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Using technology to explore mathematical relationships: a framework for orienting mathematics courses for prospective teachers

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Abstract The technological revolution that has finally permeated K-12 education has direct implications for modern teacher educators whose "Hippocratic oath" is to best prepare future teachers for twenty-first-century classrooms. The goal of this article is to suggest that the heart of sound technological implementation is to encourage students to use whatever tools are available to explain the mathematical relations that underlie what they observe on the screen. We suggest ways in which Mishra and Koehler's construct of Technological Pedagogical Content Knowledge may be customized to provide a framework for guiding prospective teachers' efforts to develop and assess lesson plans that use technology in novel and effective ways. Data are presented in the form of two contrasting case studies to illustrate the differing degrees to which prospective mathematics teachers leveraged technology to teach themselves and their future students to explain the mathematics behind various topics.

Keywords TPACK · Technological knowledge · Content knowledge · Technology

Introduction

In his visionary book, Papert (1993) lamented that while time travelers from the turn of the twentieth century would have been puzzled by the advanced technologies used in hospitals, they would have recognized all of the antiquated tools found in any typical U.S. classroom. We are happy to report that this scenario no longer characterizes classrooms in the twenty-first century. For example, interactive whiteboards are now predicted to be in at least one of every six classrooms *across the world* by 2012 (IGI Global 2010). These changes are not surprising; they reflect shifts in society's view of the value and convenience of technology and the ubiquity of its use among both the adult and student populations. According to Project Tomorrow's recent survey of more than 370,000 students, parents, prospective and

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practicing teachers, and administrators nationwide, “Today’s students are rapidly assimilating and adapting new technologies used in their personal lives to drive increased productivity in their learning” (Project Tomorrow 2010, p. 1).

These projections offer mathematics teacher educators exciting opportunities to enhance the ways in which prospective teachers learn about mathematics and mathematics pedagogy. First, the ubiquity of technology in schools enables teachers to realize Papert’s (1993) vision of capitalizing on students’ informal learning style and their natural propensity to teach one another. Second, by adopting new perspectives on the importance of integrating advanced communication and visualization technologies into the teaching of mathematics, we have the opportunity to align teacher preparation curricula with what prospective teachers want (and believe they need) to learn (Project Tomorrow 2010). This shift will prevent institutions of teacher education from becoming marginalized in the eyes of the consumers and the public as well.

The question this article addresses is: how can we realign mathematics classes for prospective teachers so that they include a focus on the effective use of technology for teaching? The overall hypothesis guiding our attempts to answer this question is based on Mishra and Koehler’s (2006) framework of *Technology, Pedagogy, and Content Knowledge* (TPACK). This frame offers a helpful way to conceptualize what knowledge prospective teachers need in order to integrate technology into teaching practices. In short, this theory suggests that effective teaching with technology lies at the intersection of Technological, Pedagogical, *and* Content Knowledge as pictured in Fig. 1.

As can be seen, Mishra and Koehler leave the specifics of what lies in each circle to disciplinary researchers. For their part, the National Council of Teachers of Mathematics (NCTM) may be seen as beginning the work of defining TPACK by writing the “Technology Principle” (NCTM 2000) which recommends, in part, that each teacher use technology in “appropriate and responsible ways”. Our view of “appropriate and responsible” is to consider TPACK as *using technology to explore mathematical relations*. The goal of modern mathematics instruction should not be to use classroom-based technologies solely for drill and practice (although these programs serve as critical tools to track students’ progress and provide skill-appropriate tasks). Instead, our vision of TPACK is to help teachers develop a technological habit of mind oriented toward using advanced computation and communication tools to help students explore and *understand* the underlying concepts and their relation to the larger world outside of school. We contend that the key to

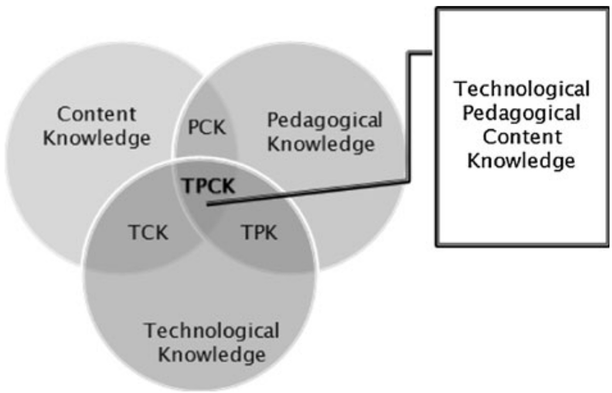


Fig. 1 Components of TPACK from Mishra and Koehler (2006)

this understanding—and the heart of TPACK for mathematics teacher educators—is to help prospective teachers learn how to support their future students' proclivity to exploit current (and future) technological affordances in order to describe the mathematical relations behind the results shown on a screen.

Literature review: defining TPACK in mathematics education

The theoretical perspective that underlies this work is based on a view of learning as a social process in which tools are seen as critical, mediating elements that have a direct effect on classroom practices and, therefore, what mathematical ideas are appropriated (Cobb and Yackel 1996; Sfard and McClain 2002). Using this perspective, we attempt to "fill in" some of the intersections of the TPACK diagram for mathematics education. First, we describe three constructs that can be seen as elaborating the technology content knowledge (TCK) intersection and then offer a vision of technology-pedagogy-content knowledge (TPACK) that could be used to prepare prospective teachers for using twenty-first century technology. In the results section, we use this elaborated framework to analyze a set of lesson plans created by prospective teachers. Our goal is not to present definitive data regarding a TPACK rubric, but rather to illustrate its pivotal role in helping teacher educators prepare prospective teachers for the future.

Overall theoretical perspective of learning

The overarching theoretical perspective that guides our work is based on Sfard's (2008) commognitive perspective of learning. Sfard has coined this term to suggest that educational researchers may do well to dispel the distinction between interpersonal communication and intrapersonal cognition. In her view, "…thinking is defined as the *individualized version of interpersonal communication*—as a communicative interaction in which one person plays the roles of all interlocutors" (Sfard 2008, p. iv [italics in original]. This perspective has been useful in guiding our efforts to elaborate TPACK for mathematics education in two ways. First, if learning is viewed as a socially situated practice, then (a) *teaching* can be seen as the practice of orchestrating mathematical discourses and (b) *learning* can be seen as the ways in which students engage in these discourses. In short, the role of any teacher (or teacher educator) can be seen as negotiating the emergence of conceptual discourse that involves the use of appropriate tools as a normative part of the commognitive process.

The role of the student is also intricately related to his or her participation in the discourse *with a focus on the ways in which tools mediate* the discussions and acceptable ways of proffering and debating mathematical ideas. As with many sociocultural theories (e.g., Wertsch 2002), the theory of commognition views tools in the broad sense to include symbolizations, language, and external tools such as technology—all of which are seen as mediators of activity. However, unlike other social theories, Sfard goes further to examine what is meant by a "mediating" activity. She maintains that tool use directly affects what is learned because doing mathematics is seen as a recurring cycle of acting on mathematical objects and then symbolizing this activity. As the symbolizations become taken as shared within the learner's immediate discursive community, they become reified mathematical objects for subsequent actions and hence produce more mathematics. Sfard also points out that this recursive process is autopoietic—it is the only system of communication that allows for self-generation and perpetuation of knowledge and hence differentiates the process of human learning from that of other species who cannot communicate what is known (or culturally accepted) from one generation to the next. For our purposes, this perspective illuminates the importance of building on stable mathematics with current—albeit transitory—tools. For example, even though Euclid did not have access to *The Geometer's Sketchpad* (Jackiw 2001), we use this tool to explore and make further connections among objects first proposed as early as 320 BC. In this way, the body of knowledge is enhanced and passed on to the next generation of learners who will reinterpret and advance the ideas with their tools and purposes of the day.

TCK: research on technology in K-12 mathematics instruction interpreted from a commognitive perspective

Of the three subsections illustrated in Mishra and Koehler's (2006) frame shown in Fig. 1, we argue that technological content knowledge (TCK) has received less research attention than the other two. In particular, technological pedagogical knowledge (TPK), which includes the broad range of all research on distance education and computers in schools in general, has been under study for many years and examined from many perspectives ranging the spectrum from purely behaviorist to primarily social. Similarly, pedagogical content knowledge (PCK), which was first described by Schulman (1987), has been extensively studied in mathematics education by many researchers including Ball and her colleagues (Ball et al. 2005, 2008; Hill et al. 2008). Therefore, we limit this examination of research to findings related to TCK and TPACK within mathematics education. In what follows, we offer three constructs from mathematics education literature that can be seen as lying at the intersection of mathematical knowledge and technological knowledge.

Representational fluency

Students who are able to move between various mathematical representations (e.g., between a graph and a linked simulation) are described as having fluent representational knowledge. Research has shown that when students are asked to explore technology-based environments that contain linked, multiple representations of abstract concepts, they build a rich web of relationships that ultimately leads to this type of generalized understanding (Bowers and Nickerson 2001; Bowers et al. 2002; Dick and Edwards 2008; Confrey and Maloney 2008; Roschelle and Kaput 1996). For example, in two different research reports, both Roschelle and Kaput (1996) and Bowers et al. (2002) observed students interacting with linked representations in the *SimCalc mathworlds* program. When teachers in both studies asked their students to explore the relations between graphs and linked simulations, the learners engaged in "what if" explorations that supported their ability to move fluently between the various representations. These explorations were mediated by both the software *and* the dialogic conventions that were established among group members. The students' fluency emerged as they engaged in the discursive practices of posing and refuting hypotheses regarding what they saw on the screen.

Looking toward the future, newer technologies such as touch-sensitive screens—now ubiquitous on many hand-held devices—further support the development of representational fluency by allowing students to share representations among group members, manipulate those representations more directly, and share their views via social media or public forums (perhaps with others not even physically present). From a commognitive perspective, these changes in dialogic practices will no doubt affect changes in students' proclivities to rely on technologies that are not exploited during typical oral interchanges in traditional classrooms.

Knowledge resulting from reification

Learners who are able to use technology-based tools to encapsulate their actions into objects and then build on these objects demonstrate reified knowledge (c.f., Sfard 1991). For example, Patsiomitou (2008) argued that when students used *the Geometer's Sketchpad* to create "personal tools" to encapsulate the series of steps required to construct algebraic or geometric objects, they were able to view the results of these operations as structural units. Patsiomitou concludes that "It is obvious that the processes in the software are more efficient when students are structuring algebraic expressions than in a paper–pencil-scissors environment" (p. 6). She also suggests that the "…tools comprising DGS [Dynamic Geometry Systems] may well constitute a channel whereby children extend their imagination and conceive mathematics like a source of mathematical models and representations" (p. 7). Goldenberg et al. (2008) actually go so far as to claim that "What we see as we watch children or adults 'play' with this software is often a change of perception of mathematics, from mathematics as a collection of rules and procedures to mathematics as an intellectual game, a response to curiosity, a human endeavor" (pp. 79–80).

Knowledge constructed from noticing

The role that noticing plays in the construction of mathematical knowledge has been studied in many areas of recent mathematics education research. For example, Meel (2003) cites Pirie and Kieren's (1994) work to describe "property noticing" as occurring when "the learner can examine a mental image and determine various attributes associated with the image. Besides noticing the properties internal to a specific image, the learner is capable of noticing distinctions, combinations or connections between multiple mental images" (p. 145). The question of how technology can support the process of noticing was elaborated by Battista (2008), who reported that students using a dynamic geometry environment noticed which properties of various shapes were preserved during various transformations with more clarity than when they compared static pictures of the same shapes before and after the transformations.

Instructional designers have been attempting to develop exploratory microworlds that enable students to notice various properties inherent in the precreated mathematical objects for a long time (c.f., Underwood et al. 2005; Kaput 1993). For example, Kaput (1993) notes that cybernetic manipulatives may be even better than their real-world counterparts (e.g., watching a car on a screen versus one in real life) because students can make real-time hypotheses and collect immediate feedback. These features are consistent with other designers such as Sinclair (2003) and Confrey (1999) who mention the importance of experimentation and surprise (i.e., perturbation) and the inclusion of multiple pathways for finding a solution. Sinclair also notes that the use of color, motion, and markings help students notice particular relations among mathematical objects. What is particularly important to note from this work is Sinclair's conclusion that when learners were able to use the tools in novel ways and were challenged to conduct open-ended explorations, they actually learned to hone their visual interpretation skills to reconstruct their ideas of dynamic change.

In recent years, learners have been able to access many of these small exploratory microworlds (i.e., online applets) via the Internet. The two largest repositories of applets are the Virtual Manipulative Library (http://www.nlvm.usu.edu/en/nav/vlibrary.html) and the

Illuminations project (http:www.//illuminations.nctm.org/). NCTM pioneered the latter project in an effort to illuminate particular recommendations published in their Principles and Standards (2000). From a commognitive perspective, research findings suggest that these types of activities have been particularly useful because they include sequencing and challenge questions that help teachers encourage fruitful discussions and explorations. In this way, the combination of the applets and the accompanying questions can be seen as tools that mediate the mathematical dialogues and practices that emerge when they are enacted in classrooms. In addition they "…help students build ownership of their own learning by explicitly perceiving their growth toward understanding mathematical learning targets" (Hart et al. 2005, p. 239).

TPACK: when knowledge forms an orientation

While it would be convenient to list similar knowledge constructs that comprise the main intersection of technological, pedagogical, and content knowledge, we have come to the conclusion that this intersection may be the empty set. In other words, we assert that there is no particular "knowledge" or skill subset that prospective teachers can be taught that will ensure their use of technology to enhance mathematics instruction. In our view, the goal for teacher educators should be support preservice teachers' development of an orientation that views technology as a critical tool for identifying mathematical relationships.

The conclusion that TPACK is an orientation rather than a subset of knowledge pieces was alarming at first. However, many research reports support this conclusion. For example, Zbiek and Hollebrands (2008) maintain that teachers need not acquire one particular expertise or pick one particular role; instead, teachers (and prospective teachers) need to become aware of how to design rich tasks that integrate technology into the classroom discourse so that technology-based conjectures and arguments become normative. Our interpretation of Zbiek and Hollebrands' work is that, from a commognitive perspective, teachers can support the emergence of these technology-enhanced classroom norms by asking more open-ended questions that demand technological explanations and encouraging students to become the ultimate arbiters of whether those explanations are clear and accurate.

We also found evidence that a TPACK orientation does not necessarily emerge by simply giving teachers access to, and training with, technology. For example, in their case study of practicing teachers in different schools, Goos and Bennison (2002) concluded that teachers' technology use was not necessarily proportional to their school's resources. On one hand, they found some teachers who worked in technology-rich environments with ample professional development who did not appear to have developed a proclivity to use computers or interactive whiteboards, even when their use was supported by the administration. On the other hand, they found other teachers who were working in poorly resourced schools who were "…very inventive in exploiting available resources to improve students' understanding of mathematical concepts" (p. 322). Given this evidence that there may not be a set of knowledge or skills that that lie at the intersection of technology, pedagogy, and mathematical content, we have begun to ask a different question: What factors *do* affect prospective teachers' development of a TPACK orientation?

One factor that has shown to be correlated with effective technology use suggests that if a teacher's pedagogical style begins with "what if…" explorations and encourages discussions and refutations based on examining how actions and reactions are explained within the environment, then the class is more likely to examine mathematical relations. Such practices locate the locus of authority within the class discussions rather than with the computer or teacher and hence support students' own thinking and learning (Bowers and

Nickerson 2001; Moyer-Packenham et al. 2008). In essence, the goal is not to teach skills but to encourage a TPACK orientation.

One component teacher educators can use to encourage preservice teachers' development of a TPACK orientation is to draw attention to successful and detrimental negotiation strategies that involve the use of technology. For example, Zbiek and Hollebrands describe two patterns of negotiation, funneling and failure to probe, that can derail successful discussions. Funneling occurs when teachers (often inadvertently) limit the types of hypotheses students make or the models they might create. This can occur, for example, if a teacher asks only closed-ended questions, relies on tutorial-based software that poses only single-approach tasks, or asks students to complete step-by-step worksheets to scaffold computer-based explorations. The problem with funneling is that students are led to accept conclusions they did not necessarily make, and hence, there is little personal investment (or perturbation) involved in resolving why a certain outcome occurred. Funneling can also diminish the probability that students will use the computer to express alternative models.

Failure to probe occurs when teachers do not ask hypothesis-generating questions at all. Teachers who have been observed missing opportunities to probe may have a narrow view of the power of technology to explore deep, underlying mathematical relations, and, as such, may lack TPACK. As Zbiek and Hollebrands (2008) point out, "Ironically, questioning patterns may hinder rather than facilitate exploration, reasoning and conceptual understanding—three often-expressed teacher goals for using technology" (p. 304).

In summary, we assert that TPACK, conceived as a subset of knowledge skills, is a red-herring. Rather than elaborate specific types of knowledge that teacher educators should "impart" to prospective teachers, our view of learning as a social process motivated by communication suggests that the best we can do is engage prospective teachers in technology-enhanced mathematical explorations with the explicit goal of discussing the ways in which technology enabled them to describe relationships among objects on the screen that could not have been developed without the tools employed.

Methods

Given our new view of TPACK, we aimed to examine two questions:

1. To what degree did students' projects demonstrate various aspects of TPACK? And, what other factors might have contributed to the distribution?
2. Can the elaborated view of TPACK (with the goal of identifying the mathematics that underlies what is seen on the screen) serve as guide for helping prospective teachers develop and implement activities for their future classes?

This qualitative study falls into the category of design-based research (cf. Brown 1992; Cobb et al. 2003). It was not designed to measure or quantify TPACK, but rather to offer ideas about how to fill in the "sections" of the model and then examine how we can help prospective teachers develop a proclivity to use technology in pedagogically sound ways.

Subjects and setting

The setting for this design experiment was a 6-week course that took place at a large, southwestern university in the United States. The goal of the course was to prepare prospective high school teachers to use *the Geometer's Sketchpad* to teach (and learn) various topics in mathematics. The class met for 3 h per week in a computer laboratory, and students received one unit of academic credit. The instructor for the course was the first author of this article.

Of the 21 students enrolled, 16 were prospective high school teachers (mathematics education), two were liberal studies majors (prospective elementary/middle school teachers), two were statistics majors who were interested in teaching, and one was a prospective science teacher. One of the unique aspects of this course was that it was offered in three simultaneous modes, any of which the students could choose to take: (1) traditional face-to-face sessions held in the computer laboratory, (2) asynchronous online version including Camtasia videos featuring descriptions of the content, pedagogy, and demonstrations of *the Geometer's Sketchpad* sketches, and (3) a combined (hybrid) version in which students were able to choose either face-to-face or in-person sessions depending on their schedules. Despite the different modes, all students were required to hand in the same assignments on the same calendar dates and enter comments on the discussion board during the same time intervals.

Data sources and methods

In order to introduce the idea of TPACK to the prospective teachers, the instructor described some of the research detailed in the section above. She then asked her students to post reactions and examples on a web-based discussion board. For example, one student posted "I learn a great deal about mathematical content by exploring examples." This post was translated as shown in the first entry of Table 1. Other students' posts included descriptions of experiences or examples from this or other classes (or any other technology experience) that they perceived as fitting in each of the subsets of knowledge shown in Fig. 1. In order to include the online students, a special Camtasia video was also posted on the discussion board prompt after the initial in-class introduction so that all points that were made during the in-class session were communicated to the online students as well.

Once the students had all posted and responded to others' posts regarding the various examples of TPACK types, the instructor worked with the students in the face-to-face class to brainstorm ways to shorten, clarify, and classify the entries within each of the circular intersections to make a framework of thinking about TPACK in the class. The results are shown in Table 1.

By Examining Table 1, we see that the students' list of criteria for TCK does reflect some of the research findings, but does not align completely. In particular, point (1) regarding dragging reflects some of the ideas about representational fluency. Point (2) regarding the creation of perfect objects based on properties rather than sketches touches on the idea of noticing in that students are exploring shapes within a confined system. Point (4) regarding the use of scripts aligns with the *process-object (reification) theory* of mediated technology use. The differences between the research findings and the students' perspectives illustrate a critical point: teacher educators need to help prospective teachers see the larger educational goals of thinking mathematically with technology rather than their immediate use in solving particular problems.

The main data source for analyzing the class's progression toward the development of TPACK was each student's final project. As might be expected in a course such as this, the assignment involved choosing a mathematical topic of interest and developing a *geometer's sketchpad* sketch to explore how technology could be used to enhance a textbook-only lesson. The written portion of the assignment consisted of answering the following questions:

1. What is the concept you are exploring? (Please describe mathematical content thoroughly, do not just use a mathematical definition or term).

Table 1 Components of TPACK as outlined by prospective high school mathematics teachers using GSP

Content knowledge	1. Learning by exploring examples 2. Using compass and straight edge to "feel" constructions before using computer—easier to learn that way
Technological knowledge	1. Ability to use *the Geometer's sketchpad* to create sketches based on written directions 2. Integration of features such as reflection that go beyond "cybernetic" construction tools
Technological content knowledge	1. Dragging to explore, check properties, and check results 2. Creating perfect shapes based on knowledge of properties rather than drawings (cannot do with paper and tools) 3. Using interface helps explain terms and steps (e.g., reflection as a step in construction) 4. Use of scripts to simplify complicated constructions
Technological pedagogical knowledge	1. Change learning process 2. Being aware of how tools can enhance communication (e.g., sharing ideas, trading sketches, computer-aided peer assessment) 3. Challenge students with open-ended tasks, not just following directions 4. Encourage multiple solutions
TPACK	1. Shift from focus on comparing measures to making logical arguments 2. Coming up with challenging tasks that "turn around" textbook questions—make tasks that you could not ask if you were not using a computer 3. Push students to first describe what is moving on the screen and then ask them to explain why (mathematical relations)

2. What is the *mathematical investigation* and *resulting insight* that you hope the project will elicit?
3. Describe the goal of the sketch and specific features that allow explorations. What misconception or visualization do you aim to enhance?
4. How does this lesson plan enhance a textbook-only exploration of the topic?

In order to share examples of their TPACK, all students were required to present their projects to the class for discussion. These presentations were recorded using Camtasia Studio so that the presenter's voice, questions from the audience, screen shots, and mouse movements were captured for later analysis.

The analysis of these projects was conducted by two graders, the instructor and a Teaching Assistant. The rubric used was based on the constructs that the class had outlined (see Table 1). The two graders were in complete agreement for 10 of the 12 students. They discussed the other two projects until 100% agreement was reached.

The basic goal of coding each project involved determining the degree to which the student demonstrated TK, TCK, TPK, or TPACK with the assumption that these types of knowledge are additive. In other words, all projects involved using *Sketchpad* and all worked admirably. Therefore, all students demonstrated at least TK. If a student demonstrated a good use of the technology to examine a particular content area but did not include any particular presentation affordances, such as use of color or scripting tools, then he or she was characterized as having knowledge at the level of TCK, but not TPACK. If, on the other hand, a student presented a project that clearly indicated a strong emphasis on the use of technology to explore the mathematical relations behind a mathematical phenomenon, then the student would have been rated at the TPACK layer. It is important to recall here that since not all students indicated a direct interest in teaching, we did not expect that all students would push for a project at the level of TPACK. Given this consideration and the fact that the students' mathematical backgrounds varied, we did not use the TPACK table or the content difficulty as a grading criteria. We felt that this would also enhance the research as well because, if students had been graded solely on the criteria they had outlined in Table 1, then the results would have been skewed by a push to get a good grade rather than a desire to create a final project that reflected the student's own views on how to best use the software for learning.

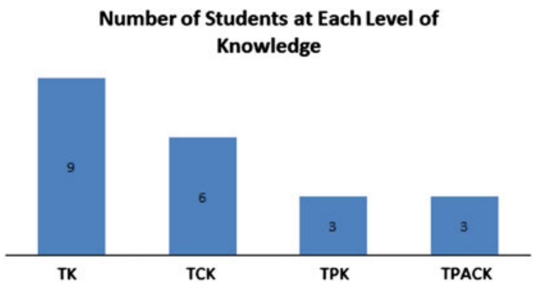
Results

Question 1 To what degree did students' projects demonstrate advanced levels of TPACK? And, what other factors might have contributed to their development of various levels of TPACK?

The results of the coding process are shown in Fig. 2. As can be seen, only three of 21 student projects reflected a TPACK orientation.

In order to identify some factors that may have contributed to these students' levels of demonstrated knowledge, we compared each student's level of TPACK with his or her attendance mode. As can be seen in Fig. 3, all of the students who chose the completely online version of the course scored at the minimal level of TPACK. This could suggest that these students did learn how to use the program, but did not engage in the discursive

Fig. 2 Distribution of TPACK codes across 21 students



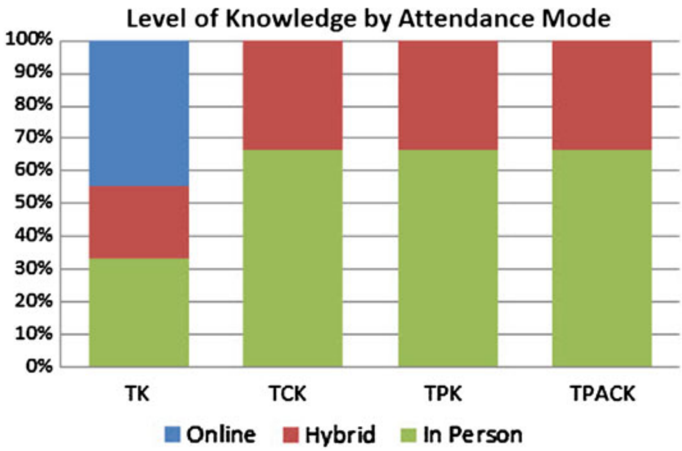
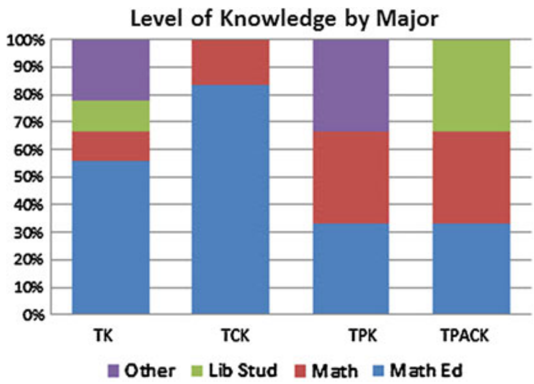


Fig. 3 Levels of TPACK based on attendance pattern

Fig. 4 TPACK by major



practices of using technology to probe either their own, or their future students' thinking in the conceptual ways that were discussed during in-person sessions.

We also decided to examine the degree to which the students' majors affected their propensity to develop TPACK-based projects. In particular, since this course was open to both prospective teachers (both elementary and secondary) and non-prospective teachers (including statistics and biology majors), we hypothesized that these educational goals might affect the students' use of TPACK in their projects. As can be seen in Fig. 4, this hypothesis does *not* appear to be supported.

In contrast, the results of the coding indicate that only two of the three students who produced fully TPACK-infused projects were prospective teachers (one mathematics education major who wants to teach at the secondary level and one "liberal studies" major who wants to teach elementary school). The third student was a straight mathematics major who had no interest in pursuing a teaching credential, but found *the Geometer's Sketchpad* to be useful in helping him model ideas he was learning more abstractly (his project involved developing geometric interpretations for the inner product of two vectors in R^2). The results at the other end of the spectrum are also interesting. Fully 55% of the projects displaying only technological knowledge were prospective secondary mathematics teachers, a finding that is both surprising and somewhat troubling. It is important to recall that these results are based on a small number of students, many of whom were taking the class online. Moreover, the goal of this preliminary study was not to quantify or prescribe a measure for TPACK, but rather to gain some insight into how the framework helped to prepare prospective teachers to use technology in pedagogically productive ways.

Question 2 How can TPACK, as elaborated, be used to design and assess lesson plans?

In order to more fully describe the difference between a project displaying TK versus one at the TPACK level, we contrast two representative projects from these sections. This analysis is also designed to explore how the overall framework helped each of these prospective teachers plan their lesson projects.

It is important to note that the actual determination of TPACK, which we have maintained results in a qualitatively different form of teaching that evolves over time, could not logically be demonstrated in any project write-up or presentation. However, the projects did include lesson plans and descriptions of how the sketch *was intended* to be used in class, and thus, if an author included specific indications or insightful questions that might perturb students to probe the mathematical relations behind what they were seeing, then the author was seen as demonstrating a TPACK orientation. In the next section, we present two example projects. The first shows how one student did anticipate integrating TPACK by modeling how she would revise a generally procedural topic with *The Geometer's Sketchpad.* She first described the mathematical ideas, then described how it might be used in a class, and then posed insightful questions that would push her future students to explore the content from several perspectives. The second example illustrates how another student *thought* she was demonstrating the concept of derivative effectively, but did not apply the mathematical relations model, and hence did not realize the true potential of engaging a TPACK orientation for her instructional plan or her own learning.

TPACK project example 1: using TPACK to plan a well-designed lesson

Student 1 developed a plan and sketch to help her future students "discover" the formulas for finding the sum of the interior and exterior angles for polygons. When describing how the sketch would enhance instruction, the student wrote the following rationale:

> This investigation involves using triangles to create different polygons (such as pentagons and hexagons) to find the sum of their interior angles. It also involves students exploring the sums of exterior angles through the definition of supplementary angles, and the angle measures they found in the first part of the investigation. Finally, it involves students exploring the differences between regular polygons and others by finding the sum of each interior angle of a regular polygon. [Student then showed screen in Fig. 5].

> The following discussion occurred during the student's presentation to the class:

> S: For the exterior angles sketch, my aim is to have students realize that no matter what polygon you have, the sum of the exterior angles will always be 360°. A lot of kids think that the since the sum of the interior angles of different polygons are

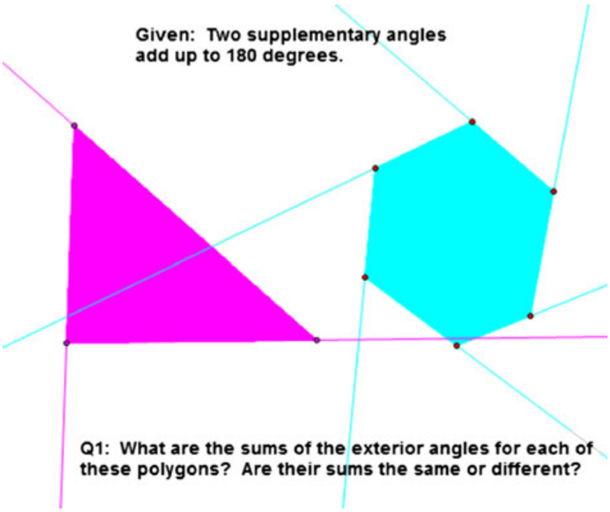


Fig. 5 Student's activity page for exploring relation between sums of exterior angles

different, the sum of their exterior angles are different. So I want to challenge this idea. First they are asked to use the measure tool to add up the angles of the two figures. Then, I ask them what it says in #2 here, 'Drag the vertices of each polygon so that they lie on top of one another. Use this to explain that your answer for question 1 is correct.'

I: And what would you expect to be a good explanation [for question 2]?

S: What I wrote for my expected answer here was, 'When dragging the vertices to lie on top of one another, a circle can be formed around all of the exterior angles. Since a circle represents 360°, this shows that the answer from question 1 is correct, and the sum of the exterior angles of any polygon is 360°.'

I: How did you come up with this idea?

S: In my math class, our instructor described how we could imagine doing this. I couldn't really imagine what would happen, so I wanted to use *the Geometer's Sketchpad* to figure it out for myself. For me, what was helpful was seeing the lines extended; otherwise the figure just goes away. Because when I did it at first, the figure just disappeared and I wanted to figure out what happened.

This student's project and presentation demonstrates TPACK in several ways. First, the exploration questions she asks demand causal explanations (e.g., Q2 asks 'how does this exploration relate to your answer in Q1?') based on prior explanations rather than simply descriptions of what is shown on the screen. Second, the student not only lists a common mathematical misconception (PCK), but also devises a technological exploration to leverage the use of dragging to confront it. Third, as described in her presentation, the idea for this exploration came from reflecting on the difficulties and insights she had when learning this material without technology. Moreover, she developed a unique way to use the technology to overcome her difficulties with the mental imagery involved. Fourth, in her presentation, all terms were clearly defined and used correctly. She did not use referents such as "this angle here" or "that thingee". Finally, this student also demonstrated her knowledge of the importance of reified knowledge and targeted noticing. For example, she planned to ask students to make a tool for triangles that could be used to mark off triangles in other polygon shapes they will make. Her goal was to ask students to explore her premade sketches, but also model their own. In this way, she is not funneling to one conclusion; she was planning to capitalize on and probe her students' observations during whole-class discussions.

TPACK project example 2: not using full idea of TPACK when planning a lesson

The goal of this second example is to illustrate how a different student, who was unaware of her own fragile conceptions of the derivative, did not use technology to fully investigate the mathematical relations that underlied her sketch (indicating she had not developed a TPACK orientation), nor did she include plans to involve her students in exploratory hypothesizing activities (indicating a lack of PCK). The lesson plan taken from her project is shown in Fig. 6:

As she began the presentation, Student 2 showed the screen in Fig. 7 and stated the following:

S: You see um, on two points of the curve you see the line passing through them. Ah, let's see… f of x is this line here and f of x plus this distance h is from here to up here. So the difference of that is f of x plus h minus f of x. As *this distance* gets smaller, as we pull this point closer to the other one, the line intersecting those two

The Limit Definition of the Derivative

$$\lim_{(h\to 0)} \frac{f(x+h) - f(x)}{h}$$

1) The concept I am exploring is that the derivative of a function can be defined as the limit as h approaches zero of (f(x+h)-f(x))/h.

2) This investigation involves having students see how this function becomes the calculated derivative when decreasing the distance between two points on a curve, and see how the whole graph of this function becomes the derivative of a trigonometric function when h is decreased to zero in both rectangular and polar coordinates. The resulting insight will be that the limit definition of the derivatives finds the derivative of polynomials and trigonometric functions.

Fig. 6 Statement of intent from student 2

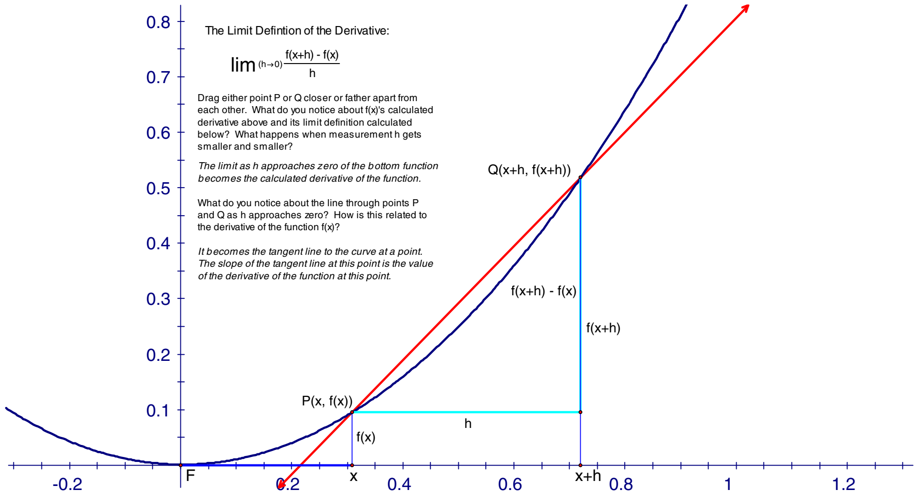


Fig. 7 Screen shot 1 from student 2

> points becomes the tangent line at the curve, and that … the delta height becomes closer and closer to 0 so that is the limit as h approaches 0 of the difference of the height over the *height*. As you can see, our *calculated derivative* becomes the same value of um, *the limit definition of the derivative*.

A review of the presentation video indicates that she was not pointing to nor appeared to be understanding where on the screen the quantities to which she referred were located. For example, as she moved the point P, she stated that the "distance gets smaller" An analysis of the video indicates that she was pointing to the geometric distance between the two points, while referring to h, the difference between the x-coordinates. She also states that the definition captures "the difference of the heights over the height." Here again, she is not sure what the "height" is, and mistakenly referring to h in this case. In short, her intention was to use the dynamic capabilities of *the Geometer's Sketchpad* to show that the secant approaches the tangent as h approaches 0, but she failed to identify what the quantity h represented. Thus, she had not developed the content knowledge to interpret the algebraic representation. This led her to a false conclusion in the next part of the activity (shown in Fig. 8).

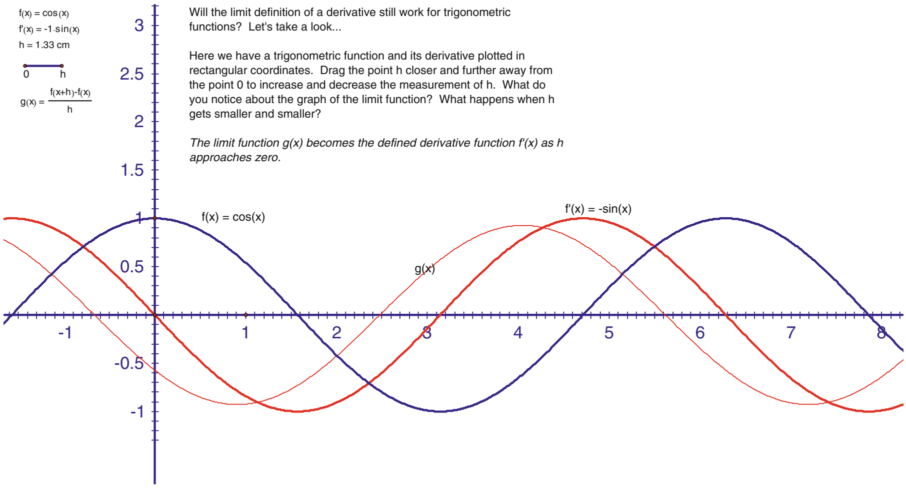


Fig. 8 Slide 3 from student 2

S: So we can see how this works in our familiar trig functions. We have our function here. This is our graphed function f of x and this is the graphed function f prime of x, the derivative of x, and this skinny red line is the *limit definition of the derivative*. So, as we change h [moving h slider toward 0], as h comes closer and closer to zero, we can see that the limit definition of the derivative becomes the regular … derivative.

As notated in the transcript, the student hesitated before concluding that the limit definition of the derivative "becomes the regular derivative." It was clear from her inflection that she was experiencing some perturbation, but she was convinced that her sketch was accurate because the two red graphs did align as her h slider reached 0 (which shows technological knowledge about sliders, plotting functions, and computing derivative functions). In order to further probe this dissonance and demonstrate the value of exploring mathematical relations, the instructor asked the following:

I: So, in this case, [shifting back to screen in Fig. 4] you are showing that the limit is basically how the secant becomes the tangent. Right?
S: Right.
I: And in the second one [Fig. 5], the blue line represents the sine function, right?… Is the thin red line sort of akin to… I mean what would you say the red line is in terms of the tangent line for the sine function?
S: I don't understand how that could be the tangent line for that.
I: It's not.
S: Right…
I: Well, where is the tangent line?
S: The tangent line still would be going along here. [Using extended hand to model a tangent traveling along sine function]
I: Right. So, what is the g of x then?
S: I am having a hard time with this… sorry.
I: No, it is a tough question. It is a very tough question.

As she worked through the perturbation further, the instructor suggested how TPACK could be used to explore this further:

I: So, you were saying that is not the tangent line. You are right; the tangent line is still going to be between two points on the blue line. And when you take the slope of that line, you are at that point, that is what this point is.
S: Hmm. Ok.
I: It might be helpful to include the points x comma f of x and x plus h comma f of x plus h in this sketch and then show the secant and tangent lines.
S: I was trying to do that actually, and I kinda ran out of time. OK, I get it…

The question of whether the student actually "got it" or not remains unanswered but we believe that had she added the tangent and secant to conduct a point-wise exploration, she would have explored the mathematical relations underlying the causal link between the tangent and the computed derivative function and gained insight into the definition of the derivative as well.

Discussion

When we compare the lesson plan and presentation given by student 2 with those of student 1, several differences emerge. First, in her introduction, student 2 does not talk about *exploring* the concepts underlying the limit definition; it is just presented in mathematical form. In contrast, student 1 was able to describe the mathematical content she was teaching in deconstructed terms and explain how the various parts of the assignment fit together. Second, student 2 did not envision future users constructing or creating hypotheses. Instead, the students were expected to "see what is happening" because something was moving on the screen. But the definition of what the components were—for example what h represented and what $g(x)$ represented—was not deconstructed or explored, and hence, the mathematical relations of how the thin red line came to lay atop the thicker derivative line was not brought to the surface. Instead, a point-wise interpretation of how the secant line lies atop the tangent line was mistakenly seen as function-wise interpretation of the limit definition of the derivative function comes to align with the computed derivative function at each point x. These features point out differences in the two students' demonstration of TC knowledge.

The project presented by student 2 might be seen as including representational fluency in that she illustrated the same concept in both Cartesian and polar coordinates. However, this reinforces the idea that technology can mediate our actions, but cannot, in and of itself, cause perturbations. Although we cannot know for sure, it appears that the first exploration (of $f(x) = x^2$) was taken from a typical textbook picture, whereas the other two sin function graphs were her own ideas. Finally, when describing how the project would enhance a textbook-only presentation, student 2 once again focused on how her future students would "see" a mathematical idea by "seeing" something changing. The question of mathematical relations, i.e., explaining why the graph of the definition of the limit approaches the actual limit graph is not probed. The remaining transcript illustrates the instructor's attempt to model probing questioning without funneling.

Conclusions

The main goal of this research was to explore the ways in which mathematics teacher educators might use Mishra and Koehler's (2006) TPACK framework to support and assess prospective teachers' knowledge of how to integrate technology into the teaching of mathematics. By elaborating the constructs more fully, we suggest that TPACK may better be viewed as an orientation than a set of subskills or knowledge constructs. This reformulation has enabled us as teacher educators to develop a greater sense of how to plan and focus instruction for prospective math teachers. In particular, we now make an effort to make explicit references to various pedagogical moves such as probing questions and unique technological features that support the need for causal explanations to support deeper mathematical understanding. It was also helpful for the instructor of the course to consider documented pitfalls when teaching with technology, such as Zbiek and Hollebrands' (2008) warning against funneling and failure to probe deeply enough (again, the goal of mathematical relations was critical here.)

A second way that the framework can inform teacher educators' work is its ability to guide students' own metacognitive processes as they reflect on their learning and development efforts. We believe the whole-class conversation wherein the students elaborated and reclassified the constructs was extremely helpful. It brought everyone present into the conversation since all of the students had made entries into the discussion board. Even those who were taking the class online (and hence were not present in-person during the negotiation) were invited to share their thoughts via the discussion forum. However, it is interesting to note that none of the students who chose to take the course fully online produced projects that represented the full ideas of a TPACK orientation.[1]

The data from this research also support the supposition that TPACK can elicit a qualitatively different approach to using technology to teach mathematics. However, with a small *n* and a loose interpretation of students' views, the results of the case study cannot be generalized. Instead, it is hoped that the reflection process might stand as a paradigmatic example for mathematics teacher educators to discuss how the TPACK framework can be used to elucidate what "appropriate and effective" use of technology in a teacher education course might include. In particular, the construct of probing mathematical relations (presented as a three-step process wherein students first hypothesize what might happen, then describe what they seen, and then work to resolve perturbations in an effort to describe mathematical relations) turned out to be a useful planning, teaching, and assessment guide.

From the students' perspective, the results of TPACK's utility may not have been as clear. Even though constructs were debated by the students and posted as a guide, the data seemed to indicate that it remained relatively elusive in some of the students' final projects. On the other hand, since this was not a specified goal of the grading scheme, it may have been effective in other ways than those measured. For example, the whole-class discussion that focused on generating the constructs shown in Table 1 did evoke a number of particular insights such as how dragging can be used as a tool to "check" not only a construction, but the way in which the independent points can be matched to the "givens" in a proof, and the dependent points can be mapped to the "prove" statements. A second insight that students mentioned was the way in which reading a script could support the generation of a proof, which led to insights about the varying roles of proof including explanation, communication, and investigation tool as well justification (c.f., De Villiers 1998, 1999). Taken together, these ideas interweave the processes of construction and proof generation and, in effect, prevent what Herbst (2002) describes as the danger separating the process of proof from the content of what is being proved, or "separating the practices of proving from the practices of knowing" (p. 308).

[1] Both of the students who were chosen for case study analysis attended all class sessions in person.

We conclude by noting that although this particular study focused only on the use of *the Geometer's Sketchpad* in the teaching of mathematics, the goal of placing mathematical relations at the heart of TPACK can be applicable to many other tools and technologies. For example, it can help guide teachers asking students to explore data sets with *Tinker-Plots* (Konold and Miller 2009) as well as teachers thinking about how to best utilize their new interactive white boards. If TPACK is pictured to be the intersection of TPK, TCK, and TPK, then the idea of mathematical relations is a whole greater than the sum of these parts. It involves conceptualizing a qualitatively different type of teaching based on the combination of different types of knowledge (Schulman 1987), using radically new communication technologies (Wertsch 2002), and engaging students who have been brought up using "apps" for just about every task at hand. However, none of these components are catalysts in and of themselves. It still remains the responsibility and purview of mathematics teacher educators to help future teachers conceptualize "appropriate and responsible" technology use. We hope that this idea of mathematical relations adds to the larger discussions of TPACK within mathematics education.

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