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Working at the intersection of teacher knowledge, teacher beliefs, and teaching practice: a multiple-case study

Charalambos Y. Charalambous[1]

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Abstract Attempts to understand what contributes to teaching quality have been channeled in different directions, with two main research streams focusing on either teacher knowledge or teacher beliefs. Few are the studies that have attended to both the cognitive and the affective domain simultaneously, trying to unpack how both jointly contribute to teaching quality. Situated at the nexus of these two domains, this study aims to understand how teachers' mathematical knowledge for teaching and their pedagogical beliefs contribute to their performance in providing explanations and selecting and using tasks, as studied in a teaching simulation. Using a multiple-case approach and examining the development of three prospective teachers' knowledge and beliefs over a content-and-methods course sequence, the study documents how limitations in either knowledge or beliefs can mediate the effect of the other component on prospective teachers' performance. Implications for teacher preparation and in-service education are drawn and directions for future studies are offered.

Keywords Mathematical knowledge for teaching · Pre-service teachers · Teacher beliefs · Teaching practice

Introduction

Understanding what contributes to teaching effectiveness has for long attracted researchers' interest, with different frameworks capturing teaching capacities proposed toward this end. In one of the most widely cited frameworks, the *Proficient Teaching of Mathematics*, Kilpatrick et al. (2001) propose five interrelated strands considered necessary for teaching

✉ Charalambos Y. Charalambous
cycharal@ucy.ac.cy

1 Department of Education, University of Cyprus, Theophanides Building, Rm 508, 11-13 Dramas St., 1077 Nicosia, Cyprus

mathematics proficiently: conceptual understanding of the knowledge needed for teaching; fluency in carrying out basic instructional routines; strategic competence in planning effective instruction; adaptive reasoning for justifying one's instructional practice and reflecting on one's work; and productive dispositions toward mathematics, teaching, and learning.

In this article, we focus on the first strand and an aspect of the last strand: teacher beliefs. Both knowledge and beliefs have attracted scholarly interest, but have largely been treated in parallel. Situated at the nexus of both components, this study aimed to understand how their interplay contributes to teaching performance, as studied in a simulated teaching environment. Capitalizing on three prospective teachers' cases, we asked:

- How do prospective teachers' knowledge and beliefs interact to inform their performance in specific teaching practices, as captured in a simulated environment?
- To what extent can affordances in one component compensate for limitations in the other?

The remainder of this article is organized as follows. In the next section, we briefly summarize what prior research has suggested regarding the contribution of teacher knowledge and teacher beliefs to teaching quality and student learning. Following that, we detail the methods pursued to answer our research questions. After presenting the results by case, a cross-case analysis is undertaken. The study findings are discussed in light of the affordances of considering both teacher knowledge and beliefs; implications for practice are drawn and directions for future research are offered.

Working at the nexus of teacher knowledge and beliefs

In this section, we outline the two components considered above and summarize what prior research has revealed about their individual contribution. We then focus on the few studies that have explored teachers' cognitive and affective domains simultaneously.

Teacher knowledge, teaching quality, and student learning

In Kilpatrick et al.' (2001) framework, teacher knowledge is conceptualized as something more than just pure content knowledge:

> The kinds of knowledge that make a difference in teaching practice and in students' learning are an elaborated, integrated knowledge of mathematics, a knowledge of how students' mathematical understanding develops, and a repertoire of pedagogical practices that take into account the mathematics being taught and how students learn it. (p. 381)

Over the past decade, different conceptualizations of this type of knowledge have been advanced, including the *Mathematical Knowledge for Teaching* (MKT, Ball et al. 2008), teachers' *profound understanding for emergent mathematics* (Davis and Renert 2013), and the *Knowledge Quartet* (Rowland et al. 2005)—to name a few. A common denominator of these conceptualizations is that to successfully undertake the work of teaching, teachers need deep, broad, and well-connected knowledge that supports decomposing and unpacking the content to make it comprehensible by students.

Largely because of advancements made in conceptualizing and measuring teacher knowledge, research over the past decade has produced favorable results linking teacher knowledge to teaching quality and student learning. For example, Baumert et al. (2010) found that pedagogical content knowledge was a more powerful predictor of instructional quality than teachers' knowledge of the content. Analyzing a series of lessons on fraction arithmetic taught by two sixth-grade teachers with different levels of knowledge, Izsák (2008) also concluded that stronger knowledge supported better use of representations and more productive interactions with students around the content. Along similar lines, synthesizing the results of a multiple-case study that examined nine middle-grade teachers teaching several topics including fraction operations, Hill and Charalambous (2012) concluded that, compared with their counterparts with lower MKT, teachers with stronger knowledge used mathematical language more precisely, provided more appropriate mathematical explanations, and were more successful at drawing rich connections among mathematical ideas.

Teacher knowledge has also been linked to student learning gains. For example, in a large-scale longitudinal study recruiting 181 German secondary teachers, Baumert et al. (2010) showed that, when controlling for other background factors, two classes taught by teachers differing in their pedagogical content knowledge by two standard deviations differed by approximately half a standard deviation in students' mean achievement by the end of the school year. Similar results were also reported in studies focusing on MKT and recruiting either elementary (Hill et al. 2005) or middle school teachers (Hill et al. 2011).

In sum, teacher knowledge appears to be a promising contributor to both instructional quality and student learning, especially if one considers the topic explored in the present study: fraction division (see below). This is because prior studies have consistently shown teachers to encounter difficulties in understanding and appropriately conveying this content to students (e.g., Ball 1990; Ma 1999; Tirosh 2000). However, even scholars focusing on teacher knowledge (e.g., Sleep and Eskelson 2012) recognize that this component alone cannot satisfactorily predict instructional quality, and recommend considering aspects of the affective domain, as well.

Teacher beliefs, teaching quality, and student learning

After the publication of Thompson's (1992) seminal work, in which she elaborated the concept of beliefs and discussed the structure of teacher belief systems, several scholars have systematically explored or discussed the impact of teacher beliefs on teachers' practice. From the gamut of teacher beliefs, in this article, we focus on *pedagogical* beliefs, namely what teachers believe about teaching and learning mathematics. These beliefs were found to inform several decisions that teachers make during instruction, including their selection, adaptation, and presentation of curriculum tasks; their questioning techniques; their tolerance of student frustration, confusion, and errors; and the manner in which they support students when the latter face momentary impasses (e.g., Manouchehri and Goodman 2000; Remillard 1999; Skott 2001; Speer 2008; Wilkins 2008; Wilson and Cooney 2002).

For example, Remillard (1999) discusses the case of two primary school teachers who both were using a reform-oriented curriculum for the first time. Believing that students learn mathematics by being offered ample opportunities for exercise and practice and by being told or shown what to do and how to do it, the first teacher focused on drill-and-practice tasks. In contrast, thinking that learning occurs when students are given opportunities to invent solutions and to explore different relationships, the second teacher employed more complex tasks. Along the same lines and also using a case study approach, Skott (2001) documents how a novice middle-grade mathematics teacher responded to different students in markedly different ways, partly because of his beliefs. Similar patterns were also identified for high school mathematics teachers (e.g., Cross 2009). Moving a step farther, Speer (2008) provides empirical evidence supporting the claim that, if analyzed at a fine-grained size, teachers' beliefs can help explain or even predict teachers' instructional decisions and actions. Providing a detailed analysis of one college mathematics instructor's beliefs and practices while teaching a calculus class, Speer showed that the instructor's pedagogical beliefs could predict his moment-to-moment decisions at several junctures of the lesson.

Teacher pedagogical beliefs were also found to be associated with student learning gains—be they affective or cognitive. Carter and Norwood (1997), for example, studied the beliefs of seven elementary grade teachers and their students. Four of these teachers held beliefs that aligned with the National Council of Teacher of Mathematics (NCTM 2000) *Principles and Standards*: These teachers valued learning mathematics from a constructivist perspective; they also endorsed conceptual understanding and reasoning rather than memorization and rote application of procedures. The teachers who held these beliefs—hereafter referred to as *Standards*-based beliefs (see also Clarke 1997)—had students who treasured more the importance of solving challenging problems and working hard on them than the students of teachers whose beliefs were inconsistent with *Standards*-based approaches. The first group of students also valued working hard and striving for understanding more so than their counterparts taught by teachers holding non *Standards*-based beliefs. This led Carter and Norwood to conclude that teachers' pedagogical beliefs influence how students view what it means to do and learn mathematics. Positive associations have also been found between teachers' beliefs and student cognitive learning gains. For instance, a more recent study focusing on 35 elementary school teachers (Polly et al. 2013) showed that students who were taught by teachers with a transmission orientation in teaching exhibited smaller learning gains in curriculum-based tests than students taught by teachers with a discovery orientation.

Working at the nexus of knowledge, beliefs, and teaching practice

Writing in 1992, Thompson warned that focusing solely on either knowledge or beliefs results "in an incomplete picture" (p. 131). A decade later, echoing her plea to attend to both constructs, Wilson and Cooney (2002) underlined that knowledge and beliefs are related in powerful ways and that they both contribute to teaching quality. Despite the efforts made to attend to both constructs, Philipp (2007) admitted that understanding how knowledge and beliefs interact in informing instruction constitutes an open problem. This lingering problem was partly due to the fact that for years researchers have worked in parallel, foregrounding either the cognitive or the affective domain, and making assumptions for the other domain—at best—or even totally ignoring it. Recent works, however, are more promising, in that scholars are increasingly attending to both domains. A critical review of this literature suggests that researchers have embraced this dual focus on knowledge and beliefs in two different approaches: either by attending to both strands in parallel or by examining interrelationships between them—thus being more consonant with Kilpatrick et al.' (2001) conceptualization. Selected examples of these approaches are briefly reviewed below.

Works clustered under the first approach are largely quantitative in nature. With the exception of one study that examines correlations between knowledge and beliefs at a fixed point in time (Drageset 2010), most studies trace *changes* in teachers' beliefs and knowledge as a result of teachers' participation in a preservice or in-service education program. For example, exploring changes in both beliefs and knowledge through a randomized experiment, Philipp et al. (2007) showed that prospective teachers who studied students' mathematical thinking exhibited more sophisticated epistemological and pedagogical beliefs and underwent more positive changes in their knowledge than their counterparts who were either simply taking a mathematics content course or taking the course *and* observing lessons. Similar results were found in studies focusing on either prospective teachers (e.g., Swars et al. 2009) or practicing teachers (e.g., Gomez and Benken 2013; Hamre et al. 2012).

Despite the obvious affordances of this dual attention to both domains, because of examining teacher knowledge and beliefs in parallel, these studies leave critical questions unaddressed: How do teacher knowledge and beliefs interact in shaping teachers' decisions and actions? Can limitations in knowledge be compensated for by positive beliefs toward the subject and its teaching? Conversely, to what extent can narrow beliefs hamper teachers' potential to offer quality instruction as their knowledge would have predicted?

By working at the intersection of knowledge and beliefs mostly through qualitative case studies, the second research strand provides some first insights into these questions. For example, Lloyd and Wilson (1998) discussed how a high school teacher's comprehensive and well-connected knowledge of functions, alongside his beliefs that were aligned with a *Standards*-based perspective of teaching (i.e., teaching based on the NCTM 2000, *Standards*), contributed to his rich enactment of a *Standards*-based unit on functions. Sleep and Eskelson's (2012) study of a middle school teacher sketched a rather different portrait of the interplay between knowledge and beliefs, showing beliefs about the importance of computational procedures to compromise the potential effect of teacher knowledge on instructional quality. A multiple-case study of four elementary grade teachers in their first year of using a reform-based curriculum (Bray 2011) went into greater depth in exploring different manifestations of the joint contribution of beliefs and knowledge to teaching quality. Teacher beliefs were found to relate more to teachers' *proclivity* to intentionally make the sharing of flawed solutions a focus of whole-class discussions; teacher knowledge, on the other hand, shaped the mathematical and pedagogical *quality* of teachers' responses to students' errors in ways that promoted student understanding.

Although not going into that depth, two quantitative studies showed teacher beliefs and perceptions to mediate the association between teacher knowledge and teaching quality or student performance. Using data from 481 Grades K-5 elementary in-service teachers, the first study (Wilkins 2008) showed teacher beliefs to mediate the association between teachers' content knowledge and the frequency with which they reported using certain practices in their teaching. Drawing on data from 266 upper-grade elementary teachers, Campbell et al. (2014) found that the effect of teacher knowledge on student achievement was influenced by teacher beliefs about organizing instruction in ways that support incremental mastery of skills.

The importance of working at the intersection of teacher knowledge and beliefs is also reinforced by another study that, unlike the previous ones, traced *changes* in these two components by following five prospective teachers (PSTs; Holm and Kajander 2012). Although focusing on the PSTs' *self-reported* changes in beliefs and knowledge, this study concluded that to help PSTs improve their practice, both knowledge and beliefs need to be simultaneously targeted.

In conjunction, the studies of the latter strand emphasize the promise of working at the intersection of teacher knowledge and beliefs to unpack the mechanisms through which these two components interact in shaping instructional decisions and actions. However, with the exception of the last study reviewed above, all other studies focused on practicing teachers and explored the interplay between knowledge and beliefs from a static perspective, namely by exploring beliefs and knowledge only at one point in time. The only study that dealt with *changes* in PSTs' beliefs and knowledge—thus adopting a more dynamic approach—drew on prospective teachers' self-reports. To address this research gap, the present multiple-case study (Yin 2009) explored how PSTs' knowledge for teaching and pedagogical beliefs at the beginning of a series of a mathematics content and methods course contribute to their instructional decisions and actions, as captured and studied in a simulated environment; it also traced changes in these components after the course culmination. In doing so, this study adopted both a static and a dynamic perspective in exploring the interplay between knowledge and beliefs.

Methods

Study participants

The study was conducted at a large Midwestern US University. The two courses under consideration (collectively identified as "coursework") were part of an intensive 1-year teaching education program leading to a Grades K-8 teacher certificate and a Masters of Arts degree in education. For the purposes of this study, we focus on three PSTs—Deborah, Vonda, and Kimberley (all pseudonyms)—which collectively help illuminate different manifestations of the interplay between teachers' MKT and beliefs. These PSTs entered coursework with different mathematics backgrounds and teaching experiences. Kimberley had a strong mathematics background. As she reported on entering the program, because of her keen interest in mathematics, she took seven content courses in high school and five courses during her undergraduate studies; she also had a minor in mathematics. In contrast, Deborah had a moderate mathematics background, as suggested by the five content courses she took in high school and a course she attended during her undergraduate studies. Even weaker was Vonda's mathematics background: She had taken only two courses in high school and two during her undergraduate studies, with the highest level course being Algebra II (which serves as a precalculus class). Among the three study participants, Kimberley also had some prior teaching experience because of having worked as a substitute teacher for algebra classes.

Coursework: context and content

Both courses sought to help PSTs develop skills, knowledge, and ways of reasoning necessary for teaching mathematics effectively. In particular, the content course aimed at helping PSTs move from simply knowing mathematics as educated adults to knowing mathematics as teachers; the methods course was designed to help them move farther along this continuum and develop mathematical knowledge and skills necessary for supporting students in learning mathematics.

Consisting of 13 three-hour meetings, the content course aimed at supporting PSTs in developing flexible understanding of important ideas and processes within the realm of number theory and operations; it also sought to offer them opportunities to practice using representations, providing explanations, and analyzing others' thinking. The mathematics methods course extended this work, but mostly focused on four domains: leading a whole-class discussion; representing mathematical ideas; assessing students' mathematical knowledge, skills, and dispositions; and planning mathematics lessons. Content-wise, the methods course—also comprising 13 three-hour meetings—mostly focused on the meaning of numbers and the four basic operations.

Data sources

Several data sources were utilized. These included an MKT paper-and-pencil test; selected items from a survey designed to investigate participants' beliefs and images of teaching; and a designed teaching environment—what we call a teaching simulation—that afforded participants the opportunity to engage in different teaching practices and reflect upon a virtual teachers' work. For each data-collection instrument, data were collected at the beginning and end of coursework (i.e., approximately 7 months later).

MKT paper-and-pencil test

Comprising of 41 items, the MKT test was based on common content knowledge (CCK) and specialized content knowledge (SCK) items developed by the Learning Mathematics "for Teaching" (LMT) group of the University of Michigan, selected to *align* with the mathematical topics and practices discussed in coursework and also considered in the teaching simulation (see more in Charalambous 2008). In particular, 34 of these items were drawn from an existing *Rational Numbers* form; the remaining seven items were developed by the LMT personnel for the purposes of the present study. The three PSTs' raw scores on the test were converted into standardized scores, based on data from the entire group of PSTs enrolled in coursework ($n = 20$).

Survey

From a survey intended to tap into participants' beliefs about what it means to do, teach, and learn mathematics (see Appendix 1), here we focus on three items that reflect beliefs about teaching mathematics that are inconsistent with the intentions of *Standards*-based curricula: "When students can't solve problems, it is usually because they can't remember the right formula or rule"; "In learning mathematics, students must master topics and skills at one level before going on"; and "If students are having difficulty in math, a good approach is to give them more practice in the skills they lack." PSTs were expected to state their level of agreement with each statement using a 7-point Likert scale, with 1 representing strongly disagree and 7 strongly agree. In addition to these statements, the survey also included an open-ended question asking participants to describe how they experienced mathematics as learners of the subject. This question aimed at exploring PSTs' "images of teaching" (cf. Lortie 1975).

Teaching simulation and associated semi-structured interviews

Comprising a series of PowerPoint slides depicting different episodes of teacher–student interactions around the content, the teaching simulation featured a sixth-grade cartoon teacher giving an introductory lesson on fraction division. At selected points, the slide show stopped and participants were asked to comment on the cartoon teacher's actions.

Participants' comments and reflections around the virtual teacher's work created an arena for further capturing their beliefs about teaching and learning mathematics; they also helped further probe into participants' images of teaching. Participants were also asked to undertake the role of the virtual teacher and *perform* rather than simply talk about certain tasks of teaching; this allowed examining their in vitro performance. Because of space limitations and to better illustrate the interactions of knowledge and beliefs in informing PSTs' work, we consider different practices for the study participants: For Deborah and Vonda, we focus on the practice of providing explanations with respect to explicating why $2 \div 3/4 = 2\ 2/3$; for Kimberley and briefly for Vonda, we discuss the practice of selecting and using tasks.[1] In the latter practice, PSTs were presented with two textbook pages and were asked to discuss how they would use them to structure an introductory lesson on fraction division. Being largely procedural, the first page (adapted from Greenes at al. 2002) presents the standard algorithm of "inverting and multiplying" and expects students to apply it to solve 16 problems. These problems are not sequenced according to their difficulty level and no explanations or representations are provided as to why the algorithm of "multiplying by the reciprocal" works or makes sense. The second page (drawn from Lappan et al. 2009) is more conceptually oriented, since it asks students to use representations and provide written explanations when solving fraction division word problems (i.e., making 1/6 yard ribbon badges and then 2/3 yard ribbon badges from given lengths of ribbon; Tasks A and B, correspondingly). Students are then expected to use the approach they developed to solve fraction divisions as if they were ribbon problems (Task C). Finally, they are encouraged generate the algorithm for fraction division (Task D).

Data analysis

For each case, we first wrote an analytic memo (Patton 2002). As a research device, this tool enabled us to not only describe PSTs' performance in the practices considered, but also summarize our thoughts about potential ways in which knowledge and beliefs interact in informing performance in the simulation. For example, in considering the choices that the participants made while offering explanations, we kept track of whether these choices were supported by their knowledge and were consonant with their beliefs (see an example below on Deborah's use of representations). We did so twice, first from a static perspective (i.e., by considering participants' performance and characteristics at the beginning of coursework) and then from a dynamic perspective (i.e., by exploring changes therein). Using these analytic memos, and employing Yin's (2009) explanation-building approach, we then performed a cross-case analysis following Thompson's (1984) classic work. As Yin (2009) clarifies, explanation building begins with an initial hypothesis, which is gradually revised and elaborated in light of new data from each case. The analyst's ultimate goal is to formulate an explanation that accounts for and accommodates the data of all cases under consideration. Our initial hypothesis that both knowledge and beliefs contribute to participants' teaching practice was further elaborated through the cross-case analysis, which also helped us understand the extent to and the ways in which teacher knowledge can compensate for limitations in teacher beliefs, and vice versa, whether *Standards*-based aligned beliefs can compensate for limitations in knowledge.

[1] A more detailed account of all three PSTs' performance in both practices is outlined in Charalambous (2013).

Results

We first start with the case of Deborah who entered coursework with beliefs that resonated with a *Standards*-based teaching approach, but whose MKT was weak. Deborah experienced the most notable changes in her MKT by the end of coursework, thus providing a venue for exploring from both a static and a dynamic perspective whether limitations in knowledge can be compensated for by affordances in beliefs. We then move to the case of Vonda, who entered coursework with low MKT and whose beliefs were inconsistent with *Standards*-based approaches. By the end of coursework, both her knowledge and beliefs slightly improved. Thus, her case allows us to explore how limitations in both domains might affect teachers' performance and how small improvements therein might be reflected in PSTs' work. We conclude with the case of Kimberley, who entered coursework with relatively high MKT, but whose beliefs were rather inconsistent with a *Standards*-based approach. Throughout coursework, Kimberley improved in knowledge and beliefs. At the antipode of Deborah's case, Kimberley's case enabled us to examine whether MKT alone is sufficient for teaching mathematics.[2]

The case of Deborah

Ranked in the last quintile according to her MKT performance, Deborah answered incorrectly several items, including those related to the different meanings of division (partitive and measurement) and the different types of units when it comes to fraction operations. Her reported school experiences were also not particularly positive, since her teachers placed emphasis on her getting correct answers. These experiences constituted avoidance models for Deborah, who strongly endorsed the idea of helping students develop conceptual understanding. These beliefs were also consistent with her strong disagreement with the survey statement "When students can't solve problems it is usually because they can't remember the right formula or rule." Taken together, Deborah's beliefs were conducive to creating rich learning environments. Yet her performance in explaining the quotient of division $2 \div 3/4$ suggested that beliefs alone do not suffice to build such environments.

Despite being a strong proponent of providing good explanations and using "pictures" for doing so, because of limitations in her knowledge, Deborah could not provide a conceptual explanation as to why $2 \div 3/4$ is equal to 2 2/3. Prompted to offer such an explanation, she first drew two circles to represent the dividend, put a comma, and then drew a third circle to represent 3/4 (Fig. 1a). She then stopped, pointing out that she was stumped and did not know how to move on. Choosing a different representation (a line to represent a yard) in hopes that this would enabled her to provide a conceptual explanation, Deborah again drew two lines, one for the dividend and another for the divisor, and then she quit (Fig. 1b).

At the end of coursework, Deborah was the PST who experienced the highest gains in her MKT. A closer look at the test items she answered correctly suggested that she now had a better understanding of division, from both a partitive and a measurement perspective. She additionally had a much better grasp of relative units when it comes to interpreting the

[2] Although we focus on only three PSTs and consider their work in a limited set of teaching practices, the results considered in this section are largely typical of the work of the entire group of 20 PSTs on five teaching practices: providing explanations, using representations, analyzing student work, selecting and using tasks, and responding to students' request for help (cf. Charalambous 2008).

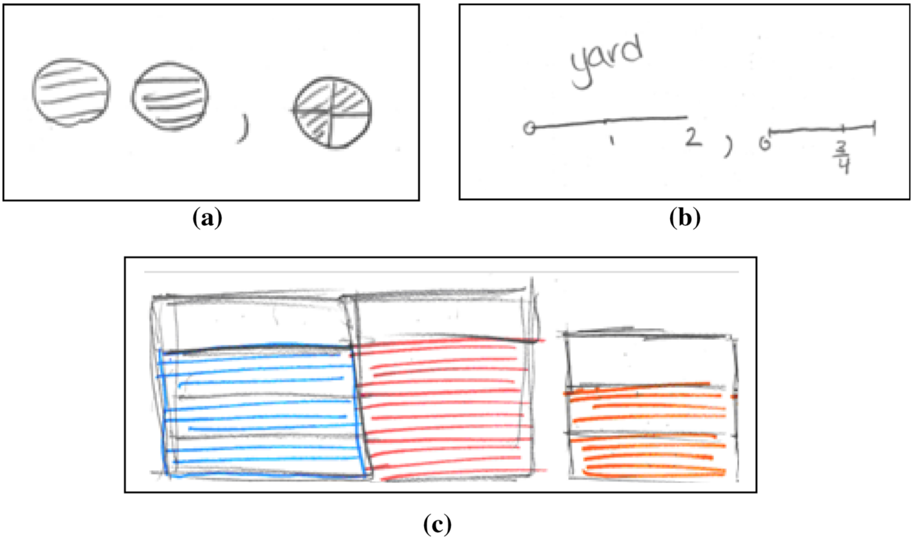


Fig. 1 Deborah's attempts to explain 2 ÷ 3/4. **a**, **b** Pre-coursework, **c** post-coursework

fractional part of a quotient. During coursework she also developed images that were consistent with a *Standards*-based approach in teaching mathematics. Talking about these images, she noted that on entering coursework, she wanted to "teach for conceptual understanding," but she did not know how to do that, arguing that she "didn't have anything in [her] toolkit to see how [instruction] would be done differently [from how she had experienced it as a student]." Coursework enriched her teaching "toolkit," since, among other things, it offered her hands-on experience in "constructing meaning through working with manipulatives." Her stronger knowledge, alongside the beliefs she originally held, coupled with more productive images of teaching, supported Deborah in offering the kind of instruction she had originally wished to provide.

For example, in explaining the above-mentioned problem at the end of coursework, Deborah first drew two rectangles attached to each other and shaded 3/4 in each (Fig. 1c). Pointing to the portions she colored in, she explained that each represents 3/4 of a whole. Next, she drew a rectangle equal to the blue portion and divided it into three parts (see smaller rectangle on the right). Pretending to be transferring the two unshaded regions into this 3/4-rectangle, she colored in two of its parts in orange and continued:

> So this [pointing to the blue 3/4-portion] is three fourths; this [pointing to the red 3/4-portion] is three fourths; this [pointing to the 3/4-rectangle] is the same three fourths [as the previous two 3/4-portions]. And if I transfer these two [pointing to the two 1/4 non-shaded pieces] over here [showing the portion colored in orange in the 3/4-rectangle] we have how much of three fourths? I broke it into three pieces and I shaded in two; it's two thirds of three fourths. … So, in total we get one [pointing to the blue 3/4-portion], two [pointing to the red 3/4-portion], and two thirds [pointing to the 2/3 portion of the 3/4-rectangle shaded in orange].

Deborah's explanation was sufficiently unpacked. As such, her performance in this practice was in stark contrast to how she had performed during the pre-coursework interview. Overall, pre-coursework Deborah was suggestive of how limitations in knowledge, despite beliefs that align with a *Standards*-based approach, can deprive PSTs from providing conceptually grounded explanations. At the same time, her post-coursework performance reveals how the confluence of stronger knowledge with beliefs and images of teaching that resonate with *Standards*-based approaches can scaffold PSTs to perform in ways that can help students attach meaning to mathematical procedures.

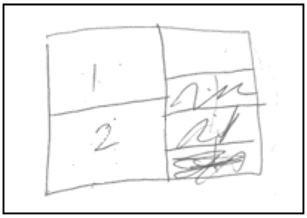
The Case of Vonda

Along with Deborah, Vonda was also ranked in the last quintile according to her MKT score. Like Deborah, she answered incorrectly several items, especially those pertaining to the meaning of division and the interpretation of remainders. Besides these limitations in her knowledge, Vonda also had very constrained images about what it means to teach mathematics, since as a learner of the subject, she was mostly expected to remember and apply rules and formulae. She also harbored rather traditional pedagogical beliefs, with central among them being that teaching and learning mathematics implies following and applying a sequence of steps. In fact, this idea was Vonda's refrain when commenting on several of the virtual teacher's instructional moves in the simulation. Supportive of this belief was also a conglomerate of other beliefs expressed during the interview. For example, Vonda believed that the teacher should transmit knowledge by walking students step-by-step through mathematical procedures. She strongly agreed with the survey statement that the teacher should help students move from easier to more complex tasks, and she was convinced that when students encounter difficulties, it is usually because they cannot remember the right rules. Altogether, these beliefs, alongside her weak knowledge and her constrained images of teaching, left little room for performing in ways that lend themselves to creating rich learning environments, as documented below.

When asked to provide an explanation for $2 \div 3/4$, after considerable thought, Vonda simply outlined the steps involved in carrying out this operation (i.e., find the reciprocal of the second fraction, "add" 1 underneath the 2 of the dividend, and multiply across the two new fractions)—a move that resonated with her beliefs about what it means to teach and learn mathematics by following and applying a sequence of steps. When encouraged to use some sort of representation to help students understand the meaning of this operation, she drew two rectangles, divided the first into two to represent the dividend, and the second into four parts, shading three of them—to represent the divisor (see Fig. 2a). She then paused and argued that she could not use a representation to explain this operation. In essence, Vonda presented the dividend units and the divisor units as different entities without any obvious connection between them. More critically, without a conceptual understanding of division, Vonda could not use this drawing to give meaning to this operation.

The influence of Vonda's beliefs on her performance was also evident in her work on selecting and using tasks: She preferred the first to the second textbook page for structuring a lesson on fraction division. She even claimed that the first page is not sufficiently supportive of student learning, because it does not "provide step-by-step instructions for dividing fractions." In using the first page, she would "call student number one to solve [division problem] one and … walk the class through the steps." Believing that "practice is the best teacher," she would follow the same approach to share the solution of each problem to "help students solidify the algorithm."

Unlike Deborah, Vonda experienced a small amount of growth in her MKT, as measured at the end of coursework, leaving her still in the lowest quintile of PSTs' MKT. Vonda now answered correctly items pertaining to the meaning of division, but she missed



(a)

(b)

Fig. 2 Vonda's attempts to explain 2 ÷ 3/4. **a** Pre-coursework, **b** post-coursework

items related to the fractional part of the quotient or the different units when it comes to fractions. In addition to this slight change in her MKT, Vonda also confessed to have experienced some changes in her beliefs and credited the coursework activities as eye openers about the importance of teaching conceptually: "I didn't think [of teaching conceptually] before; and I think that's because of my own learning experiences. They're so antiquated." This argument resonates with her answer to a survey statement, with which Vonda now disagreed: that when students cannot solve problems, it is because they cannot remember formulas and rules. Despite these changes, some deeply held beliefs Vonda held on entering coursework persisted, given that she still (strongly) agreed with the remaining survey items.

Armed with a somewhat better grasp of the concept of fraction division, Vonda was now able to explain the division problem 2 ÷ 3/4 as fitting 3/4-portions into the two wholes represented by the dividend. To do so, she drew two rectangles and divided each into fourths; she then shaded in 3/4 in each rectangle, pointing out that this represented the two 3/4s (Fig. 2b_1). However, her knowledge of relative units (i.e., divisor units) was not particularly solid, thus preventing her from seeing that the 2/3 in the quotient corresponded to 2/3 of 3/4. Pressed to explain what this fractional part represented, Vonda drew two additional rectangles, partitioned each into three parts, and shaded two parts in each, arguing that they represented 2/3 (Fig. 2b_2). After doing so, however, she realized that she "lost the fourths" and considered reverting to her original representation, concluding that she was not sure if she could explain this problem.

With respect to selecting and using tasks, even at the end of coursework, Vonda insisted on using the first page for developing an introductory lesson on fraction division, for this page was envisioned helping students "work through each of the steps involved with solving fraction [division] problems." However, in contrast to her original emphasis on developing students' procedural competence, she now wanted to give meaning to this procedure, which, as suggested above, she was not particularly capable of doing.

Overall, Vonda’s better grasp of division, alongside some changes in her beliefs and images of teaching, helped her make some progress in both practices at hand. However, limitations in her knowledge—to a large extent—and some more traditional beliefs she still held—to a smaller extent—prevented her for engaging in these practices in ways that resonate with a *Standards*-based teaching approach.

The case of Kimberley

Kimberley entered coursework with a strong mathematics background. Her pre-coursework MKT score ranked her in the upper end of the second quintile; Kimberley answered correctly several items corresponding to the meaning of division, but missed items related to relative units when it comes to fraction operations. Unlike pre-coursework Deborah, whose case helps demonstrate the extent to which affordances in beliefs can compensate for limitations in knowledge, Kimberley’s case provides an arena for exploring how certain beliefs and images of teaching can mediate the positive effect of knowledge on participants’ performance. This was particularly evident in her performance in selecting and using tasks.

As Kimberley mentioned throughout the pre-coursework interview, she largely experienced mathematics as a matter of learning and applying rules—“learning the mechanics”—but not their conceptual underpinnings. These limited, and thus limiting, experiences in terms of supporting a *Standards*-based approach in teaching appeared to color her pedagogical beliefs. Kimberley believed that repetition is important for learning the content; she also agreed with a pertinent survey statement that when students have difficulties learning the content, a good approach to scaffolding their thinking is to give them more practice. She also thought that the teacher should show and tell, and offer students tricks and shortcuts. Just like Vonda, Kimberley believed that when students cannot solve problems, it is usually because they cannot remember the right formula or rule.

Acknowledging the importance of supporting students to develop conceptual understanding, Kimberley would choose the second textbook page for her lesson. Despite this preference, however, the manner in which she thought of using the textbook tasks was consonant with her beliefs of showing, telling, and offering students ample opportunities for practice. Therefore, it was not surprising that Kimberley found the task sequence weird and considered reordering the tasks. Kimberley thought of starting with Task C, which includes numerical exercises, to help students practice the “mechanics” before solving word problems. If she had to follow the page’s sequence, she would start with a word problem to motivate the fraction division operation, and although she would elicit student ideas, she would then show students how to solve the problem. Next, she would shift to Task C to familiarize students with the mechanics involved in dividing fractions. She would also offer them opportunities to practice these mechanics by working on Task B or on some of the exercises of the first textbook page, thus ignoring Task D, which pertains to developing an algorithm for dividing fractions. Overall, Kimberley would not use the second page to its potential to scaffold student understanding.

Post-coursework Kimberley helps us explore how improvements in knowledge and beliefs, alongside some persisting images of how mathematics should be taught, play out in PSTs’ performance. Kimberley’s post-coursework MKT score was higher than her original score. The additional items that Kimberley answered correctly suggested that she had a much better grasp of the relative and absolute units (i.e., divisor and dividend units) when it comes to fraction division.

Kimberley’s post-coursework comments also pointed to a shift in her thinking about teaching and learning mathematics. Although she originally considered this subject mostly from a narrow, student spoon-feeding perspective, at the end of coursework she reported endorsing a more conceptual teaching approach:

> The major change that's taken place is my exposure to … a more conceptual kind of mathematics, which I was never exposed to as a child. So … that was a pretty major paradigm shift for me…. Initially I came in more used to a computational framework where you basically show the students the algorithm and then practice until they get it right.

Kimberley's post-coursework performance was reflective of her increased attention to issues of making meaning. Similar to her pre-coursework preference, Kimberley again endorsed the second page for an introductory lesson on fractions, now identifying three of its affordances: engaging students in explaining the fractional part of the quotient; asking students to present their work in multiple modes (drawings, written explanations, and number sentences); and expecting students to invent the invert-and-multiply algorithm.

Explaining the fractional part of the quotient was a pivotal aspect of the page that Kimberley would emphasize during instruction—a decision that should not be dissociated from her own capacity to explain this fractional part during the post-coursework interview, but not during pre-coursework. Kimberley's enactment of the other two affordances of the page reflected the enrichment in her beliefs, as well as the influences of her long apprenticeship in procedurally oriented instruction and some latent "die-hard" beliefs and practices—as she called them—associated with minimizing the possibility for student confusion. For example, Kimberley had reservations about asking students to represent their work in all three modes: She would ask them to employ only the first two. Similarly, although acknowledging the importance of students inventing the algorithm themselves, if none would propose such an algorithm, she would "show them the technique" and "have them practice it." Similar to her pre-coursework approach, thinking that the second page does not afford students enough application opportunities, she considered supplementing it with application problems from the first page.

Cross-case analysis

The portraits of Deborah, Vonda, and Kimberley help develop two propositions about the interplay between teacher knowledge and pedagogical beliefs. Following Yin (2009), we present these propositions as hypotheses warranting further exploration.

The first proposition holds that *limitations in either teacher knowledge or pedagogical beliefs can mediate the effect that the other component can have on PSTs' performance, at least as studied in a simulated environment*. Pre-coursework Deborah, for instance, wanted to use "visuals" to explain the quotient in $2 \div 3/4$. Despite this strong desire, limitations in her knowledge prevented her from doing so. With stronger knowledge, but more traditional beliefs, pre-coursework Kimberley also ended up ignoring the affordances of the second page and using it largely from a procedural perspective. That knowledge and beliefs are both important for quality practice—a corollary of the first proposition—is reflected in post-coursework Deborah and Kimberley (from an affordance perspective) and pre-coursework Vonda (from a constraint perspective). With a more solid understanding of the content and its teaching, and with beliefs that aligned with *Standards*-based teaching approaches, by the end of coursework, Deborah was able to provide a sufficiently unpacked explanation. Making a step toward seeing mathematics and its teaching from a more conceptual perspective, post-coursework Kimberley also capitalized on some of the affordances of the second page and offered ideas for enacting it as intended by the textbook designers. In contrast, constrained by both her knowledge and beliefs, pre-coursework Vonda would simply rely on describing “the steps”—as she had experienced the content as a student of mathematics. Hence, Vonda’s performance suggests that left with no other devices—in terms of either knowledge or beliefs—PSTs are likely to resort to their images of teaching and teach in ways that reflect how they themselves experienced the subject matter (cf. Ball 1990).

Can either teacher knowledge or beliefs compensate for limitations of the other? Collectively, the three cases provide evidence in the negative, because each component appears to make a distinct contribution to PSTs’ decisions and actions. *Pedagogical beliefs that align with* Standards-*based curricula seem to equip PSTs with the propensity to work in ways that are conducive to structuring mathematically rich environments,* but inclination alone does not suffice. For instance, although being a strong proponent of using visuals to explain mathematical ideas, pre-coursework Deborah did not have the means for materializing her will. Similarly, despite her desire to teach more conceptually, post-coursework Vonda was limited in what she could do, thus providing an incomplete and rather perplexing explanation. *Strong knowledge,* on the other hand, *sets the background and provides the means for quality instruction.* It appears to help PSTs identify the affordances of different curriculum tasks; it provides them with ideas as to how they can capitalize on these affordances to structure productive learning environments; and it affords them the tools (e.g., representations, examples, and analogies) to support students ascribe meaning to different mathematical ideas. However, having the means without the will can also impair PSTs’ performance. For example, although post-coursework Kimberley was able to identify several of the affordances of the second page, because of her attempts to minimize complexity, she did not capitalize on them to their full potential.

Discussion and conclusions

Prior studies have documented the individual effect of teacher knowledge and teacher pedagogical beliefs on teacher performance. Few studies have explored how knowledge and beliefs interact in informing teachers’ work. Even fewer are the studies that have attempted to do so from both a static and a dynamic perspective, let alone by focusing on prospective teachers. Seeking to address this gap, this multiple-case study explored PSTs’ knowledge and pedagogical beliefs at the beginning of coursework; it also traced changes in PSTs’ knowledge and beliefs after coursework. In contrast to other similar studies that examined prospective teachers’ *self-reported* knowledge (e.g., Holm and Kajander 2012), this study *measured* their knowledge using a multiple-choice test. To better capture the contribution of teacher knowledge to PSTs’ performance, we also focused on a single mathematical topic, which supported aligning the measures obtained for teacher knowledge with those corresponding to PSTs’ performance in the simulated environment. By asking PSTs to comment on the virtual teachers’ decisions and actions, the simulated environment also offered another venue for exploring PSTs’ beliefs, besides that afforded by the survey. These design features enabled studying the complex association at hand in some depth. Admittedly, however, the study is limited in that PSTs’ performance was studied in vitro; as such, their work is suggestive of PSTs’ *potential* to engage in the practices under consideration in real-classroom settings. This limitation notwithstanding, both propositions advanced above support Thompson’s (1992) argument that focusing only on either knowledge or beliefs yields an incomplete picture of what might contribute to teaching quality.

In line with quantitative studies that explored the joint effect of knowledge and beliefs on teacher performance (Campbell et al. 2014; Wilkins 2008), this multiple-case study also provides evidence suggesting that the effect of one component is mediated by limitations in the other component. Strong knowledge alone cannot ensure that teachers engage in work that lends itself to creating mathematically rich environments, as the case of pre-coursework Kimberley—to a greater extent—and post-coursework Kimberley—to a lesser extent—implies. Collectively, Kimberley's case suggests that beliefs that are inconsistent with *Standards*-based curricula can impede teachers from performing in ways that their knowledge could otherwise have supported. As such, her case resonates with that of Marie, the sixth-grade practicing teacher discussed in Sleep and Eskelson (2012), who, holding more traditional pedagogical beliefs, did not capitalize on her *Standards*-based curriculum to its full potential. From this respect, Kimberley's case not only corroborates prior findings, but also indicates that the patterns of mediation documented in prior research for practicing teachers appear to hold for PSTs as well.

Extending prior research findings, this study also suggests that pedagogical beliefs that align with *Standards*-based approaches in and of their own do not suffice to support teachers' work. This was clearly indicated by Deborah's case. Although holding beliefs that resonated with helping students develop conceptual understanding, pre-coursework Deborah could not offer a conceptually grounded explanation. With more solid knowledge at the end of coursework, she was more capable of doing so. At a surface level, this pattern seems to contradict other related findings concerning PSTs' performance. For instance, in Corcoran (2008), Bríd, a PST teaching a lesson of sharing six pizzas among eight people, is originally shown to approach the task from a more conceptually oriented perspective by asking students to use drawings and explain their reasoning. Although this approach resonates with her commitment to inquiry-oriented instruction, when faced with providing explanations, orchestrating students' work, or helping students overcome their misconceptions, Bríd encountered significant difficulties. From this respect, her case resembles post-coursework Vonda, who, despite her relatively productive images of teaching and her willingness to engage students in conceptual work around fraction division, could not offer a conceptual explanation for this operation. Collectively, then, these cases suggest that beliefs and knowledge afford teachers different capacities for engaging students in mathematically rich environments.

In particular, consistent with Bray's (2011) study, the exploration undertaken herein supports the claim that productive beliefs provide PSTs with the *willingness and inclination* to engage in certain teaching practices. How teachers engage with these practices, however, depends on their knowledge. It is this latter component that furnishes teachers the toolkit needed for *effective work*. Such a toolkit includes an understanding of the concepts at hand; representations, examples, and analogies to structure one's work; and proper language to unpack the content to support student understanding. Post-coursework Vonda brings this idea home. When her knowledge fell short of supporting her and when her beliefs could not provide any scaffolds, Vonda figured out the answer to the problem and tried to *show* this answer rather than use the representation to derive the answer. Thus, Bray's patterns regarding how knowledge and beliefs enable practicing teachers to approach the content more conceptually seem to be applicable to PSTs as well—and, interestingly, for different practices than those examined in Bray's work.

The extent to which the propositions generated in this study are applicable to other contents and contexts is an issue that warrants further investigation. Future studies may also attempt to specify the relative weight that the two components examined in this study have for teachers' instructional decisions and actions. Such studies could also examine the joint contribution of these components when teachers are asked to perform in actual teaching settings, under several time pressures and other contextual limitations.

In terms of its practical implications, the study suggests that teacher preparation and in-service education programs should not attempt to induce changes in either teacher knowledge or their beliefs in isolation. Knowledge and beliefs interact in informing teachers' behaviors, and therefore, both need to be targeted to equip teachers with the inclination *and* the toolkit needed for structuring mathematically rich environments. Hence, developing teaching proficiency should not be perceived from an additive perspective: We cannot simply help teachers "add" productive beliefs to their strong knowledge or, vice versa, help them strengthen their knowledge to complement limitations in their beliefs. Such an additive perspective is unrealistic, because, as this study suggests, the relationship between these two components seems to be dynamic and complex. This dynamic relationship renders the task of inducing changes in beliefs and knowledge difficult—as the comparison of pre- and post-coursework Vonda suggests. At the same time, however, Vonda's case implies that, regardless of how formidable this task might be, the price of not taking any action may be too high for student learning.

Appendix 1

The survey statements used to explore PSTs' pedagogical beliefs

1. It is confusing to see many different methods and explanations for the same idea
2. A good mathematics teacher is someone who explains clearly and completely how each problem should be solved
3. Teachers should not necessarily answer students' questions but let them puzzle things out themselves
4. Students learn mathematics best if they have to figure things out for themselves instead of being told or shown
5. When students can't solve problems, it is usually because they can't remember the right formula or rule
6. When students solve the same mathematics problem using two or more different strategies, the teacher should have them share their solutions
7. It is important for students to master the basic computational skills before they tackle complex problems
8. If students are having difficulty in mathematics, a good approach is to give them more practice in the skills they lack
9. To do well, students must learn facts, principles, and formulas in mathematics
10. In learning mathematics, students must master topics and skills at one level before going on
11. Doing mathematics allows room for original thinking and creativity
12. The most important issue is not whether the answer to any mathematics problem is correct, but whether students can explain their answers
13. Basic computational skill and a lot of patience are sufficient for teaching elementary school mathematics
14. Teachers should try to avoid telling
15. Doing mathematics is usually a matter of working logically in a step-by-step fashion
16. A lot of things in mathematics must simply be accepted as true and remembered
17. Students should never leave mathematics class (or end the mathematics period) feeling confused or stuck
18. If students have unanswered questions or confusions when they leave class, they will be frustrated by the homework

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