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From preschool teachers' professional development to children's knowledge: comparing sets

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Abstract Recently, there have been increased calls for enhancing preschool children's mathematics knowledge along with increasing calls for preparing preschool teachers to meet the demands of new preschool mathematics curricula. This article describes the professional development program Starting Right: Mathematics in Preschools. Focusing on four episodes that revolve around the topic of equivalent sets, it demonstrates how engaging preschool teachers with challenging tasks may promote their subject matter knowledge as well as their pedagogical content knowledge, leading to an interweaving of knowledge and practice. Assessment of the program included assessing the children's knowledge. Results indicated that students of participating teachers learned to be more flexible and to give more possible answers than students of non-participating teachers.

Keywords Preschool teachers' knowledge · Comparing sets · Equivalent sets · Assessing professional development

Introduction

It is challenging to plan a professional development program for teachers. First, one needs to consider the content of the program, what the teachers will be learning. For example, Schwan Smith (2001) claimed that the content of professional development should be: (1) focused on students' learning as a goal, (2) grounded in mathematics, (3) designed to support the teacher's day-to-day practice, and (4) appropriate to the participants' contexts.

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Equally important is the process of effective professional development, the manner in which the teachers will learn. In this article, we present the content and process of a segment of professional development for preschool teachers. There is also the issue of assessment, both short-term and long-term outcomes of professional development programs. Sometimes, it may take as long as 5 years before teachers become proficient in new practices (Ferrini-Mundy and Schram 1997). Ultimately, we need to recall that "student achievement is undoubtedly the main goal of the vast majority of efforts to change classroom practice. Yet, there are few studies that link amount or type of professional development to improved students achievement" (Tirosh and Graeber 2003, p. 678). In this article, we describe a professional development program for preschool teachers that attempted to meet this aim (at the end of the program, knowledge of children who learned in participating preschools was assessed and compared with those of children learning in non-participating preschools).

Starting Right: Mathematics in Preschools is an integrated program where both preschool teachers and the children learning in the preschools are participants.[1] In this program, teachers participate in professional development, while the children participate in a mathematically enriched environment created by their preschool teachers. A major aim of this professional development program is to increase teachers' mathematical and pedagogical content knowledge and in turn hopefully increase the children's mathematical knowledge. Thus, in line with Tirosh and Graeber (2003), assessing the effectiveness of this professional development includes evaluating children's knowledge. In this article, we focus on the professional development aspect of the program, describe a case that exemplifies some of the principles of our work with teachers, and point out a way of increasing preschool teachers' knowledge through engagement with tasks.

Professional development for practising teachers varies in many aspects including duration, form, and content. A program attended by teachers may range from a 1-day summer meeting followed by eight workshops during the year, to a semester course given on a weekly basis (Tsamir 2008), to an intensive 2-year program (Graven 2004). The program may take the form of university courses immersed in theory or workshops immersed in practice. A joint position paper recently published in the United States by the National Association for the Education of Young Children (NAEYC) and the National Council for Teachers of Mathematics (NCTM) related to this variance and offered some guidelines: "Inservice professional development needs to move beyond the one-time workshop to deeper exploration of key mathematical topics as they connect with young children's thinking and with classroom practices" (NAEYC and NCTM 2002, p. 6). This recommendation stresses the need to focus on the content of professional development programs for teachers of young children.

All too often, preschool teachers receive little or no preparation for teaching mathematics to their young children (Ginsburg et al. 2008). Instead, it is left to in-service professional development programs to fill this gap. What added knowledge must a practising preschool teacher gain from such a program? What are some of the tools that may facilitate this learning? At what point in the program should knowledge gained be implemented in practice?

We believe that mathematics can be a focal point of professional development for preschool teachers and that it does not have to be only hidden within other subjects, important though they may be. The knowledge necessary for teaching mathematics at any

[1] In Israel, preschool children 4–6 years of age often learn together and are taught by the same preschool teacher.

level is multi-faceted and difficult to define. Shulman (1986) differentiated between subject matter knowledge and pedagogical content knowledge. Subject matter knowledge refers to the knowledge of facts, rules, procedures, and concepts required of a specific domain. In line with the Principles and Standards for School Mathematics (NCTM 2000), mathematics instruction should include not only mathematical content but also mathematical processes such as reasoning, communication, and problem-solving. Schoenfeld and Kilpatrick (2008) added that proficient teachers should know school mathematics in depth and breadth. Deep knowledge of a subject includes knowing what "students are likely to have when they enter a classroom and where they will be going mathematically in future years" (p. 326). Broad knowledge includes knowledge of the mathematics taught at the current grade level and seeing connections to other topics at the same level.

"Teachers of young children should learn the mathematics content that is directly relevant to their professional role. But content alone is not enough. Effective professional programs weave together mathematics content, pedagogy, and knowledge of child development" (NAEYC and NCTM 2002, p. 14). Hence, developing teachers' pedagogical content knowledge is an integral part of professional development. Pedagogical content knowledge includes "an understanding of what makes the learning of specific topics easy or difficult: the conceptions and preconceptions that students of different ages and backgrounds bring with them to the learning of those most frequently taught topics and lessons" (Shulman 1986, p. 9). Accepting the need to develop teachers' subject matter knowledge and pedagogical content knowledge still leaves us with the question of how teachers' learning may be facilitated.

The tools used to implement professional development are varied. They include tasks that engage the teacher (Watson and Sullivan 2008), analyzing teachers' narratives (Chapman 2008), focusing on thought provoking cases (Markovitz and Smith 2008), and theories that promote teachers' mathematical knowledge (Tsamir 2008). Recently, the use of tasks as a tool in mathematics teacher education has become the subject of extended research. Zaslavsky (2007), in her review of the special issue in this journal regarding mathematical-related tasks, noted that engaging with tasks results in several kinds of impact concerning "mathematics, pedagogy, and sensitivity to students, as well as interrelationship between these domains, teacher practice, attitudes, and dispositions" (p. 438). Watson and Sullivan (2008) pointed out that having teachers engage in tasks may have multiple purposes:

1. To inform them about the range and purpose of possible classroom tasks.
2. To provide opportunities to learn more about mathematics.
3. To provide insight into the nature of mathematical activity.
4. To stimulate and inform teachers' theorizing about students' learning (p. 110).

We take the view that learners come to know as they engage in, discuss, and reflect upon different activities that afford the opportunity to learn. For engagement to be genuine and for learning to take place, tasks must have a degree of unfamiliarity in the content, task type, and/or expected ways of engagement. We agree with Watson and Sullivan (2008) who claimed that "such learning may be mathematical, pedagogic, or a mixture of both depending on how task goals, discussion and reflection are structured in the teaching session" (p. 110). In our program, teachers engaged in tasks that promoted their knowledge of mathematics as well as mathematical processes. Teachers implemented suitable tasks in their preschools and recorded their work with children. The narratives and cases were then discussed in order to develop the teachers' understanding of how to implement tasks as well as to increase their understanding of children's thinking.

The implementation of tasks by the preschool teachers brings us to the question of teachers' practice and its place in a professional development program. We take the stand that an integral part of professional development is teachers' practice. In line with Clarke (1994), we believe that a program for practising teachers ought to "allow time and opportunities for planning, reflection, and feedback in order to report successes and failures to the group, to share 'the wisdom of practice' and to discuss problems and solutions regarding individual students and new teaching approaches" (p. 38). In addition, professional development that combines the study of content with thinking about classroom practice may foster among teachers an orientation to study teaching on a continual basis (Sullivan 2006).

The study described in this article relates to four sessions of the professional development program which revolved around the concept of comparing sets including the notion of equivalent sets. These sessions typified our work with the teachers. Teachers engaged with challenging tasks, addressing first subject matter knowledge, and then related pedagogical content knowledge issues. This led to a gradual interweaving of knowledge and practice. A rich variety of tasks and activities were implemented with preschool teachers. Some of the tasks were meant only for implementation with the teachers in order to strengthen their subject matter knowledge and pedagogical content knowledge. Others, in collaboration with the teachers and in consideration of their experience working with children, were adapted during the professional development program and implemented with children.

According to the mandatory Israel National Mathematics Preschool Curriculum (INMPC 2008), the preschool teacher has a key role in the mathematical development of the children.

> Young children also learn by mimicking the teacher. Therefore, it is important for the preschool teacher to use correct mathematical language when speaking to the children so that they may become used to mathematical language and repeat it. The use of correct mathematical language may prevent or minimize misconceptions later on (p. 15).

In this article, we bring to light the possibility of working with preschool teachers on mathematics that is not only based on real-life contexts but includes the precision, accuracy, and notation of mathematics. This is in accordance with the spirit of the Israel National Mathematics Preschool Curriculum (INMPC 2008), which gives the following example:

> Some of the shapes and solids that children meet in their everyday lives are not 'precisely' [geometrical] shapes or solids, such as when children position their bodies in order to form geometrical shapes. When this is the case, it is advisable to say to the child that the shape is similar to a triangle or rectangle, for example, rather than say it is a triangle or rectangle (p. 48).

As this article focuses on the professional development aspect of the program, we describe the teachers' reflections on their practice which came to light during the program, which then became a springboard for furthering both their subject matter knowledge and pedagogical content knowledge. Finally, as mentioned at the outset of this article, in our view, part of assessing the effectiveness of the program is assessing the children's knowledge. In the following section, we provide background on the participants, the program format, and the mathematical context.

Background

Eleven preschool teachers ranging in experience from 1 to 36 years attended the 2-year professional development program—*Starting Right: Mathematics in Preschools*. All teachers taught preschool in the same low socioeconomic neighborhood. The first and second authors were co-instructors of the program. Teachers met with the instructors on a weekly basis (4 h per week), either at a local educational center or in one of the preschools. Each session was video-recorded and transcribed. Quotes that appear in this article are translations based on the transcriptions.

Throughout the program, each teacher was visited by a member of the professional development staff who came to the preschool on a weekly basis, sitting with children and guiding the teacher in her endeavor to create a mathematically enriched environment for her children. Attention was given to both teachers' and children's affect. The aim of the program was for *all* children to exhibit competency in all mathematical areas taught (such as geometry, numbers, and operations).

Four sessions of the professional development program revolved around the notion of comparing sets, including sets that have the same number of elements. Several mathematics curricula around the world mention set comparison as part of the preschool curricula. For example, the Curriculum Focal Points (NCTM 2006) for prekindergarten and kindergarten mention activities such as counting objects in a set, creating a set with a given number of objects, comparing and ordering sets. In England, the Practice Guidance for the Early Years Foundation Stage (2008) suggests that young children engage in activities such as matching and comparing the number of objects in two sets, including the empty set as an introduction to the concept of zero. Moreover, the mandatory Israel National Mathematics Preschool Curriculum (INMPC 2008) states that preschool children should be able to compare the number of objects in different sets as well as be able to divide one set of objects into an equal number of objects. Thus, knowledge related to comparing sets was considered relevant for preschool teachers. During these sessions, only finite sets without repeating elements were considered. For the sake of clarity, equivalent sets were referred to as 'equal-number' sets during discussions with the teachers.

While the focus of these sessions was indeed comparing sets, this topic was also used as a catalyst for introducing the preschool teachers to some of the global issues of mathematics such as definitions, critical and non-critical attributes, applying and adapting a variety of appropriate methods to solve problems and the issue of problems that may have more than one solution or no solutions at all. In this article, we use the word *method* to describe the process by which one looks for results to a given problem. We use the word *outcome* to denote the results of applying the method to the problem. Taken together, a method and an outcome of a problem comprise a problem solution. The number of solutions to a problem is determined by the number of outcomes (and not the number of methods).

The professional development program

The sessions described here proceeded through stages, beginning with addressing teachers' subject matter knowledge related to sets, equivalent sets, and mathematical processes, continuing with discussing pedagogical content knowledge issues, and finally relating to practice. This section is organized according to these stages. We examine how the described interactions address the different purposes of teachers' engagement with tasks, analyze the different aspects of teachers' knowledge apparent during the discussions, and point out the relationship of practice and teachers' learning. The final stage was assessing children's knowledge related to equivalent sets.

Stage 1: Subject matter knowledge—what are equivalent sets?

Assessing teachers' knowledge of equivalent sets: the following question (taken from Tsamir 1999) was posed to all participants

Question 1

Here are two sets A and B:

$A = \{1, t, \alpha\} \quad B = \{7, w\}$

Is the number of elements in sets A and B equal? Yes/No

How did you reach this conclusion?

The first aim of this question was to highlight that when comparing the number of elements in two sets, one must disregard the types of elements within the sets and pay attention only to the number of elements in each set. It is not trivial for young children to ignore the size, color, or shapes of objects when asked to consider the number of objects in a set. On the other hand, adults know implicitly that when asked "Is the number of objects in one set equal to the number of objects in a second set?", the sole criteria for comparison is the quantity of objects in each set. Thus, as the instructors of a professional development program, we faced the challenge of making this criterion explicit for the preschool teachers such that they would appreciate the complexity of this issue for children.

The concept of number is quite abstract. By presenting to the teachers tasks using set notation and by relating to 'elements' as opposed to 'concrete objects', we were able to demonstrate to the teachers the abstraction of the concept of number and the inherent complexity of this concept. It may seem that the question above is artificial and that perhaps an everyday context would be more appropriate for preschool teachers. However, a more concrete task might not have afforded the teachers the same opportunity to experience children's difficulties and to discuss critical and non-critical attributes in a meaningful way. In addition, familiarizing teachers with set notation and the use of brackets allowed us later on to discuss sets in more general terms, rather than always refer to specific examples. We also note that in Hebrew, the word for a set of concrete objects is the same word used for the mathematical notion of sets. Thus, although it may seem to the reader that the language of sets may seem formal for preschool teachers, it is actually quite a familiar term. This familiarity allowed the teachers to approach the mathematical concept of sets without fear. On the other hand, the mix up between the everyday notion of sets and the mathematical notion of sets may have caused misunderstandings, as we soon demonstrate.

Teachers' written answers were collected by the instructors. Eight teachers answered correctly that the number of elements in set A was greater than the number of elements in set B. Three teachers wrote they could not determine the answer to this question. Two explained why:

1. It is not clear what the letters stand for. For example, w can stand for one thing and t for something else, so it is not clear and we are missing information.
2. If we compare the letters, for example t and w, then according to alphabetical order, t is smaller than w. So, if we are supposed to compare t and w and also 1 and 7, then it

> turns out that B is bigger. If we are supposed to compare the elements in each set, then A is bigger. So, it is hard to choose.

The above explanations highlight that indeed some teachers were more concerned with the nature of the set elements than the number of elements in each set. The first teacher's inquiry, as to what the letters stand for, indicates that she needs to know what the elements are before she can answer the question. If we had set before her concrete objects, this problem may not have arisen. We may not have had the opportunity to demonstrate to the teachers the process children go through in order to understand that three pencils and three bears represent the same abstract concept of three. By discussing elements in a set, the issue of the abstraction of number concepts is raised. Numbers are not bound by the physical elements being counted or knowing what these elements are. It is also unclear how the first teacher would compare the sets if the elements were known. The second teacher is confused as to the basis for comparison, if she should be comparing the value of each element (1 is smaller than 7) or the number of elements. In order to clear up the confusion, we immediately offered two additional pairs of sets for comparison, as described below.

Clarifying the misunderstanding

In order to clarify the situation, the following two questions are posed out loud for all of the participants to think about and discuss:

1. Here are two sets, $D = \{\text{Danny}, \text{Tommy}, \text{Bill}\}$ and $E = \{\text{Jim}, \text{Ron}\}$. Is the number of elements in sets D and E equal?
2. Here are two sets, $F = \{a, b, c\}$ and $G = \{y, z\}$. Is the number of elements in sets F and G equal?

The first question puts the context of sets into real-life. Both sets contain names of boys. We know for sure what the sets contain and that both sets contain the same type of elements. The second question is a mediator between the task of comparing sets with various types of unknown elements (sets A and B from the first question) and the task of comparing sets where the elements are both known and of the same type (sets D and E). Sets F and G both contain only letters. However, within the context of mathematics, letters are also used as variables representing some unknowns. So, on the one hand, both sets F and G contain only letters. On the other hand, there is a certain degree of doubt and uncertainty which these sets raise. This is in line with Zaslavsky (2005) who reported that "the tasks that were found particularly worthwhile in terms of meaningful learning were all associated with elements of uncertainty" (p. 2). In general, the progression of sets allowed for a gradual disengaging from the concrete to a more abstract notion of number.

All the teachers agreed that D had more elements than E. Joy explained:

> Here (pointing to set D) there are three children and here (pointing to set E) there are two. The names make it clear that we are comparing the number of children.

The second question sparked the following discussion:

> Joy: Here (pointing to sets F and G) there are letters so it's clear and defined. It's clear that the amount of elements in each set is different. F has three letters and G has two letters.
> Valerie: But it depends on the criteria (for comparison). Maybe the value of y and z is bigger than a, b, and c.

Samantha: It depends on the question.
Instructors: The question was do the sets have the same number of elements?
Joy: What is the definition of an element?

At this point, Valerie is not relating to the letters as variables but is inferring to the practice of assigning numerical values to letters. Indeed, each letter of the Hebrew alphabet has an assigned numerical value and the days of the week, the days of the month, as well as the verses in the bible all use this notation (e.g. א (aleph) = 1 and the first day of the week may be called "day aleph"). Valerie's comment regarding the value of y highlights once again the need to be explicit about criteria for comparison. We note that it is Valerie who raises the issue of criteria for comparison, and then Joy raises the issue of definitions. These issues are aspects of the formal nature of mathematics and were part of the subject matter knowledge we aimed to develop throughout the program. They are also related. The mathematical definition of a concept determines the critical attributes that then determine whether an instance is an example or non-example of some concept. This knowledge is also applicable to other topics taught in preschool. For example, when identifying shapes, in order to determine whether a shape is a triangle, it is important to first know the definition of a triangle which then determines the critical attributes for all triangles.

At this point, the instructors clarify the difference between the fundamental mathematical concept of a set and the everyday concept of a set. The instructors stress that comparison is based on the critical attribute of the number of elements in the sets and that number is an abstract concept not dependent on the type of elements in each set.

Summary

The tasks used to promote mathematical knowledge consisted of comparing the number of elements in different sets. Linchevsky and Vinner (1988) found that elementary school teachers as well as some mathematics coordinators found it difficult to accept the concept of a set as an arbitrary collection of elements. "The mathematical set concept has a number of formal properties and aspects which contradict the initial collection model" (Fischbein and Baltsan 1999, p. 1). Similarly, the preschool teachers in our study easily compared the number of elements in two sets when both contained names of children. They struggled to accept that the elements t, 1, and α may all be members of the same set. The different contexts of these tasks challenged the teachers, ultimately helping them gain a fuller understanding of the concept of sets. Knowing that equivalency is contingent on the number of elements and not the type of elements in each set is important to the broad knowledge necessary for teaching mathematics in the preschool. Young children acquire the concept of number by manipulating and counting objects. Number becomes an abstract concept when children come to realize that it is irrelevant what is being counted (Clements et al. 2002).

Stage 2: Furthering subject matter knowledge—methods for comparing the number of elements in two sets

Counting: is it always possible? Is it always preferable?

Referring back to the original question posed regarding sets A and B (see Question 1), a second aim of this question was to determine what methods teachers use when comparing the number of elements in two sets. Eight teachers claimed that the sets did not contain the same number of elements. The only explanation given by the teachers referred to counting the elements in both sets. As one teacher wrote:

> Set A has three elements and set B has two elements. In order to determine if two sets have the same number of elements, you need to count the number of elements in each set.

At this point, it is uncertain whether the teachers are familiar with other methods for comparing the number of elements in two sets. Teachers were then presented with an additional four questions (taken from Tsamir 1999) (Questions 2–5).

2. At a dance party, all the students danced in couples, a boy and a girl in each couple. No pupils were left without a partner.

$$Z = \{\text{The boys}\} \quad W = \{\text{The girls}\}$$

Is the number of elements in set Z equal to the number of elements in set W? Yes/No
How did you reach this conclusion?

3. Given the sets:

$$X = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\} \quad Y = \{a, b, c, d, e\}$$

Is the number of elements in set X equal to the number of elements in set Y? Yes/No
How did you reach this conclusion?

4. Given the sets:

$$P = a, b, c, d, e, f \quad V = \{a, b, c\}$$

Is the number of elements in set P equal to the number of elements in set V? Yes/No
How did you reach this conclusion?

5. Dan was ill. The doctor prescribed one green tablet every 3 h for the first week. Then, in the second week, he was ordered to take a red capsule every 3 h.

$$G = \{\text{The green tablets}\} \quad R = \{\text{The red capsules}\}$$

Is the number of elements in set G equal to the number of elements in set R? Yes/No
How did you reach this conclusion?

These questions challenged the teachers' assumption that counting the number of elements in each set is the only method that can be used when comparing the number of elements in two sets. In fact, it is not always possible to count the elements of a set. This was highlighted by the paired dancers of Question 2 (see Question 2 above) where a one-to-one correspondence between men and women indicated equivalence between the two sets.

Although counting may be applicable, it is not always preferable. This point was highlighted when discussing the two sets in question 5.

> Instructors: Did anyone count the number of green and red pills taken each week?
> Collective answer: No.
> Instructors: Even though you can count them, is it the most efficient method?

As the instructors pointed out, although it is possible to calculate how many green tablets and how many red capsules were prescribed for each week, it is clear that one-to-one correspondence may also be used. For each green tablet taken during the first week, a red capsule was taken during the second week.

Can we "see" which set has more?

When discussing questions 3 and 4, some teachers claimed to just "see" which set has more. This method was accepted but with constraints. It was agreed that when two sets have a dramatically different number of elements, then one may "see" that they are not equivalent. However, as the instructors pointed out, "In order to be sure that two sets have the same number of elements one needs to check." The instructor's comment brings the discussion back to valid methods of comparison such as counting, one-to-one correspondence, and the subset method, which is introduced below. The teachers were then invited to take a closer look at the sets in question 3.

Lilly: The bigger one (set *P*) includes in it the smaller one (set *V*)
Instructors: So, we have another method. When one set is a subset of another, then which has more?
Valerie: The subset has less. (Note that Valerie's reply refers to the case of finite sets only. The case of infinite sets is not discussed at this stage.)
Joy: So, (the method of) seeing is called the subset method.
Eve: The subset method is not appropriate for the sets in question 3.

The last comment by Eve suggests that she is beginning to differentiate between the methods for comparing the number of elements in a set and that she is aware that not every method is appropriate for every situation.

Summary

Engaging with the tasks in the above episodes provided an opportunity for teachers to learn more about mathematics and the nature of mathematical activities. The teachers, with guidance from the instructors, came up with four different methods for comparing the number of elements in two sets: counting, one-to-one correspondence, "seeing", and the subset method. This is an important aspect of the content knowledge necessary for teaching about equivalent sets. Although the teachers may later choose not to teach their children specifically about sets, teachers should be familiar with the different methods in which one can compare the number of elements in two sets. Furthermore, it is important to know mathematical concepts in breath, including knowing that the principle of one-to-one correspondence has many applications in the preschool. For example, Joy connected what she had just learned regarding comparing sets by using one-to-one correspondence and noted how she often paired up the children in her class for some activity. One of the principles of counting items is assigning each item exactly one number. This is an application of the one-to-one correspondence concept.

Beyond the mathematical content discussed in this episode, teachers were exposed to mathematical processes such as the idea that there may be more than one method to solve a problem. This is an important aspect of deep mathematical knowledge and was developed throughout the 2-year program. It also became apparent that certain situations elicit the problem solver to use different methods. The two sets presented in the first question encouraged the problem solver to count. However, the sets presented in question 5 did not.

Another issue that was raised was the need for certainty and exactness in mathematics. The method of "seeing" may be acceptable if the difference between the numbers of elements is vast and one is only interested in knowing which set has more elements. However, if one wants to show that both sets have exactly the same number of elements, one needs to verify this by a more precise method. The teachers believed at the outset of this episode that counting was the only method which could be used and that it could always be used when comparing the number of elements in two sets. By the end of this episode, they realized that "always" is a word that should be used with caution that often there is more than one method for solving a problem.

Stage 3: From subject matter knowledge to pedagogical content knowledge—solutions include methods and outcomes

Demonstrating a task

Discussion began with the following task played out by the instructors:

> Three brown rectangular blocks are placed on one side of the table and five identical blocks are placed on the opposite side of the table. The eight rectangular blocks are all the same size and represent bars of chocolate. One instructor asks the other: "Can you make it such that there is an equal number of chocolate bars in both sets?"

Initially, the instructors demonstrated three different solutions to the teachers. It is important to note that after each demonstration, the original arrangement of five items in one set and three in another was restored. Solution 1: Method—Two items from the set of five are removed. Outcome—three items in each set. Solution 2: Method—One item from the set of five is transferred to the set of three. Outcome—two sets with four items in each set. Solution 3: Method—Four items from the set of five and two items from the set of three are removed. Outcome—one item in each set.

Finding "different" solutions

After the instructors' demonstrations, the teachers were asked, "Is there another solution?" Valerie approaches the table, transfers one item from the set of five to the set of three creating sets of four items in each set. She then lines up each set of four in a row one underneath the other. Valerie's outcome was already presented by the instructor. The difference here is the arrangement of the items on the table, in two neat rows, one underneath the other, as opposed to leaving them on two opposite sides of the table. For Valerie, it seems that a different outcome may entail a different spatial arrangement of the sets. Rachel offers another idea, "Gather them (the items) all together and give them out. One for me, one for you. Until it's equal." Rachel offers the same outcome, but here, the method for creating sets with an equal number of items was different. Nancy comments, "Just think that someone would come and eat it all up and say that now the two sets are equal." Nancy is the only one actually presenting a different solution. If both sets are empty, then they are equivalent.

Outcomes or methods?

At this point, the instructors ask the teachers to consider, "How many solutions are there to the question?" The following discussion ensues:

> Joy: Four.
> Instructors: What are they?
> Samantha: To transfer, to take away…

Samantha's reply does not indicate what the specific outcomes are (i.e. four items in each set). Rather, Samantha indicates how she goes about creating sets with an equal number of elements.

Orit: We thought about three (outcomes) where one has two methods.
Joy: I made a mistake. Maybe it's not four (outcomes) because two are the same.

Joy has begun to think about the difference between the outcome and the method used to reach the outcome. She realizes that if two different methods lead to the same outcome, then it is actually only one solution, and therefore, her first response (four solutions) was likely incorrect. Joy is then requested to demonstrate her solutions:

Joy: First, I'll transfer one from here (the set of five items) to here (the set of three items). A second option is to zero out the sets (Joy removes all the items from the table). Let's see… remove two from this set (Joy removes two items from the set of five items creating two equivalent sets of three items each).

At this point, we are unclear whether Joy differentiates between the number of methods in which she can find an outcome and the number of outcomes.

Rose: Now remove another two (this would create two sets of one item in each set).
Joy: What? I should again do removal?

We are still not sure whether Joy differentiates between methods and outcomes or perhaps, Joy is unsure whether she is allowed to use the same method twice and therefore removal should not be done again.

Orit: That's what I did.
Samantha: But that's the same method.
Orit: It's an additional outcome using removal.

Orit differentiates between outcomes and methods. Two sets with one item in each is an outcome which was not previously given. Removal is a method used to reach this outcome.

Instructors: We need to summarize. The word solution brought up images of methods as well as outcomes. (The instructors then demonstrate all five solutions, creating alternatively equivalent sets of four items, then three, two, one, and finally zero items in each set.) So, how many outcomes are there?
Orit: Five.

Changing the problem parameters, changing the number of problem outcomes

After discussing the importance of presenting to children problems that have more than one outcome, the question arose as to how the number of chocolate bars presented at the outset would affect the number of outcomes. The instructors arranged one set of four brown rectangular blocks on one side of the table and three brown rectangular blocks on the other side of the table and asked the teachers to set it up so that each had an equal number of "chocolate bars":

Fern: Take one away.
Lilly: But didn't we say that all together there has to be an even number of items? Because otherwise, one person always has more.
Instructors: So, can't you make them so they have an equal number of items?

Lilly: Not with whole items (which can't be broken). You can do it only by removal but that's the same method each time and you want different methods. It seems that if there is an even number of items, there will be more options.
Nancy: Perhaps if the kids start with an even number (of items) it will be easier for them later to do a problem with an odd number of items.

In the above discussion, teachers are beginning to realize how the different parameters may affect the number of outcomes as well as the number of methods available to reach an outcome. If there is an odd number of items all together, then transferring one item from one set to another set is not an option. This problem sparks another possibility:

Sarina: If a child cannot figure out how to create equal-number sets with an odd number of items, then … adding one (item) we will get equal-number sets.

Sarina's suggestion raises a possibility which had not been brought up previously, that of adding items from a reserve stash. Of course, if one can add items from a reserve stash, this may affect the number of outcomes. This question is posed by one of the instructors, "What do you think? Will there be more possibilities with the stash?" Collectively, the teachers respond that allowing a stash will increase the number of outcomes to a set problem.

Linda: So, when I have a stash of items, then anyway it encompasses …the other options. And if I have a bigger stash, that has even more items, then I have even more options.
Instructors: So, we would like to create equal-number sets without and with a reserve stash. The conclusion that has come up is that with the addition of a stash, the number of outcomes always increases. Who agrees?
All: (All the teachers raise their hands.)

First, we note that the instructors, although replying to Linda, readdress the question to all the teachers, bringing everyone into the discussion. The instructors then consider how the number of items in the stash will affect the number of outcomes.

Instructors: I think it's best when the stash is bigger. Because, as the stash grows, the number of outcomes, what?
Lily: Increases.
Instructors: Always? Are there always more outcomes? Raise your hand if you agree.
All: (Everyone raises their hands.)
Instructors: Let's say that before there were five items in the stash and now there are four. Then what happens to the number of outcomes?
Orit: It decreases.

The instructors leave the teachers with the following questions to think about, "Does having a reserve stash always increase your options? What would happen if we started with equal-number sets and wanted to create unequal-number sets?" Leaving open-ended questions at the end of a session was frequently done in order to encourage the teachers to continue thinking on their own about new mathematical ideas raised during the program.

Summary

Knowledge of mathematical processes was at the heart of this stage, illustrating the ways in which knowing in depth and breadth is intertwined. The original task centered on creating equivalency from two inequivalent sets. This seemingly simple task served as a catalyst to discuss the existence of problems that have more than one outcome, differentiating between the outcome of a problem and the method used to reach an outcome, and reaching the same outcome by different methods. In turn, discussing these issues fine-tuned teachers' knowledge about the meaning of equivalent sets. Equivalency is based solely on the number of elements in each set and not on the spatial arrangement or method used for comparison. These issues were readdressed as the parameters to the original problem were changed.

Changing the parameters of the original task also encouraged a discussion related to pedagogical content knowledge. At least one teacher (Nancy) began to consider how the level of difficulty may be affected by changing parameters and in which order different problems should be presented to the children. Engaging with the tasks in these episodes not only provided insight into the nature of mathematical activity but also began to increase teachers' awareness of the children's point of view. As Sarina pointed out, a child who reaches an impasse may come up with a novel idea for solving a problem. It was hoped that this discussion would help prepare the teachers for implementing similar tasks with their children.

We note that the issue of subitizing was not raised at this point. Knowledge of how young children may compare small collections readily and possibly subconsciously may also influence the types of comparison tasks teachers might pose, the expected responses from the children, and the way teachers might respond to those responses. The issue of subitizing did come up during later sessions with teachers, not discussed in this article.

Stage 4: From pedagogical content knowledge to implementation and back again—working with the children

Planning a task

The following task was proposed by Rose and Orit:

> Rose: Since it is autumn, we thought of hanging two trees on the board. Two trees that look the same. On one tree we thought of hanging five leaves and on the second tree we would hang three leaves that are the same. We would tell the child to organize the leaves such that on each tree there will be the same number of leaves.
> Instructors: What does it mean "look the same"?
> Rose: The same color, the same size, the same type. Later on it can change.
> Instructors: Can I transfer (leaves) from one tree to another?
> Rose: Yes. They have Velcro on the back (meaning that they can be stuck and unstuck with ease).
> Instructors: Is there a stash?
> Orit: No.

The instructors' role during this discussion was to clarify the specifics of the task and make explicit all the issues that need attention.

At this point, the teachers discuss the possible solutions to this problem and how the context might help or hinder the children. Recall that previously teachers discussed creating equivalent sets within the context of chocolate bars. Most children like chocolate. If they think someone else has more, it would seem natural to take from the other person in order to "even out" the treats. However, as the instructors point out, "In the matter of transferring leaves from one tree to another, it is not for sure that the children will see this as an option." The instructors are pointing out that the context of autumn leaves may not be suitable for this particular task, and in general, the context of a task is a parameter that needs to be taken into consideration. As we shall see below, the teachers do not heed this warning and choose to use this context anyway. Teachers also discuss the ramifications of starting with equal amounts of leaves, such as four leaves on each tree. One teacher is curious if the children will bring up the option of zero leaves on each tree.

Reporting on the children

During the next session, teachers reported and discussed the implementation of equivalent-related tasks which they enacted in their preschools including how the children performed on these tasks. Most of the teachers reported that they implemented the task described previously by Orit and Rose. Four teachers reported that their children transferred a leaf from one tree to another in order to create equal-number sets. Three teachers excitedly claimed that their children zeroed out the leaves. Samantha added, "The children actually said that now the amount is equal." In some cases, the children added their own constraints to the task. For example, Joy reported that one child thought of every option except for taking away two leaves from the tree with five leaves, "It is as if he thought he had to do something each time with both trees. It was very strange to me". Eve reported on one child who said "You have to add two (to the set of three)." When Eve would not let him do so, he could not come up with any solutions.

Even when teachers claimed to implement the same task, it became apparent from the discussion that this was not always so. Orit described segments from her interaction with one child. She started with two trees, five leaves on one tree and three leaves on the other. And then:

> He transferred a leaf from one tree to another and there were four and four. And then I asked him (the child) if there was another solution and he said we can remove one from here and one from here and we're left with three and three.

This child created two equivalent sets of three elements each from two equivalent sets of four elements each. The difference between the planned task and the implemented task was pointed out by both Nancy and the instructors. As Nancy pointed out, "He (the child) did not start from an initial state of two sets with a different number of items". This prompts an open-ended discussion on which situation is easier for children, creating equivalence from inequivalent sets or creating equivalence from equivalent sets. The instructors themselves could not attest to which task would be easier for children to solve. This question, like many others that arose during the program, is indeed a worthwhile research question. It was agreed among the participants that this question would be interesting to investigate, and indeed, some of the teachers later implemented such tasks with their children.

Linda took the initiative and developed her own task that she then implemented in her preschool:

> I used rounded candies of different colors. At first, the children looked at the colors. Blue with blue. Red with red. And they removed the two (candies) that didn't have matching colors. What was interesting was when I asked them if there were other options. Then they realized that they can't relate to the colors but have to start counting.

Linda's experience once again prompts a discussion regarding the parameters which may cause a problem to be easier or more difficult for young children to solve. The instructor also points out that the use of different types of elements, for example different shapes, or as was the case with Linda, the same shape but with different colors, may hinder a child from using the method of one-to-one correspondence.

Summary

Pedagogical content knowledge was explicitly addressed when talking about activities enacted with the children. Teachers' pedagogical knowledge includes knowledge about what makes a concept more difficult or easy to learn and how to present the concept within different contexts. Both of these aspects were discussed as teachers grappled with planning appropriate tasks raising questions that were not necessarily answered by the instructors. Implementation was integral to the learning process of the teachers. As teachers reported on their activities with children they collectively and cooperatively reflected and became attentive to the subtle differences between tasks that were similar but not quite the same, how these differences may be reflected in the children's work, and how the parameters of a task may impact on the ease or difficulty of a task.

Engaging with tasks prompted the teachers to reflect on the children's thinking. Following implementation of the tasks with their children, teachers were able to see and compare how different children solved the same problem as well as what some of the obstacles were. Teacher also acknowledged the relationship between children's emotions and their ability to solve problems, noting when children became excited. Being aware of the relation between affect and learning is another important aspect of teachers' knowledge.

Stage 5: Assessing children's knowledge

As noted in the beginning of this article, part of evaluating the professional development program included assessment of the children. Two groups of children participated in this stage. The first group consisted of 81 children, 5–6 years old, who were taught by the teachers participating in our program. The second group consisted of 82 children, 5–6 years old, who were taught by teachers who did not participate in our program. All children learned in the same town and had similar socio-economic backgrounds. We compared the performance of children who learned with participating teachers to the performance of those who did not.

Each of the 163 children was interviewed individually in a quiet corner of the class. The child was presented with two distinct sets of bottle caps—three bottle caps were placed on one side of the table, and five bottle caps were placed on the other. All bottle caps had the same shape, size, and color. The child was asked: "Can you make it so that there will be an equal number of bottle caps on each side of the table?" After the child rearranged the bottle caps, the interviewer returned the bottle caps to their original arrangement (three in one set, five in the other) and asked the child, "Is there a different way to make the number of bottle caps on each side equal?" This was repeated until the child said that there is no other way to create equal amounts of bottle caps on each side of the table.

This task has five outcomes. Table 1 indicates that the percentages of program children who suggested each outcome were greater than those of non-program children from other preschools. The percentages in Table 1 may also point to the level of difficulty of each outcome: the outcome (4;4), meaning four bottle cops on each side of the table, was the easiest, (3;3) was somewhat harder, (2;2) and (1;1) were evidently harder still. The cognitively problematic outcome, consisting of empty sets (Linchevsky and Vinner 1988), was employed by only 27% of program children and 9% of non-program children.

Table 2 presents percentages of the number of outcomes each child found. The table shows that 56% of program children came up with at least four outcomes as opposed to 45% of non-program children who offered no more than one outcome. In general, results

Table 1 Percentages of the outcomes provided by the children

Children	Outcomes				
	(0;0)	(1;1)	(2;2)	(3;3)	(4;4)
Program ($N = 81$)	27	52	65	80	88
Non-program ($N = 82$)	9	38	39	67	72

Table 2 Percentages of the numbers of outcomes per child

Children	No outcomes	One outcome	Two outcomes	Three outcomes	Four outcomes	Five outcomes
Program ($N = 81$)	2	6	15	21	37	19
Non-program ($N = 82$)	7	38	12	16	20	7

indicated that students of participating teachers learned to be more flexible and to give more possible answers than students of non-participating teachers. For a more detailed discussion, see Tsamir et al. (2010).

Summing up and looking ahead

In our program, enhancing teachers' subject matter knowledge and pedagogical content knowledge was done mostly through their engagement with tasks. In the spirit of the mandatory Israel National Mathematics Preschool Curriculum, some of these tasks placed a strong emphasis on accuracy and precision. We acknowledge that enhancing preschool teachers' knowledge of set comparison may also take place by engaging the teachers with tasks that place greater emphasis on real-life situations. It is up to professional development providers to enhance teachers' knowledge in accordance with culturally accepted norms, using national curriculum standards as guidelines.

Some of the tasks from the professional development program and some variations of these tasks were implemented by the teachers with their children. The enactment of the tasks by the teachers in their own preschool then served as a basis for reflection which further informed teachers' subject matter knowledge as well as their pedagogical content knowledge. Learning and practice informed each other. In addition, as Sullivan and Mousley (2001) suggested, the dilemmas that were raised as teachers chose, planned, implemented, and reflected on the tasks provided a way for the instructors to address and highlight the complexity of teaching.

Another important aspect of the program, which was then reflected in teachers' practice, was creating a supportive learning environment in which the instructors listened to the teachers and where mistakes became an opportunity for deepening the discussion of mathematical issues, without forfeiting mathematical correctness (Borasi 1996). Similarly, Rose commented on how important it is to "interview individual children before beginning to teach in order to ascertain their previous knowledge." Samantha reflected, "I learned to listen to the children."

The interweaving of teachers' knowledge, tasks, and practice may have additional benefits. First, it conveys an expectation that part of what is learnt in a professional development program is to be implemented in class even though other parts may not. In turn, teachers feel that knowledge gained is relevant to their everyday practice and may be further motivated to enhance their knowledge. Second, sharing ideas and experiences with other teachers in a supporting environment may help to create a professional community of teachers that will extend beyond the program. This is especially important for preschool teachers who, in Israel, are often isolated in their classrooms with little opportunities to meet with other teachers and professionals. Third, in discussing their practice together, teachers may increase their repertoire of activities, learning how to adapt the level or implementation of a task according to children's needs.

A lot of time and effort is invested in professional development programs. How can the success of such programs be evaluated? The answers to this question depend on the goals of the program. In our program, a major goal was to increase preschool children's mathematical knowledge by increasing their teachers' mathematical and pedagogical content knowledge. Thus, we reported on children's results as they worked on a mathematical task. Informal reports by the teachers also attested to the knowledge they gained. As Orit claimed, "in the past I was not aware of correct mathematical language and critical attributes. Today my knowledge in the subject is rich." To be sure, evaluating professional development is a non-trivial problem. As a mathematics education community, we should see this as a challenge.

Finally, we would like to point out that preschool teachers are professionals. They come to professional development programs with their own unique knowledge gleaned from their practice with young children. In our program, the teachers often suggested materials for constructing tasks. It was also the teachers who suggested the time and place for implementing the tasks within their daily routines. We believe that one of the principles of professional development is to regard the teachers as true partners in the process of change. Allowing teachers to bring in their expertise and accepting their suggestions can increase their sense of ownership as well as their confidence and ability, important traits for all teachers.

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