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Prospective elementary teachers' development of fraction language for defining the whole

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Abstract This article examines the ways in which prospective elementary teachers' develop an understanding of language use for defining the whole throughout a 9-day rational number unit. Student work samples and classroom conversations are used to illustrate their difficulties and growth with defining the whole and corresponding language use for describing fractional amounts. The results indicate that three mathematical ideas became *taken-as-shared* by the class. The first was that fractions depend on a group or whole. The second included defining an *of what*. The third was developing language in terms of what the denominator represents. Difficulties prospective teachers had conceptualizing language included distinguishing among the phrases *of a*, *of one*, *of the*, and *of each.* Implications for mathematics education courses and future research studies are also discussed.

Keywords Elementary teacher education · Fraction language · Defining the whole

Past research illustrates that prospective teachers' conceptions of fractions are primarily based on misunderstood procedures (Graeber et al. 1989; Simon 1993). With the *Curriculum Focal Points* (National Council of Teachers of Mathematics 2006) and other reform efforts advocating for students to develop fluency with these processes, it is increasingly important to support prospective teachers' development of these topics as well. This is for them to have the knowledge to accurately assess and foster students' thinking once they enter the classroom.

Classroom learning constitutes a reflexive relationship between individual and social domains. Individual students contribute to classroom discussions and at the same time classroom discussions contribute to individual students' learning (Cobb and Yackel 1996). Studies that have analyzed classroom learning at the elementary (Cobb et al. 2001) and college (Stephan and Rasmussen 2002) levels have illustrated the importance of documenting social activity. This type of research provides a detailed account of students'

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conceptions with and development of the mathematics being taught. Though several studies document prospective teachers' conceptions regarding fractions (Ball 1990a, b; Graeber et al. 1989; Simon 1993), few have analyzed the ways in which their understanding develops.

Fraction language

Some difficulties students have with fractions result from incorrectly applying whole number language understandings to fraction situations (Lamon 2005; Mack 1995; Ni and Zhou 2005; Streefland 1991). For example, in the context of subtraction, problems such as $3 - 2$ can be stated as starting with three objects and taking away two of them. When the situation involves fractions, such as $3 - 1/2$, it is incorrect to interpret this as starting with three objects and taking away half of them. Other language issues stem from students answering *how many* instead of *how much* (Lamon 2005). As will be discussed later, when asked to share four pizzas equally among five people, the prospective teachers in this study tended to answer four pieces (see Table 6) or *how many* pieces each person would get instead of answering 4/5 or *how much* pizza each person would get.

Other language difficulties stem from defining incorrect wholes. Conceptualizing the whole is important for contextualizing situations, understanding procedures, and interpreting solutions (Lamon 1996, 2005; Mack 2001; Simon 1993; Tobias 2009). Unlike with whole numbers, wholes for fractions may not necessarily be a discrete set of objects (Mack 1993). When defining an incorrect whole, misinterpretations result from not understanding phrases, such as *of the* (Lamon 2005). When asked to find how much *of the* pizza was eaten when 14 slices were eaten from two pizzas cut into 12 equal slices each, *of the* refers to a whole of two pizzas. Students incorrectly using the whole as one pizza will give solutions such as 14/12 *of the* pizza. This results in a solution describing that more was eaten than what was started with (Davis and Maher 1990). Likewise, teachers without this understanding may inaccurately assess correct solutions of 14/24 *of the* pizza as being a result of students incorrectly adding denominators (Davis and Maher 1990).

If language development with defining the whole is not addressed, students may use incorrect language to describe situations and not understand why or how those descriptions are incorrect. For example, when asked to share four pizzas among three people, students may know each person receives 1 1/3 but not understand the differences in describing this amount as 1 1/3 pizzas versus 1 1/3 *of all the* pizzas (Lamon 1996). Though students may get the correct answer of 1 1/3, they must also understand how to define the whole.

Correct mathematical language use and understanding are important for students and teachers to describe and represent various mathematical situations. With topics such as defining wholes being foundational to understanding and working with fractions successfully, correct language use with these same topics is also important.

Prospective teachers' understanding of defining the whole

Research documents that prospective elementary teachers have difficulties with conceptualizing the whole (Ball 1990a; Luo et al. 2011; Simon 1993; Tobias 2009). Simon (1993) found that when presented with the fraction division situation of finding how many cookies 35 cups of flour will make when 3/8 *of a* cup of flour are required per cookie, several prospective teachers described the remainder of 1/3 as being the amount of flour leftover

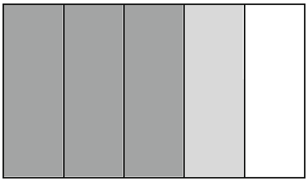


Fig. 1 Correct picture for $3/4 \times 4/5$ chosen as incorrect

rather than the number of cookies which can be made. Similarly, Tobias (2009) found that when presented with a situation of finding how many 1/4 pound servings of dough can be made from 1 7/8 pounds of dough, prospective teachers tended to find the incorrect solution of 7 1/8 by referring the remainder to the whole pound as opposed to the serving. Ball (1990a) noted that when given the problem $1\,3/4 \div 1/2$, prospective teachers could use a procedure to find the correct solution of 3 1/2. However, when asked to write a word problem for the situation, several wrote problems representing $1\,3/4 \div 2$ which resulted in the solution of 7/8. When relating the two solutions, prospective teachers said that the answer of 7/8 is 3 1/2 fourths, so the word problem is correct. In a cross-national study, Luo et al. (2011) asked prospective teachers from both the United States and Taiwan to identify an incorrect model for $3/4 \times 4/5$. Participants tended to choose the model that was actually correct for the problem as the incorrect picture. When asked why they chose that option, participants from both countries responded that they thought 3/4 and 4/5 should be out of the same whole. Therefore, they chose the picture representing 3/4 of 4/5 (see Fig. 1) as incorrect.

Prospective teachers' inability to define the whole affects their ability to interpret remainders, determine remainders, and conceptualize situations and models (Ball 1990a; Luo et al. 2011; Simon 1993; Tobias 2009). Whether these misconceptions stem from misinterpreting problems or solutions, this lack of understanding appears evident particularly when prospective teachers encounter harder fraction topics, such as multiplication and division.

Prospective teachers' limited understanding of defining the whole has been confined in the literature to fraction operations. However, conceptualizing the whole is fundamental to more than just fraction multiplication and division. This article seeks to extend the literature by documenting prospective teachers' difficulties and growth with defining the whole starting from the first day of fraction instruction.

Course design and structure

To support prospective elementary teachers' development of defining the whole and language use with describing wholes, activities were designed out of a combination of past research with children and adults (Gravemeijer 2004; Streefland 1991; Wheeldon 2008). The activities included an initial representation for the problem, either circles for regular pizza or rectangles for dessert pizza, and situations that would result in answers both less than and greater than one. Problems focused on the part-whole understanding of fractions through partitioning contexts to provide a foundation for language skills to develop (Kieren 1980).

Participants were first presented with a context in the form of a picture, a word problem, or both (Gravemeijer 2004). They then solved the problems either individually or in small groups. This was followed by whole-class discussions of selected problems. Social norms, such as explaining and justifying solution and solution processes and sociomathematical norms, such as determining what constitutes a different solution (Cobb and Yackel 1996) for whole-class discussions were established and reinforced throughout the duration of the course (Tobias 2009). In addition, participants were encouraged to develop their own solution strategies and to use pictures to aid in the explanation and justification of their solution process (Gravemeijer 2002).

By providing prospective teachers with the opportunity to formulate their own solution processes for each problem, whole-class discussions focused on the validity of students' strategies, explanations, and justifications for their solutions. As a result, classroom conversations provided an opportunity for student development to be documented. This allowed the following research questions to be investigated:

- What fraction understandings do prospective elementary teachers have with respect to defining the whole?
- How does prospective teachers' understanding of defining the whole develop?

Methodology

Thirty-three prospective elementary teachers participated in a semester-long classroom teaching experiment conducted at a large metropolitan university in the southeastern part of the United States. The study was conducted in a content course focusing on mathematics for teaching elementary school. The course met twice a week for 1 h and 50 min each day. All participants were women, in at least their sophomore year of college, and either majoring in elementary or exceptional education. All students enrolled in the course agreed to participate in the study.

Rational numbers constituted nine class days. It was the second topic taught in the course following a unit focusing on whole number concepts and operations. Language for fractions was introduced on the first and second day and continued throughout the duration of instruction.

Data collection

The data collected included video recordings and transcripts of whole-class discussions, and student work from in-class activities, two homework assignments, and an end-of-unit examination. Research team field notes and reflective journals were also collected for each class session (Cobb and Gravemeijer 2008). The course activities, homework problems, and examination questions were designed such that students could choose their own solution methods for solving the tasks with the expectation that they would be asked to explain and justify their thinking in the process.

Data analysis

The data were analyzed through three phases using Rasmussen and Stephan's (2008) method for documenting collective activity. First, each class session was video recorded and transcribed. A team of at least two researchers analyzed the transcripts from each whole-class conversation by coding student responses in terms of one of the four components (claim, data, warrant, and backing) of argumentation (Toulmin 2003). Claims are solutions to a problem. When a claim is contested, data consisting of evidence to back up the claim are provided. If the data are challenged, warrants are used to justify why the data are valid. Finally, if warrants are questioned, backing is provided for why the warrant holds authority (Toulmin 2003). For example, when asked to determine how much pizza each person receives when sharing four medium pizzas equally among five people, a claim would be that the answer is 4/5. Data could include that the solution is out *of one* pizza so each person receives 4/5 *of a* pizza. A warrant for the argument might be that each pizza was split into five pieces and each person receives four pieces. Backing for the argument could include that all of the pieces are equal.

Researchers first worked individually to code the transcripts in order and one class day at a time (Cobb and Gravemeijer 2008). After coding for a particular day was finished, the team met for approximately 3 h to compare where they determined claims, data, warrants, and backings to be in the conversation (Glaser and Strauss 1967). Researchers' codes agreed with 87% reliability. When disagreements occurred, they were always only one argumentation level apart. For example, one person would categorize a statement as data and someone else would have it as a warrant. To resolve disagreements, each researcher gave reasoning for why they analyzed the responses the way they did. This discussion continued until a consensus was reached. Once a transcript was finished this process was then done for the next class session until all 9 days were completed.

Coded conversations were then organized into an argumentation log (see Table 1). The argumentation log consisted of ideas that were only generated by students. None of the ideas in this study were introduced or developed by the instructor. The argumentation log was analyzed to determine where warrants and backings were no longer used, or when ideas shifted in an argument and were no longer questioned (Rasmussen and Stephan 2008). Table 1 illustrates an idea shifting in the conversation. As seen in the table, fractions have equal parts was used as backing in one problem, but as a warrant in another problem. Since equal parts shifted in function and was not questioned, it was said to be *taken-as-shared.* The *taken-as-shared* ideas did not imply that everyone in the class

Table 1 Sample argumentation log—idea of equal pieces shifts in argument

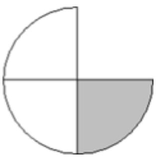
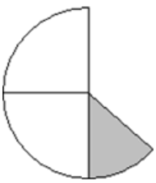
Name shaded amount	Claim	Data	Question	Warrant	Backing
	1/3	I looked at the whole piece is a half. The top part would be one, the bottom part would be two, and then the shaded part would be three to get the 1/3. I kind of undivided them	What do you mean?	I meant if the whole piece, it's into four pieces and to make that two pieces I erased one of the lines	I just looked at them as equal parts
	1/5	I broke the other two slices up in half. One-fifth of the leftovers is mushroom		She wants all the pieces to be equal	

Table 2 Sample mathematical ideas chart

	Taken-as-shared	Keep an eye on	Additional notes
Day 1	Fractions have equal pieces	Fraction equivalence	Circles introduced as a tool

understood the idea. Rather, these are ideas that appeared to be understood by the class as a whole.

Once the argumentation log was analyzed, the *taken-as-shared* ideas were used to create a mathematical ideas chart for each class day. The chart included ideas that appeared to be *taken-as-shared*, ideas to look for in future discussions to become shared, and additional notes (Rasmussen and Stephan 2008). See Table 2 for a sample of the chart. The ideas charts from each day were then compared with one another to determine which ideas shifted throughout instruction and became established.

Results

Analysis of the transcripts and student work indicated that three mathematical ideas became *taken-as-shared* as the class developed an understanding of language use for defining the whole. The first was that fractional solutions depend upon a group or whole. The second included defining an *of what*. The third consisted of developing language in terms of what the denominator represents.

The order by which the mathematical ideas became *taken-as-shared* overlapped throughout the rational number unit (see Table 3). All three ideas were introduced during the first day of rational numbers and established on different days during instruction. An idea that was discussed first was not necessarily *taken-as-shared* first. Such was the case for defining an *of what*, which was introduced first but established second.

The three mathematical ideas are discussed in terms of Toulmin's argumentation model and the social context of the course. Though the ideas were *taken-as-shared* by the class, this did not indicate individual students' development with the same topics. For illustration purposes, each idea is discussed separately and in the order by which they became established.

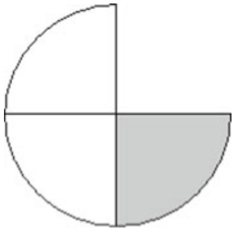
Solutions depend upon a group (whole)

The first mathematical idea which became *taken-as-shared* was that solutions depend upon a group or whole. This was introduced during the first fraction activity where students had to name a fraction to represent the shaded amount (see Fig. 2). The intent of the activity

Table 3 *Taken-as-shared* ideas

<table>
<tr><td rowspan="4">Start of day 1</td><th colspan="3">Day 1</th><th>Day 2</th><th>Day 8</th><td rowspan="4">End of day 8</td></tr>
<tr><td></td><td colspan="2">Solutions depend on a group or whole</td><td colspan="2"></td></tr>
<tr><td colspan="4">Defining an of what</td><td></td></tr>
<tr><td colspan="2"></td><td colspan="3">Develop language in terms of what the denominator represents</td></tr>
</table>

Fig. 2 1/3 and 1/4?



was for prospective teachers to develop the need for defining a whole, develop an understanding that the same shaded region can represent different amounts, and to emphasize correct mathematical language.

In this activity, all of the pictures shown represent the amount of pizza leftover on tables. The shaded amount represents the part of the pizza that had mushrooms on it. The question asked students to name a fraction to represent the mushrooms. Instead of just naming that amount as 1/3, the intent of the activity was to have students label that region as 1/3 *of the* leftovers. This was to introduce students to the idea of providing enough information when labeling fractions so that an exact amount could be determined as opposed to an arbitrary amount.

As expected, two claims for the first problem were 1/3 and 1/4. Some members of the class questioned the validity of the solution of 1/4. In response to the question of 1/4 being correct, Violet provided data and a warrant for the argument that 1/4 is correct because the answer depends on how the problem was looked at or grouped.

Violet: It depends on how you looked at it. ***(Data)***
Instructor: What do you mean?
Violet: Because I looked at it as the whole pizza. ***(Warrant)***

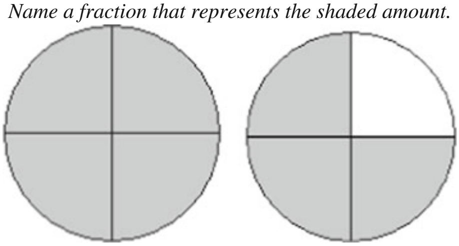
Violet's response was the first instance where prospective teachers started developing the idea that fractional solutions depend upon a group. By specifying that she looked at the whole pizza, Violet expanded the idea to include that a group may not necessarily be everything shown in a picture.

The idea that a solution depends on how pieces are grouped shifted to data in subsequent problems. When the class moved on to the situation presented in Fig. 3, claims of 7/8 and 1 3/4 were given.

When discussing the differences between 7/8 and 1 3/4, Alex provided data that both solutions depend on how the shaded pieces are grouped.

Alex: It's just a question of how you group your problem. I grouped mine into eight individual groups, so I have seven of the eight that are shaded. ***(Data)***

Fig. 3 7/8 and 1 3/4?



Instructor: Okay
Alex: Kassie did hers in fourths
Instructor: Okay
Alex: So what hers is one group of four, two groups of four. Her one group of four is an entire mushroom pizza and three-fourths is the second group of four that she worked with. So *[Kassie]* was looking at it, but just grouped it differently. ***(Data)***

Alex provided data for the argument that the two solutions of 7/8 and 1 3/4 depend on how the pieces within the situation are grouped. Alex then noted in the conversation that for a solution of 7/8 the grouping is in terms of slices and for 1 3/4 the grouping is in terms of pizza. Other students in the class, such as Beth, expanded Alex's idea by realizing that for 7/8 the pizzas are grouped together, and for 1 3/4 each pizza is viewed individually.

Beth: Well you're splitting it. You're just looking at it one at a time instead of looking at them both together. ***(Data)***
Instructor: So this one *[7/8]* you're looking at of two pizzas?
Beth: Yeah

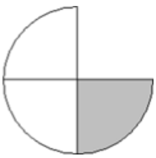
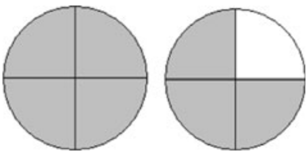
The grouping ideas that Alex and Beth discussed were presented as data in the conversation and warrants were no longer needed. In addition, the idea was not questioned indicating that it became *taken-as-shared*. This is summarized in Table 4.

By the end of the first day, the idea that solutions depend on a group or whole became *taken-as-shared*. Violet introduced the idea, and provided a warrant to further describe how she "looked at" the problem. When the class moved on to subsequent problems, Alex and Beth provided data for describing the group or whole and warrants were no longer used in the conversation and the idea was not questioned.

Defining an *of what*

The second mathematical idea of defining an *of what* became *taken-as-shared* over the course of the first 2 days. Discussions about labeling an answer started during the

Table 4 Solutions depend on a whole—warrants no longer needed

Name shaded amount	Claim	Data	Question	Warrant	Backing
	1/3 or 1/4	It depends on how you looked at it	What do you mean?	Because I looked at it as the whole pizza	
	7/8 or 1 3/4	It's just a question of how you group your problem You're just looking at it one at a time instead of looking at them both together			

conversation regarding solutions to the first table situation (see Fig. 2). Though this idea was introduced first in the class it was second to become established.

As illustrated, the class found two solutions for this problem to be 1/3 and 1/4. Initially, these solutions were generated because students argued that the directions for the activity were not clear enough to determine a single correct answer. By the end of the conversation, the class agreed that both 1/3 and 1/4 sufficed as answers. However, 1/3 or 1/4 alone was not enough information, and Claire introduced the idea of including an *of what* as a warrant for the claim that both solutions are valid.

Instructor: So here I am, I've got 1/3 and 1/4. How can they both be right? How can I just leave it? You say they both can be right. I need more information. What else would need to be here? Yeah

Claire: You could write you need to fill 1/3 *of what*. So 1/3 *of the* leftovers is mushroom or 1/4 *of the* whole pizza was leftover. ***(Warrant)***

Claire's comment of including an *of what* was the first instance where the class started to develop the idea of defining the whole. The *of what* idea allowed students to arrive at multiple answers as a result of defining more than one whole within the same problem.

Throughout the first class session, although Claire introduced the idea of including an *of what*, the instructor had to continually ask the class to justify their claims in terms of the *of what*. This was evident even during the following conversation regarding the last problem of the activity.

Instructor: What did you get?
Claire: 1/5
Instructor: What?
Caroline: 1/8 or 1/5
Instructor: 1/8 or 1/5
…
Instructor: Okay. How did you get it?

By the end of the first day, the idea that an *of what* needs to be defined was introduced. However, this idea was not yet established as students were not automatically providing an *of what* within their solutions.

On the second day, the class was still developing how to label an answer in terms of the whole. The idea was again used as a warrant in the following classroom episode. The discussion took place when the class answered how much *of a* pizza each person would get if they shared four pizzas among five people. One student in the class, Kassie, got the answer 4/20 from cutting each pizza into five equal pieces and giving everyone one piece of each pizza. Everyone received four pieces out of the 20 pieces total. In order to help students make sense of this answer, the instructor introduced a scenario of each piece being worth six points. The instructor then asked the class how to make sense of 4/20 in terms of how many total points it would be. Claudia responded with a warrant for the 4/20 noting that an answer of 4/20 does not include enough information. Katherine then replied with another warrant that the 4/20 was out of four pizzas total.

Claudia: 4/20 doesn't give her enough information because all we know is how much one slice is worth, which is six points. So if we just say 4/20 *of all* of it, well we can't say how much you know we won't have enough information to figure out how many points it is total. ***(Warrant)***

Kassie: But you do because you have the four out of the 20 slices

…

Katherine: 4/20, which is 20 *of the four* pizzas total. 20 *of all the* slices put together. ***(Warrant)***

Instructor: Now would we know how many points?

Claudia: Yeah

Katherine: 24

Instructor: So it sounds to me like if we knew it was 4/20 *of four* pizzas, it would be correct

Katherine: Right

Claudia started the conversation that a fraction written alone is not going to include enough information to determine exactly how much that fraction is worth. Katherine then used Claudia's idea to determine that 4/20 represented 4/20 *of the* four pizzas.

Both Claudia's and Katherine's arguments were warrants in the conversation. This is similar to the argument on the previous day where Claire used a warrant for the idea that an *of what* needs to be defined. Claudia restated Claire's idea by noting that a fraction alone is not enough information. Katherine then replied that the 4/20 would be the amount out *of four* pizzas. Though Claudia and Katherine expanded Claire's idea, this idea was still a warrant and did not shift in function. In addition, the instructor had to present the class with a secondary scenario of making each slice worth six points for the class to determine that 4/20 alone was not enough information. Thus, this idea was not yet *taken-as-shared*.

After this discussion, students labeled the whole in their solutions automatically. When discussing the problem of sharing three dessert pizzas among four people, Mindy provided her answer in terms of the whole as data without being prompted to do so or being questioned by other students in the class.

Mindy: And I got 3/4 *of a* dessert pizza. Any questions? ***(Data)***

When Mindy gave her answer of 3/4, she included the whole automatically in her solution and presented it as data in the argument. The idea shifted from a warrant in the previous conversation and was not questioned. The way in which defining an *of what* became established is summarized in Table 5.

Defining an *of what* became *taken-as-shared* by end of the second day as the class no longer needed to be prompted to do so. In addition, the idea shifted from warrant to data over the course of 2 days and was no longer questioned. Though this idea was the first idea that was introduced in the class, it was established second.

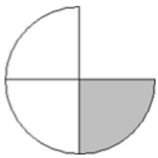
Developing language in terms of what the denominator represents

Though the class understood the need to define an *of what*, they had difficulty using appropriate language for describing this especially when more than one pizza was given in the problem. The third idea of developing language in terms of what the denominator represents was also introduced on the first day. The introduction of this idea occurred during the conversation regarding the third problem of the first activity (see Fig. 3).

As previously discussed, some members of the class arrived at the solution of 7/8. Students, such as Mary, had difficulty in describing this as 7/8 *of two* pizzas.

Mary: Assuming there are two pizzas, 7/8. ***(Data)***

Table 5 Defining an *of what* shifts in function and is not questioned

Problem	Claim	Data	Question	Warrant	Backing
Name shaded amount 	1/3 or 1/4		So here I am, I have got 1/3 and 1/4. How can they both be right? How can I just leave it? …What else would need to be here?	You could write you need to fill 1/3 of what. So 1/3 *of the* leftovers is mushroom or 1/4 *of the* whole pizza was leftover	
Share five pizzas equally among four people	4/20			4/20 does not give her enough information 4/20, which is 20 *of the four* pizzas total	
Share three dessert pizzas equally among four people	3/4	3/4 *of a* dessert pizza			

Instructor: So what would you like to follow this? 7/8…
Mary: Wait. What?
Instructor: How could I make this answer so it assumes that there are two pizzas in there?
Mary: I don't know. I would just write assuming that there are two pizzas, the answer I would get is 7/8
Alex: 7/8 of the remaining slices are mushroom pizza. ***(Data)***

During the conversation, Mary and Alex both provided data that the 7/8 was in terms of two pizzas or the remaining slices. Even with the instructor prompting Mary on what could follow the solution, she and others in the class had difficulties stating the solution as 7/8 *of two* pizzas. In addition, students such as Alex discussed the whole in terms of remaining slices, leaving the solution as an arbitrary amount.

By the end of the first day, language use for defining a whole was introduced. This idea was the third one introduced to the class that day, and it was the one that they initially had most difficulties with. Although they understood the amount of 7/8 to be from two pizzas, the class struggled with defining the 7/8 in terms of the two pizzas.

The sharing activity presented on the second day was designed to have the class continue to develop language use for describing wholes. Unlike on the first day where any whole could be used, the directions for the sharing task were written specifically for finding an answer in terms of a whole of one. The sharing activity consisted of six different situations such as the example presented in Fig. 4.

When solving the sharing tasks, many members of the class started to formulate language for describing wholes within their solutions. Prospective teachers' descriptions of the whole could be categorized into one of three groups. The first group did not define a whole and only provided a fraction answer. The second used incorrect language for defining the whole. The third used correct language for defining a whole of one.

Though the task specifically asked for the solution to be in terms *of a* pizza, wholes from individual slices to *all four* pizzas were defined. Inconsistencies among solutions were evident in the fraction of pizza each person would receive as well as the language used for

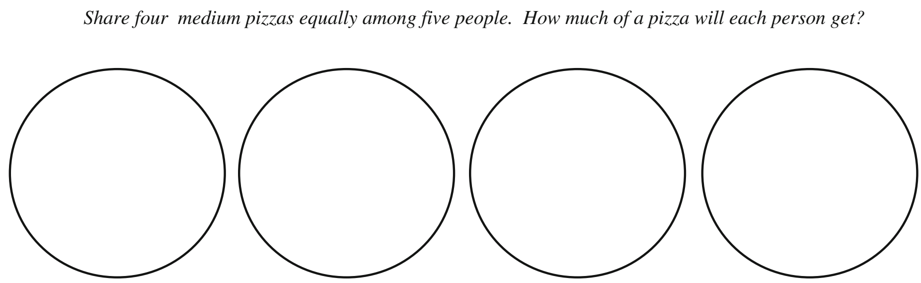


Fig. 4 Sharing task

describing the whole. As illustrated in Table 6, correct language use for defining a whole of one did not necessarily lead to a correct answer. Such was the case for four prospective teachers who arrived at the solution of 1/5 *of a* whole pizza. Likewise, arriving at the correct fraction of pizza did not necessarily lead to a correct whole being defined. Though several prospective teachers answered 4/5, some did not define a whole and others defined the whole to be *each* pizza or *all the* pizza. In addition, other solutions were mathematically correct, such as 4/20 *of four* pizzas, however incorrect in terms of satisfying the constraints of the problem.

Conversations quickly evolved around how the whole was understood. With the directions asking for the fraction *of a* pizza, prospective teachers were confused by the phrasing *of a*. Some students, such as Kassie, understood *of a* pizza to mean *of all the* pizza and arrived at an answer of 4/20. Other members of the class, such as Mary, understood *of a* pizza to mean *of each* pizza and arrived at an answer of 1/5. In the following conversation, Mary and Kassie both provided data for the argument explaining how they arrived at their solutions. Kassie then provided a warrant stating that she looked at all four pizzas as the whole to get her answer.

Table 6 Solutions to a sharing task

Share four medium pizzas among five people. Sharing equally determine what fraction of a pizza each person received

Level of defining the whole	Defined whole	Solution	Number of students who gave that solution
Only numerical	No whole defined	4/5	3
Using incorrect language for defining a whole of one	Pieces or slices	Four pieces	10
		1/5 *of a* piece of pizza	1
		1/5 slices per person	1
	All the pizzas	4/20 *of four* pizzas	4
		4/5 *of the* pizza	1
	Each pizza	1/5 *of each* pizza	4
		4/5 *of each* pizza	1
Using correct language for defining a whole of one	A pizza	1/5 *of a* whole pizza	4
		4/5 *of a* pizza	2
	One pizza	4/5 *of one* pizza	2

Kassie: I got 4/20, because together it was 20 pieces and four for each person. Questions? ***(Data)***

Mary: Well it says determine the fraction *of a* pizza each person will get. So I did 1/5 because when I divided each pizza into five pieces, each person would get one piece. ***(Data)***

Kassie: Oh I looked at *[the group of four pizzas]* as a whole. ***(Warrant)***

Mary: Yeah. I don't know. That's how I looked at it

Evident from the conversation, the language *of a* was difficult for prospective teachers. Some students interpreted this to be *of all the* pizzas, whereas others interpreted this to be *of each* pizza. While the class discussed the solutions of 4/20 *of four* pizzas and 1/5 *of each* pizza, other solutions such as 4/5 *of one* pizza were introduced into the conversation. Moments later, Mary provided backing for the argument stating that the directions are confusing in terms of what is meant by the phrase *of a* pizza.

Mary: It's just how the question asks you it's confusing because you don't know if they're talking about *each* pizza, or *one* pizza, or *all the* pizzas. ***(Backing)***

Evident from Mary's comment, difficulties the prospective teachers had with finding a solution stemmed from the language issues within the wording of the question. Distinguishing between the language *of a, of one, of each,* and *of the* seemed especially difficult. After Mary's comment that the directions were confusing, the class started developing ideas on what is meant by *of a* pizza.

Jackie was the first person to introduce the definition that *of a* pizza means *of one* pizza. Her comment came as backing for the argument that the answer of 4/20 is not valid because 4/20 is *of four* pizzas, not one. She also notes that each pizza has five pieces, not 20.

Jackie: Could it maybe be that in the question it says sharing equally determine what fraction *of a* pizza each person receives?

Instructor: Why is that important?

Jackie: Well, because with the 4/20 it's like what fraction *of all four* pizzas. But reading the question I see it as what fraction *of one* pizza does each person get? And in that case then there are only five pieces in each pizza. Not 20. ***(Backing)***

Jackie's backing for the argument was the first comment made where the class started to formulate a definition for what is meant by *of a* pizza. Her comment was also the first instance where the class started to develop the idea of relating the denominator in the solution to the whole described in the *of what*. By understanding *of a* pizza to mean *of one* pizza, Jackie noted that the 4/20 is incorrect because one pizza has five pieces and not 20. Thus, she introduced the idea that the denominator is related to the whole described in the solution.

Several students agreed with Jackie's comment that the language *of a* pizza refers to one pizza and that the denominator should be five and not 20. Those students, including Jackie, then believed that the correct solution would be 1/5 *of a* pizza. Responding to whether or not 1/5 is correct, Edith provided data that the question would have asked for the amount *of each* pizza each person would receive if the answer were 1/5. Other students, such as Katherine felt that a solution of 1/5 *of each* pizza resulted in everyone receiving 4/5 *of a* pizza total, which still fit the conditions of the problem.

Edith: Well I thought that at first and I guess I just kind of had a realization, because if it was going to be 1/5 wouldn't it say a fraction *of each* pizza not *a* pizza? Because *[in]* total they do get 4/5 *of a* pizza. So I'm maybe I'm reading too much into it. ***(Data)***

Instructor: Katherine?

Katherine: Well, that's what I did. I put that each person would get one slice from *each* pizza and then *[a]* total of 4/5. Because that's still out *of a* pizza according to the instructions. ***(Data)***

Evident from Edith's comment, if the question would have asked for the amount *of each* pizza a person received then the answer would be 1/5. Since the question asked for the amount *of a* pizza, the solution would be 4/5. Katherine made a similar comment regarding everyone getting 1/5 *of each* pizza which would be 4/5 *of a* pizza altogether.

Prospective teachers' difficulties in determining the answer to the sharing task resulted from not understanding the language *of a*. Multiple solutions were presented and the class interpreted *of a* to mean *one* pizza, *each* pizza, or *all the* pizzas. As a result, the prospective teachers were unsure of the whole and confused on what the question was asking.

Throughout the remainder of the second day, the prospective teachers continued to have difficulty determining which whole to define and what language to use. To help them distinguish between the language *of the, of each, of one*, and *of a*, an extra activity was created that was related to the sharing task. The activity was designed specifically for the class to continue language conversations.

The class was presented with two circles and asked to determine what the shading would look like if 1/2 *of each* pizza, 1/2 *of one* pizza, 1/2 *of a* pizza, or 1/2 *of the* pizza were shaded. The class agreed on the shading for 1/2 *of each*, 1/2 *of one*, and 1/2 *of a*. Disagreements arose for the shading of 1/2 *of the* pizza. Claire interpreted this as the referring to *all the* pizzas.

Claire: I said just shade one whole pizza because it's half of everything up there

Alex interpreted the problem as referring to either one pizza or the other.

Alex: The word pizza is singular. So it's one half of the pizza. Which pizza are you referring to the first one or the second one? I would agree with her if she said one half *of the* pizzas and you added an s on the end

Though the language *of the* refers to all the pizza given in the problem (Lamon 2005), difficulties for the prospective teachers resulted from their understanding of the singularity of the word pizza. Even though language understandings for *of a, of one*, and *of each* had developed, the class still had difficulty with *of the*. Language issues with *of the* focused around the word pizza. Some students felt that *of the* included *all the* pizzas, whereas others felt *of the* referred to one pizza.

In response to understanding the language *of the*, Alex commented that the number of pizzas should be labeled in the solution.

Alex: I would say one half *of the two* pizzas. One half *of the three* pizzas. One half *of the four* pizzas. How many pizzas there are. ***(Data)***

With Alex's comment regarding the singularity of the word pizza, she provided data for the solution by discussing that to be clear, the number of pizzas can be labeled in the phrase as well.

By the end of this activity, the class developed an understanding of the language *of a, of one*, and *of each* for fractions less than one. This was evident from student responses for shading 1/2 *of a* pizza, 1/2 *of one* pizza, and 1/2 *of each* pizza. Those solutions did not require warrants or backings or were questioned by others in the class. For 1/2 *of the* pizza, some students believed this meant *all the* pizzas whereas others such as Alex believed that this meant *of one* pizza because pizza was singular.

Language did not become the focus of conversations again until the class moved on to fraction operations. This occurred on the seventh day and came up only when the solution to a problem was greater than one. The first problem the class was presented with was an addition situation where the solution was greater than one (see Table 7).

The solutions of 11/8 or 1 3/8 were agreed on by the class however students such as Mary thought that the whole was two pizzas.

Claire: She ate one pizza and 3/8
Instructor: Is that okay?
Class: Yeah
Instructor: Okay… Mary
Mary: Can't you just write 11/8 *of two* pizzas? ***(Claim)***

Mary presented a claim that the 11/8 is *of two* pizzas and other students immediately questioned her solution. Edith provided data for the argument that two pizzas is incorrect because the denominator of eight in the solutions refers to the number of pieces in one pizza.

Caroline: So if you write 1 3/8, that's of *two* pizzas? I'm confused
Instructor: So 1 3/8. Which one is it Mary? Of…
Mary: Two pizzas
Caroline: How?
Instructor: How?
Edith: The denominator represents the unit and because the denominator still represents one pizza. You wouldn't say 1 3/8 is *of two* pizzas. Because it's not. ***(Data)***

Edith's comment was the second time the class discussed the idea that language for describing wholes is related to what the denominator represents. Edith described the denominator as the unit and stated that the denominator of eight represents one pizza. This idea shifted from the conversation that occurred on the second day when the class discussed the solution of 4/20 for sharing four pizzas equally among five people. During the second day, the idea was presented as backing that 4/20 was incorrect because the directions *of a* pizza referred to *one* pizza. The idea shifted to data but not until five class days after it was introduced.

Though this idea shifted, it was not *taken-as-shared* yet because the class still questioned its validity. Students, such as Caitlyn, felt that the language *of one* was incorrect for solutions greater than one.

Table 7 Fraction addition problem

Martha came into the pizza parlor and ate 3/4 of a medium cheese pizza. Then she ate 5/8 of a medium pepperoni pizza. How much pizza did Martha eat altogether?

Caitlyn: Yeah but why would it be one if you're eating more than one?
Jackie: Vocabulary sounds funny
Instructor: It does
Jackie: Wouldn't it be one pizza you said?
Caitlyn: Yeah one slice *of one* pizza. It seems like 1 3/8 are hard

Caitlyn's comment that 1 3/8 is hard referred to the way in which the whole should be described. Her difficulties stemmed from the idea that if you are eating more than one, the language for defining the whole should be in terms of more than one as well.

By the end of the seventh day, the idea that the denominator relates to the language for describing wholes shifted in function. This was presented as backing during the second day and as data during the seventh day. Though the idea shifted from backing to data, it was questioned because the language sounded incorrect for situations with fractions greater than one.

During the eighth day, the class continued working with addition situations and still struggled with explaining fractions greater than one. When given the problem 5/6 + 5/8, the class determined that the solution was 70/48; however, Caitlyn commented that they she still had difficulties labeling the answer. Edith used the idea that the whole is related to the denominator as data for the argument. Caitlyn then provided more data that the whole is out *of one* pizza which has 48 pieces.

Caitlyn: If we're using pizzas would that be 70/48 *of a* pizza? Because I still don't understand that whole pizza thing
Instructor: So we had 5/8 *of a* pizza leftover and 5/6 of the same size pizza leftover. How much pizza do we have altogether? We have 5/8 *of a* pizza and 5/6 *of a* pizza. Edith?
Edith: Wouldn't it be 70/48 *of one* pizza because one pizza is 48? ***(Data)***
…
Caitlyn: But is the reason that it's one pizza because you're using 48 as your whole out of 48 slices? Like that's only one pizza. ***(Data)***
Class: Yes
Caitlyn: I see now

Caitlyn's question was the last instance where students questioned the language used for labeling an answer. Since this idea shifted in the argument and was no longer questioned, it became *taken-as-shared* (see Table 8).

Discussion

The results of this study indicate that three mathematical ideas became *taken-as-shared* as prospective elementary teachers developed an understanding of language use for defining the whole. The first was that fractional solutions depend on a group or whole. The second included defining an *of what*. The third was developing language in terms of what the denominator represents.

Difficulties prospective teachers have in developing fraction language result from not understanding the language *of a*. The distinction among *of a, of one, of each,* and *of the*, appeared to be inherently difficult for them to understand. This is similar to research with children's conception of naming fractional amounts in terms of a specified unit (Lamon 1996). Lamon (1996) notes that children interchange language when defining wholes

Table 8 Language *of a* shifts and is no longer questioned

Problem	Claim	Data	Question	Warrant	Backing
Share four pizzas equally among five people			… what fraction *of a* pizza each person receives?		…Reading the question I see it as what fraction *of one* pizza does each person get? And in that case then there are only five pieces in each pizza. Not 20
3/4 + 5/8	11/8 of two pizzas	The denominator represents the unit and because the denominator still represents one pizza	Yeah but why would it be one if you're eating more than one?		
5/6 + 5/8		… it's one pizza because you are using 48 as your whole out of 48 slices? Like that's only one pizza			

without realizing the discrepancy with using different language. Within this study, prospective teachers held similar conceptions by not understanding what is meant by *of a*.

Other difficulties the prospective teachers have are distinguishing between the questions *how much* and *how many*. When the question *how much* is asked toward the end of the discussion of sharing four pizzas among five people, the class understands the solution to be 4/5.

Instructor: They are wanting to know, *how much* did each person eat?
Mary: Well, then it's 4/5

Mack (1990) found that when tasks are presented in familiar contexts, such as pizza, children are able to answer *how much* pizza. This study presented similar results. When the question "*How much* did each person eat?" was asked, prospective teachers knew the answer was 4/5. However, when the question was phrased as "*How much of a* pizza did each person eat?" prospective teachers did not understand what was meant by *of a*. Initially, 30% of the class gave a response of four, which answered the question *how many* not *how much*. Lamon (2005) notes that children often do this as well. Mack (1990) suggests that a familiar context aids in children's development of fractions by bridging the gap between their informal knowledge and ability to define the correct whole. Though this study incorporated a familiar context, prospective teachers struggled with understanding the language for defining a whole. Thus, for the prospective teachers, a familiar context did not seem to be enough to aid in their understanding. Their understanding also relied on the language used in the question.

Implications

This article described prospective teachers' conceptions of and development with language use for describing fractions. The results provide insight into the types of language understandings prospective teachers bring to mathematics teacher education programs and documents how language understanding develops. The results have several implications for teacher education programs and future research studies focusing on mathematics content courses.

Toulmin's (2003) argumentation model was shown to be effective in documenting classroom activity. By knowing when an idea shifted position in an argument and/or was no longer questioned, an analysis could be done which illustrated when prospective teachers developed fraction language and how. This understanding did not develop linearly as one might expect. The analysis illustrated that communal learning overlaps and is non-sequential. This is similar to Stephan and Rasmussen's (2002) findings in that an idea that is introduced first does not necessarily become *taken-as-shared* first. In this study, the mathematical ideas intertwined to the extent that at one point during the first day all three mathematical ideas were emerging before any one idea became *taken-as-shared*.

Language learning is complex and the tasks used in this study were successful in eliciting valuable and informative conversations. By discussing how to group pieces students were able to formulate the idea of defining and describing an *of what*. This then provided them with a foundation to start developing the idea that fractions depend on a referent whole (Conference Board of the Mathematical Sciences 2001) and lead to an understanding of what is meant by the phrases *of a, of one*, *of the*, and *of each*.

The findings also indicate that when prospective teachers develop an understanding of language for fractions less than one, this does not signify their understanding of language for fractions greater than one. In this study, language for fractions less than one was developed six class days prior to language for fractions greater than one. This gap may have resulted from the fact that all of the tasks used in this study incorporated fractions less than one, but not every task included fractions greater than one. Further task development and studies are needed to determine whether prospective teachers' development is more simultaneous when both types of fractions are continually used throughout instruction.

Though three mathematical ideas became *taken-as-shared* by the class, this does not indicate that the mathematical ideas were *taken-as-shared* by every student (Rasmussen and Stephan 2008). Individual students' development may not follow the same route of development as the whole class. This aspect of classroom learning was beyond the realm of this study, but an avenue of instruction which needs to be studied in future research.

This article examines prospective elementary teachers' difficulties and growth with language for defining the whole. With the need for teachers to have a deep understanding of mathematics, they also need to be able to communicate that knowledge effectively. By fostering language, education programs can simultaneously support prospective teachers' development of and communication skills with the mathematics they are to teach.

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