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Upper primary school teachers' mathematical knowledge for teaching functional thinking in algebra

Karina J. Wilkie

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Abstract This article is based on a project that investigated teachers' knowledge in teaching an important aspect of algebra in the middle years of schooling—functions, relations and joint variation. As part of the project, 105 upper primary teachers were surveyed during their participation in Contemporary Teaching and Learning of Mathematics, a research project funded by the Catholic Education Office, Melbourne (2008–2012). Analysis of the survey responses revealed that two-thirds of teachers demonstrated content knowledge on a pattern generalisation task appropriate for upper primary levels of schooling (8- to 12-year-old students), but less than half demonstrated reasonable pedagogical content knowledge (PCK). On a paired variable (function machine) task, only one quarter of teachers demonstrated appropriate PCK. Although two-thirds of the teachers indicated that they currently taught content from the "Patterns and Algebra" strand of the new Australian Curriculum, less than half were able to provide examples of appropriate learning experiences for students. More than two-thirds of teachers expressed concern about their ability to teach this area of mathematics. Implications for the professional learning of teachers to improve their mathematics knowledge for developing students' functional thinking are presented.

Keywords Teacher professional learning · Functional thinking · Content knowledge · Pedagogical content knowledge · Algebra · Mathematics education · Middle years of schooling

Traditionally, algebra has been viewed as a difficult abstract subject (e.g. Greenes et al. 2001; Lee and Freiman 2004) relegated to the secondary years of schooling—"late, abrupt, isolated and superficial" (Kaput 2008, p. 6). The practice of teaching arithmetic in the early years of schooling and postponing algebra teaching can create significant obstacles to

K. J. Wilkie (✉)
Mathematics Teaching and Learning Research Centre, Australian Catholic University, 115 Victoria Parade, Fitzroy, Locked Bag 4115, Melbourne, VIC 3065, Australia
e-mail: karina.wilkie@acu.edu.au

further learning in mathematics (Kieran 2004), particularly in those areas, such as Calculus, where the ability to reason algebraically is important. The teaching and learning of algebra is considered "a major policy concern around the world" (Hodgen et al. 2010). Resistance to algebra in the secondary years of schooling might be reduced if arithmetic and algebra were not misconceived as distinct or disjoint subjects and if students were able to develop algebraic thinking at early levels of schooling (Cai and Moyer 2008; Carraher et al. 2006).

Teaching algebra has often taken the form of symbol-manipulation techniques which can promote a narrow and instrumental understanding of algebra, rather than algebraic thinking which pervades all dimensions of mathematics. Yet, how do teachers, who themselves were schooled via narrow procedural approaches to algebra, develop the ability to "teach a more powerful and general mathematics for understanding"? (Blanton and Kaput 2008, p. 361) The challenge is to "create a body of knowledge that is learnable and useable by teachers" (Stacey and Chick 2004, p. 18) which includes an understanding of student conceptions and effective teaching strategies. Attention to teacher professional learning is needed, both for beginning and experienced teachers (Lins and Kaput 2004). The issues of teachers' content knowledge and their awareness of students' difficulties in learning algebra are of increasing importance (Saul 2008). Ball et al. (2008) similarly refer to aspects of "subject matter knowledge—in addition to pedagogical content knowledge (PCK)—that needs to be uncovered, mapped, organised and included in mathematics courses for teachers" (p. 398).

The aim of this project was to investigate teachers' current knowledge in teaching an important aspect of algebra in the middle years of schooling—functions, relations and joint variation. Its purpose was to consider how to provide Australian upper primary school teachers with targeted professional learning on developing their students' functional thinking. The need to consider strategies to achieve this arose in the context of the recent introduction of a national curriculum in which the teaching of algebra is explicit right from the early years of schooling, and pattern generalisation is a key focus for the upper primary years (8- to 12-year-old students) (Australian Curriculum Assessment and Reporting Authority 2009). A large-scale research and professional learning project provided the means to focus on practising upper primary teachers and their knowledge of algebra for these year levels. Called Contemporary Teaching and Learning Mathematics (CTLM), the project was conducted by the Mathematics Teaching and Learning Research Centre at the Australian Catholic University over 5 years (2008–2012) and funded by the Catholic Education Office, Melbourne. It involved teachers from 82 Catholic primary schools in Victoria who each participated for a 2-year period. The project's major aim was to enhance teacher PCK in mathematics. The study described here is a sub-project of CTLM that focused on the knowledge of upper primary teachers in a specific domain of mathematics (algebra). The central research question of this sub-project was what is the nature of teachers' current mathematics knowledge for teaching functions, relations and joint variation at upper primary levels of mathematics? It is intended that insights gained as a result of this research will contribute to accessible strategies and resources that support teachers' professional learning in algebra, address effective implementation of the content and proficiency strands of the new Australian Curriculum for algebra and support students' continued learning of algebra to prepare them effectively for learning at secondary levels of schooling. Thus, implications of teachers' current knowledge and suggestions for the professional learning of functions, relations and joint variation will also be discussed.

The following section presents details on the context for the project by situating the aspect of algebra that is the focus of the study—functions, relations and joint variation—

and by providing an overview of the knowledge that the literature considers teachers *ought* to have for teaching it, using a theoretical framework for different types of knowledge.

Context and background

Algebra is foundational to mathematics, and the development of algebraic thinking from the early years of schooling has emerged as a central theme in contemporary mathematics curriculum (Greenes et al. 2001). There are differing views on what algebra actually is and what defines algebraic thinking (Kaput 2008; Kieran 2004), but *generalisation* is widely accepted as the cornerstone, the building block of mathematical structure (Kruteskii 1976). It is the ability of a mathematics learner to see the general in the particular (Mason 1996). Generalisation is suggested as the route to “deep, long-term algebra reform” where students learn with understanding and access the power of algebra, rather than memorise symbol-manipulation procedures and (as has happened traditionally) learn to hate algebra (Kaput 1999, p. 134).

The two core aspects of algebra are described as the expression of generalisations using conventional symbol systems and the actions on generalisations (Kaput 2008). Smith (2008) associated these two aspects with different types of thinking—“representational thinking” and “symbolic thinking”, respectively (p. 133). These have been embodied in three strands of algebra, one of which is the study of functions, relations and joint variation (Kaput 1999). It involves a particular kind of generalising: “describing systematic variation of instances across some domain” (p. 13). Usiskin (1988) described algebra as providing “the means by which to describe and analyse relationships” (p. 18). He conceptualised one important aspect of school algebra as the study of relationships among quantities and of variables as quantities that have *variability* (the original meaning of the term *variable*). Such exploration of systematic variation and pattern generalisation can form the basis for later study of functions and the use of variables as arguments (“domain value of a function”) or parameters (“a number on which other numbers depend”). Formulas describing a pattern among variables, e.g. $y = 3x + 5$, lead to function notation, e.g. $f(x) = 3x + 5$, where x is the argument and f is the parameter (p. 14).

Smith (2008) defined *functional thinking* as a type of “representational thinking that focuses on the relationship between two (or more) varying quantities, specifically the kinds of thinking that lead from specific relationships (individual incidences) to generalisations of that relationship across instances” (p. 143). Many real-world applications are modelled as functions, and significant emphasis is placed on functional thinking in mathematics courses in the later years of schooling. Calculus, to which an understanding of functions is foundational, underlies innovation and economic success across many science and engineering domains, and there is a need for expertise in this area of mathematics (e.g. Mullis et al. 2004).

Since there is an ongoing need for such expertise in our society, researchers continue to ask the fundamental question, “What do teachers need to know and be able to do to teach algebra effectively?” To conceptualise the different types of knowledge it is believed that upper primary teachers ought to have for developing their students’ functional thinking, a theoretical framework by Hill et al. (2008) was chosen. It has been used in this article in outlining the literature on teachers’ knowledge of algebra and in discussing the findings from this study. It is briefly described in the following sub-section.

A framework for conceptualising the different types of mathematics knowledge for teaching

Shulman (1986) described a knowledge beyond *content* (*subject matter*) *knowledge* and *pedagogical knowledge* that includes “the ways of representing and formulating the subject that make it comprehensible to others”, an understanding of the conceptions, preconceptions and misconceptions of students of different ages and “knowledge of strategies most likely to be fruitful in reorganising the understanding of learners” (p. 9). This knowledge has been termed PCK and has become part of the research lexicon on teaching and teacher education. In the domain of mathematics, it includes mathematical knowledge but of a different kind to that used in everyday life by adults and to that used in other mathematically intensive occupations (Ball and Bass 2000). Shulman (1986) additionally highlighted the importance of *curricular knowledge*, familiarity with the full range of programmes, instructional materials and tools available for teaching particular concepts at different levels.

Hill et al. (2008) built on Shulman’s (1986) definitions of different types of knowledge to develop more specified descriptions through ongoing efforts to conceptualise, develop and test measures of teachers’ knowledge. In their model, they proposed three types of content knowledge and three types of PCK:

CONTENT (SUBJECT MATTER) KNOWLEDGE

- *Common Content Knowledge* (*CCK*)
- ***Specialised Content Knowledge*** (***SCK***)
- *Knowledge at the mathematical horizon*

PEDAGOGICAL CONTENT KNOWLEDGE

- ***Knowledge of Content and Students*** (***KCS***)
- ***Knowledge of Content and Teaching*** (***KCT***)
- ***Knowledge of Curriculum*** (***KC***)

The four types of knowledge highlighted in bold are those which this study sought to investigate in a survey of upper primary teachers about functional thinking: one type of content knowledge and three types of PCK.

Common content knowledge (CCK) relates to the mathematical knowledge used in everyday life by adults and “is used in the work of teaching in ways in common with how it is used in many other professions or occupations that also use mathematics” (Hill et al. 2008, p. 377). *Specialised content knowledge* (SCK) is still conceptualised as a type of content knowledge but is seen as specialised knowledge that enables teachers to “accurately represent mathematical ideas, provide mathematical explanations for common rules and procedures and examine and understand unusual solution methods to problems” (Hill et al. 2008, p. 378). Both CCK and SCK, however, do not entail knowledge of students or of teaching. In this study, SCK has been used to describe knowledge about pattern generalisation and functional thinking that implies a relational or conceptual understanding of the mathematics rather than only an instrumental or procedural knowledge.

Pedagogical content knowledge in Hill et al.’s (2008) model is divided into three categories. The first type *knowledge of content and students* (KCS) is defined as “content knowledge intertwined with knowledge of how students think about, know, or learn this particular content” (p. 375). Teachers with this type of knowledge are able to attend to how students typically learn a concept and to mistakes and misconceptions that are common. It implies an understanding of students’ thinking and what makes the learning of particular concepts easy or difficult, but does not include “knowledge of teaching moves” (p. 378) which is conceptualised as a second type termed *knowledge of content and teaching* (KCT). KCT entails knowledge about how to *build* on students’ thinking and how to address student errors effectively. The third type of PCK is conceptualised as *knowledge of curriculum* (KC) and relates to Shulman’s (1986) previously mentioned *curricular knowledge*. Ball et al. (2005) additionally emphasised the importance of teachers not only knowing the content of curriculum, but also judging how to *utilise* it to present, emphasise, sequence and instruct. In this study, KC has been used to describe teachers’ self-perceived comprehension of relevant content in the new Australian curriculum as well as their knowledge of how to apply this curriculum content to appropriate learning activities for students.

What teachers ought to know for teaching functional thinking?

The following four sub-sections review the literature about the types of knowledge considered as necessary for teaching functions, relations and co-variation in terms of the four categories of knowledge used to frame this study (i.e. SCK, KCS, KCT and KC). This provides a basis for interpreting these categories in relation to those areas of algebra that are the focus of this study.

Specialised content knowledge (SCK)

Researchers assert that growing patterns “offer a powerful vehicle for understanding the dependant relations among quantities that underlie mathematical functions” (Moss et al. 2008, p. 156). These are also known as *geometric patterns* or as Rivera (2010) preferred to call them, “figural patterns” (p. 298). Early patterning activities are seen as necessary precursors to other types of generalisation in algebra (Greenes et al. 2001). The development of functional thinking is seen as starting with an understanding of linear functions, which extends naturally from counting experiences involving repeated addition (Smith 2008). The *Principles and Standards for School Mathematics* stated that “systematic experience with patterns can build up to an understanding of the idea of function” (National Council of Teachers of Mathematics (NCTM) 2000, p. 37). In the middle years of schooling, it is considered important to know how to:

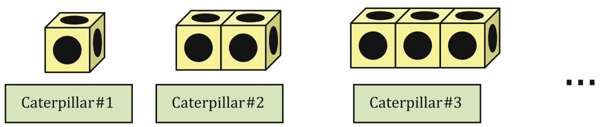
- “Describe, extend and make generalisations about geometric and numeric patterns”;
- “Represent and analyse patterns and functions, using words, tables and graphs”;
- “Represent, analyse and generalise a variety of patterns with tables, graphs, words and, when possible, symbolic rules”;
- “Relate and compare different forms of representation for a relationship”;
- “Represent the idea of a variable as an unknown quantity using a letter or a symbol”;
- “Express mathematical relationships using equations”; and
- “Model problem situations with objects and use representations such as graphs, tables and equations to draw conclusions” (NCTM 2000, p. 158, 222).

The literature describes a number of approaches to pattern generalisation. Stacey (1989) referred to finding the next item in a growing pattern using step-by-step drawing or counting as “near generalisation” and finding the general rule as “far generalisation” (p. 150). Confrey and Smith (1994) described these two ways of approaching functional situations as *co-variation* and *correspondence*. Co-variation describes the relationship between successive items in a pattern—also known as *recursive generalisation* or a *local* *rule* (Mason 1996)—whereas correspondence perceives the relationship between two quantities or variables (the item/term position number in the pattern/sequence and a quantifiable aspect of the item/term itself—also known as *explicit generalisation* or a *direct* or *closed* or *relational rule*). It is this correspondence approach that enables the description of relationships between variables and which Usiskin (1988) saw as an important aspect of school algebra study. Figure 1 provides an example of co-variation and correspondence solutions for generalising a linear growing pattern (the same example used in this study's teacher survey). Figure 2 represents the same growing pattern in a table of ordered pairs.

Knowledge of content and students (KCS)

Despite significant research over the past few decades, algebra in the earlier years of schooling is not considered to be a well-understood field; little is known about students' generalising ability or use of algebraic notation (Carraher et al. 2006). Earlier studies found that students struggled with moving beyond perceiving and describing patterns to generalising them and finding function rules or algebraic representations (e.g. English and Warren 1998; MacGregor and Stacey 1995; Stacey 1989). In the Australian context, students in the early years typically learn about repeating patterns but little if anything about growing patterns (Warren and Cooper 2008). Confrey and Smith (1994) found that students preferred the co-variation approach since it was "easier and more intuitive" (p. 33) and that moving from co-variation to correspondence approaches was a challenge. Warren (2000) also found that students tended to focus on recursive (co-variational) rather than relational (correspondence) strategies. Wright (1997) found that many students lacked strategies for finding relationships between variables and had little awareness of how algebraic symbols could represent such relationships. Kaput (2008) pointed out that one-dimensional patterning activities obscure the variable on which the pattern depends (the item number), thus keeping its structure as a function well-hidden and the students' focus on the relation between consecutive items (recursive thinking rather than relational thinking). Students may even work correctly with a table of values—where the two variables are in fact listed—without reference to the functional dependence between the variables by merely extending each number pattern and still relying on recursive strategies (Carraher et al. 2006).

Although researchers have found that encouraging students to look for a relationship between quantifiable aspects of a geometric growing pattern supported their functional thinking and their ability to write symbolic representations of a generalisation (e.g. MacGregor and Stacey 1995; Markworth 2010; Warren and Cooper 2008), it is still



Co-variation: Caterpillar #1 has 6 stickers and each caterpillar has 4 more stickers than the previous caterpillar– the total numbers of stickers are 6, 10, 14, 18…

Correspondence: Each caterpillar has the same number of stickers on each of its four side lengths as its item number – the total number of stickers is 4 times the item number plus 2 for the end stickers ($t = 4n + 2$)

Fig. 1 Two approaches to understanding functional relationships in a growing pattern

Item position number	Item (e.g., number of stickers)
1	6
2	10
3	14
4	18
5	22

Co-variation: *"When the item position number increases by 1, the item increases by 4."*

Correspondence: *"Four times the item position number and add 2 equals the item."*

Fig. 2 Two approaches to generalising functional relationships using a table of ordered pairs (adapted from Smith 2008, p. 147)

possible to merely follow a procedure that produces the equation but without necessarily understanding its representation of the pattern's visual structure. For a linear functional relationship, such a procedure might typically involve creating a number sequence from the quantifiable aspects of a geometric growing pattern, using the difference between consecutive terms as the co-efficient of the independent variable (often x or n) and then adding or subtracting a constant to create the full equation. It is also possible to use trial and error to find the equation, particularly if the general form is known to be "some number times the x plus or minus some other number". For students, learning to think functionally rather than to follow procedures is considered important for later success in algebra and Calculus.

Knowledge of content and teaching (KCT)

Several teaching approaches to developing students' functional thinking are described in the literature. Confrey and Smith (1994) proposed "the interaction of context, multiple representational forms, and technological tools" as helping students develop functional understanding (p. 32). Teachers help students to generate functional relationships within a context, use multiple representations to create and represent solution processes and therefore think about functions in diverse yet legitimate ways. Kaput (1999) also advocated a multi-representational approach that involves providing students with meaningful experience in familiar contexts and representing situations with diagrams, tables of values, language, equations and graphs. Examples of familiar contexts include: heights of plants or people, temperatures, numbers of people changing over time, and the cost of a product as a function of the number bought. The idea of a function embodies multiple instances, all collected within a single entity (e.g. a list, table, graph), a process that also involves generalising—answering the question, "What is it that all these instances have in common?" (Kaput 1999, p. 146)

MacGregor and Stacey (1995) found that encouraging students to describe the features of a geometric pattern verbally and to then express these algebraically was an important step in learning to recognise a function. Warren and Cooper (2008) referred to the effectiveness of concrete materials in teaching patterns and sequences, and of specific questioning that makes explicit to students the relationship between an item in a pattern and the item's position number and assists them to reach generalisations about unknown positions. Moss et al. (2008) studied Year 4 students and found that certain teaching strategies seemed to facilitate the students' development of functional thinking and the ability to find generalised rules for patterns and sequences. These included: the use of function machines to explore relationships between variables; building geometric patterns with pattern blocks and using position cards to highlight the item position number; and using two colours for geometric growing patterns to represent the constant (the blocks that "stay the same") and the co-efficient (the blocks that increase by a set amount at successive positions—the rate of change or gradient in a graphical representation). Friel and Markworth (2009) provided examples of several types of geometric patterns of increasing levels of complexity. These examples included "linear, direct variation relationship[s]" in which the total number of blocks is a multiple of the item position number ($y = mx$), linear functional relationships which involve the addition of a constant ($y = mx + c$), and nonlinear relationships (p. 30).

Knowledge of curriculum (KC)

The state curriculum prescribed for teachers in this research project was the Victorian Essential Learning Standards (VELS). It referred to upper primary students constructing and using "rules for sequences based on the previous term, recursion (for example, the next term is three times the last term plus two), and by formula (for example, a term is three times its position in the sequence plus two)". It also referred to students being able to "identify relationships between variables and describe them with language and words" (Victorian Curriculum and Assessment Authority 2007).

The introduction of a national curriculum has brought algebraic thinking to the attention of teachers by referring to the content strand "Number and Algebra" and the proficiency strand "Reasoning" right from Foundation to Year 10. At upper primary levels, explicit reference was made in the sub-strand "Patterns and Algebra" to students being able to describe, continue, and create patterns (Year 5) and sequences (Year 6), and to describe the rule that creates a sequence (Australian Curriculum Assessment and Reporting Authority 2009). Teachers in this study were in a period of transition to the newly implemented national curriculum.

Research on what teachers actually know for teaching functional thinking

This sub-section reviews the literature on research about teachers' *actual* mathematics knowledge for teaching algebra. Although there is substantial literature that considers the knowledge teachers *ought* to have for teaching mathematics (e.g. Adler et al. 2005; Ball and Bass 2000; Carpenter et al. 1989; Nathan and Koellner 2007) there seems to be far less literature specifically on what knowledge teachers actually *do* have, and even less literature on the specific domain of knowledge for teaching algebra in the middle years of schooling. A few studies have investigated prospective teachers' knowledge for teaching algebra. Two large British studies ($n = 154$ and $n = 201$, respectively) of prospective primary teachers researched their content knowledge of mathematics, including algebra. Items on generalisation using words and symbolic notation were completed correctly by less than half of participants (Goulding et al. 2002). Another study of 58 US elementary prospective teachers researched their content knowledge in understanding algebraic generalisations and linking symbolic equations to visual growing patterns. The most common difficulties were found to be interpreting what the variables actually represented and identifying the pattern (Rule and Hallagan 2007).

Nathan and Petrosino (2003) researched 48 prospective *secondary* teachers' content knowledge and PCK of students' likely difficulties in algebra. They found that prospective teachers with higher levels of algebra content knowledge were more likely to believe that worded story problems would be more difficult than symbolic equations to solve, the opposite of what was found in empirical research of students' learning in algebra. This has been termed the *symbol precedence view*, as compared to the *verbal precedence view*. They suggested that teachers with advanced content knowledge of algebra but who lack PCK on how novices learn "tend towards views of student development that align more closely with the organisation of the discipline than with the learning processes of students" (p. 906). It seems that content knowledge of algebra is not sufficient in itself for being able to teach it effectively.

A recent large-scale international 6 years comparative study called the Teacher Education and Development Study in Mathematics (TEDS–M) investigated the preparation of primary and lower secondary teachers for teaching mathematics in 17 countries (Tatto et al. 2012). The research investigated the content knowledge and PCK of prospective teachers at the end of their teacher education in four sub-domains, one of which was algebra and functions. Senk et al. (2012) reported on results related to the prospective primary teachers who completed 74 content-knowledge items (29 % algebra) and 32 pedagogical content-knowledge items (multiple-choice and constructed-response formats). They found that those participants who performed at a higher level demonstrated "some familiarity with linear expressions and functions" yet "had limited success applying algebra to geometric situations" (p. 8). Unsurprisingly, they found that prospective teachers who had undertaken courses to become mathematics specialists performed at higher levels on both content-knowledge and pedagogical content-knowledge items than those preparing to become generalist teachers. A report of primary-level released items from the study included two content-knowledge items related to functions, relations and joint variation. Internationally, 77 % of prospective teachers were able to predict the number of matchsticks in the 10th figure of a growing pattern, and 54 % were able to find the rule for the number of people that could be seated around n tables. One pedagogical content item that required the selection of an equation to match a growing pattern was completed correctly by only 31 % of prospective teachers (Australian Council for Educational Research 2010).

A few studies researched practising secondary teachers' knowledge for teaching algebra. Menzel and Clarke (1999) conducted classroom research on secondary teachers' PCK of algebra and reported that it was difficult to find examples of such knowledge from the data set. They noted that each of the teachers observed in the research encouraged rote learning of algebraic procedures and were seldom able to identify specific algebraic concepts with which students were most likely to struggle. They were more likely to refer to "lack of readiness to learn abstract concepts, lack of attention, or lack of practice" (p. 371). The researchers speculated that such algebra teaching could be carried out with minimal PCK and minimal reflection on reasons for students' difficulties. Hadjidemetriou and Williams (2002) compared 12 teachers' PCK of teaching functions using graphical representations with the actual results of their students ($n = 425$). Teachers were asked to judge the difficulty of items, propose a learning sequence and diagnose likely errors and misconceptions. They found that teachers' lack of content knowledge interfered with their judgements and that there was a mismatch between their perceptions of students' difficulties and the actual difficulties demonstrated by their students.

Hill et al. (2008) in their research on measuring teachers' knowledge, focused on KCS. Using multiple-choice items in the domains of Number and Algebra, they surveyed hundreds of US elementary teachers. Although their findings supported their conceptualisation of this type of PCK as distinct from content or pedagogical knowledge, they reported experiencing significant problems in measuring it. They related these difficulties to the multi-dimensionality of "KCS" and to the limitations of multiple-choice items in the survey. They believed that such questions may not distinguish between teachers' use of test-taking skills, mathematical reasoning (SCK), or KCS, in answering items correctly. They suggested that open-ended response items may be more appropriate for researching teachers' PCK, even though considerably more expensive for large-scale studies. They asserted that such topic-specific empirical research is important to further our understanding of mathematics knowledge for teaching.

Research design

Informed by empirically and theoretically grounded research described in the literature, the overall project, of which this article describes a part, sought both to investigate and improve teachers' mathematics knowledge for teaching algebra to upper primary students so as to develop their functional thinking. It aimed to "address both the pragmatic and highly theoretical issues simultaneously" to achieve "reflexivity between theory and practice" (Cobb 2000, p. 308). Since there was little research found in the literature on practising teachers' actual knowledge in teaching functional thinking, an initial survey of 105 upper primary teachers was undertaken as a precursor to an in-depth collective case study of 10 teachers. The teachers who completed the survey were participating at that time in the previously mentioned CTLM project. This included 10–12 full days of professional learning (workshops, professional reading, and between-session tasks) over a 2-year period. At the beginning and end of each year, different cohorts of teachers (Prep to 2, Years 3/4, and Years 5/6) would complete a questionnaire of items developed by Roche and Clarke (2011) during their attendance at the professional learning workshops. These surveys aimed to assess teachers' mathematics knowledge for teaching and to measure changes over time.

Data collection

Early in 2012, 105 upper primary teachers completed a questionnaire on functional thinking developed by the author, and provided data for the findings discussed in this article. The survey explored different aspects of the teachers' knowledge and practice for teaching functions, relations and joint variation and for developing students' functional thinking. The questionnaire was pre-trialled with six upper primary teachers (who were not participants in the CTLM project) to refine the structure and wording of items. The final version used in the study is presented in the Appendix.

Informed by the previously mentioned recommendations of Hill et al. (2008) on researching teachers' knowledge, the questionnaire contained several open-ended response items rather than multiple-choice questions. These items sought data about teachers' understandings in four of the previously described knowledge domains conceptualised by Hill et al. (2008) and are summarised below:

Specialised content knowledge (SCK)

In order to investigate this knowledge relevant to functional thinking, the questionnaire contained open-response items on generalising a geometric growing pattern in a variety of ways, writing a functional relationship between variables using a symbolic equation, and identifying co-variation and correspondence approaches to generalisation in a function machine task. Teachers were asked to provide four possible correct student solutions to questions from a growing pattern task (Appendix—Q3). These responses were intended to give an indication of teachers' ability to generalise growing patterns in a variety of ways. Each solution was analysed and assigned a rubric score from a learning progression for functional thinking (Table 1). Teachers were also asked to compare differently arranged tables for the same function machine (Appendix—Q4) as an indication of their ability to identify recursive and explicit generalisation strategies.

Knowledge of content and students (KCS)

This knowledge relevant to functional thinking included familiarity with the different types and levels of sophistication of possible correct student responses to pattern generalisation, and comprehension of a student's mistake in pattern generalisation. For the caterpillar task, teachers were asked to provide examples of (up to) four possible correct student responses. Besides assessing the *highest level* of their responses as an indication of their SCK, these four responses were further analysed to consider teachers' knowledge of the *range* of different types of possible student solutions, as an indication of their KCS: an awareness of the different types of responses using recursive and explicit generalisation, and different levels of sophistication of likely student responses.

Knowledge of content and teaching (KCT)

This knowledge relevant to functional thinking included addressing a student's mistake in pattern generalisation, and identifying appropriate teaching strategies for exploring functional relationships. Teachers were given a student's description of an incorrect solution for

Table 1 A learning progression framework of the development of functional thinking with growing patterns (adapted from Markworth 2010, p. 253)

1.	Extend a growing pattern by identifying its physical structure, features that change, and features that remain the same (*figural reasoning*)
2.	Identify quantifiable aspects of items that vary in a geometric growing pattern
3.	Articulate the linear functional relationship between quantifiable aspects of a growing pattern by identifying the change between successive items in the sequence (*co-variation* or *recursive generalisation*)
4.	Generalise the linear functional relationship between aspects of a growing pattern by:
4.1	Describing the relationship between a quantifiable aspect of an item and its position in the sequence (*correspondence* or *explicit generalisation*)
4.2	Using symbols or letters to represent variables; or
4.3	Representing the generalisation of a linear function in a full, symbolic equation
5.	Apply an understanding of linear functional relationships between variables to further pattern analysis and multiple representations

generalising the caterpillar growing pattern to find the number of stickers on caterpillar #37:

Caterpillar #37 will have 37 times 4 spots for the top, bottom and sides of the caterpillar, which is 148.

Teachers were asked to describe what they as a teacher might say or do in response to the student. The student has used the four spots per caterpillar body part (also the difference between consecutive numbers in the sequence) but has not included the constant (two spots on each end). To be able to address the student's incorrect strategy, teachers would need to be able to connect the student's multiplication of the number of body parts by 4 to the total stickers on each side of the caterpillar ($4n$) and to attend to the missing constant ($+2$). Teachers' written responses were categorised according to their recognition of the response being incorrect (KCS) and to the level of understanding in effectively addressing the student's misconception (KCT), using a 4-level rubric (Table 2).

Three further items in the questionnaire investigated teachers' knowledge of teaching strategies for developing students' functional thinking. Teachers were asked to indicate from a list of language terms (relevant to functional thinking concepts) those they used explicitly in their teaching, to suggest an appropriate activity involving the use of function machines, and to explain how they might use different types of input/output tables in their teaching to build on students' thinking.

Knowledge of curriculum (KC)

This knowledge relevant to functional thinking included understanding and applying curriculum content to appropriate types of learning experiences for students. Given the recent introduction of the new Australian curriculum, it was considered important to investigate teachers' perceived comprehension of the relevant content on functions, relations and joint variation, and to compare this with their descriptions of how they would/do apply the curriculum when providing learning experiences for their students. Teachers were asked to respond to the wording of the Year 6 content description in the "Patterns and Algebra" sub-strand of the *Australian Curriculum: Mathematics* (Australian Curriculum Assessment and Reporting Authority 2009) using a Likert scale from 1 to 6 ("easy to understand" to "difficult to understand"): "Continue and create sequences involving whole numbers, fractions and decimals. Describe the rule used to create the sequence (ACMNA133)".

In relation to the purpose of the overall project, the questionnaire also sought additional data from teachers on their attitudes towards implementing relevant algebra content in the new Australian curriculum, and on their suggestions for their professional learning. This information was used to inform the subsequent design of an in-depth case study of 10

Table 2 A generic rubric for assessing the level of knowledge about the teaching or learning of functional thinking (adapted from Downton et al. 2006)

1	Evidence of irrelevant response or incorrect understanding of teaching/learning of functions, relations and joint variation
2	Evidence of partial but limited grasp of teaching/learning of functions, relations and joint variation
3	Evidence of understanding of teaching/learning of functions, relations and joint variation, but some key ideas missing or not communicated clearly
4	Evidence of comprehensive understanding of teaching/learning of functions, relations and joint variation, with reasoning clearly communicated

teachers and their experience of a professional development programme over a 1-year period (to be reported elsewhere). Only the data directly relevant to the teachers' mathematical knowledge for teaching functional thinking are considered in this article.

Data analysis

A "descriptive and interpretive" approach (O'Toole and Beckett 2010, p. 43) to data analysis of the survey was implemented. Data were analysed using content analysis, with the use of Excel spreadsheets for quantitative analysis and NVivo 9 qualitative analysis software to support line-by-line coding of responses, the refinement of coding, and the adaptation of themes (Creswell 2007). A process of check-scoring teachers' responses to assess their ability to generalise a growing pattern and to assess their knowledge of co-variation and correspondence (function machine input/output tables) was undertaken by pairs of researchers to increase the reliability of results. A cyclic process of scoring a set of 10 responses to Question 3a and discussing results to reach consensus was undertaken for just over half of the 105 teachers' responses. Development, testing and refinement of the rubric for Question 4b and subsequent check-scoring of a selection of responses to reach consensus were also undertaken by two researchers.

To analyse teachers' responses to the questionnaire items, two different frameworks were used as rubrics for assessing teachers' levels of understanding in each of the four knowledge domains. The first framework was based on research by Markworth (2010) who developed an empirically-substantiated instruction theory on students' development of functional thinking with geometric (or figural) growing patterns. She used a design-based research methodology with students in the sixth grade who were anticipated to have had little prior experience with growing patterns or functional thinking. Markworth's subsequent learning trajectory was adapted to create the learning progression framework presented in Table 1.

This learning progression was used as a rubric in the analysis of teachers' written responses when information was sought about teachers' knowledge related to pattern generalisation and representation of functional relationships. Detailed examples of teacher responses and subsequent scoring using this framework are presented in relevant tables in the Results section. Teachers' responses to questions relating to their knowledge of the teaching or learning of functional thinking were scored using a generic 4-point rubric, presented in Table 2, which was adapted for each relevant item. Each of the tailored rubrics is presented in relevant tables in the Results section.

Results

The results of the teachers' knowledge are presented in terms of the four categories of knowledge used to frame the study as previously discussed, that is, SCK, KCS, KCT, and KC.

Teachers' specialised content knowledge of functional thinking (SCK)

The findings of the teachers' SCK of functional thinking in algebra are presented based on their performance on the caterpillar task and the function machine task. The former focuses on the teachers' ability to generalise in relation to the highest level of understanding appropriate for upper primary grades and the latter on their ability to identify the two types of generalisation (recursive and explicit).

Caterpillar task

Table 3 provides results of the teachers' SCK of functional thinking based on their scores for the caterpillar task. The scores represent the level of the teachers' SCK based on the learning progression framework (Table 1), that is, a score of 1 indicates low SCK and a score of 4.3 indicates high SCK for functional thinking. Students at upper primary levels of schooling are meant to be able to generalise recursively (earlier stage of understanding—score of 2 or 3 on learning progression) and then also explicitly (later stage of understanding—score of 4.1/4.2/4.3). The score of 5 in Table 1 is not applicable for these grade levels and in relation to the caterpillar task, so does not occur in Table 3. It is therefore expected that an upper primary teacher with high SCK for functional thinking should be able to achieve a score of 4.3, that is, in this case, be able to generalise a functional relationship by writing a full, symbolic equation such as "$y = 4x + 2$" for the caterpillar growing pattern. Table 3 also includes descriptions of each score based on Table 1, the percentage of teachers at each level of scores of the learning progression (i.e. their highest level of SCK), and illustrative examples that are representative of their responses.

Based on Table 3, the teachers' SCK ranged across all of the levels of the learning progression with nearly three quarters scoring in the high range of 4.1–4.3. The distribution and nature of the teachers' approaches to the task consisted of the following (percentages are rounded): 6 % of the teachers (score 1) added up the stickers; 10 % (score of 2) wrote numeric sequences (caterpillar number, number of stickers), usually in a table format, and used these to generalise recursively (co-variation); 72 % (score of 4.1, 4.2, or 4.3) were able to generalise the linear functional relationship between the item number in the pattern (the caterpillar number) and the item itself (the number of stickers on the caterpillar) using explicit generalisation (correspondence); 31 % (score of 4.2) used a symbol (typically a letter) to represent their generalisation in an expression, e.g. $4n + 2$, and $6 + 4(n-1)$ where $n =$ caterpillar number; and 2 % (score of 4.3) wrote a full symbolic equation using symbols for both variables, e.g. $s = 4n + 2$ ($s =$ number of stickers, $n =$ caterpillar number). Eight percent were unscored because the solutions were incorrect or inappropriate and 3 % made no response.

These findings indicate that 30 % of the teachers were not within expectations of SCK for functional thinking and only 2 % achieved the highest level. Specifically, 6 % of teachers demonstrated very low SCK (score of 1) and could only extend the geometric pattern by drawing and adding up the stickers. A further 10 % of teachers who used numerical reasoning to create a table of paired numbers or a number sequence also demonstrated inadequate SCK (score of 2) since upper primary students need to be able to generalise explicitly, not just recursively. Just over 70 % of teachers demonstrated at least a reasonable level of SCK (score in the 4 range) which means they were able to make explicit generalisation in some form (words, symbolic expression, or full equation). Finally, 11 % of the teachers demonstrated a complete lack of SCK of functional thinking based on this task and received no score on the learning progression.

Function machine task

The function machine task also sought to investigate teachers' SCK about the two types of generalisation—to see if they could identify recursive and explicit generalisation. The

Table 3 Teachers' levels of SCK (scores) for caterpillar task

Score on learning progression	Description	Percentage of teachers (%)	Illustrative example
1 (low SCK)	Extend pattern by drawing or adding up stickers on caterpillar	5.7	"1 + 4 + 4 + 4 + 4 + 4+ 4 + 4 + 4 + 4 + 4 + 4 + 4 + 4 + 4 + 4 + 4 + 4 + 1"
			"5 + 4 + 4 + 4 + 4 + 4 + 4 + 4 + 4 + 4 + 4 + 4 + 4 + 4 + 4 + 4 + 5"
2	Create sequence or table of quantifiable aspects of caterpillar	9.5	"Caterpillar # 1 2 3 4…
			Stickers 6 10 14 18…"
			"Adding up—skip counting 4 s. 6, 10, 14,…"
3	Explain recursive generalisation by referring to change in caterpillar structure (co-variation)	1.9	"Start with 6 spots and add 4 more for each new caterpillar"
			"Keep adding 4 for each caterpillar"
4.1	Explain explicit generalisation in words or with a calculation (correspondence)	39.0	"Number of caterpillars multiplied by 4 then add 2 for each end"
			"2 blocks on each end have 10 stickers. Take away those 2 blocks from number of blocks then multiply by 4"
			"$17 \times 4 + 2$"
4.2	Represent generalisation using symbols/letters	31.4	"$4\,n + 2$"
			"$(5 \times 2) + (c - 2) \times 4 =$ answer"
			"$6 + 4(n - 1)$"
4.3 (high SCK)	Represent generalisation with full, symbolic equation	1.9	"$(a \times 4) + 2 = b$, where $a =$ no. of blocks, $b =$ no. of stickers"
			"$4x + 2 = y$"
Unscored response	Incorrect or inappropriate student response	7.6	"$(6 \times 17) - (2 \times 17)$"
			"Look for the pattern and develop an algebraic equation"
No response		2.9	

wording for the task was deliberately open-ended to see if teachers were familiar with and able to describe the different representations of the functional relationship between variables (in this case, input and output numbers arranged consecutively and then non-consecutively). It was found that 17 % of the teachers attended to either approach in their written response (mostly co-variation; only four teachers referred to correspondence), and nearly 25 % attended to both approaches. Nearly one-third of teachers made a vague, incorrect or irrelevant response, indicating a very low SCK for this type of task. A further 13 % made a response that could not be scored because it indicated a lack of understanding of the task, and 12 % made no response at all. These findings indicate that only one-quarter of the teachers demonstrated an expected level of SCK about the two types of generalisation. It is likely that this perhaps more abstract type of representation of functional relationships, although used widely in mathematics at secondary levels of schooling, relies on a higher level of SCK to understand the different arrangements of the variables and how they elicit different types of generalisation strategies. Some teachers may have only experienced the use of such tables at secondary levels of schooling for graphing ordered pairs of variables (x, y) rather than for generalising functional relationships.

Overall these results indicate that most teachers were able solve a simple linear growing pattern generalisation using a correspondence approach but less than a third were able to express this symbolically. Less than a quarter of the teachers attended to both co-variation and correspondence approaches to generalising a functional relationship presented as a table of pairs of variables (function machine or input/output table). This indicates perhaps more familiarity with representing generalisations in words rather than with tables or symbolic rules. Or it could be that the geometric growing pattern in the questionnaire was conceptually easier to generalise than pairs of variables in a table of values.

Teachers' knowledge of content and students for functional thinking (KCS)

Table 4 provides results of the teachers' KCS for functional thinking based on the number of possible correct student responses to the caterpillar task they were able to provide—the teachers' knowledge of the *range* of different types of possible student solutions, as an indication of their KCS: an awareness of the different types of responses using recursive and explicit generalisation, and different levels of sophistication of likely student responses.

Approximately two-thirds of teachers provided two, three, or four solutions at different levels of sophistication, which means that they were aware of different strategies students might use when working with growing patterns. This indicates at least some KCS for pattern generalisation, and yet only 18 % gave at least one example of recursive generalisation (co-variation) *and* one example of explicit generalisation (correspondence). It was more likely that teachers would give two solutions that used the same type of generalisation approach, for example an extended picture and then a number sequence, or a rule in words and then as a symbolic expression. This means that over 80 % of teachers did not demonstrate an important aspect of KCS: familiarity with how students learn to generalise patterns and the progression from continuing patterns to using recursive and explicit strategies. Nearly 23 % could make only one correct student response, indicating

Table 4 Teachers' number of different types of student responses for caterpillar task ($n = 105$)

Number of different types of responses (using learning progression levels)	Percentage of teachers (%)
One type	22.9
Two types	44.8
Three types	21.0
Four types	1.0
Unscored response	7.6
No response	2.9

extremely low KCS, and only 1 % of the teachers presented four different correct student responses to the task (high KCS). A further 3 % made no response.

For those teachers who demonstrated KCS about *explicit* generalisation, by far the most common expression related to visualising the structure of the caterpillar was 4 stickers per body segment plus 2 stickers for each end. It was interesting to find that five teachers (just under 5 %) were able to show two *different* ways to visualise the structure of the caterpillar and used each of these to find the rule, for example, 4 *stickers for each block and* 2 *for the ends* ($4n + 2$), and 5 *stickers for each end block and* 4 *stickers for the blocks in between* [$10 + 4(n - 2)$], or 6 *stickers for each block take away the dots on the inside where the blocks join* together [$6n - 2(n - 1)$]. This high level of SCK about the different ways students might visualise the structure of a growing pattern is crucial for interpreting various student solutions.

Teachers were also asked to choose the most mathematically sophisticated of their student response examples and to explain the reasons for their choice. Their responses were analysed and assigned a score using the previously presented rubric (Table 2), as a way of describing their KCS about the *level of difficulty* of the types of generalisation and students' likely learning progression from continuing patterns to recursive and explicit strategies. The results and illustrative responses for each level are presented in Table 5.

It was found that less than 30 % of teachers appropriately referred to correspondence concepts and/or to the use of variables or unknowns—key concepts for developing functional thinking (rubric score of 3 or 4). One-fifth of teachers did not make a written response at all to this particular question. Just over 13 % were unable to identify correctly the most sophisticated response (i.e. lack of KCS). A further 7 % identified the correct response but gave no justification. Nearly 8 % of responses could not be scored because their solutions to the caterpillar task itself were incorrect. Overall, these findings indicate that 70 % of the teachers were not within expectations of KCS for functional thinking and only 6 % achieved the highest level.

Teachers were also asked to *interpret* a student's incorrect response to the task, which is another aspect of KCS, and then suggest an appropriate way to address it, which is considered to be KCT. Although these two aspects are conceptualised by Hill et al. (2008) as different types of knowledge, the focus of the task was intended to be on the teachers' KCT and is discussed in the following sub-section.

Teachers' knowledge of content and teaching of functional thinking (KCT)

Four items in the questionnaire investigated the teachers' KCT, one item about the caterpillar task, one item about terminology for teaching functional thinking, and two items about teaching with function machines.

Caterpillar task

The first of these items asked teachers to make an effective response to a student's error with the caterpillar pattern task (but they were not told that the student response was in fact incorrect). These results are presented in Table 6.

These results indicate that more than 25 % of the teachers were unable to clearly recognise the student's error and therefore were unable to demonstrate an effective response to it. All of those teachers who did clearly identify that the student's solution was incorrect were able to demonstrate at least some KCT in responding to the student. Overall, only 43 % of teachers described a clear and appropriate response to help the student relate

Table 5 Teachers' explanations of the most sophisticated mathematical solution to the caterpillar task and illustrative responses ($n = 105$)

Score on PCK rubric	Description	Percentage of teachers (%)	Illustrative example
Incorrect choice		13.3	
Correct choice but no explanation		7.6	
1 (low KCS)	Vague, incorrect or irrelevant explanation	6.7	"Probably the 1st or last example because it involves 2 operations"
2	Reference to relevant terms or concepts but key ideas missing or not communicated clearly	16.2	"4th Representation is clear, making a pattern more apparent"
3	Reference to explicit generalisation (correspondence) e.g. to rules, to efficiency, *or* to variables/unknowns	21.9	"3rd—All the student needs to do is change the number of blocks and will have an answer every time. It is similar to $4x + 2$"
4 (high KCS)	Reference to explicit generalisation (correspondence) *and* to variables/unknowns)	5.7	"$4n + 2$ involves representing an unknown number with a letter/symbol i.e. Algebra. Shows understanding of pattern and calculating any size caterpillar"
Unscored response	Incorrect previous student example	7.6	
No response		21.0	

the structure of the caterpillar to the missing number of dots (on the ends), which indicates an appropriate level of KCT for functional thinking, but only about 10 % achieved the highest level. A further 20 % of the teachers did not make any written response to the item on the student's error.

Terminology task

The teachers were asked to indicate from a list of terms (concepts) relevant to functions, relations and joint variation, those which they used explicitly in their teaching practice. This was intended to provide some insight into the teachers' familiarity with terminology and the use of related teaching strategies for developing functional thinking. It was found that approximately one-third of teachers indicated that they used the term "function" in their teaching. One-third indicated use of the term "variable" (used in the state curriculum) but nearly 60 % indicated that they referred to "unknown amounts or quantities" in their teaching. Nearly 80 % indicated use of the term "rule" (used in both state and national curriculum) and nearly 93 % used the term "sequence" in their teaching, but only 43 % used "growing pattern". Overall, more than half of teachers indicated a lack of familiarity with the use of geometric growing patterns for teaching generalisation (low KCT) and

Table 6 Teachers' response to student misconception in the caterpillar task and illustrative responses ($n = 105$)

	Score on PCK rubric	Description	Percentage of teachers (%)	Illustrative example
Not recognised as incorrect	N/A	Interprets student response as correct	1.9	"That it is correct"
Unclear if recognised as incorrect	1	Vague, incorrect or irrelevant explanation	18.1	"Can you prove it? Tell me how you came to that answer"
	2 (low KCT)	Reference to relevant terms or concepts but key ideas missing or not communicated clearly	7.6	"How could you prove to me that your formula/answer is correct? Without drawing 37 blocks? Could you try your calculations for a smaller number of blocks and find out if they work?"
Recognised as incorrect	1	Vague, incorrect or irrelevant explanation	0.0	
	2 (some KCT)	Reference to relevant terms or concepts but key ideas missing or not communicated clearly	9.5	"Maybe get them to visualise and remember that the blocks are connected so you do not always count the sides. Also ask how many sides does a cube have?"
	3	Reference to the missing end stickers	33.3	"I think you may have missed the ones at the end"
	4 (high KCT)	Reference to missing end stickers *and* to appropriate strategy for student, e.g. look at structure of caterpillar, look at smaller caterpillar to check	9.5	"Firstly, praise that they are close, but need to get some blocks to check. So get connector blocks and join 37 or a small group such as 10 + together, count how many sides and then how many spots. Hopefully from the model they will see the 2 extra spots on the ends"
	No response		20.0	

more familiarity with using number sequences to find the rule. As previously highlighted in the literature review, a teaching approach that relies only on number sequences can lead to students merely following procedures for finding a rule rather than developing functional thinking.

Function machine task

Two further items investigated teachers' KCT about the use of function machines. The teachers were asked *if* they used function machines in their teaching and if so, to provide an appropriate activity they might use in their teaching of upper primary students. Nearly 50 % of the teachers indicated that they taught with function machines, but only 16 % of the activities they suggested for teaching these concepts were appropriate and demonstrated reasonable KCT. These activities were related to the concepts of input and output, and finding a function or rule for the relationship between them. For example, a response deemed as appropriate was “Simple in and out functions, e.g. Put 2 in, machine adds 3—outcome 5. Figure out the function if…” The following response was considered to be ambiguous and did not demonstrate a clear understanding of the use of function machines: “Tune in activities—students use their mental calculations to begin the lesson thinking about whole numbers, decimals or fractions (depending on the lesson focus).” Overall, the result for this item indicates a low KCT for the majority of teachers.

Teachers were also given two examples of input/output tables from the same function machine and asked to explain how and why a particular way of arranging variable pairs in a table would support their teaching of functional thinking. Table A (Appendix—Q4b) contained consecutive input numbers, highlighting co-variation whereas Table B contained non-consecutive input numbers, encouraging correspondence approaches to finding the relationship between input and output numbers. Both tables contained ordered pairs where the input number is n and the output number is $2n + 1$. Table 7 presents data on the scoring of teachers’ responses using the previously described adapted rubric (Table 2).

From the results for this item, it can be seen that only 25 % of the teachers demonstrated at least some KCT by attending to both co-variation and correspondence concepts in their responses. Less than 10 % were able were able to clearly relate the differences between the arrangements of the two function machine tables and the concepts of co-variation (recursive generalisation) and correspondence (explicit generalisation) to helping students generalise the relationship between input and output numbers. Nearly one-third of the teachers demonstrated very low or no KCT on this task by providing vague, incorrect or irrelevant explanations. A further 25 % either made no response or wrote that they did not understand the question.

Teachers’ knowledge of curriculum on functional thinking (KC)

Teachers were given an excerpt from the new national curriculum and asked to indicate their perceived level of comprehension of the content on a scale from easy (1) to difficult (6) to understand. They were also asked if they currently teach this kind of content. Table 6 presents the percentages of teachers who selected each level of comprehension of the content description and who stated that they currently teach this kind of content. For example, of the 105 teachers in total, just over 25 % gave a score of 1 (easy) for the content description, and just over 80 % of *these* teachers also stated that they teach this type of content. Nearly 4 % of the teachers found the content difficult to understand (score of 6) and 50 % of these nevertheless indicated that they currently teach it.

The median score for the level of comprehension was 2, indicating that in general the teachers found the description of the curriculum quite easy to understand, although from the second column in Table 6, it can be seen that 20 % of teachers either gave a score of 5/6 (difficult to understand) or did not make a response at all (Table 8).

The accumulated totals showed that overall, two-thirds of the teachers reported that they currently teach this kind of content. The content description used in the survey was from the recently introduced Australian Curriculum but the prescribed curriculum for Victorian schools over the past several years (VELS) contained similar (even more detailed) descriptions and with examples of the two types of generalisation, so it is perhaps surprising that a significant proportion of teachers have excluded these concepts from their teaching programme, which are deemed as essential for upper primary students.

The teachers were also asked to give an example of an appropriate activity they might use to teach this kind of content, which is an important aspect of KC—the ability to *apply*

Table 7 Teachers' explanation of differences between co-variation and correspondence using function machine tables and illustrative responses ($n = 105$)

Score on PCK rubric	Description	Percentage of teachers (%)	Illustrative example
1 (low KCT)	Vague, incorrect or irrelevant explanation	32.4	Table A: "Result numbers are smaller; therefore, they are easier to work with." Table B: "Input numbers are larger so they give a more accurate output/result. That is, the sample is larger."
2	Attendance to either co-variation (recursive generalisation; consecutive arrangement of input numbers) or correspondence (explicit generalisation to find rule)	17.1	Table A: "This table perhaps would be helpful in observing patterns in which the input pattern grows by one." Table B: *left blank* Table A: "Shows function of $2n + 1$. Origin of 1 so it is better." Table B: "Shows function of $2n + 1$."
3	Attendance to both co-variation and correspondence but separately	16.2	Table A: "Benefits—you can see the sequential relationships. You can use it to predict the next number in the sequence." Table B: "Benefits—you can see that the rule is consistent no matter what your input number is."
4 (high KCT)	Connection between concepts of co-variation and correspondence and students' learning	8.6	Table A: "Can see pattern in output clearly (adding 2 each time) i.e. have 2nd step of relationship." Table B: "Greater range of numbers and focuses children to look at relationship between input and output rather than between numbers in just ip or op"
Unscored response	Written response indicating lack of understanding of question	13.3	"Not sure" "No idea"
No response		12.4	

their understanding of curriculum to the development of learning experiences for students. The results showed that just over 20 % of the teachers did not make a response to this item. Of the nearly 80 % of teachers who did provide an example, only 48 % of their examples were deemed as relevant to patterns and algebra. For example, the following response was judged as relevant: "Create a growing pattern demonstrated visually and in a table and have students continue the pattern and table of values. Extension is students will write a

Table 8 Teachers' scores for their understanding of "Patterns and Algebra" content and stated teaching practice ($n = 105$)

Score on Likert scale (1–6: easy to understand to difficult to understand)	Percentage of teachers who gave score at each level (%)	Percentage of teachers at each level who stated they taught this type of content (%)
1 (easy)	25.7	81.5
2	32.4	73.5
3	13.3	57.1
4	9.5	60.0
5	10.5	36.4
6 (difficult)	3.8	50.0
No score given	4.8	40.0

rule to describe the growing pattern." The response "Doubling/halving ingredients to cater for different numbers of people" was not considered to be relevant. In other words, more than 60 % of the teachers were unable to demonstrate the expected KC for teaching upper primary students.

Discussion and implications

The effective teaching and learning of algebra is considered a priority internationally and the recent introduction of a national curriculum in Australia has explicitly brought algebra to the attention of teachers at primary levels of schooling. Although there is considerable theoretically grounded research in the literature on the knowledge teachers *ought* to have for teaching mathematics, there seems to be far less empirical research on what mathematics knowledge teachers actually *do* have, and even less on teachers' knowledge for teaching algebra in particular. This study sought to investigate upper primary teachers' actual knowledge of functions, relations and joint variation—an important area of algebra that requires considerable conceptual development and algebraic reasoning particularly in the middle years of schooling. The intent was to understand more about what teachers might need specifically in their professional learning to help them teach algebra effectively. This section summarises and discusses areas of strengths and weaknesses in the teachers' knowledge, relationships among the four types of knowledge investigated, and implications for teacher professional development.

Teachers' strengths and weaknesses

The findings indicate that, for the most part, the teachers' knowledge of functional thinking was below the level expected for teaching middle-school algebra. This provides further evidence of teachers' inadequate understanding of mathematics for teaching. More importantly, it contributes to our understanding of the nature of the different types of knowledge they hold in terms of their strengths and weaknesses that could inform middle-school-teacher learning of algebra for teaching. Table 9 provides a summary of key findings for each of the four types of knowledge investigated in this study and what a majority of the teachers were able to do (i.e. strengths) and not able to do (i.e. weaknesses).

Table 9 Strengths and weaknesses in teachers' mathematical knowledge for teaching functional thinking

Knowledge type	Strengths	Weaknesses
SCK	Identify and represent generalisation of growing pattern in words or with a calculation	Represent generalisation symbolically
KCS	Provide possible correct student responses	Provide recursive and explicit examples of generalisation
		Use appropriate algebraic terminology
		Interpret students' mistakes
KCT		Provide appropriate response to a student's mistake
		Use algebraic terminology in teaching
		Use consecutive and non-consecutive pairs of variables for teaching generalisation
KC		Apply curriculum to appropriate activities

Just over two-thirds of the teachers were able to demonstrate a reasonable SCK by completing a geometric growing pattern task suitable for upper primary students using explicit generalisation (correspondence), with a majority expressing their generalisation in words. This result is similar to or higher than figures reported in studies on prospective teachers, in which less than half completed a generalisation task correctly (Goulding et al. 2002), or struggled to identify the visual pattern (Rule and Hallagan 2007). A recent international study of prospective teachers found that 77 % used recursive generalisation correctly, but only 54 % could generalise explicitly (Australian Council for Educational Research 2010). One-third of the teachers could represent their generalisation with a symbolic expression, but only 2 % used a full symbolic equation with both variables included. Nearly two-thirds of them were able to demonstrate at least some KCS by providing correct student responses at different levels of understanding for the growing pattern task, but less than 20 % gave both a recursive example and an explicit example of generalisation, which is of concern since upper primary students are expected to be able to use both strategies. An underlying concern is that they may also be unfamiliar with the likely learning progression for students and that explicit generalisation is considered to be at a higher level of understanding than recursive generalisation. This implication relates to a study of secondary teachers' knowledge for teaching functions using graphical representations which found that teachers had incorrect perceptions of an appropriate learning sequence for students (Hadjidemetriou and Williams 2002).

Regarding KCT, although half of the teachers indicated that they used function machines in their teaching practice, only one-quarter demonstrated at least some knowledge of the use of consecutive and non-consecutive pairs of variables for teaching functional relationships (KCT). These results are considerably lower than those for the caterpillar task in the survey, suggesting that input/output tables as another way to represent a functional relationship may be unfamiliar or not well-understood by most of the teachers. Driscoll (1999) highlighted the importance of understanding functional relationships using this type of representation. The limited number of language terms that teachers indicated they used in their teaching (lack of KCT) also suggests that they may have not had exposure to the variety of terminology and teaching strategies possible for this aspect of algebra. The results regarding KCT indicate overall that few teachers may have the necessary knowledge of how students develop functional thinking or the difficulties students have in using correspondence approaches (KCS) to be able to make effective use of tables of paired variables or function machines in their teaching (KCT), which were highlighted by Moss et al. (2008) as valuable for middle primary students in developing functional thinking.

Finally, regarding KC, the findings highlighted that a considerable proportion of the teachers may not even be teaching algebra content related to functions, relations and joint variation as described in state and national curriculum documents. Most of them believed that they understood relevant content in the new curriculum, yet the majority could not apply it to appropriate activities for teaching it (KC). This suggests that they may not comprehend the actual content in the curriculum on algebra but without realising it and that merely reading the prescribed documentation does not necessarily mean that teachers have improved their KC.

With a majority of the teachers being able to generalise a growing pattern explicitly themselves (SCK) and less than half able to interpret a student's mistake on the same task (KCS) and provide an appropriate response (KCT), it is possible many of them may have employed algebraic procedures that enabled them to find the correct rule for a functional relationship without understanding it conceptually, thus making it difficult for them to diagnose and address student difficulties. Kuchemann (2010, p. 248) stated that "teachers may not be in the habit of looking for mathematical structure and will thus need experience of thinking in this way." These findings highlight the difference between knowing *how* to do the mathematics for oneself and knowing the *why* so as to be able to teach it effectively. To teach students to develop functional thinking and a conceptual knowledge of algebra, teachers need to have developed considerable SCK based on *relational* understanding (Skemp 2002). The typical symbol-manipulation approach to learning algebra typically experienced by teachers when they were at secondary school is unlikely to provide a solid foundation on which to build their KCS and of content and teaching (KCT). This result suggests the value of professional learning that encourages teachers to apply their SCK to student work analysis and to become familiar with a variety of common student visualisation strategies and generalisation difficulties (such as inappropriate use of proportional reasoning). Suitable teaching strategies to address these would also be important.

Relationships among knowledge

The findings of the study also suggest possible relationships among some of the different types of knowledge that were inconsistent with what one might expect. For example, while the teachers' SCK was strong, their KCS was weak. This suggests that SCK does not necessarily support KCS or vice versa, that is, there is not a direct relationship between the two, or in terms of the framework of Hill et al. (2008), there is a difference between SCK and KCS. Specifically, this finding indicates that teachers are more likely to be able to solve the pattern generalisation task for themselves (i.e. indicating high SCK for it), but without necessarily understanding students' thinking or the process by which students learn to generalise (i.e. weak or lack of KCS). It corresponds to the findings of Nathan and Petrosino (2003) that prospective secondary teachers could apply their content knowledge of algebra for themselves but without knowing the processes by which novices learn. In their study of prospective and practising elementary teachers, Jacobs et al. (2010) found that some teachers' difficulties in interpreting students' solutions may have been reflective of a lack of familiarity with students' strategies (KCS) or to a lack of their own content knowledge (SCK). They explained, "To interpret children's understandings, one must not only attend to children's strategies, but also have sufficient understanding of the mathematical landscape to connect how those strategies reflect understandings of mathematics concepts" (p. 195). They did find that sustained and targeted professional development supported the improvement of teachers' expertise in noticing students' mathematical thinking.

Similarly, the teachers' KCT was weak in relation to their SCK, again suggesting that the latter does not necessarily support the former. While a majority of the teachers were able to find the correct rule for the total number of dots for any number caterpillar (SCK), less than half of them could not provide appropriate student response (KCT). The considerably lower percentage of teachers who demonstrated an effective teaching response seems to illustrate the difference between "doing" the mathematics for oneself (SCK) and "teaching" the mathematics effectively using relational understanding and functional thinking (KCT). This indicates that in general, teachers may have a higher level of SCK than KCT. The gap between 72 % of teachers with reasonable SCK and 43 % with reasonable KCT about the same pattern generalisation task raises the issue of the difference between their *relational* and *instrumental* understanding of mathematics (Skemp 2002). In order to understand the students' errors, teachers would need a *relational* understanding of pattern generalisation rather than just the ability to use a procedure for finding the rule.[1] It is possible that some teachers may have used a procedure rather than the visual structure of the caterpillar to generalise. This would still produce the correct rule but relies on instrumental understanding—"rules without reasons" (Skemp 2002, p. 2)—rather than on relational understanding, in this case how the variables relate to the visual structure of the caterpillar. Relying on such a procedure would subsequently make it difficult for teachers to interpret a student's mistake (KCS) and respond effectively to it (KCT), because they would not have understood the pattern's structure for themselves. They would therefore demonstrate more SCK than KCT.

Implications for teacher professional development

The findings have highlighted some key issues worth considering when developing professional learning for teachers in algebra based on the strengths and weaknesses highlighted in Table 8. The following presents some particular suggestions for improving the different types of mathematics knowledge needed for teaching functions, relations and joint variation.

The findings for SCK suggest professional learning that focuses on representing generalisations in a *variety* of ways so that teachers can connect their verbalised understanding of the structure of growing patterns with ordered and un-ordered pairs of variables, with graphs of variables and with the use of letters or symbols in an equation. For KCS, this suggests professional learning that helps teachers to understand more about the process by which students learn to generalise (e.g. the learning progression in Table 1), the variety of responses students might make to generalisation tasks, likely difficulties and the teaching strategies that encourage students to move from recursive to explicit generalisation as conceptually appropriate. The definition and use of language terms to help teachers understand and communicate their knowledge would also be valuable. These could include

[1] Such a procedure might be as follows: 'the 'jump' between consecutive numbers in the sequence is the number that goes before the 'n' (4 for the caterpillar pattern) and then you add or take away another number to adjust the 'n' term get the rule' ('+2' in the case of the caterpillar pattern to get '$4n + 2$'). The author has seen this procedure outlined in secondary mathematics textbooks.

terms such as unknown quantity, variable, constant, generalisation, correspondence (explicit), co-variation (recursive), growing pattern, geometric or figural or numeric, ordered pairs, function, functional relationship, rule, equation, input, output, sequence, graphs and Cartesian plane.

Regarding KCT, the findings suggest the need to help teachers develop deeper understanding of the two types of generalisation (recursive and explicit) and effective teaching approaches that provide students with opportunities to explore functional relationships in a variety of contexts, such as geometric growing patterns, function machines with input/output tables and number sequences. Regarding KC, it is worth considering the inclusion of a greater range of curriculum documentation in professional learning programmes so that teachers are able to develop a more comprehensive knowledge of content descriptions and a larger repertoire of terminology and ideas for students' learning experiences. Looking at algebraic concepts in the early years of schooling and later middle (early secondary) years might also contribute to upper primary teachers' *knowledge at the mathematical horizon* (Hill et al. 2008) so that they are able to support students at different levels of understanding.

Finally, the study has implications for further research on teachers' knowledge of functional thinking. For example, the description of each type of knowledge and the rubrics for analysing data can provide insights to researchers about these constructs and tools. Other topics in algebra also need to be explored in addition to participants in other contexts, such as lower secondary mathematics teachers, to get a wider scope of teachers' knowledge for teaching algebra in middle school.

Conclusion

This study contributes to an under-represented area in the research literature for an important area of mathematics education and mathematics teacher education. In general, it suggests that it would be worthwhile, when considering the professional learning of upper primary teachers, to pay attention to their SCK of functions, relations and joint variation so that they develop a relational understanding for themselves and their own ability to use functional *thinking* rather than simply follow learnt procedures without necessarily understanding them. This would then provide a solid foundation on which to build teachers' knowledge of how students learn and effective teaching strategies they can use. The use of a wider range of curriculum documentation in their professional learning would contribute to a greater familiarity with key concepts and terminology and a larger repertoire of learning experiences and teaching strategies appropriate for algebra. In order to add further depth to this work, the author is engaging in ongoing research using data from an in-depth collective case study of 10 teachers and their classes to continue to investigate ways to address upper primary teachers' professional learning needs in this important area of mathematics.

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Appendix: Questionnaire

1. ***Patterns and Algebra*** **content description in the Australian National Curriculum (ACARA, 2009)**

Continue and create sequences involving whole numbers, fractions and decimals. Describe the rule used to create the sequence *(ACMNA133).*

a) For me, this phrase is (*please circle*):

EASY TO UNDERSTAND DIFFICULT TO UNDERSTAND

1 2 3 4 5 6

b) I currently teach this kind of content in my class (*please circle*): YES / NO

c) If I was teaching this kind of content, an example of an activity I would use is:

(There is more room for writing on the back of this questionnaire.)

………

………

………

2. Language I use (explicitly) in my teaching (*please tick*)

Terms	√	Terms	√	Terms	√
generalising		unknown amount/quantity		function machine	
growing pattern		variable		input/output	
sequence		function		rule	

3. Teaching with growing patterns

Here is a task that explores the continuation, description and generalisation of a geometric growing pattern:

One day my little niece saw a clump of wriggling spotted caterpillars on the branch of a tree. Later she made her own collection of caterpillars with linking blocks and stickers. The first caterpillar was made with 1 block and 6 stickers. The second caterpillar was made with 2 blocks and 10 stickers. She continued to add to her collection:

i) How many stickers are needed for the next caterpillar's spots (Caterpillar #4)?

ii) How many stickers are needed for Caterpillar #7?

iii) How many stickers are needed for Caterpillar #17?

iv) For any caterpillar number you are given, how do you find the total number of stickers needed for its spots?

(*Task adapted from Markworth, 2010*)

a) Please give two examples of *correct* responses you expect students might make to parts iii and iv of this caterpillar task (previous page). Please draw these examples as if you have taken a photocopy of the student's work. *(There is more room for writing on the back of this questionnaire.)*

iii) How many stickers are needed for Caterpillar #17?

One possible correct student response:

Another possible correct student response:

iv) For any caterpillar number you are given, how do you find the total number of stickers needed for its spots?

One possible correct student response:

Another possible correct student response:

b) Which of your four response examples (above – 1st, 2nd, 3rd or 4th) do you believe is the most sophisticated mathematically? Please explain your reasoning.

………

………

………

c) A student in Year 6 explains that Caterpillar #37 will have 37 times 4 spots for the top, bottom and sides of the caterpillar, which is 148. What might you as the teacher do or say in response?

………

………

………

4. Teaching with function machines

a) Have you used function machines in your teaching? If so, please describe the types of activities you used.

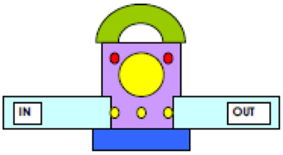
………

………

………

………

b) Here are two input/output tables for the same 'function machine':



Picture created by Matthew Sexton

TABLE A

INPUT	OUTPUT
1	3
2	5
3	7
4	9
n	?

TABLE B

INPUT	OUTPUT
2	5
5	11
8	17
10	21
n	?

What would be the mathematical benefits for students in using one table over the other in a lesson on functional relationships?

TABLE A:..

....

....

TABLE B:..

....

....

5. How do you currently feel about implementing the *Patterns and Algebra* content of the Australian National Curriculum at your year level?

....

....

....

6. What would be the most helpful form of support for you in implementing the *Patterns and Algebra* content of the Australian National Curriculum?

....

....

....

....

Many thanks for completing this survey.

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