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Definitions of mathematical knowledge for teaching: using these constructs in research on secondary and college mathematics teachers

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Abstract The construct "mathematical knowledge for teaching" (MKT) has received considerable attention in the mathematics education community in recent years. The development and refinement of the MKT construct, including the components of common content knowledge (CCK) and specialized content knowledge (SCK), came from research into elementary teachers' practices. In this article, we argue that various issues arise as these constructs are used in research on secondary and post-secondary teachers. For example, elementary teachers typically differ from teachers of higher grades in their content preparation. What then is the relationship of CCK to SCK for those holding a bachelors degree or higher in mathematics? The MKT construct is based on CCK being knowledge held or used by an average mathematically literate citizen and that SCK is different. However, among those teaching in secondary and post-secondary contexts, what should be considered CCK? Is conceptual understanding of the CCK among those with bachelor's degrees or higher level mathematics the same as SCK? We examine these questions as well as others that arose from our examination of definitions of CCK and SCK as we attempted to utilize those definitions to characterize the nature of MKT at secondary and undergraduate levels. We illustrate these issues with data from two instructional settings.

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Introduction

Over the past several decades, there has been an increased focus on the nature of knowledge needed to teach mathematics. Derived primarily from analyses of elementary school teachers and their practices, researchers have developed and refined categories of "mathematical knowledge for teaching" (MKT), developed assessments of this knowledge, and demonstrated connections between measures of such knowledge and student achievement (Hill et al. 2004, 2005, 2007). Given current needs for more and better-prepared teachers, the mathematics education community is understandably eager to extend this work to grade levels beyond elementary school. In the process, however, relatively little attention has been paid to the ways in which MKT theory is or is not applicable to teachers at secondary and post-secondary levels. This issue, however, has been noted by researchers involved in the development of MKT theory. They have acknowledged the possible influence that characteristics of elementary school teachers have had on the definition of MKT (e.g., Ball et al. 2008; Hill et al. 2007) and have expressed concerns about the theoretical underpinnings of work at higher-grade levels and the diversity of approaches used to develop assessment instruments:

> It is in the arena of underlying theory, however, that these instruments differ most. Despite claiming to cover roughly the same terrain, these projects have strikingly different approaches to specifying domains for measurement – in essence, different approaches to organizing what is "in" mathematical knowledge for teaching. …to those interested in building theoretical coherence around mathematical knowledge for teaching, the variety of approaches is distressing (Hill et al. 2007, p. 131).

Other researchers have also noted the elementary-specific context in which work on MKT has been conducted and called for attention to efforts in other areas:

> … research has only started to map out the details of mathematical knowledge in teaching. There is scope for a more comprehensive research programme to extend scrutiny beyond the particular phases, systems and topics that have received most attention to date: *to examine mathematical knowledge in secondary and tertiary teaching as much as primary*, beyond a small group of anglophone cultures, *and in relation to areas and aspects of mathematics other than arithmetic*" (Rowland and Ruthven 2010, p. 291), emphasis added.

To date, however, the theoretical and practical implications of characteristics of elementary school teachers on the generalizability of the definitions of MKT to secondary and post-secondary contexts have not been examined explicitly. In this article, we explore the extent to which the similarities and differences between populations may influence the framing of research on MKT at secondary and post-secondary levels.

We assert that these issues merit serious consideration from the research community. As the MKT definitions and findings from research that utilize such definitions expand, it is prudent to examine these conceptualizations of knowledge and consider how transferable they are from one context to another. In this article, we offer our examination of definitions of subject matter knowledge, referred to as common content knowledge (CCK), and specialized content knowledge (SCK) and consider instantiations of these definitions in the context of secondary pre-service teacher preparation and post-secondary mathematics instruction. This theoretical exploration was inspired in part by the fact that distinctions between CCK and SCK for elementary teachers have been recognized and accepted in the mathematics education community, but these distinctions may be less compelling and clear at higher levels as few examples have been presented at the secondary and post-secondary levels. We increase the robustness of the theory overall, or define its limitations better, if we know whether these categories generalize or generalize only with modifications to the definitions.

We begin by summarizing research on teachers' knowledge and the development of the MKT knowledge categories, including the role of the elementary school context in which MKT-related constructs were developed. Next, we survey efforts to examine these issues at non-elementary school levels, and we compare and contrast the characteristics of the knowledge of teachers working at different levels. After describing our inquiry methods, we present classroom vignettes to illustrate our findings. The findings come in the form of questions that arose as we endeavored to apply MKT constructs to our research into high school and university-level teachers.

Research on teachers' knowledge for teaching

Many researchers have examined teachers' content knowledge and the roles knowledge plays in shaping teaching practices (Borko and Putnam 1996; Rowland and Ruthven 2010; Schoenfeld et al. 2000; Schoenfeld 2000; Sherin 2002; Shulman 1986). In such research, much of which is conducted from a cognitive theoretical perspective, knowledge is typically seen as one of the several factors influencing teachers' decisions and their approaches to accomplishing their instructional goals as they plan for and enact instruction. While it is undoubtedly the case that mathematics teachers need knowledge of mathematics content, researchers have found it challenging to establish definitive relationships between measures of teachers' content knowledge and student achievement (Ball et al. 2001; Wilson et al. 2002). One's intuitive belief is that secondary teachers should have substantial knowledge of mathematics; however, empirical research provides little evidence about what mathematics teachers should study in college mathematics courses (Begle 1979; Monk 1994). In addition to many studies that suggest a need for elementary teachers to have more robust mathematical knowledge (e.g., Ma 1999), evidence suggests that prospective high school mathematics teachers, who earn a mathematics major or its equivalent, do not have sufficiently deep understanding of the mathematics of the high school curriculum. For example, in Ball's (1990) study of prospective teachers' understanding of division, fewer than half of the intending secondary mathematics teachers who also were pursuing a mathematics major could provide a meaningful explanation for why division by 0 is undefined, though they could produce the correct rule. In general, in a synthesis of the research on prospective teachers' understandings of mathematics, these students were able to master rules and procedures, but had weak understandings of core concepts in the high school mathematics curriculum (Floden and Meniketti 2005). Floden and Meniketti (2005) argue that, "completion of advanced college-level mathematics courses with passing grades does not imply mastery of the concepts of the K-12 curriculum" (p. 272).

These and other findings have directed researchers' attention to other kinds of content knowledge and more refined measures of this knowledge. Various categorization schemes have been developed to delineate the types of knowledge that appear relevant to the teaching of particular subject matter. For example, Petrou and Goulding (2011) cite three such frameworks: the MKT framework described here (Ball et al. 2008), the Fennema and Franke framework (Fennema and Franke 1992), and the Knowledge Quartet (Turner and Rowland 2011). The MKT model described here is the dominant such model in use in the USA and its influence internationally is also clear, with researchers often using it as a basis for comparison or validation of their own frameworks even where those frameworks may be different. For example, the Knowledge Quartet framework, which is quite prominent in the UK, examines mathematical knowledge in teaching (MKiT), a closely aligned construct to MKT (Turner and Rowland 2008). Although their approach is quite different, authors of this framework situate it in the field in part by describing the ways in which it is complementary to the MKT framework. Likewise, Krauss et al. (2008) place their own framework in the same space as the MKT framework, comparing definitions directly and, in fact, using the degree of overlap in how their sub-constructs behave compared to those of the MKT framework as one method of establishing construct validity.

In the frameworks of Ball et al. and other researchers, the construct “mathematical knowledge for teaching” is used to describe the multiple components of knowledge used in the work of teaching mathematics. Researchers, building on Shulman’s (1986) formulation of teacher knowledge, often distinguish between knowledge that is purely mathematical in nature and knowledge that is the knowledge that applies content to the tasks of teaching, the former being referred to as “content knowledge” and the latter “pedagogical content knowledge” (PCK). MKT refines both content knowledge and PCK. Content knowledge is framed as several categories of knowledge consisting of:

- Common content knowledge (CCK), the mathematical “knowledge of a kind used in a wide variety of settings—in other words not unique to teaching; these are not specialized understandings but are questions that typically would be answerable by others who know mathematics” (Ball et al. 2008, p. 399);
- Specialized content knowledge, “the mathematical knowledge ‘entailed by teaching’—in other words, mathematical knowledge needed to perform the recurrent tasks of teaching mathematics to students” (Ball et al. 2008, p. 399); and
- Horizon content knowledge (HCK), knowledge of the mathematics that follows or could follow the mathematics being taught.

PCK is reframed to focus on knowledge of content and students, knowledge of content and teaching, and knowledge of content and curriculum.

History of mathematical knowledge for teaching

The development and refinement of the MKT construct came from research into teachers’ practices and, in particular, practices of elementary mathematics teachers. Both of these features of MKT’s history have shaped the nature of the construct in important ways. This situation is no different from the interdependence of theory, data, and findings in all research:

> Whether it is tacit or explicit, one’s conceptual model of a situation, including one’s view of what counts as a relevant variable in that situation, shapes data-gathering – and it shapes the nature of the conclusions that can be drawn from the data that are gathered. … Whether and how those factors are taken into account in formulating a study and gathering data for it will shape how that study’s findings can be interpreted (Schoenfeld 2007, p. 71–72).

As discussed in the subsequent section, the origins of MKT likely impact efforts to utilize the construct at higher-grade levels.

The first discussions of sub-categories of knowledge such as SCK and CCK grew out of Ball and colleagues' research on teachers of elementary school mathematics and the nature of classroom practices (Thames 2009). Ball et al. (2001) called for an analysis of practice to understand better how teachers both use what they know and know what is needed to use in practice. This work sparked the larger work by Ball and colleagues to both specify the tasks of teaching and the mathematical knowledge used in these tasks (Thames 2009). Beginning with close examination of teaching, the Learning Mathematics for Teaching (LMT) research group then sought to develop measures of this mathematical knowledge needed for teaching, embedding many of these questions in teaching contexts (for more detail on the measure development and validation, see the special issue of *Measurement: Interdisciplinary Research and Perspectives*, Volume 5, Issue 2–3). In reviewing the literature on studies that have sought links between teachers' content knowledge and student achievement, Thames (2009) notes, "In sum, studies attempting to link student achievement to their teachers' content knowledge consistently suggest that mathematical knowledge more closely related to practice is more likely to have a positive effect on teaching and learning" (p. 15). This means, for example, that the knowledge identified from an analysis of a teacher's successful interpretation of the mathematical ideas contained in a student's contribution to a class discussion may be more influential for the learning opportunities the teacher creates than the knowledge derived from other kinds of analyses (e.g., mathematical experts' assessments of knowledge entailed by particular curricula). Researchers have found connections between elementary teachers' possession of SCK, the mathematical quality of instruction (MQI), and elementary students' achievement (Ball and Bass 2000; Hill et al. 2004, 2005, 2008).

Less work exists at secondary and college levels

Research into MKT generally, and SCK specifically, has yielded important and useful findings that contribute to the applied efforts of teacher preparation and professional development. In addition, some efforts have been aimed at refining theories to explain how components of knowledge interact with one another and how knowledge shapes teachers' practices. These constructs and their uses have taken hold in the mathematics education community as evidenced by the considerable (and increasing) number of publications addressing these topics and assessments of this knowledge. It is now generally accepted that the type of mathematical understanding required to teach even the content of the lower grades is very different from the type of mathematical knowledge the average adult would have (Hill et al. 2007), and this shift in thinking has allowed researchers to focus more closely on the sub-components that make up that mathematical understanding. It has also come to be understood that the types of mathematical knowledge that seem most important for elementary teaching are not likely to be gained from traditional college mathematics coursework (Conference Board of the Mathematical Sciences 2012; Hill et al. 2005).

What happens when researchers look instead at secondary or post-secondary teachers? Do these descriptions of various types of knowledge fit the phenomena as well in non-elementary contexts? These questions are the central foci of our analysis. There are reasons to believe that these constructs apply to teachers of all levels—after all, the intellectual work required is similar, the tasks of teaching are similar, and so it is reasonable to presume that the cognitive processes will work in similar ways. On the other hand, approaches to teaching vary across grade levels and teachers' practices may be more varied at higher-grade levels than they are in elementary school. Elementary teachers also differ from teachers of higher grades in some potentially significant ways. Those ways include, for example, that elementary school teachers typically have received less content preparation in mathematics subject matter than their secondary school colleagues (Conference Board of the Mathematical Sciences 2001, 2012). In the USA, the No Child Left Behind Act of 2001 requires secondary teachers to have a major or its equivalent and be certified in order to be considered highly qualified, while most states require elementary teachers to complete a competency test in the four core academic areas such as Praxis II for elementary teachers.

Most state certification requirements suggest teacher preparation programs in the USA require two or three mathematics courses to obtain certification to teach at elementary school grade levels. These courses are typically ones designed around core content taught in elementary school such as number, operations, and geometry. Professional recommendations suggest that these courses should not be part of the sequence required of those who are pursuing a mathematics major (Barker et al. 2004; Conference Board of the Mathematical Sciences 2001, 2012).

For secondary mathematics majors, programs include some or all of the courses in the sequence designed for mathematics majors, plus, perhaps, courses tailored to the needs of mathematics teachers (Barker et al. 2004; Conference Board of the Mathematical Sciences 2001, 2012). In 2008, 63 % of mathematics teachers in the USA in public high schools held both a mathematics major or its equivalent and a secondary teaching certificate (Hill 2011). Whatever the specifics of the program, it is typically the case that secondary school teachers have had opportunities to learn both more and more advanced mathematics than their elementary school counterparts.

The contrast is even sharper when the teachers considered have been trained as research mathematicians. In addition to majoring in mathematics, they have devoted many additional years to the study of mathematics and are both continually expanding their knowledge as well as contributing to that knowledge base via their research. As a group, mathematician-teachers differ substantially from elementary school teachers both in terms of their knowledge of mathematics and the mathematical work they do as part of their profession.

Summary of work at secondary level

Although researchers have not explicitly examined the potential impact that the elementary school level context has had on the development of MKT theory, there are a number of researchers who have attempted to examine MKT at secondary and university levels. The existence of projects exploring MKT at secondary and university levels demonstrates that there is widespread interest in such an endeavor. In addition, researchers appear to be particularly responsive to the idea of SCK and what this knowledge would look like for post-elementary level teachers as is seen in the adoption of the framework for use when considering secondary teachers. The theoretical development, however, has been limited in various ways.

Approaches that these projects take to framing MKT vary, with some borrowing an existing framework from other projects and some creating new frameworks. However, one can see the influence that Ball and colleagues' approach has had on many of them. This has resulted in many different frameworks, consistent with Hill et al.'s (2007) concern about fragmentation in the field. Some efforts, such as the Knowledge of Algebra for Teaching (KAT) project (McCrory et al. 2012), see refinement of these frameworks as one goal of their research efforts, but most take the framework as given and focus more directly on other outcomes, like test item development (e.g., the Teacher Education and Development Student in Mathematics (TEDS-M), the Praxis Exam), evaluation of classroom interventions (SimCalc), building curriculum for courses for secondary teachers [e.g., the High School Mathematics from an Advanced Standpoint (HSMFAS) Project, (Usiskin et al. 2001)] or validation of other instruments constructs (the COACTIV project (Krauss et al. 2008) and Classroom Video Analysis (Kersting et al. 2012). Some are focused exclusively on middle school level mathematics (e.g., the Diagnostic Mathematics Assessments for Middle School Teachers), others on 9–12 secondary mathematics (HSMFAS, COACTIV), and others on content that appears at both the middle and high school levels (KAT, TEDS-M).

Characteristics of its development

The approach to development taken by many projects differs from that used in Ball and colleagues' work. For example, most studies approach the task of identifying elements of knowledge by conducting some sort of content decomposition of a mathematical idea or task. This work relies on the knowledge possessed by those doing the decomposition (who typically have quite substantial backgrounds in mathematics), and the components/categories of knowledge that emerge from the *mathematics itself*. This stands in contrast to the original MKT development work that relied heavily on analyses of teachers' classroom instructional practices as sources for the mathematical entailments of the work of teaching.

In keeping with the trend set by the LMT project at the elementary level, most of these researchers attempted to focus directly on the mathematics encountered in teaching, characterized that knowledge as multidimensional, and made an effort to connect their work to teaching practice (Hill et al. 2007). However, they are inconsistent with respect to the methods and degree of grounding in classroom teaching. Some researchers focus instead on various artifacts of practice such as curriculum materials and student work, but fall short of grounding their frameworks in classroom instruction. As Hill et al. (2007) point out, these researchers approach theory development differently, resulting in a proliferation of different frameworks and no consensus in the field about the structure of secondary MKT. Hill et al. (2007) see this as problematic, "at least to those interested in building theoretical coherence around mathematical knowledge for teaching" (p. 131), and as potentially undermining to the validity of the teacher knowledge tests that many research groups are designing. "The construct of mathematical knowledge for teaching is still emergent. Without better theoretical mapping of this domain, no instrument can hope to fully capture the knowledge and reasoning skills teachers posses" (Hill et al. 2007, p. 136).

Work at the college level

In general, there has been considerably less research on mathematics teaching and teachers conducted at the college level than at elementary or secondary levels. In fact, a recent review of the literature (Speer et al. 2010) located only five instances of empirical investigations of college-level mathematics teachers and their practices. Of these, only two include analyses of teachers' knowledge (Speer and Wagner 2009; Wagner et al. 2007). In both cases, the focus of the research was on a mathematician's efforts to implement inquiry-oriented instructional materials and practices for the first time. The earlier piece characterized the types of challenges the mathematician encountered, with a particular focus on the knowledge he needed to enact this type of instruction and how that knowledge differed from what he had needed when using more traditional materials and instructional practices. The focus in the analysis was on pedagogical content knowledge and how absence of some knowledge made certain kinds of instructional decisions extremely difficult.

In the later, related piece (Speer and Wagner 2009), however, the authors examined both pedagogical content knowledge and SCK. In this work, the focus was on whole-class discussions and the knowledge (PCK and SCK) that was needed to recognize student contributions as potentially productive for advancing the mathematical discussion. Findings indicate that the students generated ideas and ways of thinking that were unfamiliar to the mathematician and that he needed to do mathematical work to make sense of and determine the validity of the student contributions. The authors characterize this "mathematical work" as drawing on (and potentially generating) SCK and cite Ball and colleagues' definition. Although attention is devoted to distinctions between PCK and SCK, there is no discussion of the characteristics that distinguish SCK from CCK.

As has been the case for most research on the teaching of college mathematics, these efforts are case-driven, and unlike the work at lower grade levels, the goals do not appear to be related to the creation of assessments. These efforts do, however, share some features with the early efforts at the elementary level. In particular, the analyses have all been based on data from teachers' authentic classroom practices. Theory development and/or refinement were not a focus of the work done at the college level, and definitions of the basic categories of knowledge were not questioned, but the efforts were grounded in practice, as recommended by Ball and colleagues (Hill et al. 2007). These efforts to utilize the SCK construct to understand mathematicians' practices stands in contrast to some efforts at the elementary level and at the secondary level where the goals are tied to developing items to assess teachers' SCK.

To summarize, researchers in elementary mathematics education have made a strong case that teachers' mathematical knowledge does matter, but that certain kinds of mathematical knowledge matter more than others and that precise definitions are needed. The theoretical development that these researchers have achieved was based on careful qualitative analysis of classroom practice and has only recently expanded into test item development. Some work has been done at the secondary level, but the underlying theoretical development has not been as thorough as that done at elementary grade levels. It has instead relied on assumptions, most often assuming that secondary MKT is similar to elementary MKT or assuming that expert opinion suffices to construct a theoretical framework. The problem with the first assumption is that the existing framework for elementary MKT may not be a useful way of thinking about secondary MKT. The problem with the second is that expert opinion as a basis for theory is no different from what has been done historically, and what was done before did not show a strong relationship between teacher mathematical knowledge and student outcomes. As noted by the researchers involved in the foundational work on MKT, the expansion of these theoretical constructs beyond the grade levels at which they were developed should be approached cautiously, with attention to the ways in which the ideas can and cannot be applied to other populations of teachers. As cautioned by Schoenfeld (2007), "assuming that the results of a study (no matter how well executed) that is conducted on a subpopulation will apply to the population as a whole is not necessarily warranted" (p. 71).

Methods of inquiry

In this theoretical endeavor, we made use of research on knowledge needed for teaching and data from two research projects. We focused our analyses on the explicit definitions of

CCK and SCK as well as their operational definitions as found in the literature on elementary teachers' knowledge. We examined those definitions and their relationships to typical characteristics of elementary teachers (e.g., their level of content preparation, their experiences doing mathematics) implicit in those definitions. Next, we compared and contrasted those characteristics with characteristics typical of secondary and post-secondary teachers, the nature of the teaching-related tasks they face and then examined potential implications of these characteristics for the definitions of CCK and SCK.

To illustrate these issues, we use data from prospective secondary school teachers' reflections on their classroom practices and from studies of post-secondary teachers' practices. The data come from two projects. The high school example comes from work with prospective secondary mathematics teachers in a masters program. The students had mathematics majors as undergraduates and were enrolled in a methods course (taught by the second author), coupled with a field experience that included independent student teaching. Students were regularly given problems that could be used as a basis for lessons in the classes they were teaching. They reported back on their use of the problems, reflecting on how the lesson unfolded, aspects of the lesson that did and did not work well, potential changes, and things they learned from teaching the lesson. The data reported come from one student's reflections on the use of a problem that she chose to use as part of a lesson in her ninth grade class.

The second project was set at the university level and involved curriculum development for an upper division undergraduate capstone course for prospective secondary mathematics education teachers who were majoring in mathematics. The data used in the current project include video recordings of classes that were gathered over three semesters, none of which were taught by authors of this article.

To understand the ways in which the MKT framework does or does not generalize to the secondary and post-secondary work of teaching, we analyzed cases from these data sources to illustrate the complexities in distinguishing among the components and types of knowledge. These cases provide illustrations that may help with the development of definitions of CCK and SCK at upper grade levels. We bounded our data (Miles and Huberman 1992; Yin 1984) to those cases that allowed the authors to conduct thought experiments about what knowledge the instructors do or do not have. We do not make the claim that the instructors have or do not have the knowledge, but that we can evaluate the appropriateness of the MKT framework for describing the class of knowledge the teacher may or may not have. For each case presented, we attempted to apply the existing MKT analytic framework to our data that would support plausible explanations and found the framework insufficient to explain our data. We do not claim that the MKT framework is insufficient in all cases, but in seeking to explore the robustness and generality of the framework, we pursued disconfirming cases that illustrate ways in which the framework may need to be refined or expanded to address mathematics knowledge for teaching at the secondary and post-secondary levels.

Findings

The results are a set of questions, illustrated by the vignettes that demonstrate central issues in generalizing the constructs of CCK and SCK to secondary and post-secondary mathematics education contexts. We discuss two major questions that arose for us in the process of this analysis.

Defining “common” and “specialized”

The first question is “What is the relationship of CCK to SCK for those holding a bachelor’s degree or higher in mathematics?” The assumptions embedded in the elementary context are that CCK is knowledge held or used by an average mathematically literate citizen and that SCK is different. However, among those teaching in secondary and post-secondary contexts, what should be considered *common* content knowledge? Might conceptual understanding of *common* content knowledge among those with a bachelor’s degree or a higher degree in mathematics be the same as SCK? For example, recognizing the mathematical accuracy of a definition, considered part of SCK for elementary teachers, is CCK for those with more mathematics education.

To illustrate, consider the following example from a secondary mathematics classroom. A teacher poses the following problem: Suppose that a staircase comprises ten steps and that you can climb the stairs one or two steps at a time. In how many different ways can you climb these ten steps? (Rubel and Zolkower 2007/2008). The teacher has the goal of using this problem to discuss combinations and counting methods and has an image of a general solution.

A typical solution of this sort might involve considering all of the cases of how the staircase can be traversed using 1 or 2 steps. Since there are 10 steps, there are a maximum of 10 total climbing moves with 10 1-steps and a minimum of 5 total climbing moves with 0 1-steps. As 10 is an even number of steps, you can climb 2 steps at a time or 1 step at a time twice and the total number of single steps depends on the total number of climbing moves. Table 1 displays a systematic list of all of the cases. For example, in the case of 6 total climbing moves, the climber might take 1 step, then successively climb 2 steps four times, then 1 step at the end. Or the climber might take 1 step twice, then successively climb 2 steps 4 times. These are distinct ways to climb the steps using 1 step twice and 2 steps 4 times. However, in both cases, the stepper is making 6 climbing moves overall. So, in this case, there are ${}_6C_2 = 15$ ways of climbing using 1 step twice.

Mathematically, if the instructional goal is to generalize such combinatoric solutions, the teacher must know how to solve this problem using combinatorics, but also how to systematically represent the cases in a way that relates the given information to the combinations of interest. He or she must be able to relate the representation in Table 1 to a similar representation that demonstrates the number of 2-steps in the center column and the complements of the combinations in the third column.

Vignette

Here, we present an example drawn from our data. Faced with the stair climbing task, a group of students presented the solution shown in Table 1. This group started with smaller numbers of total steps to build their solution for 10 steps. The students concluded that the pattern of growth of the number of ways as one increases the number of steps is the Fibonacci sequence. So for 10 steps there are 89 ways. The teacher needed to decide how to respond to the students.

This example presents several questions about the mathematical knowledge teachers need to make the next pedagogical decision in this situation. The teacher must consider whether this sequence really is the Fibonacci sequence and why. We contend that the knowledge needed to determine whether the sequence is really Fibonacci is CCK for the population of similarly situated mathematical knowers—those possessing at least an undergraduate mathematics major. However, the definition used by Ball and colleagues

Table 1 Number of ways to climb 10 steps using 1 or 2 steps by total number of climbing moves

Number of total climbing moves	Number of single steps	Number of ways
10	10	${}_{10}C_{10} = 1$
9	8	${}_{9}C_{8} = 9$
8	6	${}_{8}C_{6} = 28$
7	4	${}_{7}C_{4} = 35$
6	2	${}_{6}C_{2} = 15$
5	0	${}_{5}C_{0} = 1$
		$1 + 9{+}28 + 35 + 15 + 1{=}89$ total ways

focuses not on similarly situated mathematical knowers, but is based on a job analysis of the work of teaching and defines CCK as "content knowledge that is used in the work of teaching in ways *in common with* how it is used in many other professions or occupations that also use mathematics" (Hill et al. 2007, p. 436). Would anyone other than a teacher or a mathematician *need* to know whether this is the Fibonacci sequence? On the other hand, the ability to unpack the students' solution, particularly interpreting their representation, may or may not be a mathematical expectation in professions or occupations of those who have an undergraduate major or substantial education in mathematics. One can imagine that anyone who has to interpret infographics built on substantial mathematics such as a journalist or policy analyst would need this kind of knowledge. Is this knowledge SCK?

In addition, to decide what to do with this solution, teachers need to be able to determine how well they can advance their mathematical goal of the lesson—combinatoric solutions—and still address this solution. While this may seem to be a pedagogical question, to formulate an answer, the teacher must consider other mathematical questions such as: How does the Fibonacci sequence solution connect to the combinatorial solution? Is this an important mathematical connection to make? The decision of what to do pedagogically, whether or not to pursue the students' alternative solution, rests on the answer to the mathematical question of the connections among the combinatorial and sequence solution strategies, the mathematical importance of making this connection, and the pedagogical importance of making this connection at this time. Thus, the content knowledge and pedagogical content knowledge issues are intertwined. Overall, in assessing the mathematical knowledge needed for teaching, do we consider the knowledge needed to deal with this solution in the context of a high school classroom CCK that everyone, or at least all mathematics majors, should know; SCK that only secondary teachers need to know; pedagogical content knowledge that is focused more on knowing students, teaching or curriculum; or something else? Why? As currently defined, there are legitimate arguments to be made that this kind of knowledge could fit in all of the above categories. But a definition that does not discriminate well is not useful or usable.

The nature of mathematicians' work

Our second question is, "What is the relationship between the type of work mathematicians do in their research and while teaching mathematics?" Researchers have distinguished between CCK and SCK by saying that SCK is "Specialized because it is not needed or used in settings other than mathematics teaching" (Ball et al. 2008, p. 396, emphasis added). In other words, there is mathematical work that is required while teaching that is not required in the other contexts. This distinction is relatively clear in the context of elementary school teaching where the work required (e.g., determining whether a solution strategy is generalizable) goes beyond what a mathematically literate person might do in their day-to-day lives.

When we consider the day-to-day lives of mathematicians, however, the distinction seems less clear. Consider as an example the teaching task of examining, evaluating, and formulating a response to a student-generated solution. This is a type of work that researchers of elementary teachers assert necessitates (and enables the development of) SCK. Now consider the nature of the work of mathematicians. In the course of their typical activities, mathematicians evaluate their peers' solutions and provide feedback about those solutions. This occurs informally as colleagues share ideas and possible solutions to problems. It also occurs more formally when mathematicians examine proofs and solutions while listening to their colleagues' presentations and while reviewing manuscripts for publication. Even for mathematicians whose jobs are outside of academia and are, for example, employed as industry analysts, their activities still include reviewing and determining the validity of mathematical solutions and arguments.

In both the teaching and research contexts, the mathematician needs to make sense of the mathematical ideas and reasoning presented by someone else and determine whether the reasoning is correct. In both contexts, the mathematician must also formulate a response about the proposed solution, either to the student or (directly or indirectly via a journal editor) to their peer.

We present two classroom vignettes to illustrate this potential similarity between tasks mathematicians engage in while doing the work of teaching and those they engage in as part of their research endeavors.

Vignette #1: evaluating the validity/viability of a method

As part of an investigation into the nature of roots of equations, students worked several problems that could be solved by factoring and then were given one that was not factorable: $2x^2 + 3x + 4 = 0$. After walking around the room looking at their work, the instructor summarized what several student said to him: "…you see pretty quickly that you cannot factor it so the only thing left is to try to use the quadratic formula." After asking students to recall the formula, the instructor wrote the following on the board below the equation:

$$ax^2 + bx + c = 0, a \neq 0 \tag{[1]}$$

$$x = -b \pm \frac{\sqrt{b^2 - 4ac}}{2a} \tag{[2]}$$

The instructor then said, "Derive this. Can you prove this? Who knows how to go about deriving it?" A student responded with, "Could you work backwards?" The instructor asked, "And just substitute in and show it works?" This does not appear to be what the student had in mind and the student further described his solution strategy, "Try to start from that [pointing to equation [2]], use if and only ifs and try to get the top one [referring to equation [1]]." The instructor responded with "Ummm" which stretched out for 5 s, followed by 4 s of silence. This suggests that he was thinking about the student's idea and working to formulate a response. The instructor then responded with, "Well, I'd rather have you start at the beginning–you might be able to do that [referring to the student's suggestion]–can you start at the beginning and what do you do? What's the tool you use?" The discussion then moved on when a student offered up the idea of completing the square.

During this episode, the instructor was presented with a potential path to a solution and needed to assess the validity and viability of that path. The student was suggesting that they show that equation [1] can only be true if its roots are given by the statement in [2]. This general approach is a fairly common when constructing proofs and one that the students have likely seen on many occasions in their earlier mathematics courses. This is not the approach typically used with this specific example, however, and was apparently not the one the instructor had in mind. In the seconds after the student made the suggestion, the instructor needed to figure out whether such an approach was a mathematically valid way of approaching this particular proof task and also determine whether such an approach was a viable option for the class to pursue (and then formulate a response based on his assessments).

This exchange has features in common with what might occur when a mathematician considers the validity of a possible solution path while working on his or her own research or when asked for feedback from a colleague. In both the teaching-based and the research-based scenarios, the mathematician must first understand the gist of the possible solution path. In the video excerpt, we see the mathematician's efforts to follow what the student is suggesting when the mathematician fills in (what he thinks would be) the next logical steps in the student's solution path ("And just substitute in and show it works?"). This comes with the intonation of a question, suggesting that the mathematician is trying to check the accuracy of his interpretation of the student's proposed solution path. This work of interpreting students' spoken and/or written mathematical ideas is what mathematics education researchers have claimed makes use of SCK. However, what occurred in the classroom excerpt parallels what might occur if the mathematician and a mathematician colleague were discussing a problem and the colleague suggested a possible approach. In that case, as in this teaching-based one, the first task for the mathematician is to make sense of what has been suggested.

Next up in both situations is the task of determining whether the suggested solution path is actually a valid approach to use in such a situation. We hypothesize that while the mathematician was saying "Ummm," he was considering whether the "working backwards" approach was mathematically valid in this situation. To make this determination, the mathematician needs to draw on his knowledge of mathematics but handling such a situation is also the kind of teaching-related work that researchers of elementary school teachers claim draws on SCK. When interacting with a colleague, the mathematician would also need to consider the validity of the proposed solution.

In both scenarios, the mathematician needs to provide a response to the student or colleague. In the research-based scenario, the mathematician's response is apt to portray an assessment of the validity and to do so in language and at a level of sophistication that is similar to how he thought about it himself. In the teaching-based scenario, the mathematician is apt to need to consider other factors and package his response differently than he would when speaking to a research colleague. For example, even if the solution path the student suggests is mathematically valid, the instructor may not want to have the class pursue it. The proposed solution might involve some mathematical ideas or techniques that the instructor believes are not going to be easily understood by the majority of members of the class. Alternatively, the solution path might be accessible to all the students but might not illustrate the particular solution strategies or techniques that the instructor had planned for the lesson. Both of these considerations make use of elements of pedagogical content knowledge. In formulating a response to the students, the mathematician may also need to engage in some "trimming" (McCrory et al. 2012) that would transform his understanding of why a solution path is or is not valid into a description that would be accessible to the students given their assumed mathematical backgrounds.

The kind of mathematical work described above is a frequent and common component of mathematicians' scholarly endeavors. As such, it seems inappropriate to characterize it as "specialized" for the work of teaching given that mathematicians engage in this kind of work and make use of this kind of knowledge routinely as part of their research.

Vignette #2: Checking the validity of a technique/way of reasoning

During a lesson about decimals and fractions, the instructor had just written the following on the board and talked through how to see that it is a valid statement:

$$.\overline{1432} = 1432(.\overline{0001}).$$

A student then asked, "Could we break it up and say that .1432 repeating is the same as .1 repeating plus .04 repeating plus .003 repeating plus .0002 repeating?" The instructor quickly responded with "Yes" but then said, "Let me make sure I understand what you're saying. This is what I think you're saying" and wrote the following on the board:

$$.\overline{1432} = .\overline{1} + .0\overline{4} + .00\overline{3} + .000\overline{2}.$$

The student then said, "But wouldn't the line have to go over both like the .04 so it would be .040404?" and the instructor said "Yes, that's right" while editing what he had written on the board so it looked like this:

$$.\overline{1432} = .\overline{1} + .\overline{04} + .\overline{003} + .\overline{0002}.$$

A different student then asked, "Wouldn't there have to be some zeros after that .1 because if it was just .1 repeating there'd be more [inaudible]?" The instructor talked over the last bit of what the student said (making it inaudible) and said, "Yes, I think you're right. In every single one of these cases I have to make sure I have the four digits, right? So I think we've got to have this, right?" and edited what was on the board so it read:

$$.\overline{1432} = .\overline{1000} + .\overline{0400} + .\overline{0030} + .\overline{0002}$$

The lesson moved on from there.

This is an example of the kind of "mathematical work" that instructors do that is outside of what might be demanded in other jobs where people know and use this content. The instructor had to listen to the first student's idea and then translate it into symbols to check whether he was understanding what she said. The focus was not necessarily evaluating its correctness at this stage. That idea was then modified by the student and the instructor had to make sense of that modification and edit the symbols to match. A version of this was repeated when the second student raised a question about the accuracy of what was on the board and offered a modification.

This work is clearly outside of what might be done by someone by drawing on only their CCK of decimals. What the instructor had to do, however, does not seem specialized for only the work of teachers. This kind of interpretation of a statement, checking to see whether it has been understood accurately, listening to modifications, creating representations that accurately capture these ideas, etc., is part of what occurs as mathematicians collaborate on research problems. Although doing what the instructor did in the lesson demands more than knowledge of decimals and is likely outside the realm of what most people do regularly who possess that knowledge of decimals, it does not appear to be specialized only to the work of teaching.

Several additional questions arise from these examples. In general, to what extent are the two types of work (teaching-based and research-based) described above the same? Do they draw on the same type of mathematical knowledge. (e.g., Does the SCK needed for the teaching overlap with the content knowledge that is "common" to research mathematicians?) Furthermore, elementary and secondary teachers generally do not examine the mathematical work of their peers. Does that mean that the knowledge used while checking the validity of student-generated solutions is CCK for mathematicians but SCK for others? Is the work the same when formulating a response to these different people (students versus colleagues) and is the knowledge used to do that work the same in both contexts? For example, the instructor in this vignette provided some reasoning when he edited the solution that was on the board. ("In every single one of these cases I have to make sure I have the four digits, right?") It seems likely that this was done to help other students in the class understand why the edit was necessary. If the discussion had occurred instead with a mathematician colleague, the instructor might have presumed the colleague's mathematical understanding to be sufficient to follow the edit without needing to hear the reason. In general, does the fact that the audience for responses is made up of students in teaching contexts make the nature of that work (and hence the knowledge it draws on) different than if the audience were made up of research mathematicians? Or might this kind of response-formulation work in both contexts draw on the same type of knowledge, making the label of "specialized" inappropriate?

Conclusions and implications for further research

The distinctions among CCK, SCK, and PCK have had implications in the mathematical education of elementary teachers, supporting the need for elementary teachers to have mathematics courses designed to develop SCK (CBMS 2012) and lessening calls for elementary teachers to take more higher level mathematics courses instead of courses that support teachers' learning of SCK. This distinction has also permitted the development of assessments that have been shown to measure SCK related to student learning (Rockoff et al. 2011). However, at the secondary level, these types of assessment efforts have been less fruitful (McCrory et al. 2012), and our vignettes suggest reasons why this framing might be less promising for a population who begin with a different level of mathematical exposure. In recent work, Phelps, Howell, and Kirui (in preparation) suggest that for the purposes of assessment development, creating an assessment framework around tasks of teaching as opposed to types of content knowledge may lead to the development of more valid assessments. They argue the more significant issue appears to be that these categories connect mathematical knowledge to teaching practice. Drawing attention to evaluating student reasoning to make instructional decisions and the mathematical knowledge this work entails may be a more important discussion to begin in departments of mathematics than parsing the distinctions between common and specialized mathematical knowledge.

Toward that end, further research is needed on the types of knowledge entailed in the work of teaching in high school and undergraduate settings, through the same kinds of careful study of the mathematical demands of teaching that sparked the early work on mathematical knowledge for teaching (Ball and Bass 2000). Parallel work in professional and assessment development would help build theory while developing practical instruments to measure the impact of various opportunities for teacher learning and the impact on student learning. This focus on research and development in Pasteur's Quadrant (Stokes 1997) has the potential to contribute to both theory and practice in understanding the nature of teachers' content knowledge, how to influence teacher learning, and its impact on classroom practice and student learning.

The development and refinement of theory in this area of research is a key ingredient for continued advancements. The tremendous importance of improving the teaching and learning of mathematics means we should pay careful attention to the fact that "Every empirical act of representation, analysis, and interpretation is done in the context of a (sometimes explicit, sometimes implicit) conceptual and theoretical model. The character of such models shapes the conclusions that are produced by subsequent analysis and interpretation" (Schoenfeld 2007, p. 80). By potentially overgeneralizing the current theoretical framing from elementary to secondary and post-secondary settings, the field is missing both an opportunity to better understand the nature of mathematical knowledge for teaching more generally and is likely to develop unproductive interventions for improvement of teacher learning that support their students' learning. For example, while the CBMS (2012) recommendations for elementary teachers' mathematical preparation draw on the results of the prior research of Ball and colleagues, there is little research support and therefore less guidance about what this means for high school teachers. Can and should this kind of knowledge for teaching be embedded in existing courses, such as geometry or abstract algebra? The answer to such questions are currently based primarily in the wisdom of practice rather than in theoretical or empirical results. Future research on the ways in which teachers of geometry or algebra use their knowledge in teaching could have implications for ways to help prospective and practicing teachers gain such knowledge and how it might be assessed. Initial studies (e.g., Howell 2012) support the lack of alignment with the current model in certain cases, particularly as the knowledge relates to mathematical practices embedded in content (National Governors Association Center for Best Practices, Council of Chief State School Officers, 2010). Similarly, research on the ways that undergraduate mathematics teachers use their mathematical knowledge to teach post-secondary courses can inform graduate student professional development programs and faculty professional development opportunities offered by universities and professional societies to improve teaching and learning.

The model created by Ball and her colleagues is powerful in that it links knowledge, teaching practice, and students' learning. The secondary and post-secondary communities can benefit from the research trajectory that created this model by attending to the mathematical knowledge used in the practice of teaching in future research and exploring the connections to improved student learning.

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