to developing MKT (Jankvist et al. 2015; Mosvold et al. 2014; Smestad et al. 2014).[2] Still, it remains an open question what role the history of mathematics might play in relation to developing MKTT, and given the potential that history of mathematics has for the development of MKT, it seemed a natural progression to examine ways in which history of mathematics contributes to teacher educator students' development of MKTT. Hence, our research question is: In what respects might the use of history of mathematics and educational literature about the use of history of mathematics in mathematics education (i.e., the so-called HPM literature) contribute to teacher educator students' development of mathematical knowledge for teaching teachers (MKTT)?

In this article we provide an "answer" by example to this question, by describing the concrete design and implementation of a short course[3] on history in mathematics education for future mathematics teacher educators. The course was structured around readings of HPM literature as well as the use of a (historical) "case," oftentimes one involving reading and use of historical primary sources, upon which to apply the HPM literature. Although our main objective of the present article is one of MKTT, we hope that the answer by example may also shed some light on the still open HPM-related questions of how to introduce future mathematics teacher educators to the discussion of history in mathematics education, and how to prepare them to use history of mathematics in their future practice (e.g., Artzt et al. 2012; Waldeana and Abraham 2014). In fact, insights from the areas of HPM and MKTT do seem to coincide in some respects. For example, within HPM research it is well known that using history of mathematics in the teaching of mathematics also requires a sound disciplinary knowledge of mathematics (Fried 2001). In their extension of Zopf's work, Castro Superfine and Li (2014) analyzed how mathematics teacher educators use knowledge in their practice and found that a solid disciplinary knowledge of mathematics was needed on the teacher educators' behalf. Indeed, this suggests a fruitful interplay between the two.

Thus, the main objective of the present article is development of teacher educator students' MKTT, and we assert that using history in mathematics education may be seen as a possible way of doing this. Still, in our forthcoming answer to the above research question,

[1] HPM refers to the *International Study Group on the Relations between the History and Pedagogy of Mathematics*, which is affiliated with the *International Commission on Mathematical Instruction* (ICMI), and concerns the role history of mathematics *in* mathematics education.

[2] Other studies of history of mathematics in teacher education include: Charalambous et al. (2009), Furinghetti (1997, 2007), Huntley and Flores (2010), and Wang et al. (2018).

[3] The course described in this paper was previously analyzed from the point of view of developing teacher competencies (Niss and Højgaard 2011). This analysis is available in Jankvist (2015). An initial discussion of the potential in relation to MKT and MKTT can be found in Jankvist et al. (2016).

we highlight special features that a use of history in mathematics education possesses and offers and which, as far as we can tell, provide for sound potentials for developing MKTT, both as part of the short course and in the teacher educators' future practice.

Theoretical background

The practice-based theory of MKT, as presented by Ball et al. (2008), is frequently cited in the field, but many seem to regard this theory as limited to a decomposition of knowledge into the categories of the so-called "egg model." To avoid an overly simplified understanding, we find it relevant to emphasize two particular aspects of the theory. First, the practice-based theory of MKT highlights a commitment to the discipline of mathematics in content knowledge for teaching (cf. Shulman 1986). Knowledge of content thus entails both substantive and syntactic structures of the discipline (Schwab 1978). Substantive structures relate to the content and organization of knowledge in the disciplines, whereas syntactic structures are ways of establishing truth and validity. Second, teaching is at the core, and teaching is often referred to as "work." The work of teaching can be decomposed into tasks of teaching, and MKT is defined as the mathematical entailments of these tasks of teaching (Ball et al. 2008). Examples of mathematical tasks of teaching are: presenting mathematical ideas, selecting examples in order to highlight a specific mathematical point, and dealing with representations (Ball et al. 2008).

Inspired by Ball and colleagues, Zopf (2010) investigated the work of teaching MKT. Where Ball and colleagues (2008) decomposed the work of teaching mathematics into several tasks of teaching, Zopf decomposed the work of teaching MKT into some core tasks of teaching (see "Method and data collection" section for an elaboration). Carrying out these tasks of teaching requires MKTT, which includes a specialized knowledge of MKT, as well as a solid knowledge of the discipline of mathematics. This "disciplinary knowledge" of mathematics includes robust epistemological knowledge of mathematics as well as "knowledge about mathematical structures such as definitions, properties, theorems, and lemmas and how these are used to do mathematics; knowledge about descriptions, explanations, justifications, and proof and how these are used for mathematical work" (Zopf, p. 199). Furthermore, Zopf described MKTT as panoramic, connected, fluent, and deliberate. The panoramic quality refers to how the work of teaching MKT demands knowledge of the broader landscape of mathematics. This includes knowledge of how fundamental mathematical concepts are developed and how they are connected with more advanced mathematics. MKTT is also described as connected knowledge, in that it entails ability to make connections within and across different mathematical domains. The third characteristic is fluency, which refers to how mathematics teacher educators must be able to navigate between panoramic and detailed mathematical knowledge without much effort. Finally, the mathematical work of teaching teachers is described as deliberate and intentional work, and this refers to how the entire process from planning through enacting the work is carried out with particular instructional goals in mind.

The MKTT framework shares the foundational understanding of teaching and learning with the MKT framework, and both frameworks consider teaching as instructional interactions of teacher and students around some content. However, the frameworks differ in at least three important ways: in terms of content, learners, and purpose. First, whereas mathematics is the content that is taught to children in school, mathematical knowledge for teaching is the content taught to prospective teachers in teacher education. Second, the learners are different—children (in the case of MKT) as opposed to adults (in the case of MKTT)—and each differ in terms of their previous knowledge of mathematics. Third, the purpose differs, in that children are taught mathematics for their own use, whereas prospective teachers are taught MKT in order to teach others (Zopf 2010).

Since Zopf (2010) presented her framework and called for further studies of MKTT, certain contributions have been made. As previously described, Castro Superfine and Li (2014) extended Zopf's work when they analyzed the knowledge mathematics teacher educators *use* in their practice. From analysis of what mathematics teacher educators were doing and saying, and from considering the purpose behind this, they identified three forms of knowledge that they argue differ from the MKT of K-12 teachers. These include knowledge of (1) "connecting student errors to instructional moves," (2) "connecting algorithms to the K-6 curriculum," and (3) "connecting research to mathematics content learning" (p. 309). Whereas Castro Superfine and Li responded to Zopf's call by investigating the knowledge demands of teaching preservice teachers, Masingila et al. (2017) investigated the development of MKTT among novice teacher educators. Analyzing their own work of teaching preservice teachers via problem solving, Masingila and colleagues argued that they developed MKTT through three parts of the work. First, determining mathematical learning goals prompted their own development of MKTT. Second, their developing knowledge was elicited in the process of selecting and using tasks for problem solving. Third, they discovered that the work of asking questions to facilitate preservice teachers' learning from problem solving required a deep and connected understanding from themselves as teacher educators.

Yet another extension of Zopf's (2010) work was made by Kim (2013). Her work was similar to the study of Castro Superfine and Li (2014) in that she also investigated the demands of teaching MKT, and, like Zopf, Kim identified tasks of teaching MKT. From her analysis, Kim suggested a further development of the MKTT framework that emphasized two interrelated entities: the mathematical work of teaching and knowledge *about* mathematics. The latter coincides with Zopf's concept of disciplinary knowledge of mathematics, and it is particularly interesting for our study since it "is the nature of knowledge in the discipline, such as where it comes from, how it changes, and how truth is established" (Kim, p. 12). In our reading, this points to the history of mathematics. Another contribution of Kim's study was in discussing how to organize a curriculum for teaching MKT. In the present study, we build upon these previous efforts and extend them in a couple of ways. We draw upon the ideas about the disciplinary knowledge of mathematics, and we investigate how history of mathematics can contribute to future mathematics teacher educators' development of such knowledge. Where Kim discussed the organization of a curriculum for teaching MKT, we take this one step further and discuss a curriculum for teaching MKTT, and we discuss how future mathematics teacher educators can be prepared for using history of mathematics in their future practice.

Educational context and setting

To become a mathematics teacher educator of primary and lower secondary teachers in Denmark, a master's degree in mathematics education is often favored,[4] of which the Danish School of Education at Aarhus University is the only provider in the country. To enter

[4] Alternatively, the educators may hold a master's degree in mathematics. Note that teacher educators in Denmark are not required to possess a PhD.

the master's program the student must already have a university bachelor's degree, typically in mathematics, or a vocational bachelor's degree, as a primary and lower secondary mathematics teacher. The 2-year master's program consists of courses in mathematics, courses in general didactics, a course in didactics of mathematics, and finally a master's thesis.

The course of interest here is *Didactics of Mathematics*, as implemented from 2014 to 2017. In this course, each of the mathematics educators within the department have the opportunity to teach a mathematics education topic of their choice. One of the ideas behind this is that students are also confronted with recent research, in which the mathematics educators themselves are involved. The course comprises 15 ECTS (European Credit Transfer System), and is made up of four short courses addressing separate topics and each consisting of six sessions of 2–3 h of instruction and supervision.[5] For each of the topics, the students must submit a mini-project report. Oftentimes students work in groups, but occasionally they may also do the mini-projects individually. Based on a random selection, at the end of the course the student groups are examined in one of the four mathematics education topics.

Design of the short course on history in mathematics education

For each session the teacher educator students read a collection of texts (primarily research papers from the field of HPM). Additionally, supplementary literature was provided. As mentioned in the introduction, the students have to decide on a "case" with which they work throughout the short course and to apply the HPM literature. The students' previous experiences with history of mathematics vary from a superficial exposure to undergraduate course work.

In session 1, students familiarize themselves with different arguments for and against the use of history (and epistemology) in mathematics education, potential dilemmas, and different approaches to including history. The purpose of the three assigned texts for this session was to enable students to more qualifiedly discuss concrete uses of history at different educational levels including teacher training. The first text was by Fried (2001), who discussed the potential dilemma of incorporating non-curriculum mathematics into the classroom through a use of history of mathematics, and in case of accommodating the history to modern day curriculum the risk of anachronism, referred to as *Whig history*. The second text was by Jankvist (2009), who categorized the different *whys* and *hows* of resorting to history in mathematics education. In particular, Jankvist distinguished between a use of *history-as-a-goal*, i.e., to see it as a goal in itself to teach students *about* mathematics and more meta-perspective issues of mathematics (*meta-issues*) in relation to its coming into being, its development, etc., and a use of *history-as-a-tool*, i.e., to use the history for cognitive, motivational, or pedagogical purposes with the focus on students learning about inner issues (*in-issues*) of the discipline, e.g., concepts, proofs, theories, and so forth. As for the hows, Jankvist identified three: an illumination approach, a modules approach, and history-based approaches (in which mathematical concepts and results are presented in the order of their historical appearance). Finally, the students read part of the report

[5] In 2016 the three other short courses were: "The Relevance Problem of Mathematics Education" taught by Tomas Højgaard; "Mathematics for All" by Lena Lindenskov; and "Technology in Mathematics Education" by guest professors Morten Blomhøj and Morten Misfeldt.

on the Danish mathematics competencies framework (Niss and Højgaard 2011). Besides listing eight mathematical competencies, this framework also operates with three types of so-called overview and judgment, a type of second-order competencies, one of which concerns the "historical development of mathematics, both internally and from a social point of view" (p. 75) and one which is aligned with a use of history as a goal.

Session 2 focused on the role and use of theoretical frameworks in empirical studies related to a use of history in mathematics education. The students read two studies (Jankvist 2011; Kjeldsen and Blomhøj 2012), and they discussed the different use of Sfard's (2008) framework of commognition within these studies. Jankvist (2011) reported on a use of history of mathematics in Danish upper secondary school, the purpose of which was to find out to which extent students were able to have meta-issue discussions that were based upon and *anchored* in the students' knowledge of related mathematical in-issues. The study made use of Sfard's (2008) framework to distinguish between meta-issue discourse and in-issue discourses in the students' discussions and potential commognitive conflicts between the two. Kjeldsen and Blomhøj (2012) presented a study of university mathematics students' encounters with historical primary source material, and their experience and noticing of a different historical discourse from that which they were used to in their contemporary mathematics studies. In particular, the study highlights Sfard's notion of meta-discursive rules of mathematics. These are rules *about* the discourse of mathematics that are "historically given; they are not necessary but contingent" (Kjeldsen and Blomhøj, p. 329). "Meta-discursive rules affect how participants in the discourse interpret the content of the discourse. Therefore, developing proper meta-discursive rules is indispensable for learning mathematics" (ibid., p. 330). The point of Kjeldsen and Blomhøj is that historical sources offer an opportunity to become aware of the existence of meta-discursive rules in mathematics.[6]

Next, session 3 addressed more explicitly the use of original sources in the teaching and learning of mathematics as well as different approaches to including such sources (e.g., Barnett et al. 2014; Jankvist 2013). Here again, one purpose was to prompt students to argue for and against a potential use of original sources in a particular educational setting. The text by Barnett et al. reported on 25 years of teaching with original source material in undergraduate education in the USA and presents the idea of providing *guided readings* of such sources to the students, where the reading of the source is "paused" in order to introduce comments, explanatory exercises, and questions, with the purpose of enhancing students' understanding. Also, Barnett et al. referred to and built on the texts of Jankvist (2009) and Kjeldsen and Blomhøj (2012), thereby providing connections for the course's teacher educator students. Jankvist (2013) introduced the idea of guided reading in the Danish upper secondary school in relation to teaching modules that simultaneously focus on the history, application, and philosophy of mathematics.[7]

The topic of session 4 was history in mathematics teacher education and teachers' potential benefits of being introduced to elements of the history of mathematics. Drawing on the topics of the previous lessons, the students compared one of the first empirical studies in the field, by Arcavi et al. (1982)—who among others argue that the "history of mathematics is neither useful or useless, relevant or irrelevant to anything or something: it all depends on what you mean by the term, and how you put it into practice" (p. 34)—with

[6] As supplemental literature for this lesson, students were encouraged to examine Jankvist and Kjeldsen (2011).

[7] Supplementary literature for session 3 included Glaubitz (2011) and Jankvist (2014).

a more recent study by Clark (2012), who argued that prospective mathematics teachers' engagement with history of mathematics has the potential to contribute to their MKT, particularly when using purposeful historical examples relevant to topics the prospective teachers will soon teach. Mosvold et al. (2014) argued for the potential benefits of using history of mathematics in teacher training in relation to each of the six subcategories of MKT. Finally, the students were briefly introduced to the discussion of using philosophy of mathematics in mathematics education through a text by Jankvist and Iversen (2014).[8]

Session 5 was designed as a workshop in which the students were to further relate the course texts to each other as well as to their chosen "case." This work eventually resulted in a submitted mini-project report (approximately 12 pages, of 2400 characters each, plus appendices) for each student group. These reports were then presented and discussed during session 6, where each student group also read the report of another group in order to provide constructive feedback and to also receive feedback themselves. (In Spring of 2015 and 2016, the second author was present at the course in sessions 4 and 5.)

Method and data collection

From approximately thirty mini-projects that have been produced thus far (see Jankvist et al. 2016 for a list), we have chosen two for in-depth analysis in relation to developing MKTT. The data of our study consist of these two submitted student mini-project reports, appendices to these (e.g., concept maps), material related to the students' presentations at the final oral examination, and interviews with the students conducted during sessions 4 and 5 of the course in Spring 2016, and follow-up interviews conducted in Fall 2017 and Spring 2018.

In the forthcoming sections we first provide accounts of the two cases and how these relied on the course literature. We augment these descriptions with interview data and descriptions of what took place after the completion of the course—and also the completion of the master's program for the students. We next analyze the two cases from the perspective of what insights these three students gained for their further careers as teacher educators. Our analyses are informed by Zopf's (2010) three suggested core tasks of teaching and associated subtasks:

1. Selecting interpretations and representations used for teaching mathematical knowledge for teaching
    - *Unpacking* details compressed with mathematical concepts and procedures
    - *Surveying* interpretations and representations that are appropriate for teaching particular concepts in ways that unpack these details
2. Selecting examples used for teaching mathematical knowledge for teaching
    - *Sequencing* identified examples (to unpack the mathematics layer by layer)
3. Managing the enactment of mathematical tasks used for teaching mathematical knowledge for teaching

[8] Supplementary literature for session 4 included Smestad et al. (2014).

- *Launching*: reviewing mathematics, challenging mathematical knowledge, modeling ways of mathematical work, or presenting new mathematical ideas
- *Developing mathematical ideas*: anticipating the incoming knowledge of the students and targeting the mathematical knowledge for teaching
- *Engaging* teachers in conversations about mathematics. (Zopf 2010, Chapter 6)

MKTT is knowledge required to carry out the tasks of teaching MKT (Zopf 2010). An underlying assumption of our analysis is thus that situations involving productive engagement with core tasks of teaching MKT also represent opportunities to develop MKTT. Upon completing our analyses, in the sections that follow we link findings to the calls for teacher educators' possession of disciplinary knowledge and knowledge about mathematics by Castro Superfine and Li (2014) and Kim (2013), respectively. It should be mentioned that when we display data from the mini-project reports, appendices, and examination materials, then these have been translated into English by us.

Case 1: Use of an original source excerpt on equation solving

The first case considers the teacher educator student *Mirabelle* (all student names are pseudonyms) who addressed the potential use of an excerpt from Viète (1646) concerning theory of equations both for pupils and for student teachers in particular with respect to provoking commognitive conflicts regarding different meta-discursive rules and with respect to developing teachers' MKT. Mirabelle asked how different approaches to using excerpts from Viète's theory of equations could potentially support student teachers in working with different in-issue and meta-issue goals. In relation to in-issue goals, Mirabelle wrote:

> As a teacher, one could be inspired by Kjeldsen and Blomhøj (2012) and work in a problem-oriented fashion with excerpts from Viète's theory of equations. Students might address the question: "Which opportunities and challenges did Viète's form of notation give rise to and why may it be relevant to work with his form of notation even today?" [… The source] could function as literature in a project with the purpose of supporting students' learning of meta-discursive rules in relation to in-issue mathematical aims. One might include digital technologies to support expressing the algebraic expressions geometrically, related to Viète's form of notation in the original source. This might give rise to discussing which potentials and challenges there are in using CAS [Computer Algebra Systems] for equation solving. (Mirabelle 2016)

At the final oral examination, Mirabelle provided an analysis of meta-discursive rules in the excerpt from Viète versus modern day meta-discursive rules (Fig. 1). In particular, she pointed to Viète's use of vowels (here, *A* and *E*) to indicate known quantities and consonants (e.g., *B* and *D*) to indicate unknown quantities. The context of Viète's example is that of two right-angled triangles with equal sized hypotenuses *B* of length 10, which give rise to solving two third-degree equations. Mirabelle also provided a representation of Viète's problem in the dynamic geometry software (DGS) of *GeoGebra* (Fig. 2). It is important to note that technology in mathematics education was one of the other topics in the *Didactics of Mathematics* course, and in particular the positive and negative effects of CAS (e.g., Jankvist and Misfeldt 2015) were debated, which may in part explain why Mirabelle drew these connections. Nevertheless, Mirabelle was the only student who actively linked the discussions of history in mathematics education to those of technology in mathematics education in her mini-project report.

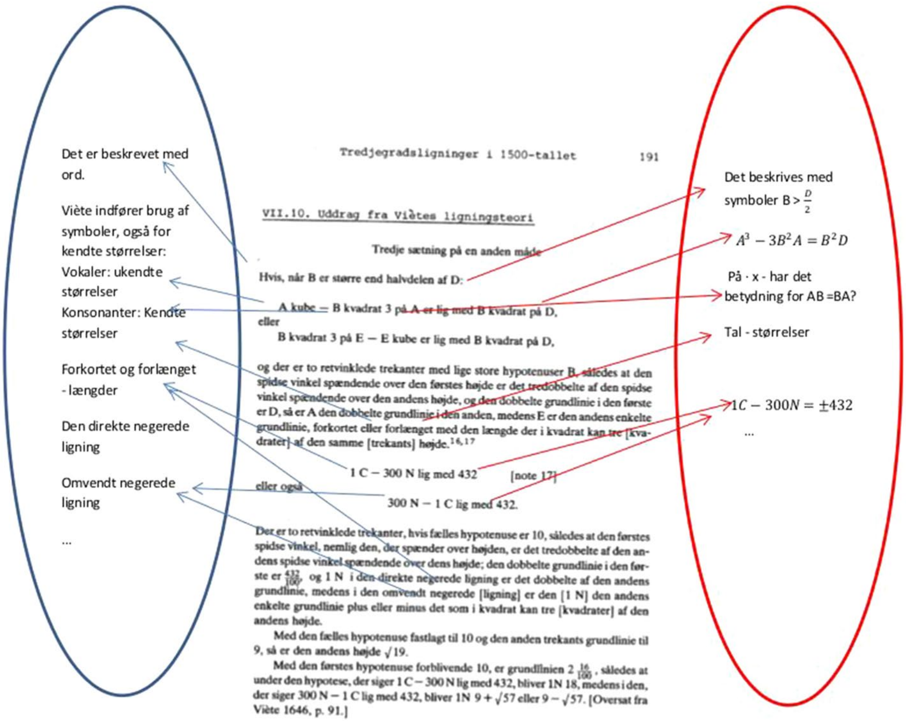

Fig. 1 Mirabelle's analysis of meta-discursive rules of Viète (left) compared to modern day (right) based on an excerpt from a published translation of Viète into Danish (Andersen et al. 1986, p. 191). The left oval states: "It is described with words."; "Viète introduces the use of symbols, also for known quantities. Vowels: Unknown quantities. Consonants: Known quantities."; "Reduced and expanded - lengths."; "The direct negated equation."; "Reverse negated equation." "…" The right oval states: "It is described with symbols $B > \dots$"; "$A^3 \dots$"; "On $\cdot$ x - does this have a consequence for $AB = BA$?"; "Numbers - quantities."; "$1C - \dots$"; "…". See "Appendix" for an English translation of Viète's text

Mirabelle also provided an analysis of the potential of working with the excerpt of Viète from an MKT perspective in teachers' training. She pointed out that some student teachers might benefit from being guided through Viète's example of third-degree equations, while others might prefer to work through it without initial help: "When you consider the relationship between content and students (and you don't know the students) it must be central to work with the possibilities of differentiating the teaching in relation to the source […]" (Mirabelle 2016, appendix). She further pointed to the possibilities of investigating the connections between third-degree equations and right-angled triangles; mathematical domains which today are not automatically associated: "One might get a sense of which mathematical domains the case [Viète's problem] spans, e.g. trigonometry, the Pythagorean theorem, irreducible equations, substitution, and partition of angles. *Meta-level rules could be brought into play here*" (ibid., italics in original). Mirabelle further stressed that focusing on Viète's form of notation can contribute to developing specialized content knowledge, not least in terms of bridge-building to the work with digital technologies and meta-issues of how mathematics develops over time: "The teacher's knowledge of Viète working with a Euclidean approach as well as an algebraic approach in his very form of

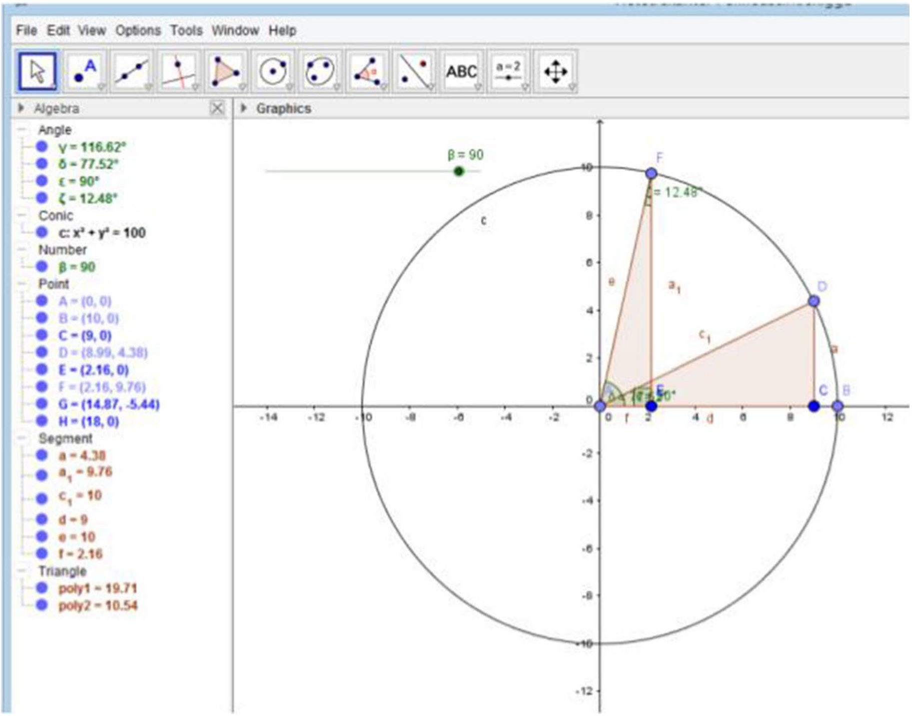

Fig. 2 Mirabelle's representation of Viète's problem in *GeoGebra*. Notice that the first coordinate of *D* should have been 9. Hence, there is an inaccuracy in Mirabelle's *GeoGebra* drawing. Still, we argue that this did not compromise the overall use of DGS to unpack the mathematical content of the original source, i.e., in this case to link the algebra with the geometry

notation may act as a specialized knowledge of content that the teacher may bring into play in relation to supporting students' reflections about their work with CAS" (ibid.).

Mirabelle went on to write her master's thesis (together with another student) on the use of original source excerpts in combination with DGS. They investigated the potential of having fourth-grade pupils read Euclid's first theorem on the construction of an equilateral triangle and discuss this construction in relation to constructions in *GeoGebra* and having them engage in discussion of what it means "to prove" something. As a basis for this work, Mirabelle and her course colleague considered the original source excerpt and the DGS as two different discourses (Sfard 2008), providing a potential for commognitive conflicts to occur, and for meta-discursive rules within these two discourses to be observed and articulated. The study of Mirabelle and her colleague indicated that it is possible for pupils at this level to have meaningful meta-issue discussions about mathematics (e.g., paper-and-pencil vs. DGS) and also learn some mathematical in-issues along the way (not only related to equilateral triangles, but also to the notion and purpose of proof in mathematics). At the time of the interview for this study, Mirabelle worked as a teacher educator of both pre- and in-service teachers at a Danish university college and she also implemented courses in schools.

Analysis and discussion of case 1

For Mirabelle, we found evidence of her engagement in all three of Zopf's (2010) core tasks of teaching MKT.

Selecting interpretations and representations In Mirabelle's mini-project, she asked the question of how different approaches to using excerpts from Viète's theory of equations could potentially support teacher students. Thus, inherent in the question guiding her work, Mirabelle engaged with the first core task identified by Zopf (2010). Mirabelle identified the details comprising the particular example from Viète, and stated that these included trigonometry, the Pythagorean theorem, irreducible equations, substitution, and partitioning (e.g., trisecting) angles. She selected the particular representation of Viète's treatment of third-degree equations (and the accompanying notation) and, by doing so, she was able to unpack the mathematical details necessary for its inclusion in her future work with teacher students. Furthermore, Mirabelle referenced the connections between language, mathematics, and technology in several instances. For her, a clear task of the work as a developing mathematics teacher educator was to identify the potential for different discourses—as a way to incorporate different representations and interpretations—while making sense of mathematics for teaching. Thus, when asked what her future teaching of pre- and in-service teachers would entail, Mirabelle claimed:

> Firstly, I think that I would bring in Sfard and the notion of discourse and work with the different discourses in play. Secondly, to work with digital technology in order to provide mathematical understanding. The connection to the source makes good sense here, e.g. to provide some conceptual understanding. Thirdly, the connections between the drawing, the equations and his language. Finally, the historical illustration approach could be applied. (Interview March 2018)

Beyond the creation of the mini-project, Mirabelle (along with her fellow student) wrote her master's thesis on combining original source excerpts and DGS, for the purpose of engaging fourth-grade pupils with Euclid's construction of an equilateral triangle and the comparable experience in *GeoGebra.* There were two aspects of the curricular intervention designed by Mirabelle and her thesis partner which provide further evidence of ways in which working with original sources contributed to her ability to survey alternate interpretations and representations of mathematical content (and resources) and make use of them in her plans for mathematical instruction (although this was intended for instruction of fourth-grade pupils). When asked to reflect on the connections between her initial mini-project work and her master's thesis, Mirabelle recalled that:

> We wanted to see if students in primary and lower secondary school can actually work with original sources. And they actually could. We chose a Euclid source about equilateral [triangles], because we knew that they were familiar with this concept. They could then use this to understand the original source and they could work within *GeoGebra*. This was a little different because we addressed the topic of 'hidden mathematics'. We wanted them to develop some understanding of how much mathematics is actually hidden, when they work with *GeoGebra*. (Interview March 2018)

The course thereby enabled Mirabelle to productively engage in selecting interpretations and representations—a core task of teaching MKT—prompted by new original source material (Euclid in this instance, not Viète). In her considerations of the mathematical tasks required to implement the primary source excerpt and DGS, Mirabelle searched for and identified a variety of representations "appropriate for teaching concepts (…) in ways that unpack mathematical details" (Zopf 2010, p. 168). For example, Mirabelle determined an original source from which to select excerpts, the constructions pupils would complete, and the details needed to perform in *GeoGebra.* Furthermore, the use of DGS was an important component of revealing the "hidden mathematics" and making the original source more accessible to pupils.

Selecting examples There were several "layers of mathematical detail" (Zopf 2010, p. 173) within Mirabelle's selected mini-project topic for teaching teacher students. Consequently, we identified evidence of "careful sequencing" of examples for which Mirabelle approached the topic of solving third-degree equations using original source material. For example, Mirabelle needed to select a sequence of examples to guide teacher students through Viète's treatment of the topic; such examples would need to account for this attention to the Euclidean and algebraic approaches. Furthermore, evidence of Mirabelle's sequencing appears in her proposal for digital technology "…to support expressing the algebraic expressions geometrically, related to Viète's form of notation in the original source" (Mirabelle 2016, p. 19). That is, in her plans to address the different layers of mathematical detail in play when teachers plan for instruction on the integration and relationship between algebraic and geometric representations, Mirabelle needed to carefully consider how to sequence such a trajectory for her teacher students. This is further highlighted by Mirabelle's focus on the "connections between third-degree equations and right-angled triangles"—noting that these two mathematical domains are not typically associated.

Finally, when we asked Mirabelle about why she selected the particular Viète source for her proposed instruction for teaching teacher students, she stated:

> At first, I was absorbed by [Viète's] history and background. … But then I also found that the source, mathematically speaking, was right for me [not too easy and not too difficult, she later stated]. I was fascinated by his story. Second, the source was accessible. And third, the potential coupling to CAS. … Initially, I thought that the source would make sense in relation to CAS. *GeoGebra* came in later when I began to work with the source—when I had to understand it. I used *GeoGebra* to try to understand what Viète actually said. (Interview March 2018)

Thus, Mirabelle's engagement with the task of selecting particular examples and approaches was at first motivated by the demands of the mathematical content, and were consequently mediated via her own learning, which was made possible through working simultaneously with a historical source and *GeoGebra*. The case of Mirabelle thus illustrates how the course enabled productive engagement with the second core task of teaching MKT.

Managing enactment of mathematical tasks In her mini-project Mirabelle participated in the sub-task of engaging teachers in conversations about mathematics, within the third core task (Zopf 2010). Although we only have evidence from what Mirabelle proposed in her mini-project in this regard, we take her plans for the activity of her course mini-project as evidence of her reflections about this core task of teaching MKT. For example, in her proposal for combining the use CAS and an original source excerpt within an activity for teacher students in which algebraic expressions are expressed geometrically, Mirabelle conjectured that “[t]his might give rise to discussing which potentials and challenges there are in using CAS for equation solving” and that “it could provide for an opportunity to discuss meta-discursive rules” (Mirabelle 2016, p. 19). In this way Mirabelle expressed, in an activity that may be used in future teacher training, plans for creating opportunities to engage teachers in conversations about mathematical tasks.

However, planning for and implementing examples that draw upon both a historical source (from Euclid, in Mirabelle’s subsequent master’s thesis work) and *GeoGebra*, provided Mirabelle with an opportunity to reflect on her work with future teachers. When asked what teachers would need to know to enact and manage this particular task, she shared that:

> … [when] we designed the intervention we deliberately formulated in-issue and meta-issue learning goals for the pupils. What are the in-issues? What are the meta-issues? What do we want to achieve with this? Pre-service teachers can also do such reflections. … We created a ‘discourse poster’ to illustrate it to the pupils. So, how to scaffold such work [sources and technology] would be an important aspect to discuss with teacher students. What are the means that enable pupils to work with something that is actually relatively difficult to grasp? I mean, there are many paths to take. We of course didn’t just give them Euclid and *GeoGebra* and say: ‘go figure out’. That would have been impossible. We had many small steps on the way. (Interview March 2018)

Case 2: Teaching the concept of function through a use of history

In our second case, *Rose* and *Sibyl* analyzed how a chapter in a textbook for mathematics teacher education used history in relation to the concept of function, and they designed a teaching activity for student teachers based on historical definitions of the concept of function by Euler (1748) and Dirichlet (1837). They asked: “How does history enter into the textbook by Schou et al. (2013) with respect to the concept of function? And how may an alternative teaching material concerning the history of the concept of function, where history is a *goal*, be designed for teacher training?” (Rose and Sibyl 2016, p. 2).

In Danish teacher education (with a duration of 4 years), the concept of function may be treated as late as third year, often in a course on “Modeling and Teaching Differentiation” as offered at one of the Danish university colleges. In the study program it is explicitly stated that the “… historical development is to be related to teacher students, instruction and school curriculum” and that the teacher students must display “overview and judgment concerning subject-related didactics”—in this case of course mathematics (Rose and Sibyl 2016, p. 4, as cited in VIA UC 2013). Further, it is stated that upon completing the course the teacher students should possess knowledge of the concept of function also in relation to various applications, and that they should be able to apply functions as a means for problem solving and modeling (ibid.). With this in mind, Rose and Sibyl began analyzing a chapter of the textbook. They found that the chapter only contained a short section related to the history of the concept of functions, and history was used for illumination purposes. The historical content, they found, was primarily a listing of key mathematicians, dates, and short interpretations of how these people contributed to the concept of function.

Furthermore, much space was spent on the concept of function in relation to history of Danish mathematics education. The section contained Dirichlet's definition of function and a more modern version of a definition, which was associated with a task designed to illustrate the difference between the two. It is unclear if the purpose of including history was either one of history as a goal or history as a tool, although the mentioned task may indicate the latter. Rose and Sibyl continued:

> Examples of Whig history are seen in the following quotations: "It has been difficult for mathematicians in previous centuries to get rid of their concrete examples of a function and *rise to a more general concept of function*" (Schou et al. 2013, p. 233, italics added). This is an example of what seems important today but was not necessarily considered important back then, where the application perspective was of higher significance than the chase for a general concept of function. The quotation is [Whiggish] because past mathematicians are considered 'off track' instead of participants in a historical discourse. Whig history is also evident in the following: "Leonhard Euler's (1707–1783) concept of function is characterized by the fact that he identified the function with the so-called analytic expression, *which is roughly what we understand by the expression today*" (ibid., italics added). This is an example of how teacher students are denied the possibility of seeing the concept of function within Euler's discourse. Rather they are urged to see the concept within the modern discourse, that is, to see history through 'modern lenses'. One pitfall of Whig history is that [teacher] students are deprived of the opportunity to experience conflicts with their own meta-discursive rules. (Rose and Sibyl 2016, p. 5)

Based on this, Rose and Sibyl argued that the chapter provided only shallow insight into the history of the function concept neither enabling the student teachers to make their own interpretations of this, nor participating in meta-issue discussions. For those reasons they concluded that it was questionable to which extent the student teachers may develop "overview and judgment concerning subject-related didactics." They stated:

> In addition, we deem that the material [the section of the chapter] does not contribute to the teacher students' [subject matter knowledge] (Mosvold et al. 2014), since they do not work in-depth with their own conceptual understanding and the different historical representations of the concept of function. Moreover, there are no invitations to didactical considerations for the teacher students, which could have provided them with the opportunity to consider how to include history in their own teaching practice. This means that they do not get the possibility to reflect upon how historical understandings of the concept of function may be mirrored in their future students' understandings. In the light of MKT, this may be seen as a limited development of the teacher students' PCK. (Rose and Sibyl 2016, p. 5)

Based on their analysis and observations, Rose and Sibyl then attempted to design a (or redesign an existing) teaching module that held the potential of teacher students developing the demanded overview and judgment of mathematics didactics.

> Since overview and judgment is an objective for the module it is essential that the teacher students obtain insight into how and why the concept of function has evolved through different discourses and is human-induced. This invites a use of history as a goal in the module. In relation to hows, Jankvist (2009) writes that a module approach is more obvious when history is used as a goal, since this combination invites the development of meta-issues with the students. […] Fried (2001)

> even argues that original sources are open to interpretation whereas in secondary literature interpretations are done beforehand. (Rose and Sibyl 2016, p. 6)

Rose and Sibyl based their teaching module on one by Petersen (2011), also reported in Kjeldsen and Petersen (2014), originally designed for upper secondary school, and adjusted this to the setting of teaching training and the specific course.

The structure of the module is that the teacher students are divided into groups, each group having its own task to fulfill and present to the rest of the class. At the end of the module each group writes a report addressing questions related to all tasks of the groups. As material for completing their task and report, the material of the module includes excerpts from original material where Euler and Dirichlet, respectively, provide their definitions of function. They are provided with a number of excerpts from textbooks, both for upper secondary level and for undergraduate level, presenting definitions of function. Also, they are provided with a list of supplementary (secondary) literature and encouraged to read this. An example of a task given to a group is to explain what a function is according to (1) Euler's original definition, (2) his extended definition, and (3) Dirichlet's definition, and describe similarities and differences between these as well as to engage into discussions of the "principle of a variable's generality." Another group is asked to explain how and why the debate of the vibrating string made Euler extend his definition of function. One group is asked to account for the contemporary concept of function, explaining also the concepts of "graph," "domain of function," and "range," as well as why proofs are necessary in the work with functions. Finally, a group is asked to account for who Euler and Dirichlet were and what the mathematical society that they worked within looked like. The final report encompasses all four tasks. As a complement to the task and report, Rose and Sibyl formulated a series of questions which the teacher students were to address in their groups:

> Discuss whether the following quote from your textbook is a fair contemplation of the authors: "It has been difficult for mathematicians in previous centuries to get rid of their concrete examples of a function and rise to a more general concept of function" (Schou et al. 2013, p. 233). (Rose and Sibyl 2016, p. 7)

Final reflections by Rose and Sibyl (2016) included a list of potentials of the teaching module, where teacher students can:

- Engage in interpretation since original sources are used.
- Experience conflicts between their own and historical mathematicians' meta-discursive rules. The conflicts may lead to a development of the students' in-issues and concept images (Tall and Vinner 1981).
- Develop their overview and judgment through meta-issue discussions about the historical development of the concept of function, since history is used primarily as a *goal*.
- Develop their subject matter knowledge, since they work with meta-discursive rules and thus may obtain a better understanding of the concept of function.
- Develop their PCK, since they must reflect upon how their historical knowledge may strengthen the ability to support their pupils' conceptual understanding.

The interview during the *Didactics of Mathematics* course revealed that both students, and in particular Sibyl, thought that studying the history of the concept of function had been rewarding, both on a personal level and in relation to the task of teacher training:

> I think it will definitely make it easier and maybe also make me aware that it's possible to find material that is already done, so that I don't have to do all the groundwork myself. Because if [I'm] being realistic, you don't have enormous amount of time to prepare your sessions. I [now] know I can find inspiration on how to do it, instead of just [starting] it briefly according to material that's in the book that we normally teach from. (Interview April 2016)

Upon completing their master's degree, Rose became an editor of textbooks with a Danish publishing house and Sibyl obtained a job as a teacher educator at a Danish university college.

Analysis and discussion of case 2

In their mini-project, Rose and Sibyl engaged with the three core tasks of teaching MKT (Zopf 2010). Furthermore, as part of her employment as a teacher educator, Sibyl implemented the teaching module with two teacher student populations in November 2016 as part of their training. We had the opportunity to interview Sibyl again in October 2017, and in the analysis that follows we highlight additional supporting evidence in our discussion on the ways in which Rose and Sibyl engaged with the core tasks of teaching MKT.

Selecting interpretations and representations Rose's and Sibyl's analysis of the textbook for mathematics teacher training revealed that the role of the history of mathematics used to inform the development of the concept of function in a textbook for mathematics teacher training was insufficient. In their analysis of the resource (Schou et al. 2013), they found the approach unsatisfactory and using their teacher education perspective informed by the HPM literature went so far as to state that "teacher students are denied the possibility of seeing the concept of function within Euler's discourse" (Rose and Sibyl 2016, p. 5). Within this aspect of their work, Rose and Sibyl surveyed available interpretations (yet those still taken from the history of mathematics) and determined which were appropriate for teaching particular concepts—in this case, that of function (Zopf 2010). Furthermore, they moved between two subtasks of the MKTT framework. Rose and Sibyl used their knowledge of the HPM and mathematics education literature from the course to analyze resources (e.g., the teacher training textbook and the teaching module they redesigned for use with teacher students), which enabled them to unpack the mathematical details relevant to the function concept, and to use this knowledge to modify an existing module for use with teacher students. In particular, for the concept of function, we found evidence of Rose's and Sibyl's desire to unravel the mathematical details contained within different definitions of function and concepts related to function, such as "graph," "domain of function," and "range."

Selecting examples As stated, Rose and Sibyl selected Petersen's (2011) module on teaching the function concept for the MKTT focus of their mini-project. The module was originally designed for upper secondary school, so in their selection of the module for use with mathematics teacher students it was necessary for Rose and Sibyl to sequence the content of the module in a way that made sense for the context. Thus, although the selection of the module for use with teacher students provided evidence of Rose's and Sibyl's work of selecting examples for the purpose of teaching MKT, they also carefully considered the structure of the module and how they should modify it to provide teacher students an alternative to the superficial illumination approach (identified in the textbook for teacher training) to teaching the concept of function using history of mathematics. For example, when Sibyl implemented a modified version of the mini-project, she reflected on her selection of the definitions of function from Euler and Dirichlet:

> Working with the original sources [of Euler and Dirichlet] was really difficult for the teacher students. I think because it was so difficult it was one of the … side effects of this module that they also learned something about functions. (Interview October 2017)

Sibyl noted during the April 2016 interview that it was easier to have such examples as the Petersen module from which to first select the overarching content for teaching the concept of function using original source excerpts, which in turn enabled them to engage in the work of sequencing the content of the module to meet the mathematical and pedagogical needs of their (future) teacher students. When asked about her teaching in which she used the redesigned module with teacher students Sibyl reflected that:

> I think because [they] had to compare a new definition [of function] with the older ones made them realize what functions are. So, I think it gave them a better understanding of functions than the traditional way of teaching functions. And, I started the module with the students where they had to write down and draw the definition of a function. They had to do this before the module. And then after the module they had to do it again, to compare. I would have to say [the definitions] changed a lot. (Interview October 2017)

Managing enactment of mathematical tasks Next, we present evidence of two managing enactment subtasks of teaching MKT, launching and engaging teachers in conversations about mathematical task, which were intricately related within the planning and implementation (with Sibyl's teacher students) of the teaching module. For example, Rose and Sibyl launched the teaching module by assigning small group discussions in which the teacher students focused on five different excerpts from the module. In their module, Rose and Sibyl planned for teacher students to discuss mathematics found in assigned excerpts. Thus, Rose and Sibyl planned for opportunities for teacher students to engage in discussions about the several dimensions of historical content that they would meet in the module (e.g., original source excerpts and their contribution to learning the concept of function, opportunities and limitations related to the inclusion of history in teaching mathematics).

During the follow-up interview with Sibyl, we asked her to discuss the implementation of the module with her teacher students, with attention to launching the mathematical and historical work of the module. Referring to her lesson notes from November 2016, Sibyl stated that:

> The entire module is about modeling and functions. I think I started […] these few weeks with the modeling and then we had a bit about variables and equations… and then after that I introduced them to this module, with the historical part. […] they had to prepare and read a bit before we started the real work with the module. So, before they went home and read the pages I found for them, we talked about Whig history, you know, Fried's [2001 article]? You know, *how* you should read historical texts. We talked about trying to use these 'blinders', you know, for a horse, and trying to not think about modern math in what you are reading. (Interview October 2017)

We regard the case of Rose and Sibyl—first as teacher educator students, and then with Sibyl as a teacher educator—as one exhibiting evidence that the course on history in mathematics education influenced their engagement with the three core tasks of teaching MKT (Zopf 2010), and that it thus provided them with opportunities to develop MKTT.

Discussion of disciplinary knowledge and MKTT

Zopf (2010) emphasized that, "[m]athematical knowledge for teaching teachers includes a robust knowledge of the epistemology of mathematics, mathematical structures and ways of work" (p. 193), also referred to as disciplinary knowledge. In the following discussion, we first address the epistemological aspects of mathematical knowledge for teaching teachers and next use our two cases as a starting point for discussing this in relation to the mathematical work of teaching.

Epistemology is the study of *how* we know things—from the perspective of this article, knowledge *of* and *about* mathematics. For Ball (1988), knowledge *of* mathematics refers to the substance, i.e., the topics, concepts, procedures,[9] while knowledge *about* mathematics entails ideas about what mathematics is, "where it comes from, what it is good for, and how right answers are established" (p. 39). This corresponds with Schwab's (1978) distinction between substantive and syntactic structures of disciplines—an epistemological perspective that is underlying both Shulman's (1986) explorations of the role of content in teaching and Ball and colleagues' conceptualization of MKT (e.g., Ball et al. 2008). According to Schwab (1978), mathematics differs from the other disciplines in two important ways. First, mathematical knowledge is certain. A true statement in mathematics is proven by natural logic, and all mathematical theorems are true. This has consequences for the evaluation of mathematical results. Second, mathematics has no material subject matter. Mathematical statements are logical rather than empirical statements, and they thus encompass all possible cases. These epistemological points equal what Rose and Sibyl found important in their redesign and use of the module on the historical development of the concept of function, and they are beautifully illustrated by the mathematical discourses of Dirichlet and Euler, respectively. Since mathematics does not have a material subject matter (Schwab 1978), speaking about a mathematical concept involves translating between different representations, e.g., from geometric to algebraic to analytic and vice versa. This is exactly the point that Mirabelle highlights in case 1, when she discussed potential use of the excerpt from Viète in relation to developing MKT with teacher students. Although closely linked to knowledge *of* mathematics—in case 1 the solving of equations and in case 2 the concept of function—these epistemological insights concern knowledge *about* mathematics and its syntactic structures. In other words, it concerns "the nature of knowledge in the discipline" of mathematics, including "where it comes from, how it changes, and how truth is established" (Kim 2013, p. 12).

From our analysis in the present study, we observed that the teacher educator students' epistemological perspectives of mathematics seem to be strongly embedded in their reflections about the three core tasks of teaching MKT. For instance, when Mirabelle reflected about the task of selecting interpretations and representations used for teaching MKT, she

[9] In her early work, Ball (1988) also included PCK, and in particular the ability to represent mathematics in different ways in order for other to understand as part of knowledge *of* mathematics.

not only reflected on how inclusion of Viète's work might contribute to developing various aspects of MKT in teacher students, she also showed how such use of a historical perspective of mathematics might stimulate understanding of how mathematics develops over time—thus relating to the broad landscape of mathematical knowledge and its development, which Zopf (2010) refers to as the panoramic characteristic of MKTT. In her reflections, Mirabelle also specified what specialized knowledge of content might look like for a mathematics teacher educator, and she coordinated the mathematical issues at play with the pedagogical perspective of supporting teacher students' reflections about their work with technology. We consider such reflections as evidence of developing consciousness about the intricacies of the special work of teaching teachers, and we suggest that Mirabelle's reflections illustrate the deliberative and connected nature of MKTT (cf. Zopf 2010). In the case of Rose and Sibyl we see indications of epistemological understanding of mathematics when they reflect on the connection between Euler's concept of function and our modern understanding. Their continued reflections on the pitfalls of Whig history illustrate the complexities of the work of teaching teachers, and it is interesting to notice that these reflections emerged from discussions of using history of mathematics in mathematics teacher education.

Furthermore, Rose, Sibyl, and Mirabelle all draw attention to mathematical discourse and potential commognitive conflicts as a means for having their teacher students notice and experience the existence of meta-discursive rules in mathematics and mathematical work. Such object-level knowledge is of course one *about* mathematics, but nevertheless one that must be activated within the teacher students' own mathematical work; this is to say in relation to their knowledge *of* mathematics. Such aspects of mathematical activity—meta-discursive rules—are rarely articulated in mathematics education, nor in mathematics teacher education. Hence, the use of history of mathematics, and in particular that of historical primary source material, seems to offer unique opportunities for developing not only MKT but certainly also the disciplinary knowledge needed for MKTT.

Conclusion

In this article we have illustrated what developing mathematical knowledge for teaching teachers might look like and how such knowledge may be developed as part of a course for future mathematics teacher educators. Our case has been that of a short course on using history of mathematics in mathematics education from a mathematics educational (didactic) perspective. In particular we have presented, analyzed, and discussed two cases from the implementation of the short course, involving three teacher educator students. We found that the design of the short course and the topic of history in mathematics education, and in particular the mathematical work with historical primary source material, provided for sound development of mathematical knowledge for teaching teachers. For two of the teacher educator students, we were able to document a continual possession and development of MKTT on their behalf.

In a teacher education context, an object of the work of teaching is to develop MKT in teacher students. This work of teaching teachers is a specialized work—similar but not equal to the work of teaching mathematics in school—that requires a specialized kind of knowledge. Zopf (2010) argued that the work of teaching MKT requires—among other things—a robust epistemological knowledge of mathematics. We argue that having teacher educator students read historical mathematical sources can stimulate this development of epistemological knowledge of mathematics required to teach MKT. The illustrations we have provided of what such kind of epistemological knowledge might look like, how it is embedded in the teacher educator students' reflections, and how their reflections about using historical sources provide a productive space for engaging with core tasks of teaching MKT, serve as warrants for this claim.

In summary, this article showcases an empirical example of how to develop MKTT, and it further exemplifies and elaborates on teacher educators' disciplinary knowledge as something also involving epistemological knowledge about mathematics.

Appendix: Center text from Fig. 1 (partial translation)

VII.10 Excerpt from Viète's theory of equations

Third sentence in another way

If, when B is greater than half of D:

If A cube $-B$ square thrice into A is equated to B square into D,

but

B square [thrice] into $E-E$ cube will be equated to B square into D,

and there are two right angled triangles with equal hypotenuse B, such that the acute angle subtended by the perpendicular of the first, is triple the acute angle subtended by the perpendicular of the second, while double the base of the first, is D, making the double of the base of the second be A…

[Let] $1C-300N$ be equated to 432;

or even let

$300N-1C$ be equated to 432.

There are two right triangles, of which the common hypotenuse is 10 [radius B] … Moreover double the base of the first is $\frac{432}{100}$ [chord D] and … the base becomes 2 $\frac{16}{100}$ … when it is said that $300N-1C$ is equated to 432, $9+R57$ [$9+\sqrt{57}$] or $9-R57$ [$9-\sqrt{57}$] will be made to be 1 N [chord A]. (adapted from Nickalls 2006, pp. 204–205)

References

Andersen, K., Andersen, I., Gram, K., Holth, K., Jakobsen, I. T., & Mejlbo, L. (1986). *Kilder og kommentarer til ligningernes historie*. Vejle: Forlaget Trip.

Arcavi, A., Bruckheimer, B., & Ben-Zvi, R. (1982). Maybe a mathematics teacher can profit from the study of the history of mathematics. *For the Learning of Mathematics, 3*(1), 30–37.

Artzt, A., Sultan, A., Curcio, F., & Gurl, T. (2012). A capstone mathematics course for prospective secondary mathematics teachers. *Journal of Mathematics Teacher Education, 15*(3), 251–262.

Ball, D. L. (1988). *Knowledge and reasoning in mathematical pedagogy: Examining what prospective teachers bring to teacher education*. Unpublished doctoral dissertation, Michigan State University, East Lansing.

Ball, D. L., Thames, M. H., & Phelps, G. (2008). Content knowledge for teaching: What makes it special? *Journal of Teacher Education, 59*(5), 389–407.

Barnett, J. H., Lodder, J., & Pengelley, D. (2014). The pedagogy of primary historical sources in mathematics: Classroom practice meets theoretical frameworks. *Science & Education, 23*(1), 7–27.

Castro Superfine, A., & Li, W. (2014). Exploring the mathematical knowledge needed for teaching teachers. *Journal of Teacher Education, 65*(4), 303–314.

Charalambous, C. Y., Panaoura, A., & Philippou, G. (2009). Using the history of mathematics to induce changes in preservice teachers' beliefs and attitudes: Insights from evaluating a teacher education program. *Educational Studies in Mathematics, 71*(2), 161–180.

Clark, K. M. (2012). History of mathematics: Illuminating understanding of school mathematics concepts for prospective mathematics teachers. *Educational Studies in Mathematics, 81*(1), 67–84.

Dirichlet, J. P. G. L. (1837). Über die darstellung ganz willkürlicher funktionen durch sinus- und cosinus reihen. *Repertorium der Physik, I,* 152–174.

Euler, L. (1748/1988). *Introductio in Analysin Infinitorum* [Introduction to analysis of the infinite book I] (J. D. Blanton, Trans.). New York: Springer.

Fried, M. (2001). Can mathematics education and history of mathematics coexist? *Science & Education, 10*(4), 391–408.

Furinghetti, F. (1997). History of mathematics, mathematics education, school practice: Case studies linking different domains. *For the Learning of Mathematics, 17*(1), 55–61.

Furinghetti, F. (2007). Teacher education through the history of mathematics. *Educational Studies in Mathematics, 66*(2), 131–143.

Glaubitz, M. R. (2011). The use of original sources in the classroom: Empirical research findings. In E. Barbin, M. Kronfellner, & C. Tzanakis (Eds.), *History and epistemology in mathematics education: Proceedings of the 6th European Summer University* (pp. 351–362). Vienna: Holzhausen.

Hoover, M., Mosvold, R., Ball, D. L., & Lai, Y. (2016). Making progress on mathematical knowledge for teaching. *The Mathematics Enthusiast, 13*(1–2), 3–34.

Huntley, M. A., & Flores, A. (2010). A history of mathematics course to develop prospective secondary mathematics teachers' knowledge for teaching. *PRIMUS, 20*(7), 603–616.

Jankvist, U. T. (2009). A categorization of the "whys" and "hows" of using history in mathematics education. *Educational Studies in Mathematics, 71*(3), 235–261.

Jankvist, U. T. (2011). Anchoring students' meta-perspective discussions of history in mathematics. *Journal of Research in Mathematics Education, 42*(4), 346–385.

Jankvist, U. T. (2013). History, applications, and philosophy in mathematics education: HAPh—A use of primary sources. *Science & Education, 22*(3), 635–656.

Jankvist, U. T. (2014). On the use of primary sources in the teaching and learning of mathematics. In M. R. Matthews (Ed.), *International handbook of research in history, philosophy and science teaching* (pp. 873–908). Dordrecht: Springer.

Jankvist, U. T. (2015). Teaching history in mathematics education to future mathematics teacher educators. In K. Krainer & N. Vondrová (Eds.), *Proceedings of the Ninth Conference of the European Society for Research in Mathematics Education* (pp. 1825–1831). Prague, Czech Republic: Charles University in Prague, Faculty of Education and ERME.

Jankvist, U. T., & Iversen, S. M. (2014). 'Whys' and 'hows' of using philosophy in mathematics education. *Science & Education, 23*(1), 205–222.

Jankvist, U. T., & Kjeldsen, T. H. (2011). New avenues for history in mathematics education—Mathematical competencies and anchoring. *Science & Education, 20*(9), 831–862.

Jankvist, U. T., & Misfeldt, M. (2015). CAS-induced difficulties in learning mathematics. *For the Learning of Mathematics, 35*(1), 15–20.

Jankvist, U. T., Mosvold, R., & Clark, K. (2016). Mathematical knowledge for teaching teachers: The case of history in mathematics education. In L. Radford, F. Furinghetti, & T. Hausberger (Eds.), *International Study Group on the Relations between the History and Pedagogy of Mathematics–Proceedings of the 2016 ICME Satellite Meeting* (pp. 441–452). Montpellier, France: IREM de Montpellie.

Jankvist, U. T., Mosvold, R., Fauskanger, J., & Jakobsen, A. (2015). Analysing the use of history of mathematics through MKT. *International Journal of Mathematical Education in Science and Technology, 46*(4), 495–507.

Kim, Y. (2013). *Teaching mathematical knowledge for teaching: Curriculum and challenges*. Unpublished doctoral dissertation, University of Michigan, Ann Arbor.

Kjeldsen, T. H., & Blomhøj, M. (2012). Beyond motivation: History as a method for learning metadiscursive rules in mathematics. *Educational Studies in Mathematics, 80*(3), 327–349.

Kjeldsen, T. H., & Petersen, P. H. (2014). Bridging history of the concept of function with learning mathematics: Students' meta-discursive rules, concept formation and historical awareness. *Science & Education, 23*(1), 29–45.

Masingila, J. O., Olanoff, D., & Kimani, P. M. (2017). Mathematical knowledge for teaching teachers: Knowledge used and developed by mathematics teacher educators in learning to teach via problem solving. *Journal of Mathematics Teacher Education*. https://doi.org/10.1007/s10857-017-9389-8.

Mirabelle, (2016). *Mini-project & appendices, Didactics of Mathematics course*. Copenhagen: Danish School of Education, Aarhus University.

Mosvold, R., Jakobsen, A., & Jankvist, U. T. (2014). How mathematical knowledge for teaching may profit from the study of history of mathematics. *Science & Education, 23*(1), 47–60.
Nickalls, R. W. D. (2006). Viète, Descartes and the cubic equation. *The Mathematical Gazette, 90*(518), 203–208.
Niss, M., & Højgaard, T. (Eds.). (2011). *Competencies and mathematical learning—Ideas and inspiration for the development of mathematics teaching and learning in Denmark*. IMFUFA tekst nr. 485/2011, Roskilde University.
Petersen, P. H. (2011). *Potentielle vindinger ved inddragelse af matematikhistorie i matematikundervisningen*. Master thesis in mathematics, Roskilde University, Roskilde, Denmark. Retrieved from http://milne.ruc.dk/ImfufaTekster/pdf/483web.pdf. Accessed 7 Jan 2019.
Rose, & Sibyl, (2016). *Mini-project & appendices, Didactics of Mathematics course*. Copenhagen: Danish School of Education, Aarhus University.
Schou, J., Jess, K, Hansen, H. C., & Skott, J. (2013). *Matematik for lærerstuderende - Tal, algebra og funktioner 4.-10. Klassetrin*. Frederiksberg: Samfundslitteratur.
Schwab, J. J. (1978). *Science, curriculum, and liberal education*. Chicago: The University of Chicago Press.
Sfard, A. (2008). *Thinking as communicating: Human development, the growth of discourses, and mathematizing*. New York: Cambridge University Press.
Shulman, L. S. (1986). Those who understand: Knowledge growth in teaching. *Educational Researcher, 15*(2), 4–14.
Smestad, B., Jankvist, U. T., & Clark, K. (2014). Teachers' mathematical knowledge for teaching in relation to the inclusion of history of mathematics in teaching. *Nordic Studies in Mathematics Education, 19*(3–4), 169–183. **(Thematic issue edited by J. Fauskanger & R. Mosvold.)**.
Tall, D., & Vinner, S. (1981). Concept image and concept definition with particular reference to limits and continuity. *Educational Studies in Mathematics, 12*(2), 151–169.
VIA UC. (2013). *Studieordningen for Læreruddannelsen i Aarhus. Modulbeskrivelser. Matematik 4.-10. Klassetrin. Matematikmodul 4: Modellering og undervisningsdifferentiering*. Aarhus: VIA University College.
Viète, F. (1646). *Opera Mathematica*, edited with notes by Frans von Schooten (Leyden, 1646). A facsimile reprint has been issued (Hildesheim: Georg Olms Verlag, 1970).
Waldeana, H. N., & Abraham, S. T. (2014). The effect of an historical perspective on prospective teachers' beliefs in learning mathematics. *Journal of Mathematics Teacher Education, 17*(4), 303–330.
Wang, K., Wang, X., Li, Y., & Rugh, M. S. (2018). A framework for integrating the history of mathematics into teaching in Shanghai. *Educational Studies in Mathematics*. https://doi.org/10.1007/s10649-018-9811-x.
Zopf, D. (2010). *Mathematical knowledge for teaching teachers: The mathematical work of and knowledge entailed by teacher education*. Unpublished doctoral dissertation, University of Michigan, Ann Arbor.

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