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Examining the task and knowledge demands needed to teach with representations

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Abstract Representations are often used in instruction to highlight key mathematical ideas and support student learning. Despite their centrality in scaffolding teaching and learning, most of our understanding about the tasks involved with using representations in instruction and the knowledge requirements imposed on teachers when using these aids is theoretical. In this study, we examine the task and knowledge demands for teaching integer operations with representations by analyzing teaching practice. Teaching integer operations is used as an intensity case, as integer operations are challenging for students, and teachers are often required to employ several representations to teach this topic. Following a practice-based approach while also taking prior literature into consideration, we first generate a list of tasks entailed in teaching with representations and then discuss the knowledge demands imposed on teachers to successfully undertake this work. We highlight these tasks and knowledge demands by analyzing and discussing an integer addition and an integer subtraction episode for each of two teachers, Bonita and Karen. Based on our analysis, we organize the generated knowledge components using the *Mathematical Knowledge for Teaching* framework. We conclude by drawing implications for teacher educators and curriculum developers.

Keywords Representations · Integer operations · Knowledge · Teaching tasks · Case study · Teaching

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Introduction

Since Shulman proposed the idea of pedagogical content knowledge (1986), researchers have attempted to understand the knowledge needed to teach mathematics (Ball et al. 2001). Researchers have mapped such knowledge, both at large (e.g., Ball et al. 2008; Rowland et al. 2009) and smaller grain sizes (e.g., Charalambous et al. 2011; Sleep 2012). Consistent with this line of research, this article explores the tasks and knowledge demands required for teachers to successfully teach with representations. We selected this topic because most mathematics K-8 curricula include numerous representations, and representations are considered a central teaching aid for supporting student learning (cf. National Council of Teachers of Mathematics [NCTM] 2000). To further focus this work, we consider the topic of integer addition and subtraction as an intensity case: one which strongly manifests the teaching practice under investigation (Patton 2002), as integer operations are well known for their difficulty to understand and teach (Gregg and Gregg 2007; Schwarz et al. 1993/4). The work reported here is expected to contribute toward the development of a framework for analyzing and understanding teaching, much needed in teacher education (Grossman and McDonald 2008).

To sketch the knowledge needed to teach integer addition and subtraction with representations, we follow a top-down and a bottom-up approach. We first consider what prior research has uncovered regarding representation use and the knowledge demands required for this work. Next, we present and analyze four teaching episodes, identifying tasks and knowledge demands needed to successfully use representations in instruction. Finally, we summarize the knowledge demands entailed in teaching with representations and offer implications for teacher educators and curriculum developers.

Theoretical considerations

Conceptualized as entities that symbolize or stand for other entities (Duval 2006; Goldin and Kaput 1996), representations can refer to the internal organization of knowledge through cognitive processes (Izsák 2003) or the external means of modeling various mental processes (Janvier 1987), such as real-world contexts, manipulatable models, pictures or diagrams, spoken languages, and written symbols (Lesh et al. 1987). Here, we focus on teachers' use of three types of *external* representations: the real-world, money context; the diagram-based, number line; and the manipulatable chips, each intended to help students make sense of integer addition and subtraction. In what follows, we first consider the affordances and limitations of external representations more generally; we then discuss the particular representations appearing in the episodes. Finally, we consider potential knowledge demands for teaching with representations as implied by the literature.

Affordances and challenges of representation use

Researchers see multiple benefits to using representations. First, representations can support students in making sense of and reasoning about mathematical tasks and concepts. They can also facilitate student learning. Pirie and Kieren, drawing on their model of eight nested layers of mathematical understanding, explain that, when challenged, the learner can "fold back" from an outer level of understanding to an inner level (see Martin 2008). Representations can support this "folding back," by both triggering this shift (i.e., challenging students' incomplete understanding) and offering a platform to "fold back," namely providing a reference point in students' future attempts to construct more advanced understanding. Representations can also assist students with organizing/sharing their thinking and constructing mental models of mathematical ideas (Dufour-Janvier et al. 1987; Schwartz et al. 1993/94). Additionally, representations can make abstract mathematical concepts more accessible (Flores 2002) and foster connection-making between procedures and concepts or between various strategies (NCTM 2000). Further, representations can be "used to 'define' operations, explain properties and algorithms, and provide a structure of problems which are solved by the use of operations" (Vest 1976, pp. 395–396).

Despite these benefits, the use of representations is not without challenges. One challenge relates to using representations without building mathematical meaning. This might happen when students are forced to imitate procedures without the opportunity to reflect on their actions or the guidance to make connections between representations and underlying mathematical ideas (Clements and McMillen 1996; Stein and Bovalino 2001). Additionally, learning a representation, with its own rules, symbols, and language, could result in a proliferation of abstract mathematical rules (Gregg and Gregg 2007), and ultimately detract from learning the mathematical idea itself (Dufour-Janvier et al. 1987). Representations can even reinforce student misconceptions when they offer an incomplete treatment of the mathematics or are too far removed from student initial knowledge or too inauthentic to their experience (Hiebert and Carpenter 1992; Solomon 1989). Teachers may also take for granted that students will become easily initiated to the representations' structure or that representations will "by default" illuminate underlying mathematical ideas. However, representations are not inherently transparent; rather transparency resides in the very process of using them, since representations, like other artifacts, are symbolic devices with cultural significance (Meira 1998). Thus, "successful development of mathematical meaning-making via representations requires opportunities for interaction and time" on the part of students (Meira 1998, p. 125).

Integer operations offer particular challenges for representation use, because unlike natural numbers, students cannot construct the meaning of integer operations by mere abstraction from real objects (Stephan and Akyuz 2011). Moreover, some of the properties of integers contradict intuitions from natural numbers (Linchevski and Williams 1999). Further, there is no single representation that can naturally convey the underlying ideas of negative numbers and their operations (Stephan and Akyuz 2011), rendering representations often counterintuitive in use. It is especially difficult to find a representation that helps students discriminate between the multiple meanings of the "$-$" sign—as an operation (take away), a value (negative), and as "the opposite of" (Lamb et al. 2012)—a critical component of instruction on this topic (Kinach 2002). Because of these limitations with existing representations, most research in this area describes attempts to develop new or modify existing representations for integer operations, although some mathematicians even recommend that none be used (cf. Linchevski and Williams 1999).

Perhaps because of disagreements about which external representations, if any, to use when teaching integer operations, and perhaps because of the importance of using multiple representations as a means to forge student understanding (Hitt 2002), reform-oriented curricula currently include various representations to support student understanding. Below, we focus on three particular representations, then consider the theoretically based knowledge demands entailed in teaching with such representations.

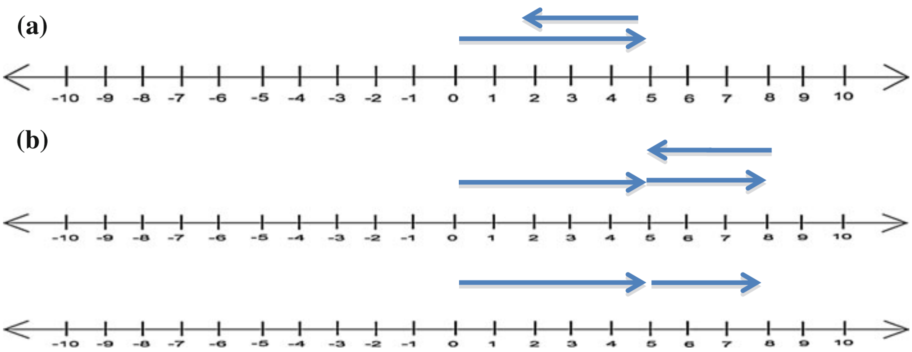


Fig. 1 **a** Vector-based, *number line* representation of $5 + (^{-}3)$; **b** vector-based, *number line* representations of $5 - (^{-}3)$

Three commonly used external representations for integer operations

The three representations under consideration—number line, chips, and money contexts—are each governed by certain conventions and used to give meaning to mathematical ideas in different ways.

Number line

In the number line, operations can be treated as movement, with the plus/minus operation sign typically indicating directionality of movement, and the positive/negative symbol indicating which way the traveling object should face (Lamb et al. 2012). For example, for an addition problem, like $5 + {}^{-}3$, one could start at 5, then turn to face left (because of the negative symbol), and finally travel forward 3 steps (because of addition). For a subtraction problem, like $5 - (^{-}3)$, one starts at 5, then switches direction to face left (because of the negative symbol) and moves backwards 3 (because of subtraction). The number line is also well-suited for the comparison approach to subtraction.

Another, less commonly taught, way to perform integer operations using the number line is the vector-based approach. For example, for addition $5 + {}^{-}3$, the first addend (5) is represented as a right-facing vector with end point at zero. To this arrow tip, we add a left-facing vector with magnitude of 3, which gets us back to $+2$ (see Fig. 1a). For subtraction $5 - (^{-}3)$, the minuend (5) is represented as above. Then, we want to remove a left-facing vector with magnitude of 3, which is impossible. However, one can represent 5 as the sum of the vectors 5, 3 and -3, then remove the vector -3, leaving 8 (see Fig. 1b).

Despite the number line's power for supporting student thinking (cf. Gravemeijer and Stephan 2002), the meanings of the "$-$" sign as both facing backwards and moving backwards (Hativa and Cohen 1995) might confuse students. These artificial rules can create difficulties for teachers as well (Kinach 2002).

Chips

The chips[1] are one example of a neutralization representation (Stephan and Akyuz 2011). It provides a simple rule for integer addition in which pairs of negative (red) and positive

[1] See http://connectedmath.msu.edu/CD/Grade7/Chip/index.html for more information.

(black) chips are removed (because $^{-}1 + ^{+}1 = 0$, according to the additive inverse property), leaving the sum. For example, representing $5 + ^{-}3$ with chips would involve placing five black chips and three red chips on a mat, pairing up 3 black with 3 red chips, and removing these pairs, which leaves two unmatched black chips—thus, the answer of $^{+}2$.

Similar to the vector interpretation of the number line, to subtract using this representation, students often have to artificially add pairs of positive/negative chips (which sum to zero and preserve the value of the minuend), a move that is not intuitive. For example, for $5 - ^{-}3$, a student has to create the minuend (5) with chips; given that subtraction corresponds to "taking away" and there are no negative chips to remove, the student must add at least 3 pairs of positive/negative chips so that 3 negative chips may be removed.

Limitations of this representation include that positive and negative numbers are differentiated only by chip color. Also, students may become confused by the idea of neutralization, seeing 3 black chips and 3 red chips as 6 chips without understanding that a quantity of chips can represent zero (Steiner 2009).

Money

In the money context, the positive sign refers to assets, while the negative sign corresponds to debt. Addition may be performed similarly to the chips model with a positive value (asset) canceling out a negative one (debt). For example, $5 + ^{-}3$ means that someone has an asset of \$5 and a debt of \$3. Three of the \$5 in assets cancel out the debt, leaving \$2 in assets. Subtraction, on the other hand, is usually defined as taking away either an asset or debt. Translating a problem like $5 - ^{-}3$ into a money context is complicated, and the mathematical problems posed to signify the subtraction of a negative are often artificial. For example, curricula often represent $5 - ^{-}3$ as taking away \$3 in debt ($^{-}3$) from \$5 in assets, which is considered to be the same as increasing one's net worth. However, the answer to the problem *I have \$5 in my piggybank, my father decided to strike the \$3 I owed him. How much money do I have*? is actually \$5 (i.e., the money I already have), not \$8. Thus, the idea of subtracting a negative is often portrayed in inauthentic ways (Schwartz et al. 1993/4), making it challenging for the teacher to present and for students to conceptualize.

In summary, the terrain of representations for teaching integer operations is challenging for students and teachers. There exist different representations, each with its own rules, affordances, and limitations. Teachers must skillfully help students use these representations and, more critically, connect them to the more abstract, procedural symbolic representation. What it takes to successfully undertake this work is the point to which we now turn.

Knowledge demands when teaching integer operations with representations

During the last decade, various theoretical frameworks have been advanced to capture the knowledge needed to teach mathematics (because of space limitations, here we omit reference to older but also important works (e.g., Thompson and Thompson 1996)), such as the *Mathematical Knowledge for Teaching* (MKT) (Ball et al. 2008), the *Knowledge Quartet* (Rowland et al. 2009), and the *Mathematics for Teaching* (Davis and Simmt 2006). All these frameworks recognize the ability to teach with representations as a critical component of teaching mathematics well.

For example, Ball et al. (2008) consider knowing "how to choose, make, and use mathematical representations effectively" a sub-component of specialized content knowledge (p. 400). This includes “recognizing what is involved in using a particular representation” and “linking representations to underlying ideas and to other representations” (p. 400). They also identify the ability to “evaluate the instructional advantages and disadvantages of representations used to teach a specific idea” as a sub-component of knowledge of content and teaching (p. 401).

Similarly, including representations in the “transformation” component of their *Quartet,* Rowland et al. (2009) acknowledge the value of choosing representations to explain and give meaning to mathematical concepts, procedures, or vocabulary, and to confront and resolve common misconceptions. Likewise, Davis and Simmt (2006) articulate the importance of familiarity of various representations to support student understanding of the interconnections that constitute a mathematical concept.

Despite recognizing the importance of representation use in instruction and offering knowledge components related to this work, the above-mentioned frameworks do not provide a detailed account of the tasks and knowledge demands of teaching with representations. This article aims to begin addressing this gap by identifying tasks entailed in teaching with representations, inferring the teacher knowledge needed to successfully engage in this work, and organizing these knowledge components along one particular framework, MKT. As explained in the last section, other works could also be used to taxonomize the knowledge demands needed for teaching with representations, once the tasks involved in this work and their knowledge requirements are identified.

Isolated (often indirect) mentions of components of the work entailed in using representations and the knowledge demands that this work requires can also be found in the literature. Most of these, however, speculate about the knowledge required to teach with representations without necessarily corroborating these speculations with empirical evidence. The few studies that utilize empirical evidence (e.g., Kinach 2002) are not based on analysis of classroom practice. We briefly review these studies for their suggestions about possible knowledge components required when teaching with representations.

Discussion of the “rules” underlying various representations suggests that teachers must understand the representations’ conventions, affordances, and limitations before teaching with them (Kinach 2002; Solomon 1989). Research also recommends that teachers understand the connections between representations and underlying mathematical ideas (Kaput 1985) and be able to structure representation use, so that these connections are highlighted. Given that representations for integer operations do not accurately or completely embody the underlying mathematical ideas (Linchevski and Williams 1999), teachers should be familiar with multiple representations and the connections between them and capable of helping students flexibly move between representations.

Finally, knowledge of students’ understanding, where they typically struggle (Ball 1992), which representations are more accessible to them, and how to justify the integer operations procedures using representations, seems to be a critical component of teacher knowledge. Teachers must carefully select which representations to use, as student learning is influenced by the representations to which they are exposed and some representations reinforce misconceptions. Teachers should also understand that students neither conceive of the material as adults do nor easily make the links between visual representations and analytic thought (Meira 1998; Sacristán Rock 2002; Solomon 1989).

The theoretically based arguments just reviewed point to the need for generating a more systematic list of the task and knowledge components involved in teaching with representations through careful analysis of classroom practice. In what follows, we describe the methods used to begin generating such a list.

Methods

Data collection and procedure for analyzing teachers' use of representations

Following the lead of Shulman, Ball, Rowland, and others, we pursued a two-pronged process (Ball and Bass 2003; Thames 2009), which involves first identifying and naming tasks entailed in teaching and then contemplating the mathematical resources required to successfully carry out these tasks. We first began with the entirety of datasets from two larger projects, one focusing on 10 elementary teachers in two US districts and another including 24 middle-school teachers in a third district. Data in these projects were collected in 2003–2004 (elementary) and 2008 (middle school) by videotaping lessons, conducting interviews, and gathering curriculum materials. Nine videotaped lessons were collected for each elementary-school teacher and six for each middle-school teacher. The videos focused primarily on teachers' moves, limiting our ability to discern student activity. After each lesson, teachers were debriefed using a standardized "post-lesson interview" which gathered information on their perceptions of the lesson and student learning (see Hill et al. 2012).

We watched all lessons on integer operations ($n = 17$) several times. While watching, each author wrote detailed analytic memos (Patton 2002) about the instructional features of each lesson. From these memos, we then identified tasks involved in teaching with representations. To do so, following other scholars' practice-based approach (Ball et al. 2008; Rowland et al. 2009), we asked, "What do teachers *do* as they teach with representations? How do teachers and students *interact* with representations? How does this interaction help (or not help) surface and communicate important mathematical ideas?" We also looked for similarities and differences in instruction across teachers, as these helped us attend to the tasks being displayed or potentially absent.

While pursuing such a practice-based approach, we did not adhere to a completely grounded-theory scheme (Corbin and Strauss 2008), since we were also mindful of what prior research has suggested. This bottom-up and top-down approach—what Grbich (2007) describes as theory directing and theory generating—helped develop a list of preliminary codes that represented entailments involved in teaching with representations. We then compared and refined our codes, merging those that seemed to be describing the same idea. Organizing these codes, we developed a set of broader categories representing instructional features pertaining to teaching with representations. The categories and their sub-codes are presented in the Appendix.

Once we identified tasks entailed in teaching with representations, we inferred the knowledge demands of this work, using Ball et al.'s (2008) question: "What mathematical knowledge, skills, and sensibilities are required to manage these tasks?" (p. 395). As customary in analyzing records of teaching practice (Ball and Forzani 2009), we did not seek to evaluate teachers' work, but attempted to *understand* the resources needed to do this work.

Selection of case studies

Because presenting the volume of data and themes generated from the above work would be impossible, we sampled two teachers—Bonita and Karen (pseudonyms)—whose instruction exemplified the categories resulting from the analysis of the broader corpus of data. Bonita's case provides insight into a teacher thoughtfully struggling with using money, number lines, and chips to represent the mathematical ideas in her lessons. Karen's case serves as a contrast; in her lessons, she largely displayed accurate use of representations and, unlike any other teacher in the larger sample, she attempted to help students generalize the mathematical procedures and ideas involved in integer operations. Because Bonita and Karen help highlight a wide range of tasks and knowledge demands, to some degree, they represent a maximum variation sample (Patton 2002).

At the beginning of the project, Bonita had taught for 10 years. Data collection occurred during the first year she used the *Connected Mathematics Project* (CMP2) curriculum (2nd edition) in the 7th grade. During her initial interview, she reported feeling overwhelmed by having to "learn [this curriculum], process it, and then take ownership of it, and then teach it." Bonita's school enrolled predominantly low-income, Spanish-speaking students. At the time of data collection, Karen had 37 years of teaching experience, and she was teaching 5th grade. Her curriculum was the more traditional *Hartcourt Brace* text. Karen's school enrolled students of low/middle socioeconomic class.

Below, we use the two cases to illustrate core demands of teaching integer operations with representations, as noted in our observations of the entire data corpus. We highlight one key episode regarding addition and one regarding subtraction for each teacher. While using these episodes to point to and discuss the categories (and their codes) generated from our analysis, we also consider the knowledge required for successfully using representations in instruction.

Results

Bonita: episode 1, integer addition

This lesson comes from Investigation 1.4 (*Accentuate the Negative*) in *CMP2* (Lappan et al. 2006). After clarifying that red chips represent negative numbers and black chips represent positive numbers, Bonita begins with the following example: "*Linda owes her sister 6 dollars for her help cutting the grass. She earned $4 delivering papers with her brother. Is she in the red or the black*?" Students disagree about the answer, and, to bring consensus, Bonita projects 6 red chips and 4 blue (in lieu of black) chips via document camera and asks again whether Linda will be in the red or the black. Students unanimously call out "red," and one explains that Linda is in the red "because she's spending more money than what she received." Bonita represents this problem by placing 4 blue chips over 4 red chips, noting that these pairs of chips cancel out. These matched pairs are left visible while Bonita points to the two red chips left over to reveal the answer. A student then reads from the textbook: "*Julia uses red and black chips to model income and expenses. Each black chip represents* $^{+}1$ *dollar of income. Each red chip represents* $^{-}1$ *dollar of income (expenses).*" Bonita elicits students' ideas as to why the answer is two, to which a student responds, "Because you take away six minus four that you won and there are going to be two left that you owe," while another adds that the two red chips left represent debt.

This initial segment shows some of the basic task demands involved in using representations for integer addition. First, Bonita must process and act upon two separate representations: the money context and the chips. Bonita successfully uses one representation to inform the other by setting up the chips, then asking students whether Linda is in the red or black; in this way, students have a concrete representation of the money context and a method for solving similar problems. Second, absent from Bonita's enactment of the chips representation is any reference to a pivotal mathematical idea: that the sum of a

In this task, students must find what is missing and write the corresponding number sentence.		
Start with	**Then**	**End with**
Three red chips	Add five black chips	?
One black and two red chips	Subtract three red chips	?

Fig. 2 Restated version of *CMP*2 Investigation 1.4 task

positive and a negative number is equal to zero. Highlighting this idea would have given more mathematical credence to the chips manipulations and would have set up the work considered in subsequent tasks, including those on integer subtraction.

Third, underlying a student's contribution in this segment could be a misconception which ought to be addressed: a student comments that "you *take away* six" chips, an utterance which conveys the idea of *subtraction* of chips rather than addition. Even though combining red and blue chips eventually results in removing red/blue chips, Bonita could have emphasized the action of pairing up (i.e., adding) chips than that of taking away chips, since the latter is associated with integer subtraction.

This analysis of tasks helps clarify aspects of the knowledge used in teaching integers with representations. First, the teacher needs to know the conventions of each representation, as described above. Besides these conventions, however, the teacher should also know what mathematical ideas to emphasize (e.g., adding a positive and a negative number yields a zero) and how these could be communicated to students when using representations. Additionally, the teacher ought to be aware of certain student misconceptions implicated in using representations (e.g., the idea of "taking away" in the context of addition). Being aware of such misconceptions, the teacher can help students distinguish between the combining and "neutralizing" act from the act of "taking away," which is associated with subtraction. As we shall see in the episodes on integer subtraction, this distinction becomes critical.

Next, the class moves to the main task from Investigation 1.4 (see Fig. 2). This activity presents different situations in which students begin with a set of chips and add or subtract a given amount/type of chips. Bonita first reminds students that red chips represent expenses/losses and blue chips represent gains and then circulates to pass out the chips. She next reads the first sub-task and immediately begins to represent it by placing three red chips on the document camera, without asking students to manipulate their own chips. Soliciting a student idea about how much to add, she then places 5 blue chips as well, and asks: "What am I going to end up? What would my result be?" Some students reply 2; others say 8. A student explains her answer of 8 ("there are 3 red ones and 5 blue ones"), apparently thinking about the *total number* of chips. To remediate this error, Bonita switches to the money context: "You're going to the store and you buy something for 3 dollars; you pay with 5 dollars, and they're going to give 8 dollars back to you?"—to which students answer, "No."

After disproving the proposed answer of eight, Bonita has two girls come to the document camera to demonstrate the answer with chips. While the students are coming up, Bonita circulates to see whether each group has represented the problem with chips, observing that some have done so, while others are still getting started. Once the student pair at the document camera successfully shows their solution with chips (3 red covered by 3 blue with 2 blue left over) and a number sentence ($^{-}3 + 5 = 2$), Bonita extends this example to the number line—a representation employed in previous lessons.

First, another pair of students draws a number line extending from $^{-}3$ to 5 and, prompted by Bonita, marks $^{-}3$ as "the starting point." Next, the students put another point on 2, but again prompted by Bonita, they revise their work and place the second point on 5 to represent the "end point." Bonita then attempts to show how the number line yields an answer of 2, by first simultaneously pointing to both $^{-}3$ and 3 on number line to "cancel" these numbers, then following suit with $^{-}2$ and 2, and finally $^{-}1$ and 1. She concludes that the answer is 2 because the numbers 4 and 5 have not been eliminated. Another student suggests starting at $^{-}3$ and moving five hops to the right, to land on 2. The student demonstrates her solution, but Bonita does not take up this promising idea. Instead, she asks the class if the proposed approach works, and, once the class agrees that it does, she concludes, "There are many ways to show this."

We pause here to again identify tasks involved in teaching with representations. We first note the importance of using accurate language and notation when employing certain representations. Inquiring about the end result, instead of about the *value* of the chips on the overhead, may have prompted some students to talk about the *total number* of chips. This lack of precision surfaces again when Bonita accepts the mathematical sentence $^{-}3 + 5 = 2$, without commenting on a subtle, yet, important, point: that the "+" sign represents the act of addition, and that a more accurate sentence would also involve the symbol denoting that the blue chips are positive (i.e., $^{-}3 + {}^{+}5 = {}^{+}2$).

This episode also speaks to the importance of drawing clear and accurate connections when using more than one representation. We see two manifestations of such connections: in the first case, such connections are legitimate and support student understanding; in the second, they are inaccurate and cause confusion. When recognizing students' difficulty with determining the answer, Bonita resorts to the money context to remediate the misconception, a move that seems to have supported student learning. However, Bonita's use of the number line was unsuccessful, as she attempted to use two representations in exactly the same manner by canceling out numbers on the number line as if they were chips.

Bonita's facilitation of student work in this segment also helps highlight what is involved in supporting students when using representations. Two such pieces become evident here largely because of their absence. The first relates to the importance of giving students sufficient time and opportunities to manipulate the materials and explore how they can be used to re-present mathematical ideas. As Meira (1998) reminds us, these representations might be transparent to the teachers who are initiated to them, but they are not equally transparent to students (e.g., chips are typically counted, not canceled out). The second component pertains to carefully listening to students and unpacking their contributions. This unpacking is necessary, so that teachers understand how students themselves are making sense of these representations and ultimately of the mathematics itself.

What knowledge resources would a teacher need to successfully undertake the tasks outlined above? First, this segment reemphasizes the importance of knowing and communicating the representations' conventions by using appropriate language and notation. For instance, in the chips representation, a teacher needs to clearly distinguish between actions on chips ("+" and "−" correspond to adding "−" and subtracting chips) and the chips value ("$^+$" for positive numbers/blue chips and "$^-$" for negative numbers/red chips). More than that, the teacher should be aware of the conventions governing the use of different representations: "canceling out" might be appropriate for chips, but it is unfounded when it comes to number lines, unless, potentially, when using a vector-based approach (see Fig. 1b).

With regards to employing different representations, this segment also points to the importance of knowing how each representation can help students overcome certain misconceptions. When Bonita employed the money context to convince students that the final answer could not be eight, she was drawing on her knowledge that this representation could make mathematics more accessible to students—an idea she articulated during the post-lesson interview ["it is easier (for students) with the money" because "they can see it more clearly"]. Her recourse to this representation also suggests knowledge of the affordances of different representations and of how one representation can be used to remediate student misunderstandings when employing another representation. Additionally, the struggles Bonita faced when working on the chips and the number line suggest that flexibly working across representations requires an understanding of connections between these representations and an awareness of how these representations in isolation and in tandem can be used to surface and discuss important mathematical ideas. Finally, as already mentioned, the teacher needs to remember that representations are not transparent in and of themselves, to give students sufficient time for exploration around representations, and to closely hear student productions for what these might suggest about student learning/ struggles. Those components are more apparent in Karen's work discussed next.

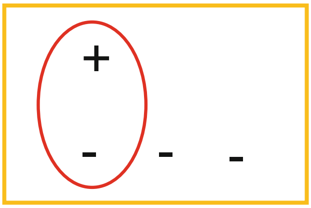
Karen, episode 1: integer addition

At the beginning of the lesson, Karen reviews absolute values and opposites of integers. She then sets up the lesson by telling students they will be using positives and negatives (plastic $+$ and $-$ symbols, possessing the same conventions as chips), along with the number line they used the previous day. She begins by asking, "If I had a positive 1 and a negative 1, what would I have?" and eliciting answers from different students. As there is some debate over whether they would have 1 or 0, Karen asks students to start at $^+1$ on their number line and then to move $^-1$. She asks what moving $^-1$ means, and a student replies that it is the same as moving backwards 1. She then has them note the ending location, which is zero, before summarizing, "Does everybody believe me when I tell you that a positive 1 and a negative 1 are the same as 0?" After this, she asks students to represent $^+1 + {}^-1$, and then $^+3 + {}^-3$, with the chips, pairing up positives and negative to make zeros.

Having established that a number and its opposite add to zero and having represented this with chips, Karen asks students to represent $^+1 + {}^-3$ with their chips, as she does on the overhead (see Fig. 3, below), asking what the circled amount is worth (0), then what is left ($^-2$). With this sample problem, as with those above, the students represent exactly what Karen does with the chips. Karen also has students do this same problem on the number line, telling them to put their fingers on $^+1$, then move them 3 to the left, to show the result is identical for both representations.

We pause here to think about the tasks evident in Karen's use of representations. For one, Karen acts upon both the chips and number line representations in accordance with their conventions. For example, unlike Bonita in the previous episode, Karen meticulously links the first addend to the starting value on the number line, the second addend to moving on the number line, and the result to the end point on the number line, while also clarifying

Fig. 3 Karen's overhead chips work



what moving $^{-}1$ means. Additionally, we see Karen employing and connecting two representations by verifying the conventions for using chips via the number line. Importantly, she also communicates the mathematical idea underlying the integer addition and subtraction algorithms: that $^{+}1 + {}^{-}1$ sums to zero. Karen also generally uses mathematical notation precisely, as suggested by the fact that she uses different symbols for the operation and for the signs of the two addends.

Karen's work in this segment surfaces another task involved in using representations: carefully choosing and sequencing examples to highlight key ideas, making them more transparent to students as they interact with each representation, and gradually building mathematical complexity. Karen starts with $^{+}1 + {}^{-}1$ to ensure that students understand an important mathematical idea just discussed. Before assigning a more challenging example ($^{+}1 + {}^{-}3$), she selects $^{+}3 + {}^{-}3$, to reinforce and solidify the idea that several pairs of negative and positive chips still yield zero. Additionally, Karen's concurrent use of both representations could help students gradually detach integer operations from certain representations, thus facilitating students' shift to a more abstract level, as we shall see below. Karen also skillfully scaffolds student work with each representation—important, as these representations are not intuitive.

To successfully engage in the tasks outlined above, several pieces of knowledge are required. First, this portion of instruction, like the previous episode, highlights that teachers' work requires knowledge of each representation's conventions. Also needed is knowledge of the mathematical ideas to be emphasized and how these ideas can be surfaced and communicated to students using one or more representations. Karen's use of examples reflects not only mathematical knowledge, but also knowledge of choosing and sequencing examples to illuminate the underlying mathematical ideas (i.e., knowing the mathematical importance of $^{+}1 + {}^{-}1 = 0$, that $^{+}3 + {}^{-}3$ is a more generalized version of this idea, and that $^{+}1 + {}^{-}3$ uses this idea with chips remaining, making it slightly more complex).

After providing some more integer addition practice problems ($^{-}2 + {}^{-}3$, $^{+}4 + {}^{-}3$) and having students imitate her solution process, Karen asks students to solve about 15 problems and prompts them to look for patterns. While they work, Karen circulates to remind them of the chip procedure. She also requires that students use their representations to explain their answers to the problems. For example, Sam (all student names are pseudonyms) knows the answer to a given problem ($^{-}3 + {}^{+}2$), but cannot show it with the chips. Karen insists that he does so to "explain why his answer is negative one." Also, during this segment, we see one student quickly solving problems without chips, while most students replicate Karen's use of chips to carry out all given problems.

After this time for student practice, Karen writes the mathematical expression $^{+}n + {}^{-}n =$ on the overhead and asks, "If I add two numbers that are positive, positive n plus positive n, am I going to get a positive or negative for my answer?" Students reply, "positive." She then asks about adding two negatives ($^{-}n + {}^{-}n =$), and students declare the answer will be negative. Finally, Karen asks what happens if she adds a positive and a negative number ($^{+}n+^{-}n=$). She calls on a student to share her thinking, who argues that “If it’s the negative one that’s higher, then it will be a negative. And if the positive number is higher, then it’ll be positive.” After having another student repeat this, Karen restates that, “whichever one is higher that’s the one we’re going to use the sign of,” giving an example of positive 7 plus negative 4, followed by negative 7 plus positive 4. In the remainder of the lesson, students are given time to practice these rules, but to also link them back to the two representations they have been using.

The teaching task described here relates to generalizing a mathematical procedure so that students eventually function autonomously and independently of the representations used (cf. Ball 1992; Goldin 2003). Despite its centrality, this work was not observed in any of the lessons in our larger dataset. To facilitate this work, Karen first affords students the opportunity to work on different problems representing all possible situations: adding two positives, adding two negatives, and adding a positive and a negative that yield either a positive or a negative sum. When eliciting the generalizations from students, we again see skillful sequencing, in that Karen begins with the two simpler conditions (adding two numbers with like signs) and ending with the more complicated unlike-signs condition. We also note the careful sequencing of examples ($^{+}7 + {^{-}4}$ and $^{-}7 + {^{+}4}$), which allows students to see how the sign differences influence the sum; of course, considering all four variations ($^{+}7 + {^{+}4}$, $^{-}7 + {^{-}4}$, $^{-}7 + {^{+}4}$, $^{+}7 + {^{-}4}$) might have better supported student understanding.

At this point, however, her work also surfaces the importance of using precise language, something that Karen has not systematically done in this lesson. While likely comprehensible to students, the expression “higher number” was not mathematically precise. Karen could have instead used the idea of absolute value reviewed at the beginning of the lesson to increase the precision of the generalization she co-constructed with students. The opportunity that Karen gave students to apply the generalizations and connect them back to the representations, however, likely solidified the generalizations developed in the lesson.

Turning to the knowledge demands that the tasks of generalizing and language precision impose on teachers, we note that a teacher must first be aware of the importance of scaffolding students to gradually detach their thinking from representational aids to develop more abstract understanding. The teacher must also know what it means to generalize to develop an algorithm—something that our analysis suggested was unclear to some of the teachers in our larger sample. For example, to generalize, a teacher must assign multiple problems of each type to properly attend to patterns. The teacher must also know how to phrase a generalization so that it clearly captures a mathematical operation in a manner both comprehensible to students *and* mathematically valid. If this generalization is co-constructed while interacting with students, the teacher additionally needs to know how to revoice students’ ill-formed generalizations to render them more mathematically appropriate. Doing so requires not only knowledge and use of precise language, but also the ability to see across and connect different lesson activities (e.g., connecting this work to the review of absolute values), so that they form a coherent mathematical story.

Although we are not sure whether Karen took notice of different patterns in students’ interactions with representations, the video footage suggests that the teacher also needs an awareness of the fact that students reach this abstract thinking at different paces: some might need several opportunities to work on representations; for others, representations might become more quickly transparent and move to abstraction more rapidly. Equally important is the teacher sensitivity and inclination to check whether students are making meaning when using representations or if they are simply imitating the steps shown by the teacher. In Karen's work, this inclination was evident by her constantly pushing students to explain why their answer to a given problem made sense.

The next two episodes consider Bonita's, then Karen's, work with integer subtraction. Teaching integer subtraction with representations is significantly more demanding than integer addition. Consequently, the episodes recounted below lend themselves to illuminating additional tasks and knowledge demands imposed on teachers when using representations.

Bonita, episode 2: integer subtraction

After working on the first activity shown in Fig. 2, the class moves to the second problem: starting with two red chips and one blue chip and subtracting three red chips ($^{-}1 - {}^{-}3$). Bonita models this task on the document camera by laying out the chips representing both the minuend and the subtrahend and writing a subtraction sign between them (see Fig. 4). She then challenges students to solve it.

Working in groups, the students determine a variety of answers for this problem ($^{-}1$, $^{+}2$, $^{+}5$). Perhaps because students disagree on the answer, Bonita returns to the money context and asks, "I have a dollar in my purse and I tell my mom, look, 'I want to buy something … that's going to cost me […] two dollars.' How much am I going to owe mom?" She then continues: "A dollar, right? And then I see another thing and I say, 'Oh man, I want to buy this and it costs three dollars.' How much am I going to owe mom?" Students offer various answers (4, 5, $^{-}4$). Bonita calls a student named Jenny to the document camera to explain the most popular answer, negative 4. Using the representation shown on the document camera (Fig. 4), Jenny matches a blue and a red chip, which leaves four red chips unmatched. Prompted by the teacher to write a number sentence to represent this problem, Jenny writes $^{-}2 + {}^{+}1 + {}^{-}3 = {}^{-}4$ to represent the whole transaction.

Here, we see some of the tasks entailed in representing integer subtraction with chips. To appropriately use the chips to model the "take-away" interpretation of subtraction, a teacher would need to abide by the representation's conventions. At the same time, she would need to clearly discuss the mathematical challenge which calls for re-representing the minuend by adding pairs of blue/red chips, emphasizing that this does not change the starting value of $^{-}2$—thus conveying another key mathematical idea: the preservation of the minuend. Clearly, Bonita's setup of the problem eliminates any opportunities for recording and communicating this key mathematical idea—an integral task in effective representation use.

Her setup also suggests that Bonita was not familiar with the conventions of this representation when it comes to integer subtraction. This lack of familiarity is illustrated in at least two respects: instead of *performing* the act of taking away chips, Bonita *depicted* this act by simply putting the subtraction symbol between the two sets of chips. Related to that, instead of representing only the minuend, as the "take-away" meaning of subtraction dictates, she represented both the minuend and the subtrahend, something that calls for the comparison notion of subtraction. Both these features of Bonita's work point to another key element in using representations successfully: that representations should be employed as tools for re-presenting operations rather than as means for simply presenting symbol manipulations and depicting final answers.

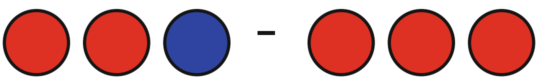


Fig. 4 Bonita's modeling of the second task with chips

Jenny's work at the end of the segment offered Bonita the opportunity to reconsider the chips' conventions. Although leading to an incorrect solution, Jenny correctly utilized the conventions she had been taught and wrote a correct mathematical sentence that represented the scenario suggested by the chips *as placed* on the document camera. To correctly assess this student's work, Bonita would have needed to examine whether the student adhered to the representation's conventions and whether her mathematical sentence was consistent with how she manipulated the chips—a critical component of scaffolding student work with representations. Had Bonita engaged in this work, she might have realized that the flaw did not lie within Jenny's work, but actually within the original modeling of the problem.

Bonita's recourse to the second representation could have offered a solution out of this deadlock and had the money context easily lent itself to representing integer subtraction; unfortunately, this was not so. While the first part of Bonita's story does correspond to the minuend (owing mom a dollar), asking mother for another three dollars corresponds to *adding* instead of subtracting a negative 3. This prevented Bonita from using the money representation effectively and flexibly moving between representations to support student understanding, a task she successfully undertook when teaching integer addition.

Bonita's struggles in this segment reinforce the importance of knowing the representations' conventions as discussed in earlier episodes. Additionally, Bonita would have needed to know the affordances and limitations of each representation, so that she chose and used these representations more successfully. For example, the chips representation lends itself better to distinguishing between two seemingly similar, yet significantly distinct symbols: the "$-$" symbol as an operation and the "$^{-}$" symbol as the value of a number. Similarly, the money context might be particularly useful when discussing integer addition, but, as discussed, it becomes convoluted when applied to integer subtraction. Paraphrasing Diezmann and English (2001), Bonita needed to have developed *representational literacy*: knowledge of *which* representation is most appropriate in a given situation, knowledge of *why* it can be used to illuminate certain mathematical ideas, and knowledge of *how* it can be used to this end. This representational literacy would have also enabled Bonita to closely attend to and unpack Jenny's work, identifying that Jenny was, in fact, abiding by the chips' conventions.

After Jenny's contribution, another student points out that subtraction is not modeled, and Bonita admits confusion. Consulting the teacher's guide, she discovers that the correct answer is $^{+}2$, and abandoning the chips, she correctly demonstrates how to solve the problem using the standard algorithm (i.e., "a negative times a negative gives a positive"). Finally, she asks the students to determine how to use the chips to represent the solution. While it is hard to see what the students are doing, most of them seem to be either "canceling out" the red and blue chips as before or looking completely stuck.

Overall, the episode recounted above exemplifies how lack of understanding of the conventions and the rules governing certain representations can lead the class to unproductive paths. In the episode that follows, we see how a more robust teacher understanding of representations and their conventions can better support students' grasp of integer operations.

Karen, episode 2: integer subtraction

Like Bonita, in this episode, we see Karen using chips to represent and solve integer subtractions such as $5 - {}^{-}4$. Unlike Bonita, she uses this work to start generalizing the procedure. To begin, Karen tells students they are starting with positive five, and they want to subtract negative four. Students already have chips in front of them to use, though most seem to be watching the overhead and not touching the chips. She places 5 positives on the overhead and asks a volunteer, Chris, to come forward and perform the subtraction. Chris takes 4 positives away. Karen asks the class, "What did Chris just take away?" A student answers, "4". Karen then replies, "He took away *positive* 4. I said take away *negative* 4," and elicits other students' ideas.

After several minutes of unsuccessful student suggestions, Karen reminds students that a pair of positive/negative chips is worth zero. Adding such a pair on the overhead, she then clarifies that this move does not change the initial value. Next, pretending to be "Chris' bank of zeros," Karen places three more "zeros" on the overhead and helps Chris decide to take away four negative chips, by carefully pointing to the correspondences between Chris' manipulations on chips and the mathematical symbols. For example, when after re-representing the starting value with 5 positives and pairs of zeros Chris attempts to remove 4 pairs of chips instead of 4 negative chips, Karen reminds him that he has to take away four negatives, not zeros. Eventually, Chris successfully removes $^{-}4$ chips, and the class concludes that the answer is $^{+}9$.

Two main tasks are evident here. The first is what Zopf (2010) describes as "provoking the stumble"—designing a problem that elicits common student confusion. Karen possibly had predicted that Chris would take 4 positives away, a hypothesis supported by the fact that she has selected numbers for which there are sufficient chips in *quantity*, but not in value (i.e., color), to remove. In other words, Chris can remove 4 chips from the 5, just not 4 negative ones. With an example like $^{+}5 - {}^{-}6$, students might immediately see that there are insufficient chips to remove 6, not making this particular mistake. This seemingly purposeful selection of examples was a teaching move we observed in other teachers in our larger dataset (Charalambous et al. 2012) and was critical for surfacing and addressing a common student misconception. The second task relates to the fact that the setup of the problem created a space for emphasizing and reinforcing two fundamental mathematical ideas: that a positive/negative pair is equal to zero and that adding such pairs does not alter the starting value.

We argue that to successfully engage in the work sketched above, the teacher has to know the mathematical ideas that undergird the chips manipulation and, equally important, be able to communicate these ideas to students in a comprehensible manner. She also needs to know how students might struggle with these ideas, and how to skillfully select examples that provoke the stumble for students. Karen's interactions with Chris also surface the importance of the teacher's capacity to carefully follow students' chip manipulations and connect them to their mathematical underpinnings. Successfully engaging in these interactions also requires knowledge of the representation and its conventions and capacity to steer discussion toward important mathematical ideas that the representation can illuminate.

Turning again to the episode, we observe Karen challenging students by asking, "How could [Chris'] answer be 9 if he started with 5 and he's taking away?" A student, James, responds, "subtracting a negative is basically adding a positive, because in algebra, a double negative is a positive." Karen initially leaves James' comment aside, offering practice problems for students to work through (e.g., $^{+}6 + {}^{-}4$; $^{+}6 - {}^{-}4$), before returning to ask students whether they believe James' thinking. To support a more grounded answer, Karen then asks students to work on several more practice problems, including both $^{-}4 - {}^{-}4$ and $^{-}4 - {}^{+}4$; she eventually, revisits James' claim to emphasize that mathematicians need to understand, otherwise if they forget, they "couldn't prove it" to themselves.

Because some students have difficulties with integer subtraction, some of the practice problems are done with the teacher manipulating chips on the overhead and students manipulating chips at their desks (e.g., "make -3, now subtract 4, do you have enough, what do you need to add, now can you take away, what is left?"), others are given as independent work. During independent work time, one can see a couple of students using the number line; most students are using the chips and imitating the teacher's exact procedure, with the teacher providing hints for those who are stuck.

In this segment, we see Karen posing a question to trigger and immediately remediate a common student misconception. By asking how, while performing a subtraction, Chris ended up with more positive chips than those with which he started, Karen ensures that even students who are not alert to a seemingly controversial fact are confronted with it: when performing a subtraction, one could end up with more chips than the starting ones. To remediate this misconception, Karen again engages in the task of skillfully selecting and sequencing examples by asking students to consider both an addition and subtraction with exactly the same numbers. She apparently selects these examples to help students see the difference between adding and subtracting ($^{+}6 + {}^{-}4$ and $^{+}6 - {}^{-}4$). This judicious selection of examples is also shown when she picks $^{-}4 - {}^{-}4$ and $^{-}4 - {}^{+}4$ to direct students' attention to instances with and without sufficient chips to remove. We also see that Karen, unlike Bonita, made sure students were familiar with the chips' conventions before releasing them to interact with the representations independently. Upon circulating the room, she seemingly assessed each student's level of understanding and provided pointed questions to assist those who seemed confused.

Perhaps more importantly, this segment (alongside others considered thus far) demonstrates an important function of the work around representations: unpacking and decompressing mathematical procedures to help students see their underlying concepts, before these procedures are eventually condensed and later turned into what Sfard (1991) calls "reified objects"—mathematical entities that are detached from the processes which led to their development. This is clearly captured in Karen's interaction with James. James' statement is a generalization at which students are ultimately expected to arrive. Yet, Karen does not directly venture into this generalization. She wants students to sit with James' promising production, and she revisits it only after the class has considered several supportive examples. In the post-lesson interview, Karen explains that not immediately providing the rules for integer operations, but rather having students try practice problems to inductively generate these rules was a deliberate move on her part; she contended that anchoring these rules by the manipulation of representations and their underlying mathematical ideas supports student understanding and recalling.

Karen's whole-class or individual interactions with students also point to the importance of skillfully scaffolding students' manipulations of representations, especially when the latter become less transparent. *Adding* chips—in order to then enable subtraction—appeared to be counterintuitive to several students who struggled with such practice problems. To support student work, the teacher might need to give students time to interact with the representations and pose pointed questions that will guide students' manipulation of representations, just like we saw Karen doing in this segment, but also listen to the ways students are making sense of the representations and the mathematics and consider ways to challenge student misconceptions.

Strong and deep knowledge is required to help students understand mathematical procedures and their underlying mathematical ideas, but also to gradually steer students to an abstract mathematical generalization. As we have seen in other segments discussed above, it requires knowledge of selecting and wisely sequencing examples, which, as

Rowland (2008) reminds us, should be suitable for and compatible with the representation at hand (e.g., the money context might overburden students' memory with the additional contextual information if used to develop generalizations). Apart from selecting these examples, it also requires knowledge of students' common struggles—especially when representations become less transparent for students—and of how to trigger them, should such misconceptions not arise while interacting with students. Additionally, this work requires knowledge of how long students should be afforded the opportunity to practice with representations before they eventually move to a generalization—or in Ball's (1992) thinking, before they remove the training wheels and cycle autonomously on their mathematical bicycles. But above all, it requires an understanding of why representations are used, an idea eloquently captured in Karen's meta-level talk: to support understanding and help students prove ideas and procedures that, in the long run, will be condensed and automated.

The preceding four episodes help illuminate from different, yet complementary, angles the task and knowledge demands entailed in teaching integer operations with representations. In the next section, we summarize and organize these demands, while also suggesting implications for teacher educators and curriculum developers.

Discussion and conclusions

We combined results from prior research with a practice-based approach based on analyzing instruction to generate a list of tasks entailed in teaching with representations. Using these tasks, we then described the knowledge required by teachers when using representations in teaching integer operations. Because integer operations are challenging to teach with representations and no representation can satisfactorily capture the mathematical ideas inherent in integer operations, this topic offered a magnifying lens for examining the tasks and knowledge demands imposed on teachers when engaged in this practice. It is in these two areas—identifying the tasks entailed in and the knowledge components required for teaching integer operations with representations—that we see the main contribution of the work reported here.

The Appendix shows the results of this work; this task compilation is likely neither exhaustive nor comprehensive. It captures, however, a significant portion of the work required when teaching with representations. Even without being comprehensive, this list is, we believe, helpful if we are to start developing a framework for analyzing, understanding, and ultimately improving, instruction (cf. Grossman and McDonald 2008; Lampert 2010).

The second contribution of the article lies in identifying knowledge components conducive for successfully teaching with representations. Several scholars (e.g., Flores 2002; Kinach 2002; Linchevski and Williams 1999; Solomon 1989) have explicitly or more implicitly suggested elements of such knowledge across many different articles; yet these elements had not been combined and codified into a formal record. Further, the original work in this field rarely reflected insights into empirically grounded evidence. To offer a more coherent structuring of these components than what already exists, we next use MKT as an organizing framework to synthesize our and others' findings. Other works, such as the *Knowledge Quartet* (Rowland et al. 2005), and the *Knowledge of Teaching Mathematics* (Tatto et al. 2008) or combinations thereof (e.g., Clivaz 2013) could also be employed. We opted, however, to use the MKT framework with which we are more familiar. Hence, what follows could be seen as an *exercise* of using and elaborating a teacher-knowledge conceptualization with respect to a certain teaching practice—that of using representations—through close scrutiny of actual instruction.

In their work on mathematical knowledge for teaching, Ball et al. (2008) list aspects of teaching with representations under Specialized Content Knowledge (SCK) and Knowledge of Content and Teaching (KCT). To this, we add aspects of teaching with representations to Knowledge of Content and Students (KCS); we also point to other sensibilities that a teacher ought to have when using representations. We describe each in turn.

SCK

Much of the literature, including the MKT work of Ball et al. (2008), suggests the importance of *understanding the conventions of each representation*. This is also demonstrated by Bonita's struggle with the number line for addition and chips for subtraction. *Knowledge of the important mathematical ideas that govern the use of each representation and which should be communicate to students while using representations in instruction* is also mentioned in reports of research on teaching with representations. Karen's use of the chips and number line to validate the additive inverse property serves as an example of the presence of this knowledge. Additionally, our work points to the *knowledge of the connections that can be made between and among representations to forge students' learning of important mathematical ideas*; this piece of knowledge is clearly manifested in Bonita's use of money to reinforce the chips' work for integer addition, but is absent in her attempt to do so with integer subtraction. Absent in the instruction of most of the sampled teachers were also two knowledge components: *knowledge of how to gradually decompose and unpack the mathematical rules and operations through the use of representations* and *knowledge of how to use representations to develop generalizations and gradually help students move to a more abstract level of thinking and operating on mathematical symbols*. If the goal of using concrete representations is to build toward more abstract, symbolic representations, as Karen begins to do, this knowledge component—which, incidentally, seems to be absent from the literature—is critical.

KCT

The developers of the MKT framework identify the ability to "evaluate the instructional advantages and disadvantages of representations used to teach a specific idea" as a subcomponent of knowledge of content and teaching (Ball et al. 2008, p. 401). We have distilled this further into *knowledge of the instructional and mathematical affordances and limitations of representations* which supports the *knowledge to determine which representation(s) is/are more appropriate for illuminating certain mathematical ideas and for developing generalizations*, *when used alone or in tandem*. Another knowledge component, seen in Karen's instruction relates to the *knowledge of selecting and using appropriate examples when using certain representations* to lead instruction to a main mathematical point, which also suggests an *understanding of the main mathematical purpose of using representations.*

KCS

Under this category, we noted that both our empirical work and other existing research mentioned *knowledge of key student ideas that can be surfaced and remediated while* *working with representations.* Another knowledge component relates to *knowledge of common errors that students can make when working on representations and how these errors can be used to help students ascribe meaning to representation manipulations.* Both these components appear in Karen's use of $5 - {}^{-}4$ with the chips, as she surfaces the common misunderstanding of the chips' conventions for integer subtraction. Two other components that our work helped surfaced and which have not been identified in prior research relate to an awareness of *the suitability of the language/notation used when working on representations for a specific student population* and *knowledge of how long and in what ways to work on a representation before helping students move to a more abstract level.* For instance, Karen's students worked on seemingly sufficient amount and variety of addition problems before shifting to the more abstract rules for integer operations.

Other sensibilities

In Ball et al. (2008) conceptualization, MKT is not defined in a strictly utilitarian way, but also includes "perspective and habits of mind … that matte[r] for effective teaching of the discipline" (p. 399). Our analyses of teachers' work, alongside prior works, point to a set of sensibilities that teachers ought to have when working with students on representations. These include an (a) *awareness of the fact that representations are not transparent in and of their own*, and therefore, teachers need to afford students ample time and space to experiment on representations and make meaning for themselves; (b) a *proclivity to inquire into students' making-meaning*, for students might be simply imitating the representation manipulations displayed by the teacher; and (c) a *propensity to provide differentiated scaffolding*, since students march at different paces when shifting from concrete manipulations to abstraction.

Although discussed in the context of two teachers' lessons, the knowledge components considered above were also visible or made visible by their absence in the lessons of our larger dataset. While our findings were robust across teachers, there may be more knowledge components that this work has not uncovered. Future studies could therefore build on the mapping of the terrain of knowledge needed to teach with representations attempted in this article by examining teaching with representations with regard to other topics and/or different grade levels.

From a more practice-oriented perspective, our evidence suggests that it is challenging for teachers to use (multiple) representations to teach integer operations. To us, this begs the question: If we think representations are important for supporting student understanding, how do we better scaffold teachers' use of representations?

While this question suggests areas for further research, the knowledge components identified above could also inform in-service and pre-service teacher training programs. For example, teacher education could better support teachers to understand the conventions governing the use of each representation and the mathematical ideas that underlie these conventions, alongside ways to build toward generalizations via representations. Teachers also need to learn the important mathematics ideas to which certain representations lend themselves and how to manipulate/use them to highlight these ideas. Helping teachers become more efficient in "translating" between multiple representations should also be on the agenda of pre-service and in-service teacher education. Discussions of the affordances and limitations of each representation and of how to build from the representation work to the mathematical procedures the representations are intended to illuminate also seem critical, including attention to skillful sequencing of examples. In addition, teachers need to be made aware of the lack of transparency of these representations for students and consider ways to allow students to productively work with these representations.

As this work suggests, teaching integer addition and subtraction might serve as a particularly rich topic for a unit on teaching with representations, for it covers many of the major tasks with representations. Of course, we recognize that pre-service and in-service programs are often constrained in how much they can involve, largely due to time limitations. Therefore, we suggest this work be complemented by curriculum materials. In previous work (Charalambous et al. 2012), we have shown that sufficiently supportive materials can scaffold teachers' work with representations, by detailing the conventions of each representation, helping teachers see the connections between representations and the mathematical ideas their use is purported to highlight, and providing detailed examples of representation use. Our findings here further underscore this necessity, if representations are to be used successfully in instruction to meet the purpose for which they have been developed: support student learning. Based on the work reported in this article, the potential of representations for meeting this goal is more likely to be unlocked by teachers who possess the knowledge components and sensibilities sketched above. Otherwise, representation use may do more pedagogical harm than good.

Appendix: Tasks entailed in teaching with representations

(A.) Representing and solving problems/carrying out mathematical operations

- Recognizing and abiding by the representations' conventions
- Using representations as a means to illuminate certain mathematical ideas involved in a procedure
- Employing appropriate language and notation when using representations
- Decomposing and unpacking mathematical rules and operations through careful use of representations
- Selecting representations that lend themselves to explaining a mathematical procedure

(B.) Creating a context for connecting multiple representations

- Identifying similarities and differences between representations
- Using one representation to help students make sense of another

(C.) Creating a context for generalizing procedures

- Using representations to build generalizations and help students move to a more abstract level
- Selecting and sequencing examples to support student ability to generalize
- Using multiple representations to help students make sense of the underlying meaning of a mathematical procedure

(D.) Scaffolding student work on representations and the mathematics

- Using representations to surface student misconceptions and emphasize important mathematical ideas
- Using representations to trigger and remediate student misconceptions
- Flexibly moving between representations to support student understanding

- Providing a balance between explaining the representation conventions and allowing students the space and time to make meaning of the representations and the mathematical ideas they are intended to illuminate
- Examining whether students correctly follow the representations' conventions and ascribe meaning to the representations' manipulations
- Pressing students to articulate the mathematical meaning they are making out of using representations
- Listening to students and unpacking their (promising) productions around using representations
- Differentiating the scaffolding provided to students depending on (a) the anticipated level of transparency of a given representation and (b) students' differential needs and their progress toward abstracting the underlying mathematical ideas the representations are intended to illuminate.

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