J Math Teacher Educ (2018) 21:63–94
https://doi.org/10.1007/s10857-016-9349-8

Assessing key epistemic features of didactic-mathematical knowledge of prospective teachers: the case of the derivative

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Published online: 12 May 2016
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Abstract In recent years, there has been a growing interest in studying the knowledge that mathematics teachers require in order for their teaching to be effective. However, only a few studies have focused on the design and application of instruments that are capable of exploring different aspects of teachers' didactic-mathematical knowledge about specific topics. This article reports the results obtained following the application of a questionnaire designed specifically to assess certain key features of prospective, higher secondary-education teachers' knowledge of the derivative. The questionnaire was constructed using a theoretical model of mathematical knowledge for teaching based on the onto-semiotic approach to mathematical knowledge.

Keywords Teacher education · Teachers' knowledge · Didactic-mathematical knowledge · Epistemic facet · Derivative

Background

Over the last three decades, the knowledge which mathematics teachers require in order to provide effective instruction in relation to specific mathematical topics has become an increasing focus of interest, not only among researchers whose work concerns the education of mathematics teachers but also among educational authorities as a whole. The

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main reason for this is that the development of pupils' mathematical thinking and competences is inherently dependent on their teachers' abilities. Consequently, a great deal of research has been conducted in an attempt to identify the components of mathematical knowledge that teachers need in order to teach efficiently and to facilitate their students' learning. In this context, the work of Shulman (1986), Fennema and Franke (1992) and Ball (2000) offered a multifaceted perspective on the development of the knowledge required to teach. However, more recent research, such as that by Llinares and Krainer (2006), Ponte and Chapman (2006), Sowder (2007) and Sullivan and Wood (2008), highlights the lack of consensus regarding a theoretical framework for describing the knowledge required by mathematics teachers (Rowland and Ruthven 2011).

The abovementioned studies have all helped to further our understanding of the different knowledge components that teachers need to acquire in order to teach effectively and to foster their pupils' learning. However, a more detailed understanding of the knowledge required to teach mathematics can only emerge by focusing on specific topics, for example, the knowledge which a secondary school teacher needs in order to teach the derivative (Badillo et al. 2011). In this regard, it is not clear whether such knowledge can be adequately identified through the use of models based on "broad" categories. Indeed, despite the progress which has been made in terms of characterizing the knowledge which a mathematics teacher needs in order to offer effective instruction on specific topics such as the derivative, there is still a lack of clear criteria that would not only enable this knowledge to be analyzed and categorized in detail, but which would also aid researchers by providing guidelines for developing and strengthening such knowledge. Different authors (e.g., Asiala et al. 1997; Baker et al. 2000; García et al. 2011) have investigated the characteristics of high school and college students' understanding of derivative. Moreover, there have also been researches on the knowledge teachers require for an effective teaching of derivative (e.g., Berry and Nyman 2003; Badillo et al. 2011; Sánchez–Matamoros et al. 2012). Part of these studies have focused on practicing teachers and others on student teachers, using various frameworks to interpret the nature of derivative as well as the knowledge and comprehension thereof. However, there are still open problems that require further research.

Given the above, this article reports some of the results obtained following application of a questionnaire which, based on the *Didactic-Mathematical Knowledge* (DMK) model (Pino-Fan et al. 2015), for assessing and developing didactic-mathematical knowledge, was designed in order to explore key features of prospective secondary teachers' didactic-mathematical knowledge of the derivative, specifically in relation to what we refer to as its epistemic facet. Specifically, we address the following aims in this article:

1. Applying the DMK model analysis categories to high school prospective teachers' knowledge of derivative.
2. Exemplifying the use of Onto-Semiotic Approach (OSA) tools (system of practices; objects and processes configuration) to analyze the knowledge brought into play when solving mathematical tasks that involves the derivative.

The article is divided into seven sections. Following this introduction, section two presents the theoretical framework on which the questionnaire is based, while section three describes the methodology used. Sections four and five present, respectively, the content analysis of the tasks included in the questionnaire and the analysis of prospective teachers' knowledge in relation to one of these tasks. Finally, section six presents and discusses the results obtained, while in section seven a number of conclusions are drawn.

Theoretical framework

In mathematics education research, there are diverse proposals of models that attempt to identify and describe the elements making up knowledge of mathematics teachers. For example, the proposal of Shulman (1986, 1987), the teacher's knowledge model of Grossman (1990), the mathematical knowledge for teaching (MKT) of Ball and colleagues (Ball et al. 2001; Hill et al. 2008); the knowledge quartet (KQ) of Rowland, Huckstep and Thwaites (2005); and the theory of proficiency in teaching mathematics of Schoenfeld and Kilpatrick (2008). These scientific works, in which the many models of mathematics teacher's knowledge are developed, show a multifaceted vision of the identification of the knowledge required for teaching. Also, although they represent a significant improvement in the characterization of the knowledge that teachers should have, there are still some limitations to our comprehension of how to explore, recognize and foster the categories of knowledge that these models propose, as stated by Silverman and Thompson (2008),

> "While the mathematical knowledge for teaching has started to gain attention as an important concept in the mathematics teacher education research community, there is limited understanding of what it is, how one might recognize it, and how it might develop in the minds of teachers". (p. 499)

In this study, we have used the model of mathematics teachers' knowledge known as *Didactic-Mathematical Knowledge* (DMK), which is based upon theoretical assumptions and theoretical–methodological tools of the theoretical framework known as Onto-Semiotic Approach (OSA) to mathematical cognition and instruction (Godino et al. 2007). The DMK model takes into consideration: (1) the contribution and development of the theoretical framework OSA, which has been developed in several research studies by Godino and colleagues (Godino et al. 2007; Font et al. 2013); (2) the development and contribution of the research by Godino (2009) where the foundations and basis of DMK are presented; (3) the findings and contribution of the several models that currently exist in the field of research of Mathematics Education—Shulman (1986, 1987), Grossman (1990), Hill et al. (2008), Schoenfeld and Kilpatrick (2008), Rowland et al. (2005); and (4) the results obtained in several empirical studies that we have conducted (e.g., Pino-Fan et al. 2012, 2013).

The DMK model interprets and characterizes the teacher's knowledge from three dimensions (Fig. 1): *mathematical dimension*, *didactical dimension* and *meta didactic-mathematical dimension.*

> "DMK's mathematical dimension makes reference to the knowledge that allows the teacher to solve the problem or mathematical activity that is to be implemented in the classroom and link it with mathematical objects that can later be found in the school mathematics curriculum. It includes two subcategories of knowledge: common content knowledge and extended content knowledge. The first subcategory, common content knowledge, is the knowledge of a specific mathematical object, which is considered as sufficient to solve problems and tasks proposed in the mathematics curriculum and in the textbooks of a certain educational level; it is a shared knowledge between the teacher and the students. The second subcategory, extended knowledge, refers to the knowledge that the teacher must have about mathematical notions that, taking the mathematical notions that are being studied at a certain time as a reference (for example, derivatives), come ahead in the curriculum of the educational level in question or in the next level (for example, integers in high school

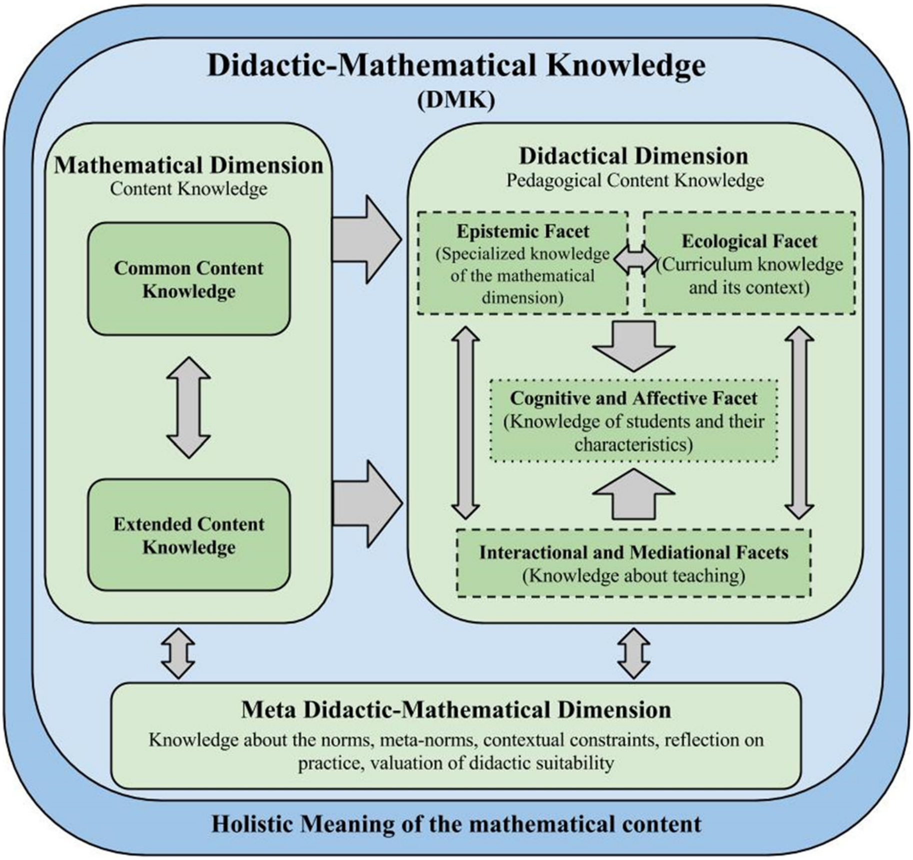


Fig. 1 Dimensions and components of the DMK and its relationships with other models

> or the fundamental theorem of calculus in college). Extended content knowledge provides the teacher with the necessary mathematical foundations to suggest new mathematical challenges in the classroom, to link a certain mathematical object being studied with other mathematical notions and to guide students to the study of subsequent mathematical notions to the notion that is being studied. These two subcategories that include the mathematical dimension of DMK, are reinterpretations of both the common content knowledge (Hill et al. 2008) and the horizon knowledge (Ball and Bass 2009), respectively. According to these authors, this interpretation is based on the need to settle the knowledge that a mathematics teacher should possess on specific topics to be taught at some specific school grades". (Pino-Fan et al. 2015, p. 1433)

It is clear that the mathematical dimension of DMK is not enough for the practice of teaching. The authors of the various models mentioned above agreed that in addition to the mathematical content, the teacher should have knowledge about the various factors that influence when planning and implementing the teaching of such mathematical content. In this sense, the *didactical dimension* of DMK includes the following subcategories of knowledge (Pino-Fan et al. 2015): (1) specialized knowledge of the mathematical dimension (epistemic facet); (2) knowledge about cognitive aspects of students (cognitive facet); (3) knowledge about affective, emotional and attitudinal aspects of students (affective facet); (4) knowledge about the interactions that arise in the classroom (interactional facet); (5) knowledge about the resources and means that may enhance the learning of students (mediational facet); and (6) knowledge about curricular, contextual, social, political and economic aspects that influence the management of students' learning (ecological facet). Regarding each of the subcategories of the didactical dimension of DMK, Pino-Fan et al. (2015, p. 1434–1436) point out:

1. Epistemic facet refers to specialized knowledge of the mathematical dimension. The teacher, apart from the mathematics that allow him solving problems which require him mobilize his common and extended knowledge, must have a certain amount of mathematical knowledge "shaped" for teaching; that is to say, the teacher must be able to mobilize several representations of a mathematical object, to solve a task through different procedures, to link mathematical objects with other mathematical objects taught at a certain educational level or from previous or upcoming levels, to comprehend and mobilize the diversity of partial meanings for a single mathematical object—that are part of the holistic meaning for such object (Pino-Fan et al. 2011)—to provide several justifications and argumentations, and to identify the knowledge at play during the process of solving a mathematical task. Thus, it is clear that this DMK's subcategory includes not only the notions proposed in the model of the proficiency in teaching mathematics of Schoenfeld and Kilpatrick (2008, p. 322) on "knowing school mathematics profoundly and thoroughly" but also the notions of Hill, Ball and Schilling (2008, p. 377–378) on "the mathematical specialized content knowledge".
2. Cognitive facet refers to the knowledge about the students' cognitive aspects. This subcategory considers the necessary knowledge to "reflect and evaluate" the proximity or degree of adjustment of personal meanings (students' knowledge) regarding institutional meanings (knowledge from the point of view of the educational setting). To this end, the teacher must be able to foresee (during the planning/design stage) and trying (during the implementation stage), from the students' pieces of work, or expected pieces of work, possible answers to a certain problem, misconceptions, conflicts or mistakes that arise from the process of solving the problem, links (mathematically correct or incorrect) between the mathematical object that is being studied and other mathematical objects which are required to solve the problem.
3. Affective facet refers to the knowledge about the students' affective, emotional and behavioral aspects. It is about the knowledge required to comprehend and deal with the students' mood changes, the aspects that motivate them to solve a certain problem or not. In general, it refers to the knowledge that helps describing the students' experiences and sensations in a specific class or with a certain mathematical problem, at a specific educational level, keeping in mind the aspects that are related to the ecological facet. The cognitive and affective facets such as are defined by the OSA (Godino et al. 2007), together provide a better approximation and understanding of the knowledge that the mathematics teachers should possess on the features and aspects that are connected to the way students think, know, act and feel in the class while solving a mathematics problem. Thus, these two facets (cognitive and affective) includes and broaden Shulman's ideas (Shulman, 1987, p. 8)—on the "knowledge of students and their characteristics", Schoenfeld and Kilpatrick (2008)—on "knowing the students as persons who think and learn", Grossman (1990, p. 8)—on the

"comprehension of students, their beliefs and mistakes about specific topics", and Hill, Ball and Schilling (2008, p. 375)—on the "knowledge of content and students".

4. Interactional facet. The study of the required features to appropriately manage the students learning on specific mathematics topics, have considered to the interactions as a fundamental component in the learning and teaching process (Coll and Sanchez 2008; Planas and Iranzo 2009). In this sense, and having in mind the ideas of Schoenfeld and Kilpatrick (2008) on constructing relationships that support the learning process, the interactional facet refers to the knowledge of the interactions that occur within a classroom. This subcategory involves the required knowledge to foresee, implement and evaluate sequences of interaction, among the agents that participate of the process of teaching and learning, oriented toward the fixation and negotiation of meanings (learning) of students. These interactions do not only occur among the teacher and the students (teacher–student), but also can occur between students (student–student), student-resources, and teacher-resources-students.
5. Mediational facet. In relation to the resources and means used to manage the learning, the proposed models by Shulman (1987) and Grossman (1990) consider the knowledge of classroom materials as part of the curriculum knowledge. Nonetheless, due to the current mathematics curriculum tendencies, these acquire an important role in the organization and management of learning. For this reason, the meditational facet refers to the knowledge of resources and means which might foster the students' learning process. It deals with the knowledge that a teacher should have to assess the pertinence of the use of materials and technological resources to foster the learning of a specific mathematical object, and also the assigning of time for the diverse learning actions and processes. The link between the interactional and mediational facets develops and enriches the notion of "knowledge of content and teaching" proposed by Ball et al. (2008, p. 401).
6. Ecological facet, which refers to the knowledge of curricular, contextual, social political, economic… aspects that have an influence on the management of the students' learning. In other words, teachers should have knowledge of the mathematics curriculum of the level that considers the study of a mathematical object, the links that might exist with other curricula, the relations that such curriculum has with social, political and economic aspects that support and condition the teaching and learning process. The features considered in this knowledge facet take into account the ideas of Shulman (1987, p. 8)—on the "curriculum knowledge", "knowledge of educational ends, purposes and values"—and Grossman (1990, p. 8)—on the "knowledge about horizontal and vertical curriculum for a specific topic", and the "knowledge of context".

The six facets described above, which shape the didactical dimension of didactical-mathematical knowledge, along with the mathematical dimension of DMK, can be considered when it comes to analyze, describe and develop the teacher's knowledge—or future teachers'—involved in the different phases of the design of processes of teaching and learning of specific mathematical topics: preliminary study, planning or design, implementation and assessment (Pino-Fan and Godino 2015).

The *meta didactic-mathematical dimension of DMK* refers to the knowledge needed by teachers to: reflect on their own practice (Schön 1983, 1987; Schoenfeld and Kilpatrick 2008), identify and analyze the set of norms and meta-norms that regulate the teaching and learning processes of mathematics, and assess the didactic suitability in order to find potential improvements in the both design and implementation stages of such processes study.

In connection with the objectives of our study, we used the mathematical dimension and some partial aspects of the epistemic facet of DMK. For each one of components of the three dimensions of DMK, the OSA provides tools to analyze and characterize the knowledge involved in such components. So, for example, for the mathematical dimension and epistemic facet of DMK, two levels of analysis are proposed:

1. *Mathematical practices*: description of the actions performed to solve the mathematical tasks proposed with the aim of contextualizing content and promoting learning. The general lines of action taken by the teacher and students are also described.
2. *Configuration of objects and processes*: description of the mathematical objects and processes involved in a set of mathematical practices, as well as those which emerge out of them. The purpose of this level of analysis is to describe the complexity of objects and meanings that form part of mathematical and didactic practices. This complexity is an explanatory factor not only in relation to conflicts of meaning but also as regards the progression of learning.

In particular, the notion of *configuration of mathematical objects and processes* can be used to offer a detailed description of the institutional knowledge that one would expect to find in an "expert solution" to a given task (epistemic configuration), as well as of the personal knowledge which student teachers or the teachers actually use (cognitive configuration).

It is worth highlighting that in the Onto-Semiotic Approach (OSA) ontology (Font et al. 2013), "object" is used in a broader sense to refer to any entity that is in some way involved in mathematical practice and which can be distinguished from others. If we consider, for example, the objects involved when carrying out and evaluating a problem-solving practice, we can identify the use of different languages (verbal, graphic, symbolic, etc.). These languages are the ostensive part of a series of concepts/definitions, propositions and procedures that are involved in argumentation and justification of the problem solution. In the OSA, these six objects (i.e., situations-problems, linguistic elements, concepts/definitions, propositions/properties, procedures and arguments) are considered as *primary mathematical objects* and together form *configurations of primary mathematical objects*.

The term configuration is used to designate a heterogeneous set or system of objects that are related to one another. These configurations of objects can be viewed from both the personal and the institutional perspective, and this means that a distinction can be made between cognitive (personal) and epistemic (institutional) configurations of primary mathematical objects. Hence, an *epistemic configuration* is the system of primary mathematical objects that, from the institutional point of view, are involved in the mathematical practices carried out to solve a specific problem. For its part, a *cognitive configuration* is the system of primary mathematical objects that a subject mobilizes as part of mathematical practices carried out to solve a specific problem.

Methods

This exploratory study uses a mixed methods approach and involves the analysis of both quantitative (level of accuracy of item answers: correct, partially correct and incorrect) and qualitative variables (cognitive configurations associated with the responses given by the prospective teachers).

The DMK-Derivative Questionnaire

The instrument used was the *Epistemic Facet of Didactic-Mathematical Knowledge about the Derivative Questionnaire* (DMK-Derivative Questionnaire). It comprises seven tasks and was designed in accordance with the DMK model for assessing and developing didactic-mathematical knowledge.

When designing the questionnaire, three criteria were considered in order to select the tasks that would be included in it. The first criterion was that the tasks should provide information about the extent to which a prospective teacher's personal understanding of the derivative was consistent with the global or holistic meaning of this mathematical object (Pino-Fan et al. 2012). This was achieved by including items that activate different meanings of the derivative: slope of the tangent line, instantaneous rate of change and instantaneous rate of variation. In this article we distinguish between "instantaneous rate of change", which refers specifically to the "quotient" between two quantities of magnitudes, and the "instantaneous rate of variation", which refers to the "quotient" of real numbers with no reference to magnitudes. The "instantaneous rate of variation" is commonly known as the limit of the incremental quotient.

The second criterion was that the items selected had to reflect the different types of representations that are activated in the three sub-processes which, according to Font (1999), are involved in the process of finding the derivative function:

1. Translations and conversions between the different ways of representing $f(x)$.
2. The step from a representation of $f(x)$ to another representation in the form $f'(x)$.
3. Translations and conversions between the different ways of representing $f'(x)$.

Consequently, the tasks included in the questionnaire bring into play the different types of representations that are involved in these three sub-processes, namely verbal description, graphical description, formulas (symbolic) and tabulation (for both the function and its derivative).

The third criterion, which refers to the kind of didactic-mathematical knowledge held by prospective teachers, considers the inclusion of three types of task: (1) those that require teachers to use their common content knowledge (solving a mathematical problem that would be set at the upper secondary level); (2) those that require extended content knowledge (generalizing tasks involving common knowledge, or making links to more advanced mathematical objects that appear in the curriculum); and (3) those that require aspects of the epistemic facet of DMK (using different representations, different partial meanings of a mathematical object, solving the problem by means of various procedures, giving a range of valid arguments, identifying the knowledge that is brought into play when solving a mathematical problem, etc.).

Subjects and context

The questionnaire was administered to a sample of 53 students enrolled in the final modules (sixth and eighth semester) of the degree in mathematics teaching offered by the Autonomous University of Yucatan (UADY) in Mexico. This is a four-year degree (8 semesters) and the Faculty of Mathematics of the UADY is responsible for educating teachers to work at upper secondary or university level in the state of Yucatan (Mexico). The 53 students who responded to the questionnaire had studied differential calculus in the first semester of their degree course, and they had subsequently completed other modules related to mathematical analysis (integral calculus, vector calculus, differential equations, etc.). They had also studied subjects related to the teaching of mathematics.

We note that along the last two semesters these prospective teachers took a course named "microteaching" where they could apply skills such as induction, communication and strengthen their ability in teaching small groups. These lessons were recorded and the students got feedback. In addition, during their (eighth) last semester, participants in our study took a course named "Professional Training Workshop" in which they showed their skills to teaching mathematics that were acquired as a result of all previous didactic courses received.

Onto-semiotic analysis

The OSA proposes two levels in the analysis of the mathematical activity carried out by a subject (person or institution) to solve a mathematical problem or task: (1) describing the system of operative and discursive practices (actions) that the subject performs; (2) describing the objects and processes involved in such practices. Both types of analysis involve content analyses of the text where the mathematical activity is reflected or expressed. These content analyses are supported by the types of objects and processes proposed in the OSA, which are described in detail in previous articles (Godino et al. 2007; Font et al. 2013). An example is introduced in the next section using Task 3 of the questionnaire.

Content analysis of the tasks included in the DMK-Derivative Questionnaire

In this section we describe the a priori analysis that is carried out for each of the tasks included in the DMK-Derivative Questionnaire, which sets out the knowledge that we would expect teachers to use when solving these tasks. For reasons of space only Task 3 is described in detail, with a summary being provided for the analysis of the other tasks.

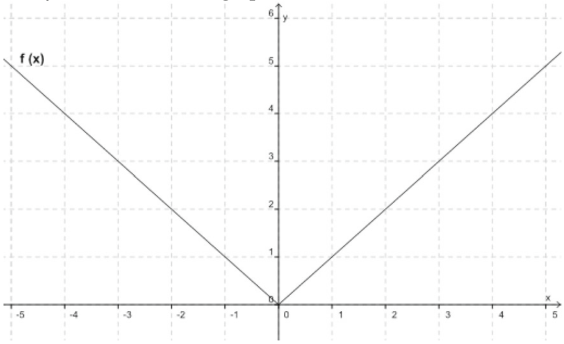
Task 1 is a classical question that has been used in a number of studies (Habre and Abboud 2006; Bingolbali and Monaghan 2008) to explore students' knowledge of the different meanings of the derivative. The question is: *What does the derivative mean to you*? As this is a broad question, prospective teachers are expected to provide a list of possible meanings of the derivative. This task therefore explores their common knowledge regarding the meanings of the derivative.

Task 2 (Fig. 2) has been the object of several studies (Tsamir et al. 2006) and explores three types of knowledge that comprise the epistemic facet of didactic-mathematical knowledge about the derivative: (1) common content knowledge [item (a)], such that the prospective teacher should be able to solve the problem without needing to use various representations or arguments; (2) specialized knowledge of the content [items (b) and (c)], where in addition to solving the problem the teacher is required to use representations (graphs, symbols and verbal descriptions) and valid arguments that justify the procedures; and (3) extended knowledge [item (d)], which entails generalization of the initial task about the derivability of the absolute value function at $x = 0$, on the basis of valid justifications for the proposition "the graph of a derivable function cannot have peaks", made by defining the derivative as the instantaneous rate of variation (limit of the increment quotient). The interpretations of the derivative as the slope of the tangent line and the instantaneous rate of variation are associated with this task.

Task 3 (Fig. 3), taken from Delos Santos (2006), explores the extended knowledge of prospective teachers, as solving it requires them to use more advanced mathematical

Task 2

Consider the function $f(x) = |x|$ and its graph.



a) For what values of x is $f(x)$ derivable?
b) If it is possible, calculate $f'(2)$ and draw a graph of your solution. If it is not possible, explain why.
c) If it is possible, calculate $f'(0)$ and draw a graph of your solution. If it is not possible, explain why.
d) Based on the definition of the derivative, justify why the graph of a derivable function cannot have 'peaks' (corners, angles)

Fig. 2 Task 2 from the DMK-Derivative Questionnaire

Task 3

For a given function $y = f(x)$, the values shown in the following table hold:

x	$f'(x)$
0	0
1.0	2
1.5	3
2.0	4
2.5	5

a) Find an expression for $f(x)$
b) Can you find another expression, different to the first one, for $f(x)$? What would it be? Justify your answer.

Fig. 3 Task 3 from the DMK-Derivative Questionnaire

objects from the curriculum, for example, the integral of a function or the fundamental theorem of calculus. The representations they need to manage in solving this task are symbolic, graphical and tabular. The extended knowledge evaluated in this task is associated with the meaning of the derivative as the slope of the tangent line.

For each one of seven tasks, we performed an analysis of the kind we will now illustrate for Task 3. The first step involves proposing plausible solutions (or what we call an "expert" solution), which is then used to carry out a content analysis by applying the theoretical tool: *epistemic configuration* (referred to in theoretical framework section). Finally, we identify the "curriculum content" that is set out in the syllabus, which in our specific case is the differential calculus syllabus for prospective teachers.

Plausible solution

Following, we present one of the possible solutions which is expected by prospective teachers when faced with Task 3. Other solutions for the task can be provided; however, for the purposes and objectives of this research conveniently, we choose the following:

(a) Based on the data shown in the table, it is possible to identify a pattern as follows:

X	$f'(x)$
0	2 (0) = 0
1.0	2 (1) = 2
1.5	2 (1.5) = 3
2.0	2 (2) = 4
2.5	2 (2.5) = 5
:	:
x	$2\ (x) = 2x$

Therefore, given that $f'(x) = 2x$ and knowing that for a function $f(x) = x^n$ the derivative is given by $f'(x) = nx^{n-1}$, then an expression for $f(x)$ would be $f(x) = x^2$.

(b) It is indeed possible to find another expression for $f(x)$, one that is different to $f(x) = x^2$. If $f'(x) = 2x$, then $f(x) = \int 2xdx = x^2 + C$. Thus, $f(x)$ can be any function of the family of functions $f(x) = x^2 + C$, where $C \in \mathbb{R}$.

Analysis of the primary mathematical objects and processes involved (epistemic configuration)

Let us now consider the mathematical objects and processes that are involved in the formulation and plausible solution corresponding to Task 3.

Process of Representation $\leftrightarrow$ Signification

As part of the duality between representation and signification, we identified various linguistic elements and concepts/definitions (both a priori and emerging) in both the formulation of and expected solutions to Task 3. Below we set out these linguistic elements and concepts, as well as their meanings.

Linguistic elements

The a priori linguistic elements include:

- The expression $y = f(x)$. This denotes an indeterminate function, in this case a function that fulfills certain conditions as set out in the table.
- The table of values (Fig. 3). Derivative function of an unknown function, of which five images for values of the variable x are known, is given in the table. It provides ordinate pairs of the kind $(x, f'(x))$.

- The expression "finds an expression for $f(x)$". This refers to the existence of a procedure for finding a function whose derivative has the values shown in the table.

The emerging linguistic elements include:

- The graphical representation of the values in the table (Fig. 4), which represents the conversion of the table of values given by the derivative function into its corresponding Cartesian graph.
- The expressions '$y' = 2x$' or '$f'(x) = 2x$'. Equation for the line that passes through the five points aligned in the Cartesian plane and this is the symbolic representation of the derivative function.
- The expression $f(x) = x^2$. Primitive (antiderivative) of the function $y = 2x$.

Concepts/definitions

The a priori concepts required to solve the task are as follows:

- Function of a real unknown variable. Function $f(x)$,, which will be determined from its derivative function, partially defined by five points.
- Ordinate pairs (original and image) of the derivative function.
- Derivative function of a real variable. Partially defined by five points whose coordinates are expressed in table form. It is assumed that the five points given evoke or represent the whole of the graph of the derivative function.

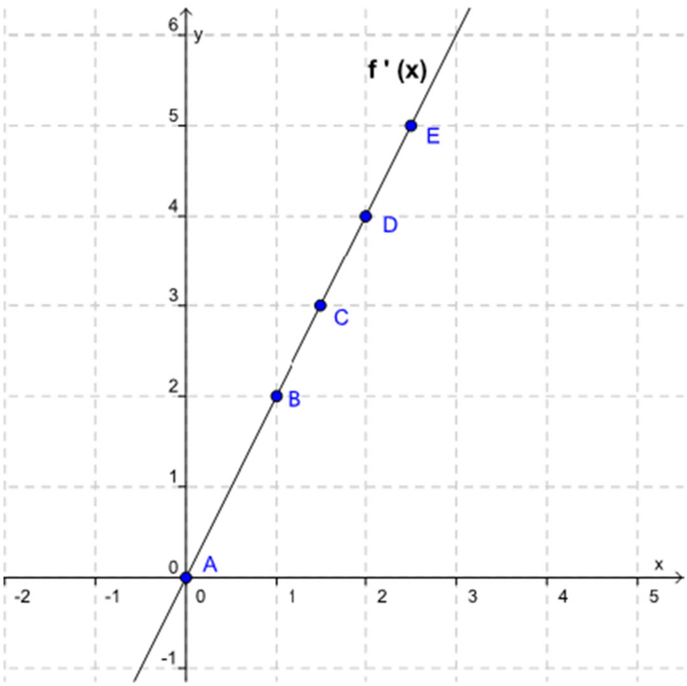


Fig. 4 Graphical representation of the known values of the derivative function

The emerging concepts identified are as follows:

- Straight line. Contains the five points of the Cartesian graph that represents the derivative. Linear interpolation and extrapolation is assumed.
- Slope. Variation in the y axis (ordinate) with respect to the x axis (abscissa) between any two points on the straight line that passes through the five points defined in the table. This enables us to find the equation of this line by means of the point-slope equation ($y = 2x$).
- Primitive or antiderivative of the function. Function whose derivative is $f'(x) = 2x$.
- Family of functions. Functions are found within the set of functions that have the form $f(x) = x^2 + c$, where $c \in \mathbb{R}$.

Process of composition

On the basis of the representational linguistic elements and the concepts/definitions identified in the previous step, it is now possible to consider the propositions/properties and procedures used in solving the task, as well as the arguments which justify their use.

Propositions/properties

The a priori propositions we identified are:

- PP1: '$y - y_1 = m(x - x_1)$, particularized here as $y = 2x$'. Point-slope equation, which enables the algebraic representation of the derivative function to be found.
- PP2: Rules of derivation. Specifically, "the derivative of a constant function is equal to zero", which enables us to determine that the function being sought is any function of the family $f(x) = x^2 + c$ where $c \in \mathbb{R}$.

The emerging propositions are as follows:

- PP3: The primitive of $y = 2x$ is $y = x^2$ (fundamental theorem of calculus). This is used to find the required function once its derivative has been found.
- PP4: The derivative function of all the functions of the type $f(x) = x^2 + c$ where $c \in \mathbb{R}$, is $f'(x) = 2x$. Solution to part b) of the task.

Procedures

We identified the use of the following procedures in plausible solutions to the task:

- P1: Linear interpolation in the Cartesian graph of the derivative function given for particular values (Fig. 4). This is used to find the algebraic expression of the derivative.
- P2: Computing the antiderivative of $y = 2x$. This can be done either through rules of derivation (derivative of the potential function) or by means of rules of integration. This procedure produces the answer to both parts of the task.
- P3: Trial and error, trying possible rules of correspondence between the values of x and those of $f'(x)$ using the values given in the table. This is a numerical–technical procedure, which only differs from P1 in the search for a pattern that would enable the establishment of the rule of correspondence which would allow the derivative function to be defined.
- P4: "Formal" procedures. Procedures of a graphical–technical or numerical–technical nature, which vary from the previous kind of procedures in that, once the algebraic

representation of the derivative function has been found (whether numerically or graphically), one then makes use of more advanced content from the upper secondary curriculum, for example, the concept of indefinite integral and the fundamental theorem of calculus in its intuitive form (PP3).

Process of argumentation

This aspect of the configuration of primary mathematical objects, one which emerges from the process of composition, can be regarded in itself as a *process of argumentation*, by means of which one gives meaning to and relates primary mathematical objects to one another in such a way that their organization constitutes a solution to a problem. The arguments we identified are:

- A1: An algebraic expression of the derivative function is $y = 2x$ because visually it can be seen that the five points given are arranged in a straight line, one which passes through the origin and whose slope is 2. This argument provides an empirical validation of the expression of the derivative function, assuming that the five points given represent the graph $(x, f'(x))$ of the derivative function.
- A2: The function being sought is $y = x^2$ because the derivative of this function is $y = 2x$. This establishes the validity of the solution given for the function $f(x)$, taking into account the rule for deriving the potential function.

Curricular content

The curricular content evaluated by means of Task 3 is as follows:

- Quadratic functions. Family of functions.
- Derivative function (tabular, graphical and symbolic representation; in its definition as the slope of the tangent line).
- Theorems for deriving functions (rules of derivation: for the constant function).
- Antiderivative or indefinite integral.
- Fundamental theorem of calculus (relationship between the derivative of a function and its antiderivative).

The detailed a priori analysis that has been performed in previous lines is of special relevance because from this analysis is possible to characterize the expected knowledge of prospective teachers. Thus, it is possible to compare and assess the degree of approximation of the knowledge of future teachers in respect of the expected knowledge.

Task 4—both item (a) and item (b)—which is taken from Viholainen (2008) explores the specialized knowledge of the content of prospective teachers, as it requires the use of various representations (graph, verbal description, formulas) and a range of justifications for the proposition "the derivative of a constant function is always equal to zero", in which different interpretations of the derivative may be employed: slope of the tangent line, instantaneous rate of change and instantaneous rate of variation.

Task 5 (Fig. 5) appears to be the sort of exercise commonly found in differential calculus books that are used at the upper secondary level, its solution being obtained by applying certain theorems or propositions about the derivative. Therefore, both item (a) and item (b) evaluate aspects of common knowledge related to the derivative, where the latter is understood as the slope of the tangent line or the instantaneous rate of change, Task 5

Given the function $y = x^3 - \frac{x^2}{2} - 2x + 3$

a) Find the points on the graph of the function for which the tangent is horizontal.
b) At what points is the instantaneous rate of change of *y* with respect to *x* equal to zero?

Fig. 5 Task 5 from the DMK-Derivative Questionnaire

respectively. However, the main objective of Task 5 is to explore the associations that prospective teachers make between the different meanings of the derivative, and as such the task evaluates aspects of specialized knowledge of the content.

Task 6 was taken and modified from Çetin (2009). Both item (a) and item (b) yield information about specialized knowledge of the content related to the derivative understood as the instantaneous rate of change. On the one hand, item (a) requires an interpretation from students regarding verbal linguistic elements, both graphic (graphs of derivatives) and iconic (images of cups), so that they may attempt to establish an injective correspondence between these graphic and iconic elements. On the other hand, students should find procedures that allow them to establish the correlation for each element and to give valid justifications for their solutions.

Finally, Task 7 (Fig. 6), which has also been adapted from the article by Çetin (2009), provides information about the teachers' extended knowledge, since it involves an approximation to the derivative of a function, described by values in the table, at point $t = 0.4$ by means of numerical values of the function. Task 7 is not the typical type of problem that would be encountered at the upper secondary level, and it requires an understanding of the derivative as the instantaneous rate of change, and specifically as instantaneous velocity. This problem can be solved by various methods; for example, Lagrange's interpolating polynomial, and this supports the categorization of this task as assessing extended content knowledge.

Analysis of the prospective teachers' knowledge that is revealed through their solutions to Task 3 on the DMK-Derivative Questionnaire

One of the variables considered when analyzing the data obtained from the DMK-Derivative Questionnaire was the *type of cognitive configuration*. By way of an example we will now present a detailed analysis of the cognitive configurations that emerged in the solutions given by prospective teachers to Task 3.

Task 7

A ball is thrown into the air from a bridge 11 meters high. $f(t)$ denotes the distance that the ball is from the ground at time *t*. Some values of $f(t)$ are shown in the table below:

t (s)	0	0.1	0.2	0.3	0.4	0.5	0.6
(t) *(m)*	11	12.4	13.8	15.1	16.3	17.4	18.4

Based on the table, at what speed will the ball be travelling when it reaches a height at $t =$ 0.4 seconds? Justify your chosen answer.

a) 11.5m/s b) 1.23m/s c) 14.91m/s d) 16.3m/s e) Another

Fig. 6 Task 7 from the DMK-Derivative Questionnaire

Cognitive configurations associated with the solutions to Task 3

In the framework of the OSA, a cognitive configuration is a type of response given by subjects when it is analyzed showing the objects and processes involved. Configurations will be distinct when they differ as regards a characteristic primary mathematical object. The analysis of the responses given by prospective teachers to Task 3 revealed four types of cognitive configuration for item (a) and four for item (b). We have labeled these configurations as follows: [item (a)] (1) graphical–technical; (2) graphical-advanced; (3) numerical–technical; (4) numerical-advanced; [item (b)] (5) advanced; (6) technical; (7) erroneous uniqueness; and (8) equivalent functions. In what follows each of these cognitive configurations is analyzed using a prototypical response.

For reasons of space these cognitive configurations of objects and processes will be described in less detail than was the case in section four, where we considered the formulation and plausible solution corresponding to Task 3.

Cognitive configuration 1: graphical–technical

In the didactic-mathematical practices of prospective teachers in which this type of configuration is activated, the data in the table (tabular representation of the derivative) are used to produce a graphical representation from which the algebraic expression for the derivative function can be obtained. This is followed by a process of argumentation based on procedures related to "rules" or "techniques" of derivation (propositions), which are used to find an expression for $f(x)$. Figure 7 shows an example (from Student A) in which this type of configuration is activated.

A breakdown of the solution presented by Student A reveals linguistic elements of a graphical nature (graphical representation of the ordinate pairs from the table), as well as verbal elements that illustrate the central feature of the teacher's practice, namely "identifying" the linearity of the ordinate pairs given in the table (which belong to the graph of the derivative of a function). Subsequently, given two points on the straight line formed by these data pairs the student finds the equation of the line or the symbolic representation of the derivative function.

By means of a process of composition, we can observe the procedure followed by the student, which is illustrated by the statement "…the behaviour of the function $g(x)$ [$'(x)$] is linear, and therefore by knowing any two points it is possible to find the function $f(x)$". In the description (process of formulation) of the procedure one can see the implicit use of propositions that are required to carry it out, for example, "given any two points on a straight line it is possible to find its equation" or "the point-slope equation can be used to find the equation of a straight line for which two points are known".

In his answer (process of communication) Student A does not contemplate using the antiderivative to find the function from the derivative, this being an aspect that is also reflected in his answer to item (b) of the task, "I can't think of another expression". Nonetheless, his proposed answer to item (a) is correct: "$\therefore f(x) = x^2$".

Those students who gave answers in which a graphical–technical configuration was activated in relation to item (a) of the task did one of two things: either they did not provide evidence of how they obtained the function from which the derivative function was derived, even though it is assumed that they could have remembered that the derivative of $f(x) = x^2$ is $f(x) = 2x$ (as in the case of Student A), or they explicitly indicated that an expression for the function that fulfilled the conditions set out in the table was $f(x) = x^2$

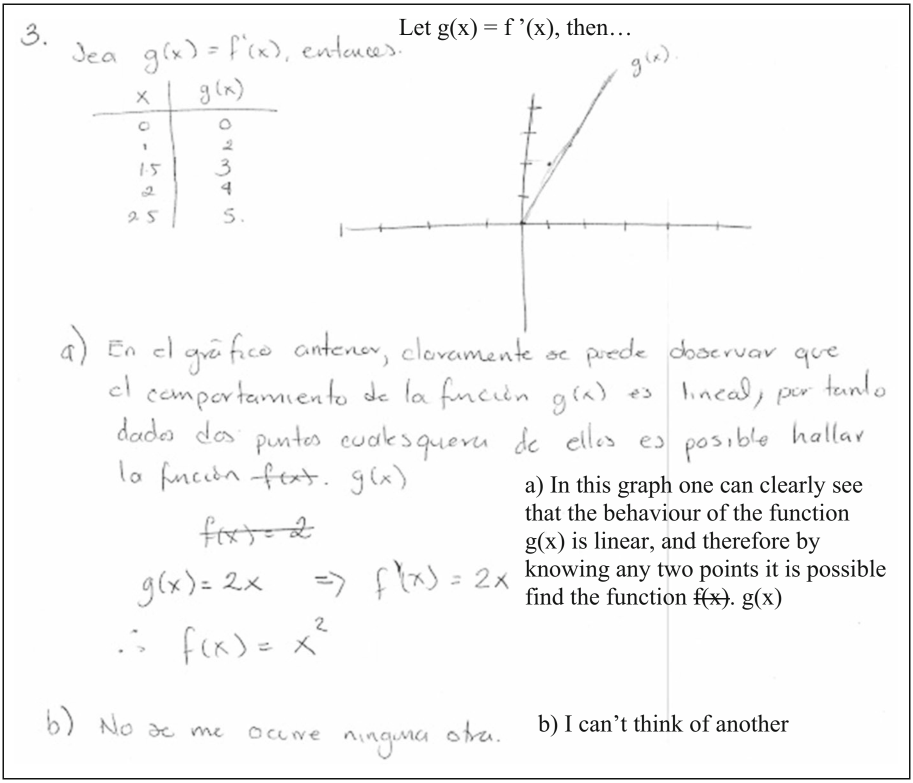


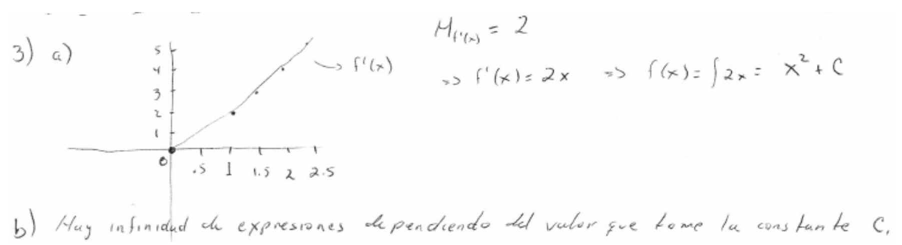
Fig. 7 Solution to Task 3 given by student A

(process of communication), because if they derived this function by means of rules of derivation they obtained the function $f(x) = 2x$ (process of argumentation that is not completely valid with respect to expected knowledge), which they had found from the values given in the table. In all cases, it is clear that the prospective teachers who produce answers associated with this type of configuration lack the extended knowledge required to solve problems such as that proposed here, since there is no process of generalization from item (a) to item (b) of the task. This could cause problems at a later date when they are required to manage the knowledge of their own students.

Cognitive configuration 2: graphical-advanced

A *graphical-advanced configuration* can be seen in those answers that use the data in the table to produce a graphical representation from which the algebraic expression for the derivative function can be obtained. Subsequently, through a process of argumentation based on procedures related to more advanced curricular topics (such as the antiderivative) the student finds an expression for $f(x)$. Figure 8 shows the answer given by Student B, which is a prototypical example of this configuration.

It can be seen in Fig. 8 that the mathematical objects used in the first part of Student B’s practice are very similar to those employed by Student A, since by considering not only the



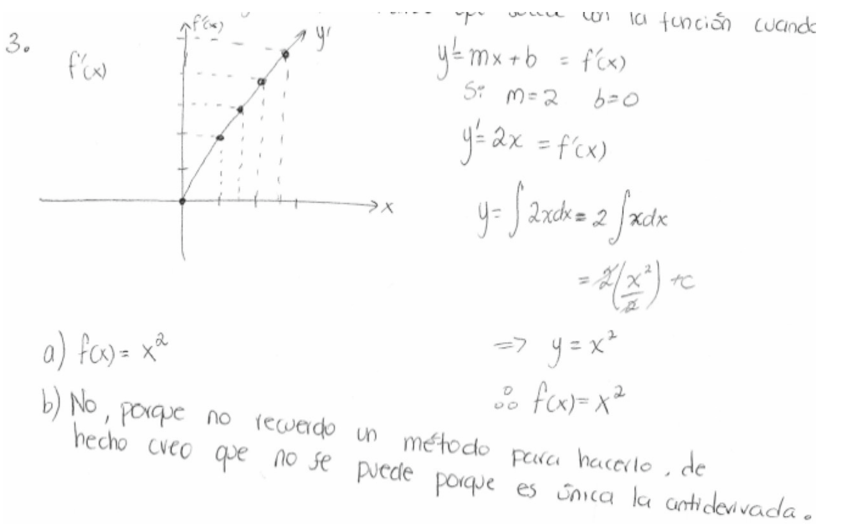
b) There is an infinite number of expressions, depending on the value of the constant C.

Fig. 8 Solution to Task 3 given by Student B

linear relationship between the points given in the table but also the concept of derivative as the slope of the straight line, the student finds a symbolic expression for the derivative "$f'(x) = 2x$". Subsequently, and based on this symbolic expression for the derivative, the student uses more advanced concepts/definitions and propositions/properties such as the antiderivative and the fundamental theorem of calculus. The extended knowledge which Student B uses to solve the task is even more evident in his response to item (b) of the task: "There is an infinite number of expressions, depending on the value of the constant C" (process of communication). Consequently, this student's answer shows a satisfactory process of generalization.

In general, the prospective teachers who used this type of configuration gave correct answers to Task 3, demonstrating that they had sufficient extended knowledge to solve the problem set. However, the fact that a student activated a cognitive configuration involving more advanced concepts/definitions, propositions/properties and procedures from the math curriculum on differential calculus did not always lead to a correct answer. The example in Fig. 9 shows that although Student C used advanced mathematical objects in his answer to item (a) of the task, his response to item (b) shows that he does not fully understand these objects. Specifically, the linguistic element (verbal expression) of which this response consists indicates a process of communication and argumentation which are not valid in relation to the problem solution: "No, because I don't remember a method for doing so, in fact, I think there can't be another expression because the antiderivative is unique". This illustrates that Student C fails to achieve a process of generalization and, therefore, of idealization with respect to the knowledge required to solve the task.

Thus, some students, as in the case of the example shown in Fig. 9, omitted the constant C (commonly known as constant of integration), in their answers to item (a), which made it difficult for them to answer item (b). These kind of results concur with studies such as Kiat's (2005) who states that students do not think of the antiderivative from a conceptual point of view, but on the contrary, they think of it as an inverse process of the integration and more concretely, as a procedure that allows them to obtain the "result of the inverse operation" directly, just like in multiplying and dividing. He also points out that the students who omit the constant of the "antiderivative" C are not aware that a constant must be written to specify the antiderivative function within a family of functions; in other words, the students are not aware that an integral is formed by a set of antiderivatives with C as a constant that varies.



b) No, because I don't remember a method for doing so, in fact, I think there can't be another expression because the antiderivative is unique.

Fig. 9 Solution given by Student C to Task 3

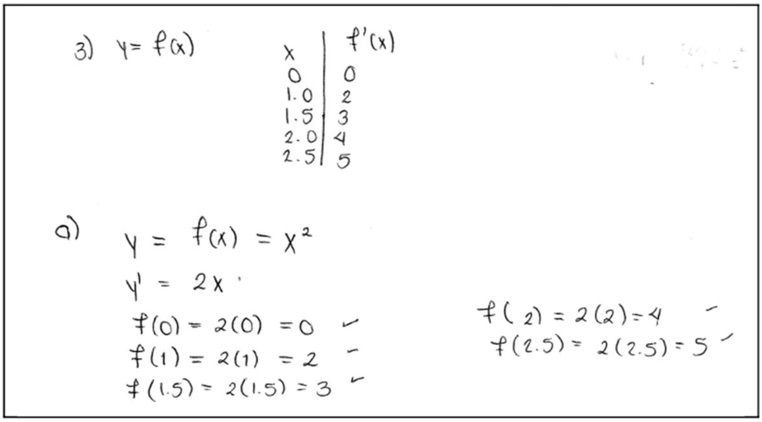


Fig. 10 Solution given by Student D to item (a) of Task 3

Cognitive configuration 3: numerical–technical

Figure 10 shows a prototypical example of answers involving a numerical–technical cognitive configuration. In this case the didactic-mathematical practices of students begin by considering the set of data given in the table so as to identify the pattern that enables them to find the rule of correspondence with which they can define the derivative function. Subsequently, they use a process of argumentation based on procedures related to "rules" of derivation in order to find an expression for $f(x)$.

Cognitive configuration 4: numerical-advanced

Figure 11 shows a typical example of answers that involve a cognitive configuration of this type. It can be seen that the practice of Student E begins by considering the set of data given in the table, identifying the pattern that enables him to find the rule of correspondence with which he can define the derivative function in symbolic form, $f'(x) = 2x$. Using

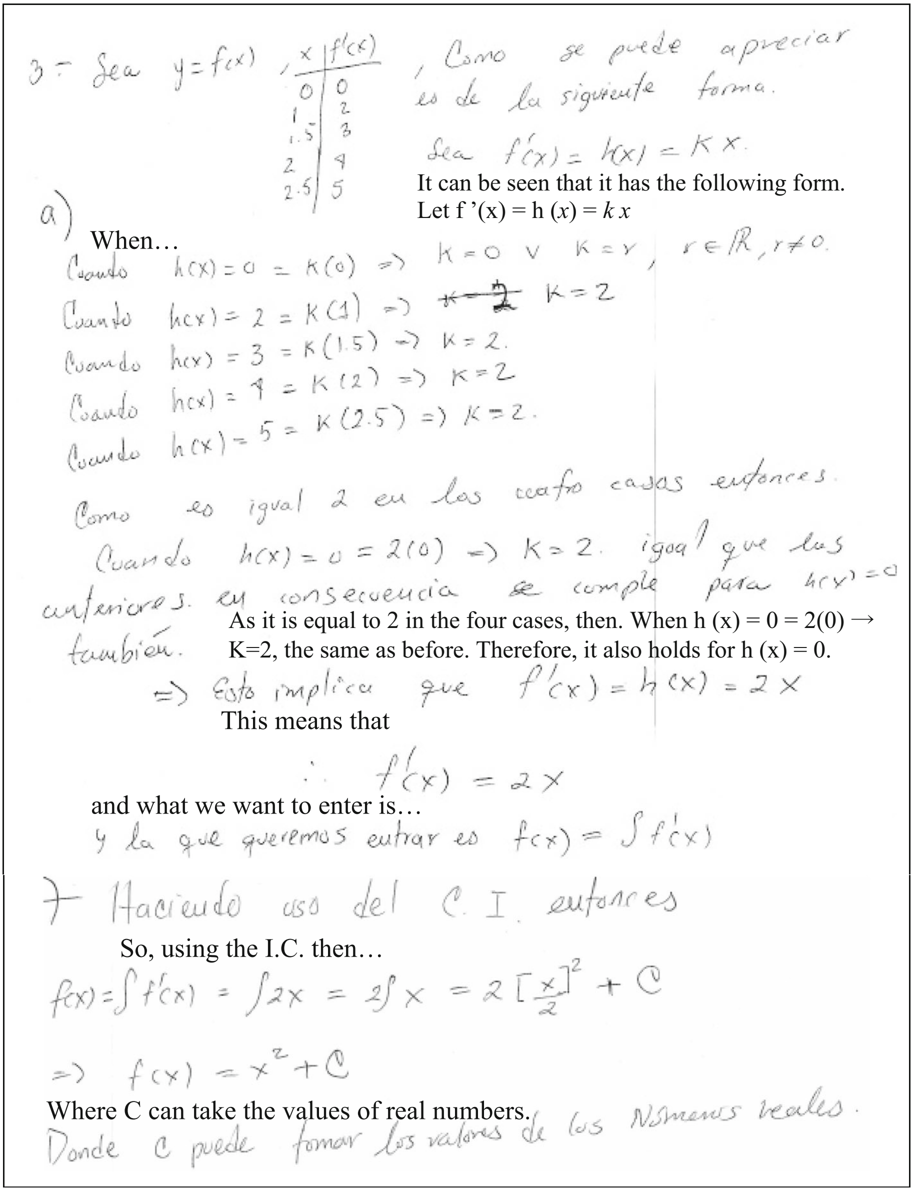


Fig. 11 Solution given by Student E to item (a) of Task 3

argumentation and procedures based on more advanced curricular concepts/definitions and propositions/properties (such as the indefinite integral and, implicitly, the fundamental theorem of calculus) he then finds an expression for $f(x)$.

Generally speaking, those students who gave answers involving a numerical-advanced configuration achieved a process of generalization with respect to their answer to item (a), and in their processes of argumentation and communication they made use of more advanced concepts/definitions and propositions/properties than were included in their differential calculus syllabus. This shows that their level of extended knowledge is sufficient to solve the task.

However, some students that mobilized this type of configuration show confusion between the notions of integral and antiderivative, as in studies such as Borasi's (1992) and Hall's (2010) in which it is shown how students (and some teachers) see the integral and the antiderivative as synonym terms. For example, in Fig. 11 the student explicitly says "So, using I. C., then $f(x) = \int f'(x)\ldots$", when stating that he will use the properties of the integral to give an answer.

Cognitive configuration 5: technical

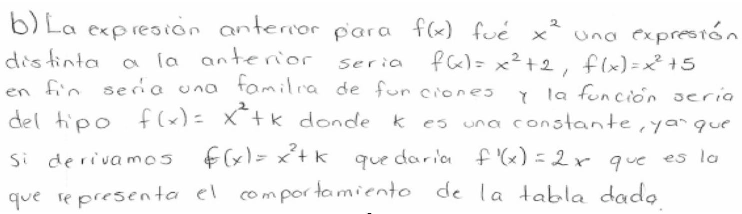
In general, those configurations which we have labeled as "technical" are those in which derivation theorems (propositions) are used to produce and justify the answers given to items (a) and (b) (processes of argumentation and communication). Figure 12 shows three prototypical examples of this type of response.

By breaking down these answers, one can see that the linguistic elements used are mainly verbal, along with a symbolic element to denote the family of functions '$f(x) = x^2 + c$', or another element of this set of functions. Note in Fig. 12 that the arguments put forward by students in support of the proposition that it is possible to find another, different expression that satisfies the conditions set out in the table are based on the use of derivation rules, and specifically on the proposition "the derivative of a constant function is zero" (process of formulating the propositions used in the process of argumentation).

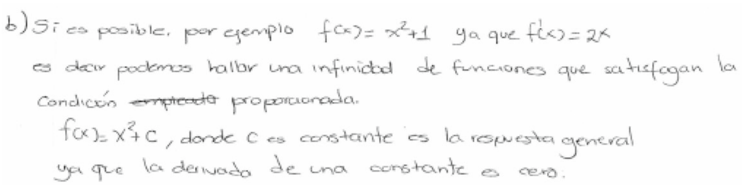
It is clear that the prospective teachers who used this kind of cognitive configuration did not take into account concepts/definitions such as the integral or propositions like the fundamental theorem of calculus, which would have provided a valid process of argumentation for their procedures, although these concepts and propositions are, in some way, implicit in the procedures used. This illustrates that these prospective teachers have a limited grasp of the extended knowledge required to solve tasks such as the one set here.

Cognitive configuration 6: advanced

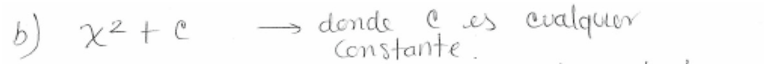
Figure 13 shows a prototypical example of this type of answer. In general, we use the term "advanced" to refer to configurations such as that which features in the answer of Student F or Student B (Fig. 8), both of whom use more advanced concepts/definitions such as the antiderivative, or propositions like the fundamental theorem of calculus, to justify their solution to both parts of the task. That these prospective teachers have the extended knowledge required to solve this kind of problem is demonstrated by the fact that they gave correct answers to both parts of the task. Furthermore, this indicates that responses, which fit this category, involve adequate processes of argumentation, generalization and communication.

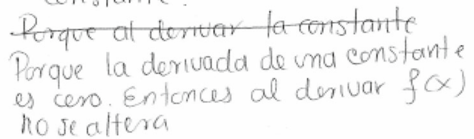


b) The previous expression for f(x) was x^2, a different expression would be $f(x) = x^2 + 2$, $f(x) = x^2 + 5$, so it would be a family of functions and the function would be of the kind $f(x) = x^2 + k$, where k is a constant, because if we derive $f(x) = x^2 + k$ we get $f'(x) = 2x$, which represents the behaviour of the data shown in the table.



b) Yes there is. For example, $f(x) = x^2 + 1$, since $f'(x) = 2x$, in other words, we can find an infinite number of functions that satisfy the condition shown. $f(x) = x^2 + c$, where c is a constant, is the general answer, since the derivative of a constant is zero.



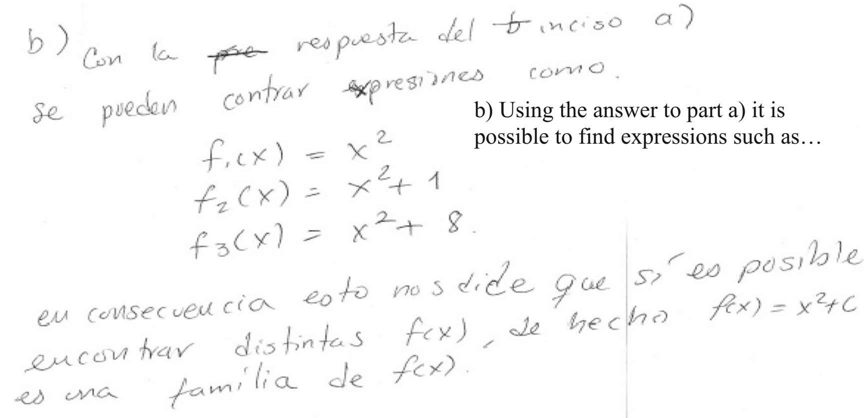


$x^2 + c$, where c is any constant. Because the derivative of a constant is zero. So when derive f(x) it does not change.

Fig. 12 Solutions given by three students to item (b) of Task 3

Cognitive configuration 7: erroneous interpretation regarding the uniqueness of the derivative

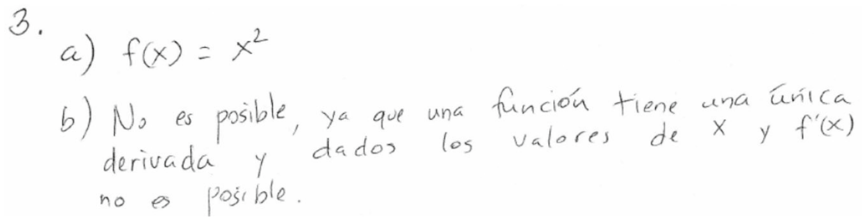
This type of cognitive configuration is associated with answers that reveal an erroneous conception of the uniqueness of the derivative function, since it is explicitly stated that it is not possible to find a different expression, other than that given in response to item (a), because "the derivative is unique" (argumentation). Examples of this kind of response can be found in Student C's answer to item (b) ("…I think there can't be another expression because the antiderivative is unique" (Fig. 9), as well as in the answer of Student G shown in Fig. 14: "There is no other expression because a function has a single derivative and given the values of x and $f'(x)$ it is not possible". In general, the students who gave similar answers to those of students G and C also failed to argue in a valid way. Our results concur with the findings of Kiat (2005), who points out that the students tend to consider the antiderivative as the inverse process of the derivative "just like dividing or multiplying"; in other words, as a direct inverse process. Most of the students that we classified in this type of configuration stated in their justifications that it was not possible to find another



b) Using the answer to part a) it is possible to find expressions such as…

Therefore, this tells us that it is possible to find different f(x), in fact, f(x) = x^2 + c is a family of f(x).

Fig. 13 Solution given by Student F to item (b) of Task 3



b) There is no other expression because a function has a single derivative and given the values of x and $f'(x)$ it is not possible.

Fig. 14 Solution given by Student G to Task 3

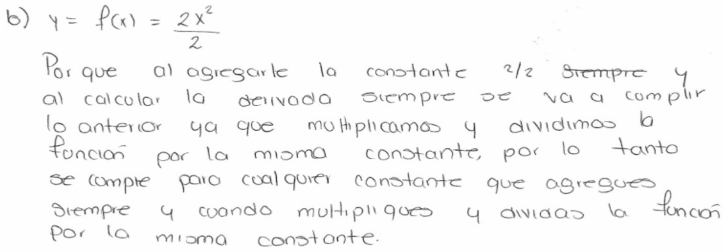
functions because "since the function that has been derived, let's say $f'(x)$, it is possible to find only one $f(x)$ because the derivative of a function is unique and vice versa". In other words, the students do not realize that the antiderivative as "inverse process" of the derivative is a rather more conceptual than procedural or operatorial process; it is a *non-direct inverse process.*

Cognitive configuration 8: equivalent functions

This type of cognitive configuration is associated with answers which explicitly state that it is not possible to find a second, different expression because the possible expressions are equivalent (equivalent functions) to that given in response to item (a). Examples of answers in which this type of configuration is activated are shown in Fig. 15: the answer given by Student H to item (b) and the answer of Student I.

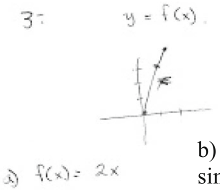
In the case of Student H it can be seen that in order to find a different expression to the one given in response to item (a) he performs certain "algebraic manipulations" with the function $f(x) = x^2$, multiplying by a constant, in particular, by the constant $\frac{2}{2}$. He then

Answer of Student H



b) Because by introducing the constant 2/2 and then calculating the derivative it will always be fulfilled, since we are multiplying and dividing the function by the same constant, and therefore it is fulfilled for any constant that you introduce, provided that you multiply and divide the function by the same constant.

Answer of Student I



b) I could only 'find' an equivalent expression, which when simplified gives the same value.

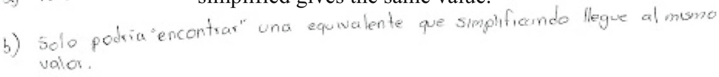


Fig. 15 Solutions given by Students H and I

generalizes this particular process as follows: "…therefore it is fulfilled [that the derivative is $f'(x) = 2x$] for any constant that you introduce, provided that you multiply and divide the function by the same constant". This linguistic element also accounts for the procedure used by Student H.

In general, the answers to item (b) that involved configurations 7 and 8 were incorrect and demonstrate that the prospective teachers in question need to improve their extended knowledge in order to solve problems such as that proposed in Task 3.

Results and discussion

In analyzing the data obtained through administration of the DMK-Derivative Questionnaire, we considered two variables: *the type of cognitive configuration* and *the level of task accuracy* (i.e. correct, partially correct or incorrect). The analytic technique used with the first variable (type of cognitive configuration) was *semiotic analysis*, which provides a systematic description of both the mathematical activity carried out by the prospective teachers in solving the problems, and the mathematical objects (linguistic elements, concepts/definitions, propositions/properties, procedures and arguments) that were involved in their practice (Godino et al. 2007).

For the variable *level of accuracy*, scores of 2, 1 or 0 were assigned to answers that were, respectively, correct, partially correct or incorrect. Thus, the maximum possible score was 26. Twenty-four of the prospective teachers (45.3 %) obtained a score higher than 13, but of these 24 only nine (17 %) responded correctly to more than 67 % of the questionnaire. This illustrates that more than 50 % of the students had difficulties with solving the questionnaire tasks. The mean score (12.4) obtained by the 53 prospective teachers, and the distribution of their scores are shown in Fig. 16.

In general, the DMK-Derivative Questionnaire presented an intermediate level of difficulty for the prospective teachers (Fig. 17). The items they found most difficult were 2-d (Fig. 2) and Task 7 as a whole (Fig. 6). Task 1 and items 2-a, 3-a and 4-a were the easiest for them to solve.

With respect to Task 3, Table 1 shows the results obtained in relation to level of accuracy. It can be seen that the prospective teachers had little problem answering item (a) of the task, with 84.9 % of them providing a correct answer. They did, however, have difficulties in responding to item (b), such that 28.3 % of them answered incorrectly and 39.6 % failed to give an answer.

The types of cognitive configurations used by prospective teachers to solve Task 3 are summarized in Table 2. The 15 students (28.3 %) who answered item (b) incorrectly used configurations of the types we have labeled as *equivalent functions* and *erroneous uniqueness* (see definitions in section five). The first of these is associated with answers which explicitly state that it is not possible to find another, different expression, since the possible expressions are equivalent (equivalent functions) to that given in response to item (a). The second refers to answers that contain an erroneous conception of the uniqueness of the derivative function: in this case, it is explicitly stated that a second and different expression cannot be found as the derivative is unique.

The configurations labeled as *technical* (Table 2) are those in which the students used derivation theorems to produce and justify the solutions given to items (a) and (b). The term *advanced* refers to those configurations in which the students used more advanced concepts such as the integral or the fundamental theorem of calculus to justify their solutions to both parts of the task. As regards item (a) of the tasks it can be seen in Table 2 that 11.3 % of the prospective teachers used a graphical–technical configuration, in other words, the derivative function was calculated by means of a graphical interpretation of the data given in the table, and by using rules of derivation they found the function $f(x)$. A numerical-advanced configuration was present in the answers of 15.1 % of the students, who used the data given in the table to identify the pattern that enabled them to find the rule of correspondence with which they could define the derivative function, and through the use of concepts such as the integral they then found the expression for $f(x)$.

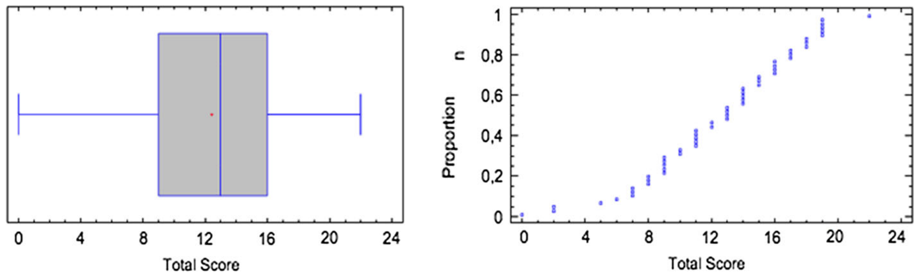


Fig. 16 Boxplot and distribution of scores obtained on the DMK-Derivative Questionnaire

	Item	Difficulty Index	%
1	I-1	0.8679	86.79
2	I-2a	0.7547	75.47
3	I-2b	0.6038	60.38
4	I-2c	0.6415	64.15
5	I-2d	0.1321	13.21
6	I-3a	0.8491	84.91
7	I-3b	0.5660	56.60
8	I-4a	0.7547	75.47
9	I-4b	0.5849	58.49
10	I-5a	0.5660	56.60
11	I-5b	0.4528	45.28
12	I-6a	0.4528	45.28
13	I-7	0.1132	11.32
Mean: 0.56			

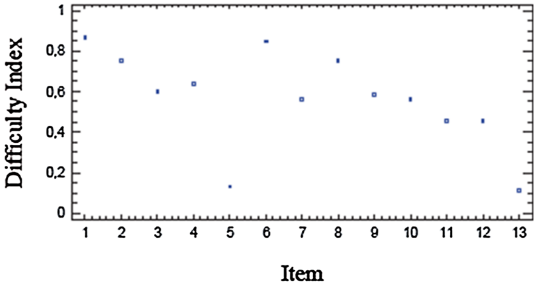


Fig. 17 Difficulty index for the items on the DMK-Derivative Questionnaire

Table 1 Frequencies and percentages for level of accuracy in relation to Task 3

Level of accuracy	Item (a)		Item (b)	
	Frequency	%	Frequency	%
Correct	45	84.9	21	39.6
Partially correct	0	0	9	17
Incorrect	7	13.2	15	28.3
Did not answer	1	1.9	8	15.1
Total	53	100	53	100

Table 2 Frequencies and percentages according to type of cognitive configuration in relation to Task 3

Cognitive configuration	Item (a)		Cognitive configuration	Item (b)	
	Frequency	%		Frequency	%
Graphical–technical	6	11.3	Advanced	8	15.1
Numerical–technical	32	60.4	Technical	21	39.6
Graphical-advanced	5	9.4	Erroneous uniqueness	9	17
Numerical-advanced	8	15.1	Equivalent functions	6	11.3
No solution given	2	3.8	No solution given	9	17
Total	53	100	Total	53	100

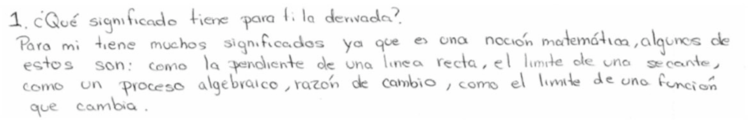
The quantitative and qualitative results obtained in relation to Task 3 support the need to improve the extended knowledge of prospective teachers so that they are better able to solve problems such as that set in this task. This is especially the case for those students whose answer to item (a) involved *graphical–technical* or *numerical–technical* configurations, as well as for those whose solution to item (b) made use of *erroneous uniqueness* or *equivalent functions*. The knowledge (primary mathematical objects and the meaning of how they are used) shown by those prospective teachers who activated *graphical-advanced* or *numerical-advanced* configurations when responding to item (a), and those whose answer to item (b) involved an *advanced* cognitive configuration are those who come closest to the required knowledge that was established in the epistemic analysis described in the fourth section of this article.

Another interesting result was obtained when analyzing the answers of prospective teachers to Tasks 1 and 5 of the questionnaire. The results obtained in relation to Task 5, and to the previous tasks in general, suggest a lack of connection between the meanings of the derivative which the students "know" or "remember" and those they actually apply in their mathematical practice on the derivative.

A good example of this is the case of Mary, who in Task 1 proposed various meanings for the derivative, such as "…the slope of a straight line…" and "…rate of change…". However, in Task 5 she was unable to establish a link between these meanings. In Fig. 18 it can be seen that Mary begins to answer item (a) of Task 5 in the same way that she then does for item (b). However, when she realizes what she is being asked in item (b) she then gives the answer "It doesn't have a horizontal tangent" for item (a). She then responds correctly to item (b), finding the points at which the rate of change of *x* with respect to *y* is zero.

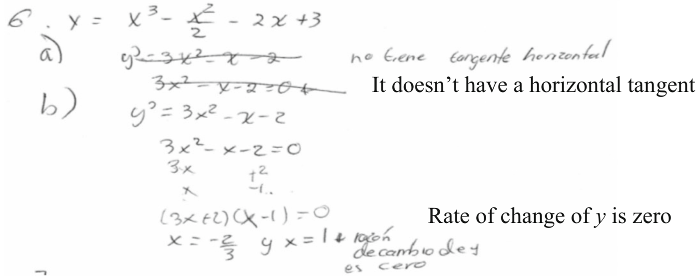
An even clearer example is the case of John (Fig. 19, Task 1), who provides meanings for the derivative such as "…slope of the line tangent to a curve…" and "…rate of change of a function…". However, he then gives a correct solution to item (a) of Task 6, but for item (b) of Task 5 he answers: "I think it's the one above, and if that's the case then I wouldn't be able to answer part a".

Answer to Task 1



1. What does the derivative mean to you?
For me the derivative has several meanings because it is a mathematical notion. These include: the slope of a straight line, the limit of a secant line, an algebraic process, rate of change, the limit of a function that changes.

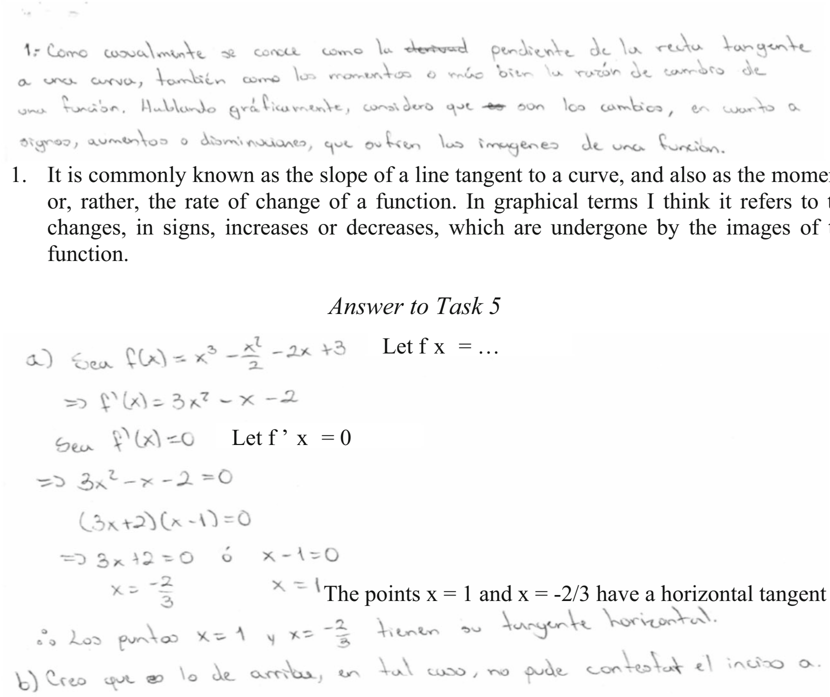
Answer to Task 5



It doesn't have a horizontal tangent

Rate of change of *y* is zero

Fig. 18 Mary's answers to Tasks 1 and 5 on the DMK-Derivative Questionnaire


Answer to Task 1

1.- Como usualmente se conoce como la ~~derivad~~ pendiente de la recta tangente a una curva, también como los momentos o más bien la razón de cambio de una función. Hablando gráficamente, considero que son los cambios, en cuanto a signos, aumentos o disminuciones, que sufren las imagenes de una función.

1. It is commonly known as the slope of a line tangent to a curve, and also as the moments or, rather, the rate of change of a function. In graphical terms I think it refers to the changes, in signs, increases or decreases, which are undergone by the images of the function.

Answer to Task 5

a) Sea $f(x) = x^3 - \frac{x^2}{2} - 2x + 3$ Let f x = …

$\Rightarrow f'(x) = 3x^2 - x - 2$

Sea $f'(x) = 0$ Let f ' x = 0

$\Rightarrow 3x^2 - x - 2 = 0$

$(3x+2)(x-1) = 0$

$\Rightarrow 3x+2 = 0$ ó $x - 1 = 0$

$x = -\frac{2}{3}$ $x = 1$ The points x = 1 and x = -2/3 have a horizontal tangent

$\therefore$ Los puntos $x = 1$ y $x = -\frac{2}{3}$ tienen su tangente horizontal.

b) Creo que es lo de arriba, en tal caso, no pude contestar el inciso a.

b) I think it's the one above, and if that's the case then I wouldn't be able to answer part a.

Fig. 19 John's answers to Tasks 1 and 5 on the DMK-Derivative Questionnaire

As can be seen in these two examples (Figs. 18 and 19), some prospective teachers find it difficult to make connections between two definitions of the derivative, and they fail to make associations with meanings such as "the rate of change of *y* with respect to *x* is zero at those points where the tangent line of the function is horizontal".

Final reflections

In the present research, we have exemplified the use of some of the categories of teacher knowledge and the theoretical–methodological tools to analyze and characterize these categories of knowledge proposed by the DMK model. To this end, we performed both the a priori analysis of one of the tasks included in an instrument designed to explore relevant aspects of both mathematical dimension and epistemic facet of DMK and the analysis of the responses that a sample of prospective teachers gave to such task.

On the basis of the analyses carried out, we illustrated the use and relevance of two tools proposed for characterizing of knowledge involved in both the mathematical dimension and the epistemic facet of DMK: *mathematical practices* and *objects and processes configuration*. Both tools can be used to analyze institutional and personal knowledge.

Thus, for example, the "configuration of objects and processes" tool can be useful to determine in detail the expected knowledge (a priori analysis of institutional knowledge or meanings), in which case we would speak of “epistemic configuration”. Similarly, said tool can be used to characterize the knowledge of teachers (personal knowledge or meanings) in which case we would speak of “cognitive configuration”.

It should also be noted that the variable and theoretical tool “cognitive configuration” associated with the prospective teachers’ responses is of considerable importance when it comes to understanding the nature of the didactic-mathematical knowledge that these teachers actually possess. Earlier in this article we argued that the “configuration of primary mathematical objects and processes” can be used as tool that facilitates the analysis and categorization of certain features of the epistemic facet of prospective teachers’ didactic-mathematical knowledge. In addition, and through the use of other tools—for example, tools to characterize the knowledge of teachers about: attitudes and emotions of students, resources and suitable means to enhance of the learning, interactions and norms that regulate the teaching and learning processes, reflection on own practice—the cognitive configuration can be used to determine the complex network of learning and mathematical knowledge of teachers. In this regard, one aim of teacher education programs should be to ensure that the cognitive configurations of prospective teachers (such as those described in the fifth section) which are used in practice are progressively adapted to fit the previously identified epistemic configurations required. The development of these teacher education programs (and the determination of such epistemic configurations) should take into account the complexity of the global meaning of the derivative (Pino-Fan et al. 2011).

Teachers and teacher educators should also be aware of the web of objects and processes brought into play in solving mathematical tasks, in order to manage learning and organize educational processes. For this reason, such teacher education programs should consider developing skills in the teachers oriented toward to identification of cognitive configurations of their future students. Furthermore, these educational programs should enhance the teachers’ knowledge regarding the determination of the “expected knowledge” (detailed a priori analysis as the presented here), since it is from these expected knowledge that can be characterized the knowledge effectively evidenced by students more precisely.

Parallel to the above, we must note that the results of the analysis of the answers provided by prospective teachers to Task 3 suggest that they need to improve their extended knowledge about the derivative, since they do not establish explicit and “fluent” connections between said mathematical object and others mathematical objects that are in subsequent levels of the high school mathematics curriculum (taking into account the context of the sample).

Broadly, and in order to give a broader view of the results obtained with the implementation of tasks, we would like to point that the analysis of the responses given by prospective teachers to the tasks included in the DMK-Derivative Questionnaire indicates that they had certain difficulties in solving the tasks related to common and extended knowledge about the derivative. The results obtained with tasks such as 4, show that these teachers performed better when solving tasks in which the derivative is understood as the slope of the tangent line. Furthermore, it was evidenced that the prospective teachers lack certain aspects not only of the epistemic facet of DMK (use of different representations, use of different meanings of the derivative, solving the problem through various procedures, giving a range of valid arguments to justify these procedures, etc.) but also of the common knowledge required to solve the task. Likewise, the results obtained in relation to Tasks 6 and 7 illustrated the difficulties which the prospective teachers experienced when they had to use the derivative as the instantaneous rate of change in a relatively complex situation. Here the DMK-Derivative Questionnaire revealed how common knowledge is in itself not enough to deal with the kind of tasks that will emerge in the teaching context, for which teachers will also need a certain degree of both extended knowledge and epistemic facet of DMK.

In summary, both the design of the questionnaire and the responses of these prospective teachers reveal the complex set of mathematical practices, objects and processes that are brought into play when solving tasks related to the derivative. Teachers need to become aware of this complexity during their education so that they will be able to develop and assess the mathematical competence of their future students. Finally, the categories of DMK model described in this article can, along with the corresponding theoretical and methodological tools, be applied to the teaching of various mathematical topics. However, each mathematical topic has specific features linked to the epistemic facet of DMK, and these need to be analyzed prior to their being used as an institutional reference.

Acknowledgments This study has been performed in the framework of research projects FONDECYT N° 11150014 (Funded by CONICYT, Chile), EDU2015-64646-P and EDU2012-31869, Ministry of Economy and Competitiveness (MINECO, Spain).

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