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Productive struggle in middle school mathematics classrooms

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Abstract Prior studies suggest that struggling to make sense of mathematics is a necessary component of learning mathematics with understanding. Little research exists, however, on what the struggles look like for middle school students and how they can be productive. This exploratory case study, which used episodes as units of analysis, examined 186 episodes of struggles in middle school students as they engaged in tasks focused on proportional reasoning. The study developed a classification structure for student struggles and teacher responses with descriptions of the kinds of student struggle and kinds of teacher responses that occurred. The study also identified and characterized ways in which teaching supported the struggles productively. Interaction resolutions were viewed through the lens of (a) how the cognitive demand of the task was maintained, (b) how student struggle was addressed and (c) how student thinking was supported. A Productive Struggle Framework was developed to capture the episodes of struggle episodes from initiation, to interaction and to resolution. Data included transcripts from 39 class session videotapes, teacher and student interviews and field notes. Participants were 327 6th- and 7th-grade students and their six teachers from three middle schools located in mid-size Texas cities. This study suggests the productive role student struggle can play in supporting "doing mathematics" and its implications on student learning with understanding. Teachers and instructional designers can use this framework as a tool to integrate student struggle into tasks and instructional practices rather than avoid or prevent struggle.

Keywords Student learning · Instructional practice · Middle school mathematics · Productive struggle · Persistence

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Introduction

Students' struggles with learning mathematics are often viewed as a problem and cast in a negative light in mathematics classrooms (Hiebert and Wearne2003; Borasi 1996). Teachers, parents, educators and policymakers routinely look for ways to overcome this perceived "problem," regarding it as a form of learning difficulty, and attempt to remove the cause of the struggle through diagnosis and remediation (Adams and Hamm 2008; Borasi 1996). Through this lens it follows that students' struggles in mathematics are not viewed as meaningful learning opportunities.

Hiebert and Grouws suggest, however, that *struggling to make sense of mathematics* is a necessary component of learning mathematics with understanding (Hiebert and Grouws 2007). The idea that struggle is essential to intellectual growth has a long history. Dewey referred to the process of engaging students in "some perplexity, confusion or doubt" (1933, p. 12) as essential for building deep understanding while Piaget (1960) described learners' struggle as a process of restructuring their disequilibrium toward new understanding. Cognitive theorists have referred to cognitive dissonance as an impetus for cognitive growth (e.g., Festinger 1957) and have identified experimentation (Polya 1945) and sense-making (Handa 2003) as important ingredients for understanding. Hatano (1988) related cognitive incongruity with the development of reasoning skills that display conceptual understanding. Brownwell and Sims (1946) argued, like Dewey, that students must have opportunities to "muddle through" (p. 40) the process of resolving problematic situations rather than be conditioned through repetition. More recently, Hiebert and Wearne (2003) stated "all students need to struggle with challenging problems if they are to learn mathematics deeply" (p. 6).

While the phenomenon we call struggle may be internal, it is also observable in most classrooms. A depiction of what a student's productive struggle looks like in the naturalistic setting of classroom instruction can provide insight into how aspects of teaching can *support* rather than *hinder* this instructional process, which research suggests is of benefit to students' understanding of mathematics (Kilpatrick et al. 2001; Hiebert and Grouws 2007).

In order to study the phenomenon of productive struggle, I focused my study on the following research questions:

1. What are the types and patterns of student struggles visible to the teacher that occur while students are engaged in mathematical activities that are visible to the teacher in middle school mathematics classrooms?
2. How do teachers respond to student struggles while students are engaged in mathematical activities in the classroom?
3. What kinds of teacher responses appear to be productive in supporting student struggles toward understanding?

In my investigation, I examined episodes of classroom instruction in which students appeared to struggle in some way, and to which teachers responded. I analyzed the types of student struggles as students worked on tasks of high cognitive demand focused on proportional reasoning in middle school mathematics classrooms in order to examine how struggle and learning could be connected. The kind of guidance and structure teachers provide may either facilitate or undermine the productive efforts of students' struggles (Tarr et al. 2008; Stein et al. 2000; Doyle 1988). By students' productive struggles, I refer to a student's "effort to make sense of mathematics, to figure something out that is not immediately apparent" (Hiebert and Grouws 2007, p. 287). A close examination of interactions in the classroom both between teacher and students and among students helped reveal the nature of the struggles that students were having with mathematics. I also observed and analyzed the features of teaching and the choices teachers made to (1) maintain the cognitive demand of the task (2) address the struggle and (3) build on students' thinking in order to guide the students. Based on these features present or absent in the interactions, I classified to what degree the struggles were productive or not productive in supporting students' understanding of their problem, the strategies students used, and the reasoning the students employed to solve the given problem.

Conceptual framework

The phenomenon of struggle mentioned above refers to the intellectual effort students expend to make sense of mathematical concepts (Hiebert and Grouws 2007) that are challenging but fall within the students' reasonable capabilities. Struggles that advance the students in their thinking can play an important role in deepening students' understanding if supported carefully toward a resolution and given appropriate time (Hiebert and Grouws 2007; Bjork 1994). My conceptual framework, thus, is built on three main components: (1) the role of struggle in learning mathematics with understanding; (2) the nature and types of mathematical tasks and their relationship to students' struggle; and (3) the ways teachers' respond to students' struggles in classroom interactions. Given my study's focus on the context of learning mathematics with understanding and the influence of teaching on the development of that understanding, it is important to consider the nature of mathematics and what it means to be competent in the discipline (Schoenfeld 1988). My study uses the perspective of mathematics as a social phenomenon, whereby people create objects, study the patterns and relationships of these objects and communicate these ideas within a social culture (Hersh 1997; NCTM, 2000). I take the view that mathematics is a dynamic discipline that involves exploring problems, seeking solutions, formulating ideas, making conjectures and reasoning carefully as opposed to a static discipline consisting only of a structured system of facts, procedures and concepts to be memorized or learned through repetition (Schoenfeld 1992; Hiebert et al. 1996).

The role of struggle in learning mathematics with understanding

Learning mathematics with understanding includes engaging in "doing mathematics" through a process of inquiry and sense-making (Schoenfeld 1992; Lakatos 1976) that by necessity involves students "expending effort to figure out something that is not immediately apparent" (Hiebert and Grouws 2007). Studies suggest that classroom environments often fall short of the ideal setting to "do mathematics" (Schoenfeld 1988; Weiss and Pasley 2004). A typical classroom environment is a mixture of "doing mathematics" with traditional classroom practices, sometimes referred to as direct teach, where students observe as teachers demonstrate and explain ways to do certain types of problems and then assign students problems using the demonstrated methods (Stigler and Hiebert 2004; Kapur 2011). While students' struggles may arise in a wide spectrum of classroom environments, studies suggest that settings that are risk-free, where students can externalize their struggle and where consequences of "wrong" answers are not seen as failures but rather opportunities to explore, grow, and learn serve to better support and motivate students to persist (Holt 1982; Borasi 1996; Carter 2008; Kapur 2011).

Research in the learning sciences provides evidence that forms of struggle or even failure have a role in students' learning. Bjork and his collaborators examined activities that contained elements of *desirable difficulties* for students and found that "…conditions that create challenges and slow the rate of apparent learning often optimize long-term retention and transfer" (Bjork and Bjork 2011, pg. 57). VanLehn and his collaborators examined student struggle in the form of *impasse-driven learning* and found not only that successful learning required students to reach an impasse but also that in general, learning did not occur without some occurrence of an impasse (VanLehn et al. 2003).

Kapur's research on instruction and instructional design models provides an additional perspective on struggle over mathematical learning and problem solving. His study on *productive failure* (Kapur 2009, 2011) compared the performance of 7th-grade students in Singapore implementing a Productive Failure (PF) task that was complex and ill-structured to students taught the same topic with Direct Instruction (DI). In a follow-up test containing complex problems based on their earlier work, students in the PF group outperformed their DI counterpart. While it appears counterintuitive that learning would occur for students who fail to resolve the problems they are engaged with, Kapur argues that the process of engagement, persistence, struggle and potential failure may be important components of "developing deeper understandings" (Kapur and Bielaczyc 2012, p. 46).

Informed by the studies mentioned above, my conceptual framework builds on the premise that deeper learning can occur at sites where impasse or difficulty arises. Whether or not the task is completed, the process of struggling to make sense of it may be beneficial in the long run (Kapur 2009, 2011) for longer-term retention (Bjork 1994; Kapur 2009; VanLehn et al. 2003) and for learning with understanding (Hiebert and Wearne 2003). Part of the purpose of this study is to identify the kinds of student struggles that occur in a classroom that are visible to teachers so that they can recognize its value and support the students in this learning opportunity.

The nature and types of mathematical tasks and their relationship to student struggles

Tasks are a central part of a teacher's instructional toolkit, and what students learn is often defined by the tasks they are given (Christiansen and Walther 1986). In order to move students toward developing a deep conceptual understanding of mathematics, classroom teaching must incorporate opportunities for students to grapple with meaningful tasks (Lampert 2001; NCTM 1991; Schoenfeld 1994). Students must also be given opportunities to make sense of important ideas in mathematics and to see connections among these ideas (Boaler and Humphreys 2005).

Hiebert and Wearne (2003) found higher performance gains on assessments that required understanding as well as computations in two 2nd-grade classrooms where students were asked more questions, emphasized making connections in their procedures and were given more time to respond as compared to students in four other classes that practiced prescribed procedures.

Those students engaged in tasks that were setup and implemented with high levels of cognitive demand showed the highest student learning gains in studies such as the QUASAR Project (Silver and Stein 1996; Stein and Lane 1996). While QUASAR researchers observed that high-level tasks do not guarantee high-level student engagement, they found that low-level tasks almost never result in high-level engagement (Smith and Stein 1998). The findings from the QUASAR Project (Henningsen and Stein 1997) strongly suggest that in order for students to "do mathematics," the classroom must provide an environment where students can engage in worthwhile and high-level activities for which struggling with problems and tasks is an expected part of the daily routine.

Informed by studies such as the QUASAR Project, high-level cognitive demand tasks have high potential for struggle precisely because they demand intellectual work. I chose the Mathematical Task Framework (Smith and Stein 1998) developed from the QUASAR Project (Stein et al. 2000) to categorize the tasks that teachers implemented in my study. The Mathematical Tasks Framework (Smith and Stein 1998) categorizes tasks into four levels: lower-level Memorization and Procedures without Connections tasks and higher-level Procedures with Connections and “Doing Mathematics” tasks. Memorization tasks are generally algorithmic and in most cases require reproducing a learned fact, rule, or definition that was memorized. Procedures without Connections tasks are generally procedural, with little ambiguity as to the steps required to complete the work. Procedures with Connections tasks require procedures but with the purpose of developing deeper understanding of or connections to underlying mathematical concepts. Multiple representations or solution paths are possible. “Doing Mathematics” tasks require students to analyze the task, explore ideas contained within and connect them to other concepts and make explanations. The cognitive demand level may provoke a level of anxiety in the students.

The ways teachers respond to student struggles in classroom interaction

As well planned as the tasks may be, students can encounter difficulty during various stages of the task enactment process from its introduction and development to its closure. My conceptual framework was informed by studies that focused on interactions among the classroom participants and related those kinds of support and guidance to resolving struggles. On the one hand, explicit actions by teachers or peers can work to build community understanding and resolve students’ struggle *without depriving students of the opportunity to think for themselves*. On the other hand, the urge by teachers to help struggling students can result in lowering or removing the cognitive demand (Henningsen and Stein 1997) by such actions as telling students the answer (Chazan and Ball 1999) directing the task into simpler or mechanical processes (Stein et al. 1996) or giving guidance that funneled students’ thinking toward an answer without building necessary connections or meaning (Herbel-Eisenmann and Breyfogle 2005).

Classroom interaction is by nature unpredictable, notably between teachers and students in the moment-to-moment workings of mathematical activity, particularly as teachers attempt to balance the complexities and constraints of classroom settings (Kennedy 2005). Once a teacher notices a student struggle, the teacher decides how to react (Jacobs et al. 2011). One category of teacher response is to provide information to the struggling student. Through this type of action, the students may then be more task-enabled than before the interaction (Maybin et al. 1992) though in the process, the response may affect the task’s level of cognitive demand.

I draw from extensive studies related to student–teacher interactions and use the lens of student struggle to draw connections between the literature and aspects of teacher responses that support students’ learning of mathematics with understanding. Studies have highlighted the important role that prior knowledge plays in students’ learning (e.g., Piaget 1952, 1960; Rittle-Johnson 2009; Bransford et al. 1999). Teacher responses with *examples* that connect students’ thinking with their prior knowledge can give students useful strategies and skills to approach the task at hand. Teachers’ use of *analogies* is another strategy used to successfully connect students’ struggle with elements of their prior knowledge

(Richland et al. 2004). O'Connor and Michaels (1993) reported teachers' use of *revoicing* as a tool to help students clarify solutions or problems, remind them of a connection to prior knowledge, animate the participants in the interaction and share in *reformulating* or *reframing* the problem (p. 328).

Responding to student struggles by asking *questions* serves various purposes. As part of a discourse interaction between teacher and students, questions can give direction to students' thinking and opportunities for students to *organize ideas* as they engage with a task (Sorto et al. 2009). Questions, particularly when carefully sequenced to develop and build on students' ideas, help assess the students' thinking while refocusing students on important mathematical points that they may have missed (Anghileri 2006; Cazden 2001). Pierson (2008) reported that the use of probing questions that demanded intellectual work resulted in a more productive exchange and increased student learning in comparison with those questions that did not. By carefully questioning and listening to aspects of students' struggles, the teacher can then make appropriate responses to build upon students' ideas and thinking.

VanLehn et al. (2003) reported two key tutorial behaviors that were linked to student learning, namely where tutors generated opportunities for impasse and where tutors gave zero-content prompts. When these tutorial behaviors were observed, the students had to "think harder" to find the right steps, explain and connect the tutor's explanations to their own thinking. Borasi (1994) found in her studies that exposing and discussing *errors* and *misconceptions* improved learning. Eggleton and Moldavan (2001) noted that by helping students confront their errors and resolve the incongruity, the mistakes are seen as a source of learning and sense-making. When teacher responses can change students' statement of "I don't get it" to a statement of "I don't get it *yet,*" the struggle shows signs of moving the student forward in his or her productive engagement with the mathematics.

The established teacher–student relationship can play an important role in what students come to value in their interactions with the teacher. Statements such as "you're on the right track" can prompt a student's sense of agency and confidence to continue or revise their ideas (Doerr 2006). A response such as "you're way off," however, serves to reject the student's effort as a whole without salvaging any portion (vanZee and Minstrell 1997; O'Connor and Michaels 1996). Gresalfi et al. (2009) refer to the construction of student competence and agency as a valuable part of how students take up opportunities to participate and learn in the classroom. Struggle can dissipate unproductively if students disengage from their task or activity. Therefore, the student–teacher interactions in a classroom environment can serve to contribute or hinder students' willingness to *persist* and struggle (Eccles et al. 1993) as well as help shape their sense of agency in a productive struggle (Gresalfi et al. 2009). A teacher response that allows more time on task acknowledges the students' effort and competence particularly in the face of difficulty and increases the quality of engagement without lowering the cognitive demand and is more apt to encourage the student to persist despite the student struggles (Ames 1992; Anderman and Maehr 1994; Dweck 2000).

The impact of supporting students in learning can vary as to what the student comes to know as mathematics (Gresalfi 2004; Ball and Bass 2003). Well-intentioned lessons can give way to the immediate needs of the student at the moment of enactment (Stein et al. 1996) and the choices teachers make in response to the situation can create different learning opportunities. The interactions between teachers and students, therefore, require constant balancing of challenges and support as the tasks unfold (Michell and Sharpe 2005).

Methodology

My investigation into the role of productive struggle in learning and teaching mathematics is exploratory in nature. The goal is to gain insight into the types and nature of student struggles that arise in middle school classrooms when students are working on tasks of higher cognitive demand and to examine the interactions that ensue between the student and teacher and the types of responses teachers use to support and resolve the student struggles. My conceptual framework suggests that learning is best supported when the teaching (1) maintains the cognitive demand of the task, (2) addresses the struggle and (3) builds on student's thinking. I designed my study to document these episodes of student and teacher interactions in response to student struggle, if they should arise, during enactments of tasks. I tried to capture the flow and content of the struggle and response interactions to gain insight into how teachers incorporate the dimensions mentioned above.

I used an embedded case study methodology (Yin 2009) with instructional episodes as unit of analysis within the larger unit of teachers. The goal was to identify and describe the nature of the student struggles and the instructional practices of teachers that supported, guided or did not guide the students' sense-making of the mathematical tasks in the lesson episodes. I used my field notes, teacher and student interviews and episode transcripts to describe and analyze those interactions.

Participants

The participants were 327 6th- and 7th-grade middle school students and their teachers from three middle schools in mid-size cities in western Texas, the southern border of Texas and central Texas. The selection of the participating teachers was not random but instead based on prior classroom observations of teacher–student interactions. I sought classrooms where students were encouraged to engage in classroom discourse and develop and express their ideas during mathematical activity. I felt this type of classroom setting was necessary for students to have an opportunity to express struggle in their mathematical work (Lampert 2001). There were four white female teachers, one white male teacher and one female Hispanic teacher. The teachers had from two years' teaching experience to 18 years' with a median teaching experience of 10 years. The students in western Texas were 33.5 % white, 55.9 % Hispanic and 10.6 % other. The students in southern Texas were 6.7 % white, 89.7 % Hispanic and 3.6 % other. The students in central Texas were 56 % white, 23 % Hispanic and 21 % other.

All six teachers taught from the same mathematics textbook, *Mathematics Exploration part 2* (McCabe et al. 2009) during the 2009-2010 school year. Several of the tasks were taken directly from the book, and all tasks involved proportional reasoning.

Procedure

Data collection

Each teacher was observed teaching six to eight classes during a one-week period in May 2010 with each class ranging from 60 to 90 min. Thirty-nine total class sessions were observed among the six teachers for a total of 52.5 observation hours. I videotaped each teacher's mathematics class with one stationary camera and with one mobile camera to capture struggle interactions. I kept field notes of the classroom activity when not using the mobile video camera and wrote reflection notes of the classroom observations after each class. Pre- and post-project interviews of each participating teacher were audiotaped and transcribed. The semi-structured interviews pertained to teachers' views on mathematics learning, how they managed struggle, and the kinds of actions they chose when students were struggling. Students who appeared to struggle were also interviewed after class.

Data analysis

As an exploratory case study, the goal of my data analysis was to identify, describe and examine the struggles students encountered during their engagement with a task and the nature of the teacher responses. I viewed all the video footage to find those struggle interactions and created an excerpt file of 186 video clips of instructional episodes guided by Erickson's (1992) methods for analyzing video data. An instructional episode for the purposes of my study consisted of a classroom interaction about a mathematical task that was initiated by a student struggle that was in some way visible to a teacher or another student, whether voiced, gestured or written. The episode ended when (1) the student acknowledged understanding by word or action or was able to complete his/her task; (2) the student overcame a hurdle or impasse and continued attempting his/her task; (3) the student continued to struggle but the teacher had moved on; or (4) there was a shift by the teacher to a different task with no resolution given by the student nor demanded by the teacher. The transcripts of the video clips and interviews were coded using the open-coding process (Strauss and Corbin 1990).

Coding struggle

In an initial examination of the episodes, struggle themes emerged along the lines of "what," "how" or "why." There were overlaps in my initial codes and lack of clarity about the classifications that failed to align with how students voiced their struggle. For example, students would say, "I don't know what to do." Others would say, "I don't know how to do the problem." Though they used the key classifying words I considered distinct, namely the "what" and the "how," the nature of the struggle appeared essentially the same after looking at their work and the stage at which they voiced their struggle.

Next, I considered examining procedural versus conceptual struggles, but these classifications were too broad for analysis. I did another iteration of the episodes using codes informed by literature on problem solving (Schoenfeld 1987; Kulm and Bussmann 1980; Polya 1945; De-Hoyos et al. 2004) which examine struggles during formulation, implementation and sense-making, and verification. This classification had promise, but the sense-making struggle could have occurred at any point in the task enactment. In my final iteration, I looked through the lens of what teachers might have noticed in the students' struggle and discovered four categories of struggle, which are described in my Findings section. My current study is limited only to examining struggles expressed externally and noticeably to the teacher and not struggles that students may experience internally. A future study that examines the nature of internal struggles by using think-aloud methods, for example, may shed light on the unique characteristics of externally expressed student struggles in contrast to internally expressed ones.

Coding tasks

I used the Mathematics Tasks Framework (Stein et al. 1996) to code both the intended and enacted tasks that served as the context for the instructional episodes. The tasks were identified using four levels of cognitive demand described earlier: Level 1-Memorization; Level 2-Procedures without connections to concepts or meaning; Level 3-Procedures with connections to concepts and meaning; and Level 4-Doing mathematics. I provided each teacher an activity booklet (see "Appendix") containing tasks suitable for classes that ranged from 60 to 90 min. Three of the four activities focused on conceptual understanding of proportional relationships. I chose tasks involving proportional relationships because proportional reasoning is a primary focal point across the middle school band (TEA 2005; NCTM 2000; Schielack et al. 2006) and is considered "the capstone of children's elementary school arithmetic;…it is the cornerstone for the mathematics that is to follow" (Lesh et al. 1988, p. 94).

In order to observe students' struggle across the spectrum of learners, the tasks were designed to be challenging but accessible, building on students' prior knowledge.

Coding teacher response

In coding teacher responses, I noticed that an initial teacher response to a student struggle elicited some action on the part of the student. A sequence of moves of varying lengths generally followed between student and teacher or among students. Although an initial teacher response had certain characteristics, such as eliciting student thinking, I concluded that it was the set of response sequences that provided direction and support for the students' struggle. Using elements of grounded theory (Strauss and Corbin 1990), I classified four categories of teacher responses along a continuum (Glesne and Peshkin 1992; Strauss and Corbin 1990), which I describe in my Findings section.

Coding resolution of student struggle

I code resolutions in three categories: productive, productive at a lower level or unproductive. In a productive resolution, the student actively worked through the struggle at the intended level of cognitive demand and either completed the task or continued to engage in it. In a resolution that is productive at a lower level, a teacher or another student intervened to remove the struggle by simplifying the task for the student. This simplification basically connected to the original task although it reduced the cognitive demand. While the student completed the task successfully or remained engaged, we consider this a passive success because the teacher rather than the student acted to remove the struggle.

Finally, an episode can end with the student giving up, continuing to be confused and/or unable to figure how to do the task or understand why something works. In other instances, the teacher fundamentally changed the nature of the task, for example by making the task procedural without connecting this to the original task. Although the student may have completed this new task, I nonetheless would classify this resolution as unproductive.

Trustworthiness

I requested two independent readers, one a mathematician and one a mathematics education graduate student to take a sample of 20 struggle episodes to determine how consistent my codes were with their coding of struggles, responses and resolutions. After several discussions, I refined my classification until the inter-rater reliability reached 90 %. In addition, a teacher participant examined 12 of the episodes and provided feedback on my coding of struggles, responses and resolutions. We were consistent in 83 % of the coding and analysis. Our conversation provided insight and additional confirmation of my preliminary analysis.

Findings

I begin by describing how the tasks were implemented to give context to the student struggles that arose. I then describe the kinds of struggles that occurred during the task enactments and provide examples of the struggle types. I next address my second goal, to describe the kinds of teacher responses observed in the student–teacher interactions about the struggles and to provide examples of the teacher responses. I examine the types of teacher responses through the lens of the three dimensions that informed my study, namely how the responses (1) maintain the task's level of cognitive demand, (2) address the student struggle and (3) build on student thinking that research suggests fosters learning and understanding (Stein et al. 1996; Borasi 1994; Doerr 2006). The third goal was to describe resolutions to the interaction episodes and provide examples of the resolutions. The following analysis informed the resulting Productive Struggle Framework that I propose to capture the elements of a productive student struggle episode.

Tasks implemented in the classrooms

In my observations, teachers set up the tasks with reasonable fidelity to the suggested teacher guide I provided. The guide included a suggested lesson sequence to allow students time for individual work, group work and whole-class discussion. For the most part, students enacted the tasks in a similar manner at the various sites. The students generally began by working individually for about five minutes per task, then discussed their work with a partner or a small group of three to four students for another five minutes or so while the teacher observed. Teachers listened or engaged in asking students questions. For example, one teacher went around the tables during student discussions and stamped a sheet of paper on each small group's table to indicate that the students at the table were engaged in discussing their problems and solutions. Some students raised their hands to gain the teacher's attention or ask a question. As a whole class, different questions and struggles surfaced that had not occurred during the individual or small group time.

Student struggles

Table 1 summarizes the kinds of struggles that emerged from my data.

The following describes each type of student struggle as students attempted to:

1. *Get started*

Students voiced confusion about what the task asked them to do ("I kind of understand it… but I'm a little confused"); claimed they did not remember doing problems of this type though it appeared vaguely familiar to them ("I have absolutely no idea….I don't remember that far"); called for help ("Mr. Baker, I need help."); gestured uncertainty and resignation (looks, thinks, sits back and then says "I don't know."); or showed no work on their paper. These struggles occurred most often when students began their work individually and sometimes when working together in small groups. The stationary video

Table 1 Kinds of student struggles and their percent frequencies

Kind of struggles	Descriptions	Frequency % (186 total)
1. Get started	Confusion about what the task is asking Claim forgetting type of problem Gesture uncertainty and resignation No work on paper	24
2. Carry out a process	Encounter an impasse Unable to implement a process from a formulated representation Unable to implement a process due to its algebraic nature Unable to carry out an algorithm Forget facts or formula	33
3. Uncertainty in explaining and sense-making	Difficulty in explaining their work Express uncertainty Unclear about reasons for their choice of strategy Unable to make sense of their work	30
4. Express misconception and errors	Misconception related to probability Misconception related to fractions Misconceptions related to proportions	13

camera captured certain classrooms where students were quicker to voice their difficulties than students in other classrooms, an aspect that did not come out from the mobile video camera transcripts.

2. *Carry out a process*

Some students who had difficulty carrying out a procedure demonstrated or voiced some plan for achieving the goal of the task but encountered an impasse. These impasses tended to revolve around an inability to implement a process such as solving for an unknown in a proportion or converting a fraction to a percent. Other issues included numerical mistakes, failure to carry out an algebraic procedure or difficulty recalling a geometry formula. These struggles occurred most often when students began their work individually and sometimes when working together in small groups.

In one example, a student appeared to have a plan but reached an impasse when the numbers to the problem became more difficult. In Task 1.5, the students were asked to compare the fullness of two different-sized containers when a gallon of water was removed from each. A student in Ms. Fine's class determined that the $\frac{3}{5}$ full water jug was now $\frac{2}{5}$ or 40 % full. She was, however, unable to determine what to do with the rain barrel that had been $\frac{24}{48}$ full but was now $\frac{23}{48}$ full. She said to the teacher, "I need help…I don't know, first I thought I would try to get it [the 48 in the denominator] as close to 100 as possible so I multiplied it by two." This student ultimately reached an impasse in carrying out her plan.

Struggles to connect procedures to concepts caused some difficulties while others occurred when students tried to carry out a procedure, as with Ms. Fine's student. Mistakes were just one of the causes of this type of struggle, particularly if students could not locate the mistakes or did not recognize that a mistake had occurred. Many of these process mistakes were brought up for discussion through the opportunities teachers provided for students to share their work (Fawcett and Gourton 2005).

3. *Uncertainty in explanation and sense-making*

Students' uncertainty in explaining their work or solution tended to occur in the latter part of an enacted task as students shared their work in small groups or with the whole class. In order for students to complete each task, they were expected to explain their work and solutions in writing and in many instances to each other or to the class. Students often struggled to verbalize their thinking and give reasons for their strategies even if their answer appeared correct on their paper.

For example, some of the students voiced their struggles to explain their work in task 1.4. One student responded, "I don't know how to explain it, it's just kinda like (pause) I don't know how to justify it." Another stated, "I know what I'm thinking, I just can't show the exact way." Many of these instances of struggle would not have surfaced had the teacher not moved around to question the students about their answers or if the students did not have the opportunity to share their work with their small groups. Listening to others explain their work prompted students to question each other's work and try to justify their thinking. Struggles arose in small groups when their thinking was challenged or did not make sense to others.

4. *Express misconception and errors*

Struggles involving the students' misconceptions appeared to be instances when deep-seated mistaken ideas were used as a basis for solving problems rather than students' confusion or possible error due to carelessness. These struggles occurred when students had to share their work and communicated to other students or to the teacher during small group or whole-class discussion. One example of a misconception occurred in keeping distinct gallon amounts and percent amounts in Task 1.6 when students determined how much water must be added to the 48-gallon rain barrel with 24 gallons to be of the same fullness as the five-gallon water jug with three gallons of water. One student correctly calculated that the rain barrel with $\frac{24}{48}$ water was 50 % full and the water jug with $\frac{3}{5}$ water was 60 % full. She concluded that the 10 % difference in the percentages was equivalent to a 10-gallon difference, saying, "Shade in 10 more squares [of her graphical representation of a 48 grid with 24 grid squares shaded; each square was to represent 1 gallon]…its like 80 %." The student then gestured, displayed confusion and fell silent.

Teacher response

I situate the four types of teacher responses on a continuum (Fig. 1).

I then examine how the following three dimensions of the student–teacher interactions were affected by the

types of teacher responses:

- The level of cognitive demand of the mathematical task;
- The attention to the student's struggle; and
- The building on student's thinking.

Table 2 summarizes the four types of teacher responses.

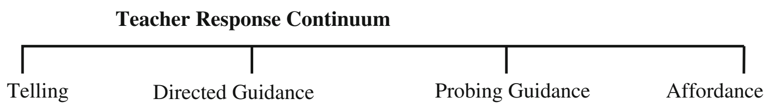

Fig. 1 Teacher response continuum

Table 2 Teacher response summary

Teacher response	Characterizations	Frequency (%)	Dimensions
1. Telling	Supply information Suggest strategy Correct error Evaluate student work Relate to simpler problem Decrease process time	27	Cognitive demand Lowered Attend to student struggle Remove struggle efficiently. Build on student thinking Suggest an explicit idea for student consideration
2. Directed guidance	Redirect student thinking Narrow down possibilities for action Direct an action Break down problem into smaller parts Alter problem to an analogy	35	Cognitive demand Lowered or maintained from intended Attend to student struggle Assess cause and direct student Build on student thinking: Used to build on with teacher ideas
3. Probing guidance	Ask for reasons and justification Offer ideas based on students' thinking Seek explanation that could get at an error or misconception Ask for written work of students' thinking	28	Cognitive demand Maintained Attend to student struggle Question, encourage student's self-reflection Build on Student Thinking Used as basis for guiding student
4. Affordance	Ask for detailed explanation Build on student thinking Press for justification and sense-making with group or individually Afford time for students to work	11	Cognitive demand Maintained or raised Attend to Student Struggle Acknowledge, question, and allow student time Build on student thinking Clarify and highlight student ideas

The following examples reveal the nature of the four types of teacher responses as they occurred in the context of a struggle interaction with a task.

Telling response

In a *telling* response, the teachers generally provided sufficient information for the students to overcome the struggle, often by removing the cognitive demand intended by the task and redirecting the struggle to a procedure without connection to the concept.
Telling Episode Task 3.1:

Two students in Mr. Baker's (T) class, Nathan and Fineas (N & F) were having difficulty getting started using algebraic relationships to formulate an expression involving a variable.

1 N: It's any number you make up.
2 T: **We're not going to make up another number.**

We're going to use ***x***.

3 N: Oh, yeah. But x is the number we want, right?
4 F: **No.** x is like in general…
5 N: I know. You can put in whatever for x. Right?
6 F: No, no
7 T: You're just getting…**but it could be whatever.**

Mr. Baker addresses Nathan's struggle in line 2, but lowers the cognitive demand by telling him what he should do. In lines 3 and 5, Nathan appears to struggle about the role of a variable and how to proceed. In response, Mr. Baker does not build on the student thinking in line 7 and seems to be back where the student began in line 1. While telling as a response may be used only to offer needed information without lowering the cognitive demand of the task, more often than not, the cognitive demand was diminished.

Directed guidance

Directed guidance responses appeared to redirect student thinking toward the teacher's thinking, narrowed down possibilities for action, directed an action, broke down problems into smaller parts or altered problems to an analogous one such as from an algebraic to a numerical one. In the example below, Ms. Fine's (T) hints center on her implementation plan that gave the student, Lisa, (L) directions on how to carry out the work.
Directed Guidance Episode Task 1.5:

8 L: I got this far.
9 T: **So what's the actual question?** How did that go for so….
10 L: You drain a gallon of water. It was $\frac{3}{5}$ so it became $\frac{2}{5}$ because you drained a gallon of water, right?
11 T: **So what percent is $\frac{2}{5}$?**
12 L: It's 40 %
13 T: Yeah, but what about the other one? How did its percentage change?
14 L: What? This one? That's the one I'm stumped on. I need help.
15 T: Okay, **how do we go from this to a percentage?**
16 L: I don't know. First I thought I would try to get it as close to 100 as possible so I multiplied it by 2.

17 T: Okay
18 L: Which is 4 off of…
19 T: Okay, **what's the other way we did this**…

In line 9, Ms. Fine attempts to solicit Lisa's thinking, but in lines 11 and 15, she redirects to a procedural question that deflects the struggle Lisa is having, narrows the focus and funnels Lisa toward a procedure without making connections, thus lowering the cognitive demand. Line 19 redirects from the student's thinking altogether to a direction the teacher has in mind. This episode is an example of an interaction where the teacher does not attempt to illuminate the student's thinking and though the student has reached an impasse, the teacher directs the student toward an answer built on the teacher's thinking rather than the student's.

Probing guidance

Teachers' use of *probing guidance* made students' thinking visible and served as the basis for addressing the students' struggles. Ms. George (T) noticed on many of her students' papers that the sales price was written as $0.25S - S$. In this interaction, Ms. George probed Amy (A) to explain the work on her paper but refrained from saying there was a mistake. Ms. George's probing questions kept the intellectual work focused squarely on Amy. *Probing guidance episode* Task 3.7:

20 T: Okay. **Tell me what that means.** [She sees $0.25S - S$ written on Amy's paper and Amy had been peering at Nathan's paper which reads $S - 0.25S$].
21 A: It means that you times it by the percent, which is 0.25, and then you have to subtract it and that's what you have to pay.
22 T: **Subtract it. What do you mean subtract it?**
23 A: Subtract the total from it.
24 T: So you're going to **find the discount, 0.25 times** ***S*** **is the discount.** Once you get your discount, you're going to subtract the discount and the total.
25 A: (Nods in agreement)
26 T: **What kind of number would you get if I told you that** ***S*** **was $12.**

In line 20, Ms. George probes Amy's thinking and asks her student to clarify the proposed procedure [line 22] without lowering the cognitive demand. Ms. George remains focused on the struggle that did not appear to be resolved and seeks confirmation about what Amy seems to mean [line 24]. Ms. George provides the opportunity for Amy to think more about what expression she is trying to write without leading her toward which quantity should be subtracted from which. Ms. George makes a move that addresses the struggle specifically [line 26] and asks Amy to make use of the expression with a numerical value. This move appears to give Amy guidance without removing the cognitive demand and requires her to think more deeply about what her mathematical expression means.

In summary, *probing guidance* responses consistently revert to students' thinking by building on their thinking and asking for explanations, reasons and justifications. Questions asked by teachers were open-ended for students to consider, discuss and respond sometimes among small groups or in whole-class discussions. Time was given to the students ranging from as short as 20 s to close to 15 min.

Affordance

Affordance type of teacher responses provided opportunities for students to continue to engage in thinking about the problem and build on their ideas with limited intervention by the teacher. Teachers were explicit in encouraging students to continue their efforts in their tasks. For instance, Ms. Torres makes an affordance response suggestion to a student struggling with uncertainty about an explanation: "Is that the same reason?…I'll let you ponder on that okay?" In another episode, Ms. Harris approached a small group with different answers to Task 1.5, saying "Justify your answer. After you've done your work, write a sentence yes or no and why. And make sure you justify it some way either with some math or a picture or someway… and compare your answers at your table."

The episode in Ms. George's (T) class provides an example of an affordance response. *Affordance episode* Task 3.3:

A student, Jeremy (J), is tentative in how to do this task.

25 J: Add 5 to bag 2?

26 T: **Well, I don't know. Can you?** You want to change bag 2 to have the same chance of getting a blue marble as in bag 1….okay, **well why don't you test it.**

27 J: That would give me…that wouldn't work. I have to take 10 away from that. So that'd be $\frac{30}{60}$ or one half.

28 T: **What would that do for you? I'm just a little confused** as to what you…So this is which bag?…**mark it for me**…and you may have to look at the marbles in there and see what you have to change. Okay? **Just think about that for a moment.** Okay. (Leaves J to work, as he seems engaged with the problem.)

In line 26, Ms. George addresses the struggle of uncertainty voiced by the student in not knowing what to do by asking the student what he is thinking and maintains the cognitive demand while helping to articulate the problem. Ms. George then addresses the struggle more explicitly in line 28 by directing the student to consider examining the number of marbles. A critical component of this type of interaction response is to give students time to attend to their thinking [line 28] and provide a motivating reason to continue to work on their task. The teachers had to monitor the progress of the students, however, as momentum in doing the mathematics was lost at times when the students could not navigate beyond their struggle.

Interaction resolutions

In most instances, a student's answer concluded an episode and appeared to resolve their struggle without evidence of their understanding of the mathematics. As Bjork and Bjork (2011) indicate, "…prior exposures create a sense of familiarity that can easily be confused with understanding" (p. 62).

I identified resolutions as *productive* if they (1) maintained the intended goals and cognitive demand of the task; (2) supported students' thinking by acknowledging effort and mathematical understanding and (3) enabled students to move forward in the task execution through student actions. My findings show that 42 % of the struggles fulfilled all three of these criteria. I classified those resolutions that were productive in point (2) above as *productive at a lower level*, with point (1) lowered somewhat in the cognitive demand of the intended task, and in point (3) the teacher rather than the students actively guided the students through the struggle and the students passively following a directed guidance.

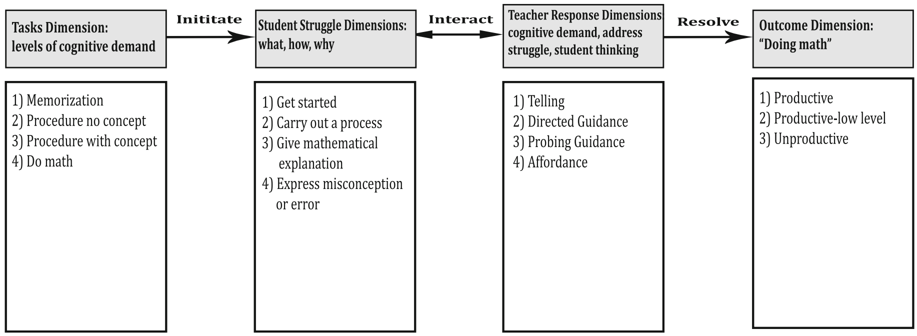

Fig. 2 Productive Struggle Framework in an instructional episode

Forty percent of the student struggles resolved at a lower level. I categorized struggles as *unproductive* if students continued to struggle without showing signs of making progress toward the goals of the task; reached a solution but to a task that had been transformed to a procedural one that significantly reduced the task's intended cognitive demand; or if the students simply stopped trying. Eighteen percent of the student struggles resolved unproductively.

The Productive Struggle Framework (Fig. 2) below captures elements of student–teacher interactions from initiation, interaction and to the resolution of struggle.

One noticeable pattern in the data showed similar student struggles over the same task across the three teaching sites. A second pattern pertained to teacher responses to similar tasks that created different productive qualities in the resolutions to student struggles. An example of different resolutions to the same task occurred in Task 1.5.

This first episode in Ms. George's class began with Drew, who gave his answer for Task 1.5 with no indication that it was incorrect: "I put no because there would be the same as before because you have taken a gallon from both." Ms. George sought confirmation from Drew about his statement and probed him to explain his answer as well as provide a mathematical way to verify his claim. She even asked, "Why would that one be fuller now, do you think?" While the student appeared confused, Ms. George asked Drew to repeat what he was saying to clarify his stance and reasoning. At the same time, Ms. George tried to slow down the pace of the dialogue and not show impatience for an answer with statements such as "Be patient" and "Keep working." Coming back, she then asked "What have you got?" and questions such as "…why do you think it changed on you?" after Drew seemed to be sure of his answer.

The above resolution is an example of productive struggle during which the cognitive level was maintained as Ms. George's pressed the student to further reflect and make sense. She then allowed the student time to consider the questions that were posed. Through the process of reflection and time, Ms. George provided the student with opportunities to represent his answer mathematically and explain his thinking.

A second episode occurred in Mr. Baker's class when a student was unable to explain her work. Mr. Baker proceeded to give an explanation of the answer without giving students opportunities to respond. When Mr. Baker asked, "but if you take a gallon out of the barrel, does it make quite as much of a difference?" the students gave no response. Rather than addressing the students thinking or struggle, he stated, "So what you can do is you can take and make fractions so, let me see if I can get this here…" A *telling* response was used with the work undertaken by the teacher and the task's level of cognitive demand was lowered.

In summary, the outcomes for the students' struggles and the resolutions to the interactions were a mix of those that were productive by maintaining the high level of cognitive demand of the task, building on the student's thinking, and attending to the struggle as a process that the students could work through, and those that were less productive when the directed guidance lowered the level of cognitive demand. Finally, unproductive interaction resolutions resulted in the students indicating neither a clear understanding of the task nor an ability to make progress in tackling the problem. Those interactions where teacher responses failed to tap into or support students' thinking and thus took the challenging aspects of the task from the students, or where tasks were simplified to procedures without connections resulted in an unproductive resolution. Therefore, while struggles are situated with the task and student, struggles can be directed more productively by teacher's careful selection of response types and support.

Discussion

Discussion of student struggles

Prior empirical research on student struggles was limited to a focus on examining the occurrence of struggle in whole-class discussion settings and did not examine in detail the nature of individual students' struggles (e.g., Inagaki et al. 1998; Santagata 2005). While Borasi (1996) and Zaslavsky (2005) looked at struggles students have with errors, misconceptions and uncertainties, most studies have made general reference to struggle (e.g., Carter 2008, Hiebert and Wearne 2003). My study provides more detail on the kinds of struggles that occur in middle school mathematics classrooms among students engaged in mathematical tasks.

I used proportional relationships as context in the implemented tasks because these concepts are an important part of middle school mathematics and because students must be given opportunities to make sense of important ideas in mathematics and see connections among these ideas (Boaler and Humphreys 2005). Proportions are often treated as procedural computational problems where the goal is to find missing values using a technique such as "cross-multiplication" (Heinz and Sterba-Boatwright 2008). As a milestone in students' cognitive development (Cramer and Post 1993), however, the concept of proportional reasoning demands a deeper conceptual understanding of dynamic transformations, structural similarities and equivalences in mathematics (Lesh et al. 1988).

Findings indicate that students struggled to find strategies and representations of proportional relationships and follow through with a plan (Schoenfeld 1992; Hiebert et al. 1996). Students also struggled to examine and explain the solutions they had produced and to connect to the original problem. In their engagement of various tasks, students voiced confusion as they tried to understand the problem, do a computation or use an algorithm such as a proportional computation or rational number representation conversion; in other words, they reached an impasse (VanLehn et al. 2003).

The ability to execute a procedure does not guarantee that students can solve a task involving procedures with connection or "doing mathematics" (Boaler 1998). For tasks that included algebraic relationships, students would commonly indicate their struggle most often by trying to make sense of what the expressions meant (Carraher et al. 1987). The struggle and the subsequent decline in the cognitive level of the task, in some episodes, may be due to task inappropriateness for a particular group of students (Henningsen and Stein 1997).

In some classes as compared to others, the relatively high incidence of struggles with uncertainties or confusion as to how to get started with a task suggests that the class as a whole had varying levels of tolerance for grappling with a problem. Some of the students went off task by socializing with each other, becoming disruptive or ceasing to do their work. In other classrooms, norms and culture seemed to be in place such that students had opportunities to discuss their solutions, and in the course of explanation try to reason and communicate their thinking mathematically (Cobb et al. 1993). The nature of student struggles seemed related to the sociomathematical norms that were in place in each class. In other words, struggles were not only cognitive in nature.

Discussion of teacher responses

Studies have documented how teachers implement a variety of moves in their interaction with students, which are often dictated by the situation, the needs of the students and their own beliefs and content knowledge (Anghileri 2006; Haneda 2004; Stein et al. 1996; Dweck 1986; Kennedy 2005). Prior research has addressed many issues of instructional practices, but none have specifically addressed support of student struggles, particularly in a productive manner. The teacher response continuum attempts to capture the broad set of practices the participant teachers implemented in response to student struggles. Some responses provided clarity for students by narrowing a field of examination while others restricted the field to the extent that it put the students' ideas out of the range of consideration. The type of response creates very different learning opportunities for the students. For example, in an interaction with a student and Ms. Norris over task 1.5, a student responded that the fullness was now "23 out of 48" in the rain barrel. Ms. Norris went on to affirm the student's response, "so that's right, but how do you get that it's more than 40 %? How do you get to the percent?" Ms. Norris narrowed the field of examination without taking the student's idea out of consideration. In contrast, Ms. Fine's response over the same task included the statement, "Okay, what's the other way we did this…" which refers not to the student's line of thinking but to teacher's.

In my observations, the teachers appear to strike a balance between trying to sustain student engagement and maintain the cognitive demand of the task (Kennedy 2005). There were varying degrees of hints, corrections and suggestions teachers provided when students needed guidance. The interactions reveal the teacher's role in trying not to overwhelm the students, who each possess varying levels of tolerance for persistence and frustration. Similar to the findings in the QUASAR study (Stein et al. 1996), I found that when teachers focused on the struggling students, they also risked losing the focus and engagement of the rest of the class. The teachers, therefore, appeared to focus on two levels: first on how to best address the struggling student with the task's goals and cognitive demand and second on how to manage the rest of the class who were not engaged or finished with their task. My analysis finds that despite these challenges, some teachers chose to afford students time; to question, probe, clarify, interpret, or confirm students' thinking; and to provide opportunities for discussion among classmates. These factors contributed to keeping the intellectual work of the tasks with the students.

If the students became stymied or showed signs of frustration or lack of resources, the teacher responses attempted to balance the probing questions with encouragement as well as the kinds of guidance that keep those struggling students engaged while focused on attending to their struggle. Most teacher responses began with an assessment of where the students were in their task. A common response was similar to Ms. Torres, who asked a student struggling to explain his solution, "I can't quite understand how you got [there]…can you explain that to me?" Other questions appeared to give direction and organizational support to students' thinking (Sorto et al. 2009; Anghileri 2006; Williams and Baxter 1996).

Studies have shown that prior knowledge plays a large role in connecting new knowledge to students' working knowledge as they engage in mathematical tasks (Rittle-Johnson 2009; Richland et al. 2004). Teacher responses made references to methods and concepts students had been exposed to and made analogies that related their current problems to problems that had "easier numbers" or were numerical rather than algebraic. Responses to student errors and misconceptions suggest the value teachers placed on reasoning and sense-making (Eggleton and Moldavan 2001; Borasi 1994). These interactions gave students opportunities to revise their thinking and not dismiss the effort they expended by acknowledging aspects that contributed to the process of problem solving (Gresalfi et al. 2009).

Three primary factors appear to influence the teacher responses: (1) Expectation of student effort characterized in the *probing guidance* and *affordance* responses that probed students' thinking and afforded time; (2) Accomplishment of the task in relation to the students engaged in the process of doing mathematics, as used in *directed guidance*; and the (3) Efficiency of task enactment as seen in the *telling* responses.

Discussion of interaction resolutions

Despite the common task, outcomes to student struggles differed significantly along a spectrum from productive to unproductive. The process leading to a resolution appears more complex than just relating struggle to response when taking into account the uniqueness of the students, their prior knowledge, fluency with skills, disposition toward "doing mathematics" and their level of motivation. What works for one student may not work for another (Gresalfi 2004). While the role of the task is to give mathematical context, the role of student engagement and the instructional practices teachers bring to the interaction is vital to supporting student learning (Kilpatrick et al. 2001).

The nature of teacher responses that addressed the cognitive demand of the task and the time afforded the students related to how productively interactions were resolved (Haneda 2004; Stein et al. 2000). The interaction resolutions appeared to depend on the student's prior knowledge and their willingness to engage in the problem (Bransford et al. 1999; Dweck 1986). The structural constraint of class time and classroom dynamic also affected the resolution. Time constraints posed a challenge for teachers as they attempted to bring closure to a task but conflicted with attempts to address student struggles. Teachers also had to balance addressing the struggles of their students with the restlessness of other students who were disengaged or had already completed their task. This and the other two factors are consistent with previous studies (Stein et al. 1996; Henningsen and Stein 1997; Lampert 1990; Kennedy 2005) that pertain to challenges to instructional practices.

The classroom culture, environment and norms were well established by the end of the school year, with clear expectations of how students discuss, question, make assertions, justify, and make meaning of mathematics. The range of student engagement and behavior even within one teacher's different classes became apparent as the students voiced various levels of deference and acceptance of teacher statements, explanations and justifications of mathematical processes, as well as the effort spent on their tasks before requesting help from their teacher. The establishment of expectations for "doing mathematics" as part of the sociomathematical norms of the class would provide a more hospitable setting for students and teachers to interact about student struggles (Ellis 2011).

Conclusion and implications

This study reveals the complexity of the phenomenon of struggle and its situatedness, in particular how these interactions occur, are resolved, and are a function of student, teacher and classroom context and norms. I note several emerging patterns across the Productive Struggle Framework that may give insight into how these codes are related. One pattern suggests that the level of cognitive demand appeared to influence how the teachers responded. The relationship between the struggle and teacher response appeared to depend on the interaction as situated between the particular student and the task. The teachers tried in most cases to balance what appeared to be tolerance levels that students displayed at a particular cognitive load with how much to press the students in their attempts to resolve their struggle at that particular cognitive demand. The teacher responses became more directed when the students appeared unable to overcome their struggle. The resolutions appeared to depend on the increase or decrease in the intended cognitive demand of the task and how students progressed in their engagement of the task, whether actively or with greater help from the teacher.

A second pattern across the codes suggests that teachers responded to different kinds of struggle in different ways. For example, when students struggled around uncertainty with explanation and sense-making, teacher responses most often involved probing guidance. The responses acknowledged students' efforts and, as in Ms. George's earlier example, focused the students toward the possible source of their struggles.

If the struggle was a procedural impasse, the teachers generally addressed that component to move the student forward in the overall task by reminding students of some prior work or even providing some information. Again depending on how much telling was involved by the teacher, the resolutions ranged from productive to productive at a lower level. However, if the struggle was more conceptual, the teacher responses encompassed a broader spectrum. Ms. Fine's example was a directed guidance but in other instances, responses ranged from telling students as with Mr. Baker in responding to the use of variables, to other more probing or affordance responses with questions that put the work of resolving the struggle back on the student. The resolutions, therefore, ranged from productive to unproductive.

In the third pattern, tasks designed and implemented to make student thinking and in particular student struggles visible appeared to help teachers be more responsive to the students' struggles. The teachers were better able to assess the students' struggle for example with misconception and errors by using what the students said or had written to inform their response. Ms. Torres, for example, responded to a struggle by asking, "…can you explain that to me?" or as Ms. George responded "well, I don't know. Can you?" The connection from task to struggle and then response to resolution was not always straightforward and often required time to extract the students' thinking. Resolutions therefore ranged from productive to unproductive. However, the struggle that seemed intractable for the student during these tasks became identifiable and thus perhaps useful in a future task (Kapur and Bielaczyc 2012).

Raising an awareness that struggling to make sense of mathematics is a natural part of "doing mathematics" can contribute to students and teachers recognizing that this phenomenon is a valuable part of learning with understanding. The encouragement to communicate with teacher responses such as “Tell me what you mean” and “Talk about it some more” or the insistence on sense-making with “Why is that?” provided opportunities for students to elaborate on what they understand and to clarify the source of their struggles. Responses that encouraged continued effort such as “Try that” and “Well, what if you do…” provided positive reinforcement for engagement without the student worrying about whether the result was right or wrong. An appropriate tempo for the interaction, one that did not rush the process or resort to shortcuts, promoted the sense that understanding both the problem and the process was more important than just finding a quick way to the answer. Posing problems of high cognitive demand gave the students opportunities to think, reason and problem-solve in ways that meant the students often had to struggle about the mathematics.

For future research, we can build on the descriptions of struggle, interaction and resolution to better understand the relationship across these phenomena and to assess how and to what extent productive student struggle contributes to student learning. In addition, tasks can be designed more explicitly to provoke student struggle with avenues for productive resolution.

Appendix

Activity 1: Barrel of fun

Suppose we have a 48-gallon rain barrel containing 24 gallons of water and a 5-gallon water jug containing 3 gallons of water.

Task	Intended level of cognitive demand
1.1 Which container has more water?	3
1.2 Which container is said to be fuller? Explain your answer	3
1.3 Use the coordinate grid below to draw a picture of the two containers and their water level. You may let each square represent 1 gallon and shade in the part representing the water. Does it matter what shape you make these containers?	3
1.4 How many gallons of water would need to be in the 5-gallon jug so that it has the same fullness as the 24 gallons in the 48-gallon barrel?	4
1.5 If we drain a gallon of water from each container, does this change your answer about which container is fuller? Explain	4
1.6 How many more gallons of water do we need to catch in the barrel in order to have the same fullness in the barrel as we have in the jug? Explain	4

Activity 2: Bags of marbles

There are three bags containing red and blue marbles as indicated below:

Bag 1 has a total of 100 marbles of which 75 are red and 25 are blue.
Bag 2 has a total of 60 marbles of which 40 are red and 20 are blue.
Bag 3 has a total of 125 marbles of which 100 are red and 25 are blue.

Task	Intended level of cognitive demand
2.1 Each bag is shaken. If you were to close your eyes, reach into a bag and remove one marble, which bag would give you the best chance of picking a blue marble? Explain your answer	3
2.2 Which bag gives you the best chance of picking a red marble? Explain your answer	3
2.3 How can you change Bag 2 to have the same chance of getting a blue marble as Bag 1? Explain how you reached this conclusion	4

Activity 3: Tips and sales

Task	Intended level of cognitive demand
3.6 A pair of pants regularly costs $40 but is on sale at 25 % off the regular price. How much will you pay for these sales pants, without computing tax? Explain how you got your answer	3
3.7 A shirt regularly costs $*S* and is on sale at 25 % off the regular price. Write an expression, using *S*, for the amount of dollars discounted. Write an expression that represents how much you will pay, disregarding tax	3

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