J Math Teacher Educ (2018) 21:237–261
https://doi.org/10.1007/s10857-016-9358-7

Can teachers and mathematicians communicate productively? The case of division with remainder

Jason Cooper[1] · **Ronnie Karsenty**[1]

Published online: 17 October 2016

Abstract Is it possible that a meeting of mathematicians and primary school teachers will be productive? This question became intriguing when one professor of mathematics initiated a professional development course for practicing primary school teachers, which he taught alongside a group of mathematics Ph.D. students. This report scrutinizes the uncommon meeting of these two communities, who have very different perspectives on mathematics and its teaching. The instructors had no experience in primary school teaching, and their professed goal was to deepen the teachers' understanding of the mathematics they teach, while teachers were expecting the course to be pedagogically relevant for their teaching. Surprisingly, despite this mismatch in expectations, the course was considered a success by teachers and instructors alike. In our study, we analyzed a lesson on division with remainder for teachers of grades 3–6, taught by the professor. The framework used for the data analysis was *mathematical discourse for teaching*, a discursive adaptation of the well-known mathematical knowledge for teaching framework. Our analysis focuses on the nature of the interactions between the parties and the learning opportunities they afforded. We show how different concerns, which might have hindered communication, in fact fueled discussions, leading to understandings of the topic and its teaching that were new to all the parties involved. The findings point to a feasible model for professional development where mathematicians may contribute to the education of practicing teachers, while they are gaining new insights themselves.

Keywords In-service professional development · Mathematicians · Elementary school mathematics · Division with remainder · Mathematical discourse for teaching · Commognition

✉ Jason Cooper
jasonc2107@gmail.com

[1] Department of Science Teaching, Weizmann Institute of Science, 234 Herzl Street, 7610001 Rehovot, Israel

Introduction

The founding fathers of mathematics education were research mathematicians, such as Hans Freudenthal and Felix Klein, who believed that “the whole sector of mathematics teaching, from its very beginnings at elementary school right through to the most advanced level research, should be organized as an organic whole” (Klein 1923, p. 24[1]). Viewing all levels of mathematics as an organic entity, it is natural to assume that mathematicians should have an important role in school mathematics in general and in the professional development of school teachers in particular. However, in the current state of affairs, this assumption is not nearly as straightforward as it was in the early days of mathematics education. In recent decades, mathematics education has established itself as a separate academic discipline, drawing on theories and methodologies from the social sciences and psychology, and the role of research mathematicians is thus no longer obvious (Fried 2014). In fact, conflicts between the communities of mathematicians and mathematics educators have been significant. For example, in the debate over mathematics education known as the Math Wars (Davison and Mitchell 2008; Schoenfeld 2004), mathematicians have tended to reject reform initiatives in mathematics, advocating the approach of traditional mathematics. Some of them have voiced this stance in no uncertain terms [see open letters written to the Secretary or Minister of Education both in the USA (Klein et al. 1999) and in Israel (Israeli Mathematicians 2010)].

Despite the changing circumstances, many contemporary research mathematicians, including Ralston (2004), Wu (2011) and Hodgson (2002), still believe that university mathematics departments can and should contribute to the education of mathematics teachers. One aim of the research reported herein is to provide empirical evidence for the feasibility of this stance. Ralston (2004) claims that university mathematicians’ deep knowledge of mathematics “gives them useful insight into what topics are particularly important in school mathematics and sometimes into good approaches for teaching these topics” (p. 404). Wu (2011), like Ralston, sees mathematical knowledge as crucial for teaching and makes a case for the seemingly obvious claim that “you cannot teach what you don’t know.” Yet, Wu is aware of two almost contradictory demands on the body of mathematical knowledge that teachers should be provided with: relevance for teaching (in the sense that the knowledge does not stray far from the material that teachers need to teach), and consistency with the fundamental principles of [university] mathematics. Such fundamental principles may constitute what Hodgson (2002) calls mathematicians’ perspective on mathematics, which he sees as their main potential contribution to mathematics education.

However, these mathematicians may be underestimating the differences between the mathematics taught and practiced at university and the mathematics required for school teaching. Ball et al. (2008), elaborating on the seminal work of Shulman (1986), have developed a theory of mathematical knowledge for teaching. Ball’s research group has designed instruments for assessing such knowledge and found that teachers tend to score higher than mathematicians on some items. Concerned with developing mathematical knowledge for teaching, Stylianides and Stylianides (2010) have elaborated a framework of “Pedagogy Related” (P-R) mathematical tasks and commented that “the pedagogical demands implicated by the design, implementation, and solution of P-R mathematics tasks [require] not only good knowledge of mathematics but also some pedagogical knowledge.

[1] Translation taken from ICMI (2008).

[…] It may be hard to require or expect that [research mathematicians] […] have knowledge of pedagogy" (p. 170). Bass, a research mathematician substantially engaged in school mathematics education, sees *mathematics for teaching* as a field of applied mathematics and claims that "the first task of the mathematician who wishes to contribute in this area is to understand sensitively the domain of application, the nature of its mathematical problems, and the forms of mathematical knowledge that are useful and usable in this domain" (Bass 2005, p. 418). This raises the question of how such an understanding may be developed. Bass's personal example—a fruitful collaboration with mathematics educators such as Ball—may not be practical for all mathematicians who wish to contribute to teacher education.

Our study, which investigates an in-service professional development (PD) course run by mathematicians, offers an alternative answer. This atypical setting[2] did not only provide an opportunity to explore the potential of mathematicians' contribution to teacher education, but it also illuminated how mathematicians teaching in a PD may develop on-the-job understanding of mathematics for teaching, which Bass (2005) considers so crucial. The PD was the initiative of a mathematics professor at a leading university in Israel and was taught by mathematics Ph.D. students, with the goal of broadening and deepening primary school teachers' knowledge of the content that they teach. This article focuses on a lesson around the topic of division with remainder, taught by the professor. In analyzing this lesson, we followed the footsteps of researchers such as Nardi et al. (2014) who are coming to view university mathematics not just as a body of knowledge, but also as a discourse, conceived as accepted modes of communication in mathematics departments. We adopt this view and extend it to school teachers, seeing the PD under investigation as the meeting of two communities of discourse. Our aim is to describe the opportunities for learning that this meeting affords. In the following, we first present a brief epistemic account of division with remainder as a mathematical topic and state the theoretical lenses through which we analyzed three episodes selected from the lesson. Details on the PD, the lesson and the specific episodes are given, followed by what constitutes the lion's share of this report—the analysis of episodes and discussion of results and implications.

Division with remainder in university and in primary school: an epistemic analysis

In abstract algebra, division is a somewhat neglected operation. Its natural domain is field theory, where every nonzero element has a multiplicative inverse, and division is conceived as multiplication by inverse. In the more general structure of rings, division cannot be defined since not every element need have an inverse, yet in some cases (Euclidean rings), division with remainder (DWR) can be defined. The problem of dividing a by b is formulated as finding q and r, such that $r < b$ and $a = qb + r$. DWR is a more advanced topic than regular division and is typically taught in elective courses on advanced ring theory.

A related topic is modulo arithmetic. Given a divisor b, $\mathbb{Z}$ can be separated into equivalence classes according to the remainder from division by b, and operations of addition and multiplication can be defined on these classes. In this context, the quotient (q) is of no significance, only the remainder is of consequence.

[2] In Israel, professional development is usually conducted by professional teacher educators or experienced teachers.

In elementary school, division is one of the four basic operations and has its typical word problem models. The most common models—partition and quotition—are first introduced in grade 1. The partitive model is described as "dividing into equal parts" (Pedagogical Secretariat of the Israeli Ministry of Education 2009, translated from p. 24), e.g., "12 pencils are divided into 3 boxes in such a manner that each box will hold the same number of pencils. How many pencils will there be in each box? (4 pencils)" (ibid., p. 25). The quotitive model is described as finding the number of parts of a given size, e.g., "12 pencils are divided into boxes in such a manner that each box will hold 3 pencils. How many boxes will we need? (4 boxes)" (ibid.). Of course, the dividend may not be divisible by the divisor, in which case there may be something left over. Division with remainder can be seen as an interim solution to the problem of defining and performing whole number division before students have been introduced (in grade 5) to "the meaning of fraction as the quotient of division" (ibid., p. 98). Accordingly, DWR nearly disappears from the school curriculum once fractions are learned.[3] Indeed, if you ask educated adults to calculate 7:2, they will probably answer *three and a half* or *three point five* (as obtained from a calculator), and not *three remainder one*.

In reality, our lives are filled with partitive and quotitive problems where items cannot be split. Such problems serve as models for DWR, even though many real-life situations are less strict about the range of solutions than abstract DWR: real life sharing need not always be perfectly fair, and parts need not always be of perfectly equal size. Thus, children are expected to engage in two different discourses of division. The abstract arithmetic discourse is about numbers—students calculate the result of a division exercise—whereas the discourse of models of division is about solving particular types of realistic problems. The discourse of models (e.g., fair sharing) shares some keywords with the arithmetic discourse, such as *divide* (a fair distribution of objects, but also an arithmetic operation on numbers[4]), and *remainder* (a collection of objects left over but also a number).

In the arithmetic discourse, students calculate the result of division problems—quotient and remainder. The notation they use in Israel is, for example, $7:2 = 3(1)$. This may appear to be quite similar to the way the problem is formulated and visually mediated in university (e.g., *find q and r such that* $7 = q \times 2 + r$); however, seen in a broader context there are significant differences. The shift from arithmetic to algebraic discourse is a central challenge in primary school. Students often experience difficulty even in simple algebraic activities such as addition with an unknown (e.g., $3 + ? = 8$). The Mathematics Learning Study Committee, in "Adding It Up" (2001), listed some of the perspectives of traditional elementary school arithmetic that need to change in the shift from arithmetic to algebra: "An orientation to execute operations" should shift to an orientation to represent relationships, and "[the] use of the equal sign to announce a result" should shift to signifying an equality (ibid., p. 270). In the school notation of DWR, the equals sign can be seen as announcing the result of a calculation. The university formulation of DWR belongs to an algebraic discourse, where the routine is one of manipulating the right side of the equation ($q \times 2 + r$) to make it equal to the left side (7). Here, the equals sign represents an

[3] DWR is briefly mentioned in the context of the long division algorithm and may be mentioned again in the related context of dividing polynomials, a procedure required in Calculus.

[4] The words in Hebrew are nearly interchangeable ("haluka" and "hiluk"), both based on the root "helek" (meaning "part"). For example, the action "hilakti" may be interpreted either as "I divided" or as "I distributed".

equality relationship, not a call to execute a calculation. As such, it belongs to a more advanced—algebraic—discourse.

Modulo arithmetic is rarely addressed in the school curriculum explicitly; however, from a university perspective it is at the foundation of some elementary topics. For example, "parity arithmetic" (odd + odd = even) is a case of modulo arithmetic, where the sum of two odd numbers is "equivalent to 0 modulo 2."

Theoretical framework

Mathematical knowledge for teaching (Ball et al. 2008) would seem to be an appropriate framework for analyzing the teaching and learning of mathematics within PD sessions. However, viewing university and school mathematics as discourses necessitated an adaptation of Ball's framework. Our approach is that knowledge (whether it is teachers' knowledge or knowledge of mathematicians) is not only reflected in communication, but is actually *constituted* in communication. Viewing knowledge and practice (such as teaching or engaging in mathematical research) as aspects of a single entity (discourse) allows us to ground PD discussions, in which mathematicians and teachers took part, in the parties' mathematical and pedagogical perspectives and practices. Moreover, it also suggests the converse, that is, how changes in the parties' discourse that began in the PD might potentially influence the teaching perspectives and practices of both the teachers and the mathematicians. The framework we use is *mathematical discourse for teaching* (MDT), a discursive version of mathematical knowledge for teaching developed by Cooper (2014, 2015a, b). The overarching epistemic framework of MDT is commognition (Sfard 2008), a discursive framework that views fields of human knowledge as communities' well-defined forms of communication. The term communication is used inclusively, to include not only speech and gesture, but also communities' practices, such as teaching and problem solving. Cognition is conceived as communication with oneself (hence the term "commognition"—combining *communication* and *cognition*), and learning is conceived as changes in one's patterns of communication. Such changes may start out as discourse-for-others—imitation of the discourse of members of the community for the sake of social approval, but should eventually change the learner's discourse-for-oneself—patterns of internal communication which we may call "thinking." Commognitive methodology focuses on four interrelated features of discourse: *keywords* (e.g., remainder, rule, divide) and their usage; *visual mediators* (e.g., remainder notation); generally endorsed *narratives* (e.g., "zero is a legitimate remainder") and the rules by which the community endorses them; and recurring *routines* (e.g., long division algorithm). We found these four features of discourse to be effective for comparing the patterns of communication of teachers and mathematicians regarding DWR.

Individuals participate in many partially overlapping discourses. The ones that we are interested in pertain to the practice, teaching and learning of mathematics. Here MDT draws its inspiration from mathematical knowledge for teaching; in fact, we view mathematical discourse for teaching as a "commognitization" of mathematical knowledge for teaching—redefining its basic concepts (knowledge, teaching) in discursive terms. Accordingly, each category of *knowledge* in mathematical knowledge for teaching has a corresponding *discourse* in mathematical discourse for teaching. In this study, our main distinction is between mathematical discourse and pedagogical content discourse—the discursive counterparts of the two main categories of mathematical knowledge for teaching: subject matter content knowledge, and pedagogical content knowledge. Following Ball and her colleagues (Ball et al. 2008), each of these discourses can be further broken down: Mathematical discourse consists of common content discourse (the mathematical discourse that is common to a large portion of educated society), specialized content discourse (mathematical discourse that is typical of teachers of mathematics) and discourse at the mathematical horizon (patterns of mathematical communication that are appropriate in a higher grade level). Pedagogical content discourse consists of discourse of content and teaching, discourse of content and students and discourse of curriculum.

Our focus in this article is mainly on the *opportunities* for learning that the PD afforded; thus, our analysis does not require the full brunt of the commognitive framework as a methodological tool. Instead, we use it as a "directive guide" that amplifies our attention to significant phenomena in the data we inspect. We present herein a short overview of learning from a commognitive perspective, which we regard as helpful for appreciating the subtleties of the analysis to be presented later on.

Learning may take place at the object level, typically adding or modifying narratives pertaining to objects that are already a part of one's discourse (e.g., "the remainder is less than the divisor"), or at the meta-level, typically introducing fundamentally new objects to one's discourse, or changing the rules by which narratives are endorsed (e.g., the shift in the use of the equal sign mentioned in the epistemic analysis, from announcing a result to signifying an equality). Meta-level learning is often triggered by a *commognitive conflict* (Sfard 2008), a term we will demonstrate in the analysis of our findings. Researchers such as Ben-Zvi and Sfard (2007) have shown that the teacher's role in such learning is crucial. Learners cannot appeal to empirical evidence to resolve the conflict; rather, they must accept the teacher as a legitimate representative of the community they are striving to become a part of, and start to use the new discourse before they can fully appreciate its merit, at first as discourse-for-others and eventually as discourse-for-oneself.

Meta-level learning poses a theoretical challenge. If new objects and new rules of discourse are learned through discursive interactions with others, how can such interactions take place before learning has occurred? One answer to this cyclic dependency is that participation in a new discourse may at first be *ritualistic*, where the goal of discursive routines is alignment with others and the social approval of a teacher. As learners engage with the new discourse, their participation should become *explorative*, where the goal of routines is the production of narratives.

We note that current understandings of learning from the commognitive perspective are based on investigation of contexts where novices are developing mathematical discourse through interactions with an expert (e.g., children with teachers, university students with mathematicians). In our research, we stretch the commognitive perspective to investigate the meeting of two established communities, mathematicians and teachers, where the discourse being developed is mathematics *for teaching*. The expert–novice interactions are replaced in this context with interactions between two professional communities—the instructors have expertise in mathematical discourse and the teachers have expertise in pedagogy.[5] The nature of learning in such a context can be expected to be different from what has been reported in commognitive research to date; hence, our research does not only draw on the commognitive theory, but also contributes to its development.

[5] Although this view may seem somewhat dichotomist, and as such an oversimplification, we found it useful to relate to each community's main strength. Taking into account that mathematicians seldom have substantial teaching experience in primary classrooms, and that most primary teachers are not graduates of university mathematics departments, we feel that the dichotomy is justified.

The study

Background

The study reported herein is part of a larger project, investigating a PD course that was the initiative of a professor of mathematics. Concerned with the mathematical knowledge of his children's teachers, he became involved in teacher education, at first informally, but in 2008 he taught a PD course under the auspices of the Israeli Ministry of Education. The course was in accordance with the declared policy of the Ministry, requiring primary teachers who were educated as "general teachers" to enroll in mathematics PD courses in order to be qualified to teach mathematics in their classrooms. To handle the high demand for his course, in 2009 he recruited a team of graduate students (mostly Ph.D. students of mathematics) to serve as instructors of different groups within the course. The teachers received professional credit for their participation, and the instructors received a small fee for their efforts. The professed goal of the PD was to broaden and deepen the teachers' knowledge of the mathematical content they teach; accordingly the classes were organized by the teachers' grade levels, lower elementary (grades 1–2) and upper elementary (grades 3–6). The teachers' expectations, expressed in anonymous questionnaires, concentrated on the issue of relevance. They expected the PD to be relevant for teaching; the most prominent form of expected relevance was to receive classroom-ready activities, and a variety of other pedagogical concerns were expressed as well, such as teaching methods for specific content, understanding student thinking and errors, and managing student heterogeneity. The very different targets and expectations expressed by the instructors and the participating teachers would seem to predict a serious clash—not only were the teachers' expectations outside the declared scope of the PD, but it is even questionable whether the instructors had the resources to address these expectations! Under these circumstances, it is quite surprising that the PD was eventually considered a success by all involved—the instructors, the participating teachers and Ministry officials—who expressed their nearly unanimous satisfaction. Understanding how this apparent success story came about constituted one of the major motivations for the research project. Data were collected in the course held during the 2011–2012 academic year, in which approximately 90 primary school teachers (grades 1–6) enrolled. The course was taught in 6 classes, approximately 15 participants in each class. Data collection was extensive and varied: audio recordings of 60 PD sessions (180 h); anonymous questionnaires submitted by the participating teachers—expectations at the outset of the PD and feedback after each session; assignments submitted by the teachers; interviews with instructors before and after most of the lessons; email exchanges among the instructors; and audio recordings of various meetings held by the instructors, including one at the end of the PD in which they summarized and assessed the year. The previously described theoretical framework of mathematical discourse for teaching (MDT) was developed in order to capture the nature of the learning that took place in this unusual setting.

Of all this extensive data, ten lessons were selected for analysis (Cooper 2016), representing a variety of instructors, mathematical topics, grade levels and various stages of the PD. Analyses of three lessons have been reported elsewhere (Cooper 2014, 2015a, b). In this report, we present an extended analysis of one lesson for upper primary school teachers. This session, the eighth in a sequence of ten lessons, took place in April 2012 (details follow).

Aim

The overarching concern of this study is the affordance of this meeting of two communities for the growth of mathematical knowledge for teaching. More specifically, we aim to understand how the interaction of two very different discourses of mathematics and teaching created opportunities for learning.

The lesson, participants, data collection and analysis

The lesson's topic was DWR, and it was taught in the upper elementary strand by the professor who ran the PD, whom we will call Rick. Fifteen teachers participated in this lesson, and the group was heterogeneous in terms of the teachers' experience and the grade levels they were teaching (3rd to 6th grade). Since the education of primary teachers in Israel takes place in Colleges of Education rather than universities, the teachers did not have university mathematics in their background. The duration of the lesson was 3 academic hours. A "lesson graph" overview is included in "Appendix".

At the time, the professor was working on a book for primary school teachers. Prior to the lesson, the teachers were assigned to read a draft of the chapter on DWR and to relate to it in writing. We do not draw on these assignments directly; however, our familiarity with the teachers' written responses to the chapter contributed to our understanding of their utterances during the lesson.

The lesson was audio recorded and fully transcribed (in Hebrew). The authors carefully reviewed and corrected the transcription, and translated the sections that are quoted in this article. The first author was present in parts of the lesson and took field notes. The three episodes selected for analysis highlight some of the conflicts between the parties' discourses and as a result were particularly rich in opportunities for learning. Our analysis, guided by our theoretical framework, focused on points of conflict in the communication. We aimed to understand how the parties' utterances were connected to their practice (at university and in school) and how conflicting perspectives fueled the discussions.

Findings: analysis of learning opportunities

In this section, we present an analysis of three episodes, in the chronological order in which they occurred during the PD. These episodes demonstrate the different nature of the parties' MDT and the learning that took place in the interplay between the parties' discourses.

In Episode 1, Rick tried to warrant the need for a new DWR notation, drawing on a combination of mathematical and pedagogical considerations. His pedagogical warrants were accepted, but his mathematical warrants were challenged. In Episode 2, teachers reflected on possible implications of Rick's ideas on DWR for teaching long division. In Episode 3, a discussion of zero remainder took place, during which mathematical and pedagogical perspectives were juxtaposed.

Episode 1: Introducing and justifying a new notation

In preparing for the lesson on DWR,[6] Rick was struck by an inadequacy of the notation used in primary school: It violates the transitive property of equality. As he wrote in an email to the other instructors: "17:5 = 3(2) [and] 11:3 = 3(2), [transitivity] would imply that 17:5 = 11:3 which is of course nonsense." For Rick, this mathematical flaw had pedagogical implications as well. He was aware of the difficulties students encounter while attempting to solve exercises with unknowns (e.g., 3 + ? = 8, an exercise for which students sometimes simply add the numbers to get 11) and felt that coming to regard equality as an equivalence relationship, rather than a call to perform a calculation, was crucial for overcoming such difficulties. From this perspective, a violation of transitivity—one of the three equivalence axioms—is a serious offense. To address this quandary, Rick consulted with the other PD instructors and they jointly came up with an alternate notation, 17:5 = 3(2:5). However, in order to justify the need for a new notation, Rick had to convince the teachers that the standard one is indeed deficient. Though it was the violation of transitivity that first bothered him, Rick began with some other inadequacies, perhaps feeling that teachers would not share his concern regarding the violation of the transitivity axiom. We will now review, through presenting and analyzing vignettes from the lesson, two inadequacies that Rick uncovered.

Inadequacy #1: What kind of creature is 3(2)? Does it have independent numerical value?

259, 261	Rick	What kind of a creature is "three remainder 2"? […] When we write something like this [4 + 2] in grade 1, it's an arithmetic expression, right? […] This arithmetic expression has numerical value […] and the equality sign tells us that the value of what's written here is the same as the value of what's written here, okay? […] And when I have this thing [17:5 = 3(2)] I wonder what's the intended meaning of "equals" here. In what sense is what's written here [17:5] the same as what's written here [3(2)]? Okay? That is, is what's written here [3(2)] an arithmetic expression like all the arithmetic expressions I'm accustomed to?

Rick is suggesting that 3(2) does not have numerical value, and therefore, it is inappropriate to use it in the context of the equality sign. However, one teacher (Dana) surprised Rick, claiming that it does have numerical value after all:

262	Dana	If it's agreed that it is a remainder
263	Rick	So you're saying it's an arithmetic expression too.
264	Dana	Yes, of course.

Seeing that this line of reasoning did not convince Dana, and perhaps other teachers remained unconvinced as well, Rick persisted:

[6] See Section 4 in the lesson graph, "Appendix".

273	Rick	When I write 4 + 2 = 6 […], this 6, the answer, has meaning outside the exercise, right? Ah, if I erase this now [17:5], does this [3(2)] have meaning?
274	Hana	No, because we don't know 3 times what.
275	Dana	Okay, but this still doesn't mean that we didn't have equivalence here.

Though not stated explicitly, Rick was appealing to the symmetry axiom of equivalence. The equality 17:5 = 3(2) does not make sense when read from right to left. It is interesting to note that although it was Rick's concern about equality as equivalence that triggered the new notation, Dana was the first to introduce the keyword *equivalence*, which had been discussed in previous PD meetings. Here too, Dana remained unconvinced. She may not have been familiar with the three axioms of equivalence (reflexivity, symmetry and transitivity), yet her claim is mathematically sound in the following sense: In the domain of DWR expressions, having the same result (quotient and remainder) is indeed an equivalence relationship, under which 17:5 and 11:3 are equivalent, and their equivalence class could be represented by 3(2). This undermines Rick's assumption that 3(2) does not have numerical value, or that its value depends on the divisor.

At this stage, after a warrant that left some teachers unconvinced regarding the inadequacy of the standard notation, Rick changed his approach.

A different perspective on DWR: fraction arithmetic is just around the corner

276–281	Rick	In a couple of years they'll learn something else. What's the connection between what they're learning right now in 3rd grade and what they will learn? […] The question was how much is 17 divided into 5. And there are two answers here.

Rick is looking for the connection to fractions, where instead of remainder 2 we say two-fifths.

284	Dana	[…] the divisor, it will be the denominator […] and the remainder will be the numerator.
285	Rick	Your answer is of course correct. It's a bit too procedural in the sense that it shows me 'put this here and that there'. I want to come back to the grandma and her [five] grandchildren[a]. She had 17 candies, let's switch to cookies now, it'll help me in a moment.

[a] See Section 3 in the lesson graph, "Appendix"

Acknowledging that Dana had understood his reference to fractions, Rick was nevertheless unsatisfied with the "technical" nature of her observation. His aim was to bridge whole numbers and fractions, both in the arithmetic discourse and in the discourse of a partitive model—hence the shift from candies to cookies, which have the potential to be divided. This bridging requires a more conceptual connection to fractions.

288, 290	Lia	At first […] here she doesn't divide all of the 17[a] […] and here she does divide all the 17!
291	Rick	Correct […] so first of all the big difference between this [17:5 = $3\frac{2}{5}$] and this [17:5 = 3(2)] is that here she divided it/them[b] all and here she didn't divide it all.
297	Rick	This remainder 2, when she decided to divide them[c] after all
298	Ann	She divided [the remainder] by five.

[a] "Divide" can refer to the abstract arithmetic operation, in which case the translation would be "divide the whole number 17" or to the concrete act of distributing fairly, in which case the translation would be "distribute all of the 17 cookies." This ambiguity continues throughout the discussion

[b] In Hebrew, pronouns such as "it" and "them" are sometimes omitted. Literally, the translation is "she divided all," which maintains the ambiguity—we are arithmetically dividing "it" (the number 17), but we are also partitively dividing "them" (17 cookies)

[c] Here, the pronoun is unambiguously "them," referring to cookies

The point here is that "at first" only 15 out of the 17 is/are divided. Rick is walking a narrow line between an abstract arithmetic discourse and a concrete discourse of a model. "At first" is ambiguous and could refer either to the grandmother's actions (at first keeping the remainder for herself) or to a point in the curriculum before children know fractions. The affordance of this ambiguity as a bridge is demonstrated by Ann's utterance (298) "divided ***by*** 5," which was drawn from her arithmetic discourse (numbers are divided ***by*** 5, cookies are divided ***into*** 5 portions[7]) in response to Rick's partitive discourse "she decided to divide *them.*"

301	Dana	[…] In fact, here every child received more [than 3]
304	Rick	The potential [to divide the remainder] remained in the grandma's hand. Maybe later she'll distribute it.

"Later," the complement of "at first", is similarly ambiguous; it could either mean when grandma brings a knife and splits the cookies or when students learn fractions (or both simultaneously). Rick has introduced a discourse midway between whole numbers and fractions, still presenting the remainder as a whole number, yet one that has the potential to be divided in the partitive model. This has implications for the discussed inadequacy of the standard notation; if we consider the potential to divide the remainder, the amount of cookies each grandchild will receive depends on the divisor; thus, the remainder does *not* have independent numeric value—its value depends on the divisor. This "potential to divide" featured prominently in what came next: The new notation was seen not only as addressing inadequacies of the standard notation, i.e., inadequacy #1 (previously discussed) and inadequacy #2 (discussed hereinafter), but also as having pedagogical merit.

Inadequacy #2: Standard notation violates transitivity

Seeing that the teachers did not accept the first inadequacy of the standard notation—3(2) does not have independent numerical value—Rick presented a second inadequacy.

[7] Hebrew distinguishes between dividing *by* and dividing *into* in much the same way as English does.

310	Rick	Something bothered us as mathematicians. We said, wait a minute, if this equality really expresses an equivalence […] apparently I must deduce that 17:5 and 11:3 are one and the same
311	Sal	Right
312	Rick	And it really isn't… it really is not.
322	Kim	Here it is two-thirds and here it's two-fifths
323, 325	Rick	[…] if I wanted to divide it, I'd divide it here into three [parts] and here into five.
327, 329	Rick	[…] Our suggestion was, let's, instead of just writing [the remainder] 2, we'll remind ourselves that it's 2 that I wanted to divide into 3.

Kim, in utterance 322, slipped into the discourse of fractions, but Rick corrected her, returning to the "midway" discourse of the potential to divide ("if I wanted to divide it…"). Nevertheless, she appears to have understood the point: From the perspective of fractions 3(2) is ambiguous, and this ambiguity is resolved in the new notation. However, Rina and Sal seemed to be aware of two distinct discourses—"before" (whole numbers) and "after" (fractions):

330	Rina	But before the division[a] it's still alright [to say that 17:5 and 11:3 are equivalent].
335	Sal	I'm just saying that […] meanwhile, before… nothing has been divided…
336	Rick	Yes?
337	Sal	Still two apples […]
338	Rick	They're apples, yes
341	Sal	Here and here
342	Rick	Right
343	Sal	That is, as far as the quantity is concerned, everything is alright

[a] Yet again "before" is ambiguous—before division is learned or before it is performed

Rina and Sal had a good point which Rick did not appreciate at the time, according to his later testimony. The implication is interesting: Rick's concern with the standard notation was primarily mathematical; however, the statement 17:5 = 11:3 can be mathematically acceptable, as long as we take this equality to mean that the two expressions have the same quotient and the same remainder. This ***is*** in fact an equivalence relationship! Yet, one can argue that it is the wrong equivalence. With the mathematical horizon in mind, equality of rational numbers is the more reasonable equivalence, and Rick has shown this without resorting to fractions.

Although his warrant was grounded in the topic of fractions, Rick emphasized that he was not suggesting that fractions be taught in grade 3. This is still whole number notation. The remainder notations (2:3) and (2:5) imply that the *whole number* remainder (2) has the *potential* to be divided by 3 or 5, and in this sense they are different. Thus, his discourse is actually a bridge between whole numbers and fractions, where fractional division is still just a potential. This bridging status was further explored:

361	Rick	Though I continue to speak the language of whole numbers […], I have written the answer here in fractions. But there aren't any fractions here, Okay? Everything is already ripe [for fractions], right?
372	Perl	[…] You can't call it a quotient.
374	Ann	It's a partial quotient…
376	Rick	[…] It will be a quotient only when we move to fractions, of course.

After a long discussion of the *mathematical inadequacies* of the standard notation as a warrant for a new notation, Rick brought in 361 a totally new kind of warrant—a *pedagogical* consideration *in favor* of the new notation. Rick's rather vague claim—that "everything is ripe [for fractions]"—left it up to the teachers to fill in pedagogical details of a transition from the new notation's discourse of "potential to be divided" to the discourse of fractions. And some teachers did indeed draw on their experience and articulated some pedagogical affordances of the new notation, as elaborated in episode 2.

Episode 2: Implications of the new notation for long division

Rick made a connection between DWR and the long division algorithm,[8] showing how each stage in the algorithm is in fact a problem in DWR, where the subtraction stage finds the remainder. Two teachers took this further, connecting their own ideas to the new notation.

The new notation resonates with long division

In Israel, long division is conducted with the divisor to the right of the dividend (see Fig. 1), and the remainder is written directly above the divisor. A teacher pointed out that extending the horizontal line above the dividend "transforms" the remainder (3) into the fraction $\frac{3}{4}$.

475	Rick	I have 3 [remainder], and now what do I do with [the remainder]?
478	Kim	In brackets
479	Rick	It comes here, right?
480	Kim	So it [the remainder[a]] is three-fourths
481	Rick	And I remind myself [with the new notation] that it's from division by four.
482	Kim	But actually that's good. You can write it under the four[b], and then it's three-fourths.
483	Tammy	Ahh!
484, 490	Rick	Oh, nice! Very clever. Did you hear? […] the 4 underneath it, and then it's so tempting to do this and wrap it up, right?
491	Kim	Put it in brackets, the three.
492	Rick	With, without, as you like. It doesn't matter

[a] In Hebrew, pronouns have different forms for different genders. "It" was used in the feminine form and thus referred to "remainder" (a feminine noun) and not to the number three (a masculine noun)

[b] Kim probably meant to say "*over* the four," as shown in Fig. 1

[8] See Section 6 in the lesson graph, "Appendix".

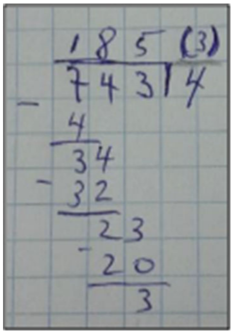


Fig. 1 Long division algorithm

Rick's pedagogical warrant for the new notation struck a chord with Kim, who was concerned with the transition from whole numbers to fractions. Her slip in 480 (three-fourths is not a remainder, it is the *rational* result of the division) highlights how the discourse of fractions may emerge from the discourse of remainders and how the new notation may support this emergence. Her "trick" of extending the division line creates a visual mediation $\frac{(3)}{4}$ quite similar to Rick's new notation, which blurs the distinction between remainder and fraction. The affordance of this notation to bridge whole numbers and fractions is reflected in Rick's comment "with, without [brackets], as you like." With brackets, it's still a whole number remainder with the *potential* to be divided; when the brackets are removed, it becomes a fraction.

The ambiguity regarding $\frac{3}{4}$, as a number or as a not-yet-completed result of division, is removed in grade 6 when long division is extended to decimals. The remainder 3 is an interim stage; for example, when students continue the long division of 743:4 (presented in Fig. 1) beyond the decimal point, they are expected to get the numeric value 185.75 (see Fig. 2). In this context, the affordance of the new notation was seen to be even more compelling, as the next subsection reveals.

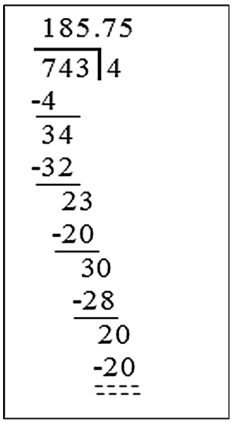


Fig. 2 Decimal long division

Another pedagogical affordance of the new notation: Transition to decimal long division

493–513	Sher	I saw the need for what you presented in division [with remainder notation] […] that in grade 6 when we moved to division, we wrote the answer as a decimal number […] Because here the remainder is three-fourths […] so I wanted to show them that right away they'll get […] point 75 […] but they saw only the three. They didn't see the three over four […] because with remainder we write 3 in brackets […] Until now. And therefore this [notation] […] it was lacking. […] They arrived at the correct answer of […] point 75 […] but I wanted them to see on their own […] what they need to arrive at.

Sher recognized an additional affordance of the new notation. In 6th grade, once students have learned decimal fractions, the long division algorithm might yield a decimal result, yet her students do not necessarily see the need to continue the algorithm once a remainder is obtained. Sher felt that if students use the new notation in simple exercises, and perhaps if they also use Kim's visual mediator in long division (185 $\frac{(3)}{4}$), it may help them overcome this potential difficulty later on in grade 6. Using the standard notation—e.g., 185(3)—"they saw only the three"; using the new notation—e.g., 185(3:4)—where the remainder is presented as an uncompleted division right from the start in grade 3, might ease the transition to decimal calculations where the operation can finally be completed.

Episode 3: The special role of zero remainder

Is zero a legitimate remainder, or is it more correct to say that when the dividend is divisible by the divisor, there is *no remainder*? This issue was debated in the lesson:[9]

530	Lara	When I taught DWR, I had some children who also wrote the zero in brackets… It seemed to me as though something was not understood regarding remainder zero… to me it looked as though they did not understand the concept of remainder.
540	Dana	They didn't understand that there was no remainder.
541	Rick	In just a moment we will write remainder zero.
542	Lara	Remainder zero?
543	Rick	I like remainder zero.

At first glance this may appear to be an issue of little significance, as Rick suggested later in the lesson (958): "As you prefer, 'remainder zero' or 'no remainder,' it doesn't matter, because obviously it expresses exactly the same thing… a number that is divisible…" Both notations, e.g., 10:2 = 5 and 10:2 = 5(0), must ultimately be accepted as mathematically correct. Lara and Dana did not claim that the second notation is incorrect. Their concern originated in their pedagogical content discourse, not in their mathematical discourse. In other words, they drew on their experience in teaching this content and

[9] See Sections 6 and 8 in the lesson graph, "Appendix".

expressed reservations about the use of remainder zero by students, not because it is incorrect (from a mathematical perspective), but because this may indicate that students have not fully understood the idea of remainder (from a pedagogical perspective). This concern about student *understanding* was elaborated when a teacher shared an idea on how to help students develop an understanding of remainder in general and remainder zero in particular:

551, 553	Sara	Like in grade 3, I was teaching my individual [teaching] group of weak children. And I prepared a table for them […] I wrote for them dividend, divisor, quotient, and remainder [at the top of the columns] […] I gave [DWR] exercises and we wrote the remainder under the remainder [heading] and the quotient […]
554	Rick	Ah… and it came out zero. Yes.
555–574	Sara	And we talked about the fact that in the place where there's no remainder, it's the zero […] When we arrived [at the realization] that it's zero […] I gave them the rule, and I also drew […] say 17 divided by 5 […] but I drew it for them in circles[a] […] I divided the 17 and they saw […] [and then] I said "now 16, now 18. What are we left with? How much?" We drew circles underneath, inside the table. And when it [the table] filled up they suddenly said "Oh! Now it's possible to divide and there's no remainder". And when we again added 1, it's 21, again with a remainder of 1. And that's how we reached that rule you made [writing remainder 0], that they really understood.

[a] Representing quotitive division of objects

Sara's description of her activity sheds light on the concerns previously expressed by Lara and Dana. The *understanding* of zero as a remainder ("they really understood") is grounded in the connections between two discourses—arithmetic and quotitive. The table reveals a cyclic pattern in the arithmetic discourse—as the dividend increases by 1, the remainder increases up to 4 and then returns to 0 (see Fig. 3). This was accompanied by a visual representation of a model, which connected remainder 0 to "nothing left over."

This table activity is very significant from a commognitive perspective, in combining two different modes of mathematizing. The visual mediation (groups of objects) provides an extra-mathematical representation of zero as a remainder, whereas the patterns in the table provide a warrant "within" the mathematics for accepting zero as a valid remainder. This second type of warrant is crucial at many mathematical junctions, where inter-mathematical warrants are the only realistic option. For example, it is nearly impossible to justify the rules for multiplying signed numbers extra-mathematically—it is difficult to see how the product of two negative numbers represents something familiar in our world. An inter-mathematical warrant is necessary, e.g., by showing consistency with the distributive rule (see Sfard 2007).

Even after this activity was discussed, the issue of remainder zero was far from resolved. Later in the lesson, the following "fill in the blanks" problem was addressed: _:3 = 7(_)[10], and teachers discussed whether $\underline{21}:3 = 7(\underline{0})$ is a correct completion.

702	Lynn	Twenty-one, twenty-two…
704	Ann	Twenty-one—no, because you have a remainder.
705	Lynn	It could be that the remainder is zero
711	Rick	Something divided by three is 7 with a remainder. What's the answer?

[10] Or _:3 = 7(_:3) if we accept Rick's new notation.

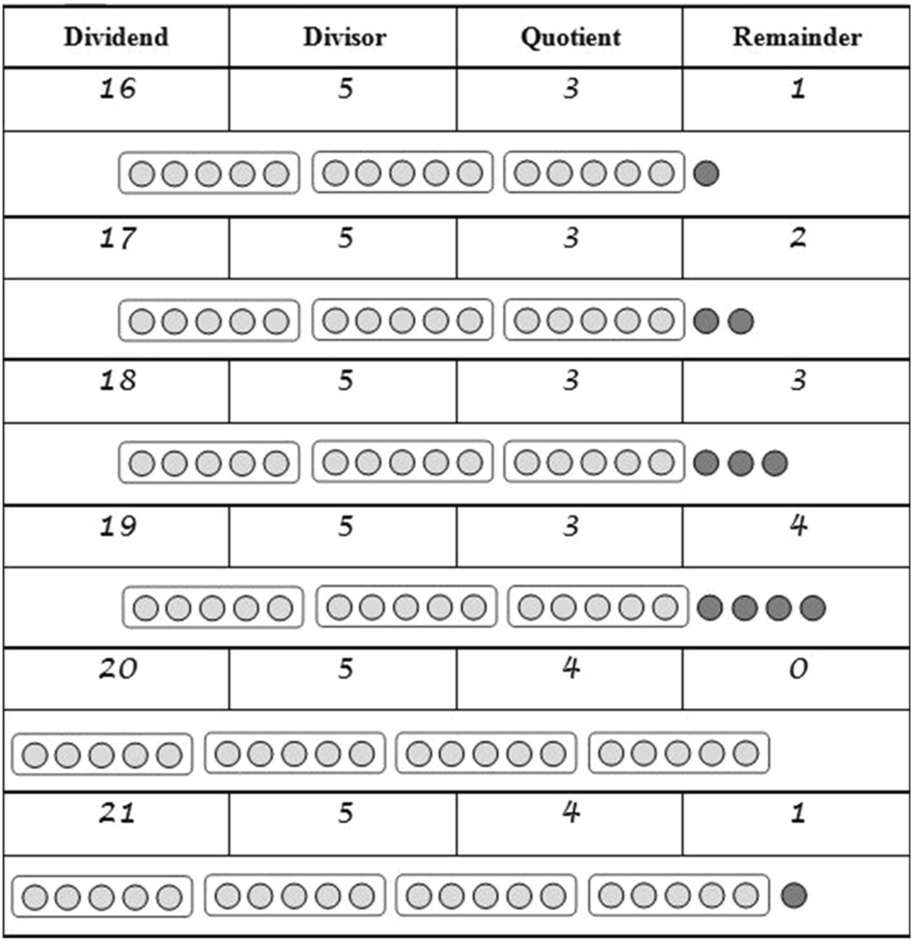


Fig. 3 Sara's table activity

Rick, by adding "with a remainder," implied that "without a remainder" might have a different visual mediation, perhaps without brackets. Dana and Gwen picked up on this:

723	Dana	When I write brackets, I mean there ***is*** a remainder. If you tell me zero…
724	Gwen	That's what I said
725	Dana	[you're saying] there ***is no*** remainder. Isn't this possibility [21] removed?
728	Rick	It's your choice…
731	Gwen	There's reason in it. What is remainder zero anyway? It stands to reason.

Rick has given teachers the agency to decide whether remainder zero should be a legitimate part of their classroom mathematical discourse. This shows pedagogical sensitivity on his part, realizing that this is not strictly a mathematical issue—that there are teachers and students involved as well. Interestingly, it was a teacher who held the torch of algebraic discourse:

732	Lucy	No, but when you build the template of the corresponding multiplication exercise, then it can be…

Lucy had the multiplicative form in mind—*find q and r such that* $a = qb + r$. From this perspective, $r = 0$ is valid just like any other value. By choosing to warrant her narrative in this way, she preferred the algebraic discourse over the arithmetic discourse of calculation.

Discussion

We have shown in our analysis the interplay of two communities' MDT. We now discuss the affordances of this interplay for learning in a PD scenario.

Teachers' and mathematicians' learning: new DWR notation

Our story begins with a research mathematician's realization that primary school DWR notation is incompatible with his mathematical discourse, in its violation of the transitive property of equality. The impetus for a new notation was mathematical; however, he was aware of two pedagogical considerations. First, he realized that coming to view equality as an equivalence relationship is important for students in making the transition to algebra, which begins with addition with unknowns. Second, he realized that a purely mathematical concern would probably not take root with the teachers. Recognizing the difference in these communities' meta-rules for endorsing notations, he emphasized the pedagogical affordance of the new notation in students' transition from the discourse of whole numbers to the discourse of fractions. However, he did not compromise his own mathematical discourse and began the discussion with a mathematical inadequacy of the standard notation.

The teachers were not passive in the mathematical discussion that followed. Their participation was *explorative*, in its goal to endorse or reject new narratives about DWR and its notation. Some challenged Rick, questioning the mathematical validity of his claims. Teachers may be experts in the realm of specialized content knowledge (knowledge that is specific for teaching; Ball et al. 2008), but we were surprised to find teachers challenging a mathematician on his home turf of advanced mathematics. Furthermore, they were correct in claiming that the equality 17:5 = 11:3 is not intrinsically incorrect. This gave rise to an interesting insight—DWR notation violates the transitivity of equality *only* with fractions in mind! Here, we see the different—yet complimentary—roles of discourse at the mathematical horizon in the parties' MDT. For both, the relevant mathematical object at the horizon is fractions, yet for the mathematician this had mathematical implications (the standard notation violates transitivity, holding fractions in mind), whereas for the teachers it was connected to pedagogical issues (the new notation may have affordances for students' learning of fractions).

Once the new notation had been presented, teachers engaged in the first stages of learning; thoughtful consideration of how this new visual mediation fits in with their own MDT is the first step in transforming it from discourse-for-others to discourse-for-oneself. This enabled two new pedagogical insights, proposed by Kim and by Sher. The need for such a notation was already present in Kim's MDT, realized in her use of a similar remainder notation $\frac{(3)}{4}$, though only as the result of long division. Thus, Kim's visual mediation took on new meaning in the PD as a transitional mediator between whole numbers and fractions. Sher recognized how the new notation could be pedagogically productive in avoiding a common student difficulty with continuing the long division procedure beyond the decimal point. This insight was born in the interaction between two communities—the teachers, for their part, would not have considered changing the notation, and the mathematician could not have recognized this pedagogical affordance of the new notation on his own.

To sum up the nature of learning that took place: The mathematician engaged in object-level learning in preparing for the lesson, developing new narratives and visual mediators pertaining to DWR. Learning continued in the lesson at the object level when teachers challenged the mathematician's mathematical narratives and at the meta-level when the parties shared rules for endorsing narratives—Rick bringing rules from his university mathematical discourse and teachers bringing rules from their pedagogical content discourse. All parties' discourse of content and teaching was enriched in the discussion of the *potential* to divide the remainder, which suggests how teachers can help students make the transition to fractions.

Teachers' and mathematicians' learning: remainder zero

Unlike the new notation, which was Rick's initiative, the issue of remainder 0 was raised by teachers, who persisted even when Rick tried to brush it off as merely a semantic issue. The discussion stemmed from Lara's discourse of content and students—her concern that children who write remainder 0 in exercises seem not to have understood the concept of remainder. Rick eventually addressed this issue from a mathematical perspective, in the context of remainder arithmetic[11] where remainder zero is a crucial member of the cyclic ring; however, it was Sara who responded first, with a description of a classroom activity. As in the case of Kim, the connection between the classroom activity and the mathematical issue at stake emerged through the interaction between communities—a teacher raising a pedagogical concern (understanding remainder), Rick's cryptic response (*I like remainder 0*), which was presumably a precursor to remainder arithmetic, and the teacher's spontaneous connection of all this to a tried and proven classroom activity, which took on new meaning in the context of the discussion. Sara's activity reveals the complexity of "understanding the concept of remainder" in her MDT, which includes connecting the arithmetic discourse (quotient and remainder) and the discourse of the quotitive model (circles around objects). Again, the parties were presented with opportunities to enrich both their mathematical and their pedagogical discourses at the object level. Furthermore, Lucy's warrant for accepting 0 as a legitimate remainder presented a strong opportunity for meta-level learning, in suggesting an algebraic discourse of DWR (find q and r) instead of an arithmetic discourse (perform a division operation).

[11] See Section 9 in the lesson graph, "Appendix".

Researchers' learning

Our discussion of learning would not be complete without considering the third community involved—the researchers. Our observation and analysis of the episodes met our own research discourse (i.e., theoretical framework)—the commognitive framework of MDT. This framework allows us to appreciate and articulate aspects of the lesson in our own ways, which are outside the realm of mathematicians' or teachers' MDT. We bring 4 examples:

1. The explicitness in the new notation of the *potential* to divide the remainder suggests a solution to the learning paradox (Sfard 1998)—how to talk about fractions (as the *potential* to be divided) using only mathematical objects from the discourse of whole numbers, cleverly taking advantage of linguistic ambiguities of words such as *divide* (cookies or number).
2. Kim's slip, in seeing the fraction three-fourths as a remainder, suggests a possible learning trajectory from DWR to fractions, where talk shifts from a division *process*, which has a "result" (quotient and remainder), to a mathematical object (rational number).[12] Stages of this shift may include: I. The whole number remainder 3; II. A whole number remainder with the potential to be divided (3:4); III. The potential to divide the remainder is realized (three-fourths), yet it is still considered a remainder; IV. The result of division, including its fractional part (three-fourths), is a *number*.

The following two examples related to commognitive conflict between narratives, which Sfard describes as "seemingly conflicting narratives [that] are originating from different discourses—from discourses that differ in their use of words, in the rules of substantiation, and so forth. Such discourses are incommensurable rather than incompatible; that is, they do not share criteria for deciding whether a given narrative should be endorsed" (Sfard 2008, p. 257).

3. Sara's table with separate columns for dividend, quotient and remainder (keeping the divisor unchanged; see Fig. 3) can be seen as a step toward new rules of mathematical discourse, where narratives (e.g., *is zero a legitimate remainder*) are not warranted empirically (*there is nothing left over, so why write it*) but rather based on mathematical consistency (maintaining a pattern). Sara sensitively included empirical considerations in her activity (visual representations of quotitive division), thus bridging this meta-level transition, which is known to create commognitive conflict, e.g., in the context of negative numbers (Sfard 2007).
4. Lucy's algebraic discourse was in commognitive conflict with the discourse that Ann, Dana and Gwen engaged in. In what sense does the exercise __:3 = 7(_) imply that the remainder is not zero? We may imagine the following state of affairs: Someone was posed a calculation exercise, e.g., 22:3 = __, which was solved: 22:3 = 7(1). Bits of the solution were obscured, and our task is to reconstruct them. Presumably, if the problem had been 21:3 = __, the solution would have been written as 7, not as 7(0), which led Ann, Dana and Gwen to rule out 21:3 = 7(0). This is actually a useful interpretation for problems with unknowns, since it allows students to engage in algebraic routines of balancing quantities, while they still associate the equals symbol with routines of calculation (Cooper and Arcavi 2013 discussed how this interpretation

[12] Such a shift from talk about process to talk about object is called by Sfard (2008) *reification* of the process.

was supported in the PD by placing a curtain over unknowns, implying that they had already been calculated). However, this interpretation of exercises with unknowns should be transient, eventually to be replaced with an algebraic discourse (find q and r), where $r = 0$ is perfectly legitimate.

We note how our findings resonate with the work of other researchers on the "practical rationality" of teaching (Herbst and Chazan 2003) and in particular with the work of Nardi et al. (2012) on pedagogical and epistemological (mathematical) considerations in teachers' warrants. Rick's warrants for a new notation were, in the terms used by these researchers, both *epistemological* (avoids a violation of transitivity) and *pedagogical* (affords a smooth transition to fractions). These types of warrants were also evident in the teachers' discourse, in their epistemological criticism of Rick's warrant (transitivity is not violated in the standard notation) and in their endorsement of his pedagogical warrant. Furthermore, they were able to add *empirical* warrants, both *professional* (Sara's classroom activity) and *personal* (Sher's belief that the new notation would alleviate student difficulties in decimal long division). It is interesting to note that *institutional curricular* warrants remained outside the scope of the discussion—teachers did not appear to be bothered by the impracticality of adopting a notation that is different from the one used in published textbooks. Nardi et al. (2012) developed their framework for the analysis of teachers' argumentation regarding student work, yet it appears to be applicable in this context of warranting a mathematical notation as well.

Concluding words

When Wu, cited in the introduction section of this article, quoted the age old maxim "you cannot teach what you don't know," he had mathematical knowledge in mind. From an acquisitionist perspective (Sfard 1998), this seems obvious—to acquire knowledge from a teacher, the teacher must first possess this knowledge. This acquisitionist perspective does not do justice to the PD lesson analyzed here. A participationist perspective (ibid.) is crucial for understanding the learning that took place. From our commognitive perspective, learning took place through participation in discourse. Yet unlike classroom mathematical discourse, where students' discourse should eventually converge to that of the teacher, in this PD it could not be expected that teachers would adopt a research mathematician's mathematical discourse for teaching, nor was Rick (the mathematician who taught the PD) perceived as the ultimate authority for such discourse. Rather, expertise was shared between the two communities, and learning was based on exploring new narratives, challenging them and becoming mutually aware of affordances and concerns drawn from the other community's discourse. Such mutual learning suggests a model for mathematicians' contribution to teacher education, in providing opportunities for the exploration of mathematical ideas, while remaining sensitive to aspects of mathematical knowledge that are useful and usable for teachers, as advocated by Bass (2005). Without forsaking mutual respect and careful listening to each other's ideas, the mathematician and the primary school teachers in this PD managed to seize opportunities for questioning and rethinking concepts, notations and procedures, making the meeting of these two communities productive. What was eventually learned amounted to more than the sum of what the parties brought to the PD.

Acknowledgments This research was supported by the Israel Science Foundation (Grant No. 615/13). The authors wish to thank Abraham Arcavi for many helpful suggestions.

Appendix: Lesson graph of the DWR session

See Table 1.

Table 1 Rick is the mathematician teaching the lesson)

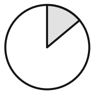
1. Introductory discussion (transcript lines 8–56[a])

Rick opened with his impression that DWR is a somewhat marginal topic in the curriculum, and asked for reasons to teach it. The teachers raised two main purposes:

- DWR represents certain kinds of real-life situations
- DWR prepares students for the topic of fractions

Rick added two more reasons:

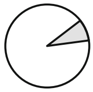
- DWR is necessary for understanding signs of divisibility
- Various challenging tasks can be built around DWR for advanced students



7 min

2. The meaning of division (transcript lines 74–128)

Rick presented the problem 18:3 = ? along with a drawing of 18 circles, and asked what is the tangible meaning of this problem. A teacher noted that there are two meanings of division, partition and quotition, and Rick applied this differentiation to the problem: If I divide 18 objects into three equal groups, how many objects will each group include; If I divide 18 objects into groups of 3, how many groups will I get. The inverse problem for the first meaning is '3 times what equals 18' ($3 \times ? = 18$), whereas the inverse problem for the second meaning is 'how many threes equal 18' ($? \times 3 = 18$), and Rick noted that the two meanings can co-exist due to the commutative law of multiplication.



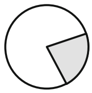
4 min

3. The meaning of division with remainder—DWR (transcript lines 129–259)

Rick presented two word problems, and asked what are the similarities and differences between them:

- 17 children arrive at a basketball match; how many teams of 5 players can be formed?
- Grandma shares 17 candies between her 5 grandchildren; how many candies does each child get?

The teachers responded that both are answered by 17:5 = ?; however the first is quotition and the second is partition. After a short discussion of pedagogical implications of the two division modes, Rick returned to the second problem, asking how students can solve 17:5 = ? in the partition interpretation, that is, $5 \times ? = 17$. One teacher suggested instructing students to color the candies one by one using the 5 different colors alternately. The result of this process, as Rick stressed, is that we get an answer of 3 candies per grandchild, *but* two candies remain uncolored. The first problem can be solved using the quotition mode, which is procedurally easier, yet again the answer invariably includes a "but" clause: 3 teams, *but* two poor children are left out of the match. Rick claimed that young students have never before encountered answers that include the word *but,* and if we wish to omit this word, we need to decide what to do with the undivided quantity. At a later stage, the answer will be $3\frac{2}{5}$ or 3.4, however at this stage of whole number arithmetic, we must resort to DWR with an agreed-upon notation, such as 3(2) as common in Israel, or 3R2 as common in the US.



13 min

Table 1 continued

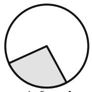
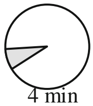
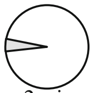
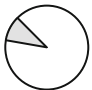
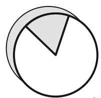
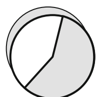
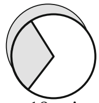
4. Notations of DWR (transcript lines 259–438) The main question raised for discussion in this part was ‘*what kind of mathematical object is 3(2), and does it have a numerical value in the sense that* $3\frac{2}{5}$ *has?’* First, teachers noted that whereas $3\frac{2}{5}$ signifies that all the quantity was divided, in the case of 3(2) we actually only divided 15. Rick then said that what really troubled him was that if we write 17:5 = 3(2) and also 11:3 = 3(2), we must as mathematicians conclude that 17:5 = 11:3, which makes no sense. The teachers engaged in a lively discussion about equivalence. Rick then introduced a new notation: 17:5 = 3(2:3) which is read “three with remainder 2 which needs to be divided by 3”. The mathematical and pedagogical affordances of this new notation were discussed: mathematical consistency; staying in the world of whole number arithmetic yet allowing for a future smooth transition to fractions; the ability to construct and check an inverse problem (e.g., 45:6 = 7(3:6) → 7 × 6 + 3 = 45)	 16 min
5. DWR and the Long Division Algorithm (transcript lines 439–528) Rick used the problem 743:4 to demonstrate how DWR is a central component of long division, where the subtraction phase calculates the interim remainder. Two teachers articulated connections between long division and DWR. One said that if we write the remainder (3) above the divisor 4, it can be interpreted as “3 over 4”, which is a bridge to fractions. Another suggested that using the new notation 185(3:4) in grade 3 may help 6th graders realize that they can and should divide the remainder, and continue the algorithm until they reach the decimal answer of 185.75.	 4 min
6. Remainder of zero (transcript lines 530–574) A teacher noted the phenomenon of students writing zero as a remainder, and wondered whether this indicates a problematic understanding of the concept of remainder. Another teacher, speaking in favour of remainder 0, explained how searching for patterns in a “division table”, with headings: dividend, divisor, quotient, remainder, can justify the value zero in cases of no remainder, and can help remedial students grasp the concept of remainder.	 2 min
7. Contextual relevance of DWR in word problems (transcript lines 575–620) Rick presented three new word problems, all connected to the same division problem 22:5, as follows: • Gili has 22 NIS (Israeli currency); how many energy bars can she buy, if one bar costs 5 NIS? • 22 people went on a trip; how many cars are needed if 5 people can fit into one car? • A man walks 22 km in 5 h, at a constant rate. What distance did he walk in 1 h? The discussion focused on how the three different answers (4, 5, $4\frac{2}{5}$, respectively) highlight the importance of context in division problems. DWR as an arithmetic operation provides the correct answer only in the first problem, and is senseless in the other two.	 6 min
8. Multiple solution DWR problems (transcript lines 622–814) Rick presented a sequence of problems that can be termed “DWR with unknowns” (returning to the standard DWR notation). The first few problems had a single solution (e.g., 25:4 = ?(?); 23:5 = ?(3); 17:? = 3(2)). Rick discussed with the teachers the level of difficulty and systematic ways to solve such problems. He then turned to problems with multiple solutions: ?:3 = 7(?); 35:? = 3(?); 26:? = ?(2), which he offered as tasks for advanced students, and the teachers elicited strategies for solving them. [10 min break]	 10 min[b]
9. Remainder arithmetic (transcript lines 823–1214) Rick positioned “rules of evenness” (e.g., odd + odd = even) in a more general context, which makes use of remainder arithmetic. The basic idea presented was to view the case of parity as a problem of finding the remainder when dividing by 2, and considering that “the remainder of a sum equals the sum of the remainders”. Rick demonstrated adding remainders when the divisor is 5, using several visual mediators: (1) ‘blocks of 5’ + a remainder; (2) skip-counting on the number line in steps of 5; and (3) using a ‘remainder clock’ with 5 signs instead of 12 (remainders 0, 1, 2, 3, 4). He then moved on to what he referred to as a topic for advanced students—multiplying remainders (dividing by 5): He used the mediator of a rectangular array to demonstrate why, for instance, when multiplying a number with remainder 3 and a number with remainder 2, the result will have a remainder of 1. He also spoke a little about the usefulness of remainder arithmetic in data encryption.	 34 min

Table 1 continued

10. Signs of divisibility (transcript lines 1215–1435) The first issue in this part of the lesson was explaining the rule for divisibility by 9 (if the sum of digits in a given number is divisible by 9, then the number is also divisible by 9), by looking at the digits as representing accumulating remainders. Later, the case of divisibility by 11 was discussed.	 19 min

Total session time: 139 min (net. 115 min)

[a] The lesson graph excludes transcript lines in which technical matters (i.e., managing slides and setting air condition) are dealt with

[b] The shaded circle in the back indicates one whole hour

References

Ball, D. L., Thames, M. H., & Phelps, G. (2008). Content knowledge for teaching: What makes it special. *Journal of Teacher Education, 59*(5), 389–407.

Bass, H. (2005). Mathematics, mathematicians, and mathematics education. *Bulletin (New Series) of the American Mathematical Society, 42*(4), 417–430.

Ben-Zvi, D., & Sfard, A. (2007). Ariadne's string, Daedalus' wings, and the learner's autonomy. *Éducation et Didactique (Education and Didactics), 1*(3), 117–134.

Cooper, J. (2014). Mathematical discourse for teaching: A discursive framework for analyzing professional development. In P. Liljedahl, C. Nicol, S. Oesterle, & D. Allan (Eds.), *Proceedings of the Joint Meeting of PME 38 and PME-NA 36. 2* (pp. 337–344). Vancouver: PME.

Cooper, J. (2015a). Combining mathematical and education perspectives in professional development. In X. Sun, B. Kaur, & J. Novo (Eds.), *Proceedings of ICMI Study 23 on Whole Number Arithmetic* (pp. 68–75). Macau: University of Macau. Retrieved from http://www.umac.mo/fed/ICMI23/doc/Proceedings_ICMI_STUDY_23_final.pdf

Cooper, J. (2015b). Growth of mathematical knowledge for teaching: The case of long division. In K. Krainer & N. Vondrová (Eds.), *Proceedings of the ninth conference of the European Society for Research in Mathematics Education (CERME9)* (pp. 2081–2088). Prague: Charles University in Prague, Faculty of Education and ERME.

Cooper, J. (2016). *Mathematicians and primary school teachers learning from each other.* Unpublished doctoral dissertation, Weizmann Institute of Science, Israel.

Cooper, J., & Arcavi, A. (2013). Mathematicians and elementary school mathematics teachers: Meetings and bridges. In Y. Li & J. N. Moschkovich (Eds.), *Proficiency and beliefs in learning and teaching mathematics: Learning from Alan Schoenfeld and Günter Törner* (pp. 179–200). Rotterdam: Sense Publishers.

Davison, D. M., & Mitchell, J. E. (2008). How is mathematics education philosophy reflected in the math wars. *The Montana Mathematics Enthusiast, 5*(1), 143–154.

Fried, M. N. (2014). Mathematics and mathematics education: Searching for common ground. In M. N. Fried & T. Dreyfus (Eds.), *Mathematics and mathematics education: Searching for common ground, Advances in Mathematics Education series* (pp. 3–22). New York: Springer.

Herbst, P., & Chazan, D. (2003). Exploring the practical rationality of mathematics teaching through conversations about videotaped episodes: The case of engaging students in proving. *For the Learning of Mathematics, 23*(1), 2–14.

Hodgson, B. (2002). The mathematical education of school teachers: Role and responsibilities of university mathematicians. In D. Holton, M. Artigue, U. Kirchgraeber, J. Hillel, M. Niss, & A. Schoenfeld (Eds.), *The teaching and learning of mathematics at university level: An ICMI study* (pp. 501–518). New York: Kluwer Academic Publishers.

ICMI. (2008). The first century of the international commission on mathematical instruction. In F. Furinghetti, & L. Giacardi (Eds.), *History of ICMI.* Retrieved September 28, 2014, from ICMI: http://www.icmihistory.unito.it/

Israeli Mathematicians. (2010, October 19). *Math-Program-letter.pdf.* Retrieved from http://www.wisdom.weizmann.ac.il/~dnovikov/MathEd/Math-Program-letter.pdf

Klein, F. (1923). Göttinger Professoren. Lebensbilder von eigener Hand. 4. *Mitteilungen des Universitätsbundes Göttingen*, *5*(1), 11–36.

Klein, D., Askey, R., Milgram, R. J., Wu, H.-H., Scharlemann, M., & Tsang, B. (1999). *An open letter to Richard Riley, United States Secretary of Education*. Retrieved 2015, from California State University Northridge. http://www.csun.edu/~vcmth00m/riley.html

Mathematics Learning Study Committee. (2001). The mathematical knowledge children bring to school. In J. Kilpatrick, J. Swafford, & B. Findell (Eds.), *Adding it up: Helping children learn mathematics* (pp. 157–180). Washington, DC: National Academy Press.

Nardi, E., Biza, I., & Zachariades, T. (2012). 'Warrant' revisited: Integrating mathematics teachers' pedagogical and epistemological consideration into Toulmin's model for argumentation. *Educational Studies in Mathematics, 79*(2), 157–173.

Nardi, E., Ryve, A., Stadler, E., & Viirman, O. (2014). Commognitive analyses of the learning and teaching of mathematics at university level: The case of discursive shifts in the study of Calculus. *Research in Mathematics Education, 16*(2), 182–198.

Pedagogical Secretariat of the Israeli Ministry of Education. (2009, 6 24). *A New Mathematics Curriculum for Primary School for All Sectors (In Hebrew)*. Retrieved 3 14, 2016, from http://cms.education.gov.il/EducationCMS/Units/Tochniyot_Limudim/Math_Yesodi/PDF/

Ralston, A. (2004). Research mathematicians and mathematics education: A critique. *Notices of the American Mathematical Society, 51*(4), 403–411.

Schoenfeld, A. H. (2004). The math wars. *Educational Policy, 18*(1), 253–286.

Sfard, A. (1998). On two metaphors for learning and the dangers of choosing just one. *Educational Researcher, 27*(2), 4–13.

Sfard, A. (2007). When the rules of discourse change, but nobody tells you: Making sense of mathematics learning from a Commognitive standpoint. *Journal of the Learning Sciences, 16*(4), 565–613.

Sfard, A. (2008). *Thinking as communicating: Human development, the growth of discourses, and mathematizing*. Cambridge: Cambridge University Press.

Shulman, L. S. (1986). Those who understand: Knowledge growth in teaching. *Educational Researcher, 15*(2), 4–14.

Stylianides, G. J., & Stylianides, A. J. (2010). Mathematics for teaching: A form of applied mathematics. *Teaching and Teacher Education, 26*(2), 161–172.

Wu, H. (2011). The mis-education of mathematics teachers. *Notices of the AMS, 58*(03), 372–384.