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Teachers attending to students' mathematical reasoning: lessons from an after-school research program

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Abstract There is a documented need for more opportunities for teachers to learn about students' mathematical reasoning. This article reports on the experiences of a group of elementary and middle school mathematics teachers who participated as interns in an after-school, classroom-based research project on the development of mathematical ideas involving middle-grade students from an urban, low-income, minority community in the United States. For 1 year, the teachers observed the students working on well-defined mathematical investigations that provided a context for the students' formation of particular mathematical ideas and different forms of reasoning in several mathematical content strands. The article describes insights into students' mathematical reasoning that the teachers were able to gain from their observations of the students' mathematical activity. The purpose is to show that teachers' observations of students' mathematical activity in research sessions on students' development of mathematical ideas can provide opportunities for teachers to learn about students' mathematical reasoning.

Keywords Teacher learning · Students' mathematical reasoning · Teacher knowledge · Practitioner–researcher collaboration

Introduction

There is a consensus in the mathematics education community that mathematics teachers need more opportunities to learn about students' mathematical reasoning. The 1999 Report of the Consortium for Policy Research in Education (CPRE) called for more opportunities for teachers to "learn not only about the subject matter, but also about how students think about the content" (p. 1). The discussion document of the Fifteenth Study of the

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International Commission on Mathematical Instruction (ICMI-15 Study) on The Professional Education and Development of Teachers of Mathematics welcomed contributions on ways in which teachers can learn more about students' learning of mathematics (Ball and Even 2004). Participants at the Meeting 1 of Thematic Group 3 of the First Conference of the European Society for Research in Mathematics Education (CERME 1) expressed interest in exploring research questions on how "teachers develop their understanding of children's ways of thinking during school practice" (Krainer and Goffree 1999, pp. 224–225).

This article reports on the experiences of a group of elementary and middle school mathematics teachers who participated as interns in the 1-year NSF-funded *Informal Mathematical Learning Project* (IML). In the project, middle-grade students from an urban, low-income, and minority community in the United States engaged regularly in after-school, classroom-based, well-defined mathematical investigations as a context for a study of the students' development of mathematical ideas, and different forms of reasoning and justification in several mathematical content strands. The teachers observed the students' mathematical activity and reported their observations in debriefing meetings with researchers held at the end of each research session. This article describes insights into students' mathematical reasoning that the teachers gained from their observations of the students' mathematical activity. The insights are based on the analysis of the teachers' observational reports in the videotaped debriefing meetings. The purpose is to provide evidence that teachers' observations of students' mathematical activity in research project on students' development of mathematical ideas can provide rich opportunities for teachers to learn about students' mathematical reasoning. The study built on research on the professional knowledge for teaching and theories about practitioner–researchers collaboration. The following research questions guided the analysis of the teachers' observational reports:

(1) What evidence is there, if any, of teachers attending to aspects of students' mathematical reasoning,
(2) What evidence is there, if any, of teachers learning about students' mathematical reasoning?

Theoretical framework

Recent developments in research on the professional knowledge for teaching mathematics, which emphasize teachers' knowledge of students' mathematical reasoning, provided the motivation for the present study. Theories on practitioner–researcher collaboration informed the collaboration between teachers and researchers.

Professional knowledge for teaching

Research on the professional knowledge for teaching mathematics indicates that effective teaching requires knowledge of students' mathematical reasoning. In particular, there is a strong positive correlation between teachers' understanding of aspects of how students' build knowledge and their students' academic performance (Rowan et al. 1997). Pedagogical Content Knowledge (PCK) requires teachers to understand the "Conceptions and preconceptions that students of different ages and backgrounds bring with them to the learning of those most frequently taught topics and lessons" (Shulman 1986, p. 9). An extended version of the construct includes the category Knowledge of Content and

Students, which requires teachers to have the ability to predict how students will likely approach specific tasks, anticipate their errors, and interpret their often-incomplete ideas (Ball et al. 2005). Ball and Bass (2003) argue that teaching involves "Interpreting and evaluating students' non-standard mathematical ideas and explanations" (p. 9), and Schoenfeld and Kilpatrick (2007) propose a theory of proficiency for teaching mathematics, which requires knowing students as thinkers.

The emphasis on teachers' knowledge of students' reasoning is grounded on research that highlights the advantages of encouraging students' reasoning in mathematics classrooms. Evidence indicates that it enhances students' mathematical understanding (Davis and Maher 1990; Francisco and Maher 2005; Baroody and Ginsburg 1990; Hiebert et al. 1997) and students' ability to construct convincing mathematical justifications (Hanna 2000; Maher and Martino 1996; Martino and Maher 1999; Thurston 1994), helps them appreciate mathematical proof as a tool to establish the validity of mathematical statements (Balacheff 1991; Yackel and Hanna 2003), and can also help address differences in academic performance associated with equity and gender (Boaler 2002, 1997; Brown et al. 1996; Lane and Silver 1999). Yet, as noted above, there is a deficit of opportunities for teachers to learn about students' mathematical reasoning. The IML project tried to address this issue.

Practitioner–researcher collaboration

There is strong evidence that effective teacher professional development initiatives involve a close collaboration between researchers and practitioners. Putnam and Borko (2000) recommend professional development programs that combine ongoing classroom support with workshops that introduce research-based ideas. Krainer (2005) defends a learning approach for teachers based on "Teachers' joint reflection and constructive support from external *critical friends*" (p. 81), who could be researchers. Teachers who regularly conduct classroom research and reflect upon their practice transfer the skills and habits of inquiry beyond the specific project they engage in, use their new skills in other classroom situations, and "become more focused on the needs and thinking of their pupils" (Randi and Zeichner 2004, p. 213).

Wagner (1997) distinguishes among three types of practitioner–researcher collaboration, depending on the degree of input allowed to practitioners into the research process. In "data-extraction agreements", practitioners have no input into the research process. They are simply research subjects or "People whose work is described and the focus of analysis" (p. 15). Researchers are the agents of inquiry and the ones who construct and report the target knowledge. In "clinical-partnerships agreements," researchers are still the agents of inquiry and practitioners are the ones whose work is the focus for analysis and reporting. However, practitioners are allowed to provide input in the research process, mostly "by assisting their researcher colleagues" (p. 15). Finally, "co-learning agreements" are similar to clinical-partnership agreements in that practitioners have input into the research process. However, practitioners now get involved in all aspects of the research process. They work jointly with researchers in conceptualizing and implementing the study. They are both inquiry agents.

Wagner argues that co-learning agreements are the best types of collaboration, as researchers and practitioners get to learn not only about the work of the other, but also about their own work. As interns, the mathematics teachers were members of the research team in the IML project. They assisted the researchers in several aspects of the research process while learning how to design and implement thoughtful mathematical activities in classrooms. Working primarily as classroom ethnographers, they conducted observations of the research sessions and students' mathematical activity, and shared them with researchers in debriefing meetings. However, they lacked research experience. Therefore, the researchers retained the role of main agents of inquiry. However, the teachers were encouraged to provide input to the research process. In particular, their observational reports in debriefing meetings were used by the researchers not only to monitor the teachers' attention to students' mathematical reasoning, but also as insights for evaluating past and planning subsequent research sessions. Overall, the collaboration between the researchers and the teachers in the IML project resembled more a clinical-partnership agreement than any of other two types of agreement.

Research setting

Participants

Nine mathematics teachers and 24 sixth-grade students participated in the IML project, which took place in a middle school [grades 5–8 and 11–14 year olds], located in an urban, low-income, and minority community in the United States. Approximately 98% of the student population was African-American or Latino. The teacher interns consisted of two elementary school teachers [K through 4 grades], four middle school mathematics teachers, and three mathematics teacher coaches. Mathematics coaches were educational specialists who worked with mathematics teachers in classrooms to help them justify their teaching. They usually had more teaching experience and a stronger mathematical background than the teachers. The average experience among the participants was 4–5 years.

Participation in the project was voluntary for both the teachers and the students. The students were selected from different sixth-grade classes in the school that hosted the project. The teachers came from different schools in the same school district. None of the students had their teachers among the interns. The decision to allow the participation of teachers of different grade levels reflected the researchers' belief that, despite the difference in mathematical background, all teachers could still attend to and learn about the processes involved in students' mathematical reasoning.

Mathematical tasks

In the IML research sessions, students worked on challenging, well-defined mathematical tasks. The tasks often involved manipulative objects and covered four mathematical strands: algebra, counting/combinatorics, fractions, and probability. Table 1 below lists two of the tasks implemented in the project. The learning opportunities on students' mathematical reasoning discussed in this article involved these tasks. One of the tasks is the 3-Tall 3-Color Tower Problem in which the students were asked to build all possible towers 3-cubes tall when choosing from unifix cubes in three colors. The task and its variations were intended to enhance the study of the students' development of different forms of mathematical reasoning and justification. The other task in Table 1 is a set of two developmental series on fractions in which students were asked to use Cuisenaire Rods to build rod models in mathematical activities involving fractions. The first series involved tasks of the type, "Find the rod X that is a/b as long as the rod Y," and was intended to help the students build an understanding of fractions as operators. The second series involved tasks of the type, "If we give rod X the number name 1, what is the number name for rod

Table 1 A sample of mathematical activities used in the IML project

Strand	Activities	Some learning targets
Fractions	"Find a rod X that is a/b as long as Rod Y" "If we call rod X the number name 1 what is the name for rod Y?"	Fractions as operator; fraction as number; equivalent fractions
Counting	*The 3-Color 3-Tall Tower Problem*: "Work together and make as many different towers three cubes tall as is possible when selecting from three colors. See if you and your partner can find a way to convince yourself and others that you have found all possible towers three cubes tall	Combinations, permutations, different types of reasoning; Indirect/direct forms of justification; generalization

Y?" and was designed to help the students gain understanding of fractions as numbers. When students start to understand the idea of fractions as numbers, they become comfortable calling rods simply "one-third," as opposed to "one-third *of*." They also come to understand that the number name "one" can be assigned to *any* as opposed to a *particular* rod and, consequently, depending on which rod is assigned the number name "one," a given rod can have one number name, in one situation, and another number name in a different situation. The idea of fraction as operator is more common in schools than the idea of fraction as numbers. However, both represent two complementary uses of fractions in mathematics.

Student sessions

In the IML research sessions took place under particular conditions, the students were encouraged to work collaboratively with one another and to always justify their solutions to the mathematical tasks to group members. The researchers received all students' contributions positively, and avoided making judgments about their mathematical validity. Instead, they encouraged the students to be arbiters of the validity of each other's solutions based on whether they thought that the arguments presented made sense to them. Students were given extended time to work on the tasks, and opportunities to work on different tasks involving similar ideas to enhance their ability to refine and make connections between mathematical ideas.

The teacher interns conducted their observations of the students' mathematical activity in groups of two to three people. Usually, on different student sessions, the teachers observed a different group of four to six students at a particular table. However, occasionally, the teachers were allowed to follow the same group of students over several sessions. The teachers were given general instructions about what to focus on in their observations. They were simply told to compile detailed observational notes of the students' mathematical activity and to be prepared to report them in the debriefing meetings. They were also asked to refrain from interacting with students. If they wanted to intervene, they were allowed to ask only clarifying questions and to limit the duration of their exchanges with students to a minimum. This was intended to prevent the research-inexperienced teachers from giving out answers or solution strategies to students, and thus "contaminate" findings about the directions in which students' thinking might proceed in the interactions with peers group.

The researchers spent most of the time trying to facilitate the students' mathematical activity. Since they had more research experience than teachers, they intervened more than teachers in the students' mathematical activity. However, the researchers avoided giving out answers, solution strategies, or evaluating the students' ideas. Standard practice was to listen to students and encourage their thinking by posing questions such as "Tell me more about it", "Why?" "I am confused," or "I am not convinced." The researchers also tried to promote the students' collaboration by frequently inviting the students to listen to and comment on each other's ideas and explanations in their groups and during students' class presentations. They used their observational notes and suggestions from teachers to make decisions about the timing, order, and duration of the students' presentations.

Debriefing meetings

The debriefing meetings were a central component of the IML project, as they provided researchers the opportunities to trace and promote teachers' attention to and understanding of aspects about students' mathematical reasoning. The meetings had three distinct parts. In the first part, the researchers used questions such as, "What did you notice, today?" "What did you see?" or "Who wants to go [report] first?" to invite the teachers to start sharing their observational notes on the students' mathematical activity. This part lasted only a few minutes.

The second part was the longest. Researchers continued to invite observational reports from teachers and to encourage participation by all teachers by asking for observational reports on students' mathematical activity in all tables. However, the distinguishing feature of this part was that the researchers engaged the teachers in discussions of their observational reports. This was intended to trace evidence of teachers' attention to aspects of students' mathematical reasoning in their observational reports. The researchers facilitated the teachers' discussions in the same way as they facilitated the students' mathematical activity in student sessions. Researchers invited observational reports from teachers, but avoided deciding on the content and parameters of the reporting for the teachers. They encouraged the teachers to decide which observations they intended to make public and how they intended to articulate them. The researchers also refrained from evaluating teachers' reports, preferring to engage the teachers in collective discussions about the merits and validity of their observations. Occasionally, the researchers contributed observational reports as a way of adding promoting the discussion around important aspects of students' reasoning. However, they avoided directly telling or naming the particular issues, preferring to introduce them indirectly by simply asked teachers to comment on particular events.

The third and final part was usually short and lasted a few minutes. The goal was to help teachers articulate and see the importance of particular insights into students' reasoning that emerged in the discussions. The overall approach was to, whenever possible, have the teachers collectively articulate those issues, as opposed to the researchers telling or lecturing the teachers about them. In the rare cases where the researchers chose to be more direct, they still made sure that whatever insights they shared with the teachers built on the teachers' observational reports. However, researchers' direct were the last option and researchers were even prepared to delay closure over several meetings until they felt that teachers had thoroughly discussed and understood the issues involved. The idea of building on the teachers' observational reports to introduce the ideas was aimed at helping the teachers build meaningful understanding of the aspects about students' reasoning.

Methods

Data collection

All research sessions, including student sessions and debriefing meetings, were videotaped with the help of five digital cameras and a crew of about 12 people. The crew included camera and sound operators, and other backup technical staff. Two people operated each camera. One person was responsible for taping and the other person was in charge of capturing sound. During student sessions, usually four cameras were stationed in different places in the room, near an equal number of tables with students. The cameras recorded the students' mathematical activity going on at the tables. The fifth camera was known as the roving camera, as it followed the leading researcher in the room. The camera recorded the researchers' interactions with students and the students' class presentations, whenever they took place. The debriefing meetings were videotaped with one stationary camera, which recorded contributions from both teachers and researchers in the meeting.

The videotapes and the transcripts of the debriefing meetings were the main dataset for the present study. However, videotapes of the student sessions and copies of the students' written work collected at the end of each student session were also useful additional datasets. These data were examined for evidence that substantiated claims made by teachers in debriefing meetings. This was particularly useful whenever the researchers sought to obtain accurate teachers' accounts of students' mathematical solutions or reasoning strategies. All of the data, including videotapes of the debriefing meetings and the student sessions, and students' written work, were collected in four cycles in the 1-year IML project. Each cycle lasted about 3 weeks and consisted of about six one-and-a-half hour student sessions followed by debriefing meetings that lasted 45 min to 1 hour. The cycles took place in the fall and spring academic semesters with breaks in the summer. A total of about 36 hours of student sessions and 24 hours of debriefing meetings were collected during the year.

Data analysis

Data analysis used qualitative data treatment procedures from particular video-based methodology for tracing learners' development of ideas (Powell et al. 2003; Erikson 1992; Davis et al. 1992). In this case, researchers traced the teachers' development of ideas about students' reasoning. The analysis involved several iterations of the following seven, non-linear, interlaced steps:

(1) reviewing each debriefing session to develop a sense of its events as a whole;
(2) partitioning each session into time-coded major parts or episodes;
(3) identifying significant excerpts in the major parts;
(4) clustering significant excerpts into themes;
(5) articulating the relation between themes and excerpts;
(6) identifying learning opportunities;
(7) describing learning opportunities in connection with each theme.

The major parts, categories, and themes were defined based on their reference to particular aspects of mathematical reasoning. The constant comparative method (Strauss and Corbin 1998) was used to help determine the themes. The analysis focused on teachers' insights into students' reasoning, i.e., the particular ways in which students engaged in reasoning.

Results

Analysis of the teachers' observational reports revealed learning opportunities for the teachers on students' mathematical reasoning around five themes: conceptual understanding, forms of reasoning, articulation of mathematical ideas, mathematical justification, and conditions that support growth of students' mathematical reasoning.

Conceptual understanding

Several teachers' observational reports attended to aspects of students' conceptual understanding. One particular report focused on the students' understanding of fractions, particularly the distinction, mentioned above, between fractions as operators and fractions as numbers. Gilberto, a middle school mathematics teacher, pointed out that the students were getting better at using rods. However, he also added that the students were starting to call rods simply "one-half" without saying what rod they were half *of*, and were also starting to call the same rod different numbers names in different situations:

Gilberto: They [students] are getting better and better at using the rods to prove their thinking, but what I noticed is [that] sometimes they get confused, because they say "one-half" but sometimes they don't say precisely "one-half of what." That is confusing. That is why one student said the white [rod] is one-half, but she couldn't justify her thinking when they were on the board. She said that red was one-half. I thought "one-half of something." Also, last time it was one-ninth. Then, it became one-tenth.

The statements, "They [students] get confused," "That is confusing," and "She [student] could not justify her thinking" suggest that Gilberto thought that the students were having difficulty in identifying the correct unit/whole. However, the researchers saw it differently. They saw it as evidence of students beginning to develop the idea of fractions as numbers. Sensing a learning opportunity for the teachers, the researcher leading the debriefing meeting asked if the teachers were aware of the distinction between fractions as operators and fractions as numbers. The teachers responded negatively, which confirmed the researcher's intuition about the learning opportunity. The researcher told the researchers that, "I want to move away from one-half of what. That's my goal here" and explained the distinction to the teachers:

Researcher: What we are trying to do here is make a transition from the operator idea of a fraction like, "This is a part of this" to a one-to-one correspondence of particular lengths and particular numbers so that they [students] can reason with lengths and numbers simultaneously. If they [students] don't move away from the operator notion, they are going to get mixed up a lot, later on. We are trying to get them to move from [the idea of a] a fraction as an operator to [that of a] fraction as a number. That is the journey they are on now. It's this nice if–then kind of thinking and the notion that you can keep changing the number names of the rods that matters.

The researcher explained that both are important and complementary ways of understanding fractions. The teachers might not have inferred the distinction between fractions as operators and as numbers from their observations. However, the observations created an opportunity for the teachers to expand their knowledge or way of thinking about fractions.

Forms of reasoning

A number of teachers' observational reports revealed the teachers' attention to the different types of reasoning strategies used by the students to solve mathematical tasks, particularly the Tower Problem. In one report, Hanna, a mathematics coach, described a strategy used by a student to solve the Tower Problem, which fit a form of reasoning by the *Fundamental Principle of Counting*. According to the teacher, while explaining his thinking to her, the student said, "Since I have nine [towers] there and there are three different colors [of unifix cubes], then there are going to be 27 [towers]." This suggests a form of reasoning by Fundamental Principle of Counting as $9 \times 3 = 27$:

Hanna: I was blown away. I was fascinated. He [the student] started with the tower 3-tall, all red. Then he said, "Okay. If I have the reds in there and I introduce a yellow, then I would get three more." He had them set up like 1-3-3 and then 2 [See Fig. 1]. So, he was saying, "If I keep the red constant, and then introduce the yellow, I would have yellow, red, red; then red, yellow, red; and then red, red, yellow. Then, if I introduce the blue, it would be blue, red, red; and then, red, blue, red; and red, red, blue. And then there [are] these two on the outside that are kind of like combos, because if I keep the red on top and on the bottom, I will just switch them and I get one that is red, yellow, blue, and another one, which is red, blue, and yellow." And then he said: "Since I have nine [towers] there and there are three different colors [of unifix cubes], then there are going to be 27 [towers]." I was like, "Wow." So, he was starting to make the connection of three tall with three colors so that's going to be 27. It may not have been three to the third – he might have reasoned that three times nine is 27 and got it that way. I just thought his methodology was really interesting and the ways he laid it out was just really intriguing to me.

In another report, Daniel, an elementary school mathematics teacher, described a student's strategy for solving the Tower Problem, which fit a form of *reasoning by cases*. The cases were towers with one color, towers with two colors, and towers with three colors, and the student called them, "solids," "single mixed," and "mixed," respectively. The teacher also noticed that the student used the "diagonal" strategy to account for all towers within the cases. This means that the student knew that she had all towers within a case because cubes of a particular color had occupied all possible positions in a tower, resulting in an ascending or descending diagonal line when towers are placed next to each other:

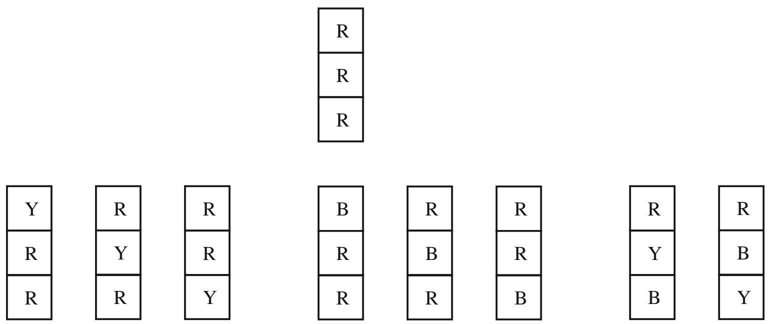

Fig. 1 The 1-3-3-2 display of 3-tall towers selected from three colors

Daniel: Their [students] work was just exquisite, very meticulous; everything was in place, the drawings… She [Justina, a student] described it [the towers] as being "solids," "mixed," and "single mixed." Then, Toby [a research assistant] asked her, "Explain to me what the 'mixed' is." She said that a mix is a combination of all of the different colors. He said, "What about the pattern?" She turned to her partner and asked, "What are the patterns? What is it?" The partner said, "It's the diagonal." She said [to Toby] "diagonal." She thought about it. She was not sure, but everything was set up very meticulously. She had three solids [towers] on the top [row], with "solids' written within parentheses. Then, she had however many towers going across the bottom [of top row], and then the "mixed" on the bottom of those, and everything done in there, with either diagonal yellow, diagonal blue, diagonal red, whatever everything was diagonal going all the way across.

Figure 2 below shows the teacher's description of the students' "case" arrangement, indicating "solids" in the first row, "single mixed" in the second and third rows, and "mixed" in the last row. A diagonal pattern is also indicated on the second and third rows.

Regarding insights into students' mathematical reasoning, statements such as, "I was blown away," and "I was fascinated," in one report and, "Their work was exquisite, very meticulous, and everything was in place," in the other report, indicate that the teachers were impressed at the students' ability to come up with such reasoning strategies. This was an insight into students' mathematical capabilities.

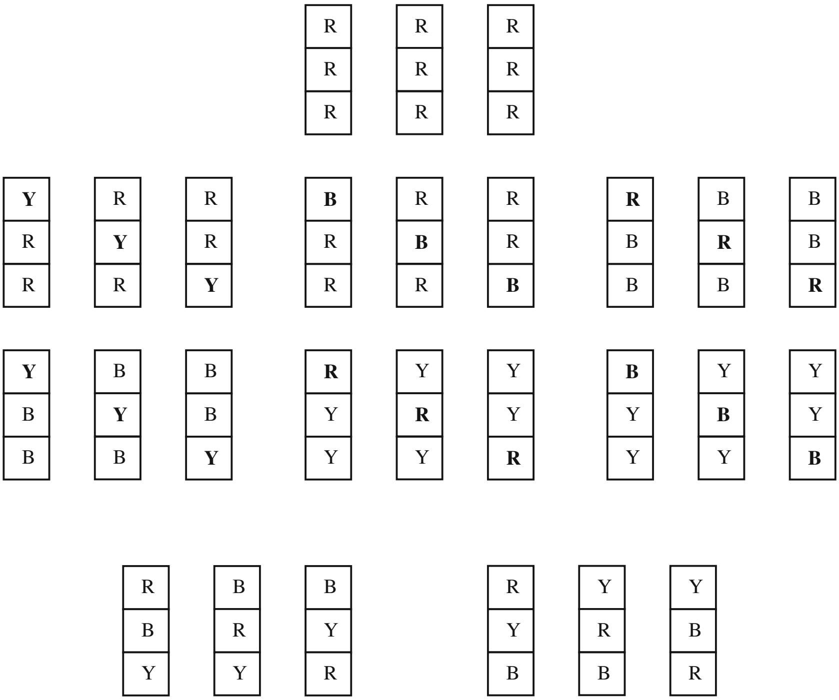

Fig. 2 Student' "case" display of 3-tall towers selecting from three colors

Articulation of mathematical ideas

Teachers' observational reports also showed attention to the particular ways in which the students articulated or made their mathematical reasoning public in the student sessions, particularly the language that students used to describe their mathematical thinking. In one report, Ms Wilson, a middle school mathematics teacher, described a student's thinking about the Tower Problem, which fit a reasoning-by-cases strategy. The teacher paid particular attention to the words that the student used to describe the cases: "typicals," for towers with two colors," "multi-colors," for towers with all three colors, "mixed match," for towers with the single color in the middle and two of the same colors on the outside, and "commons" for towers of one color only:

Ms Wilson: Chanel and Danielle gave their names [to the cases]. They had four groups [of towers] with four different names. They had the "typicals," which were two colors mixed with one by itself. They had twelve of those; and then they had the "multi-colors," which included all three colors and they had six of those. Then, they had the "mix match," where the single color was in the middle and two of the same colors were on the outside. And then they had the "commons"– all of one color; they had three of those. They did a good job explaining in their groups. David [a student] on the other end, he had the diagonal going, but he wasn't able to explain what he was doing, even though he could work with those multi-cubes very well. But he was having trouble explaining it. They did a good job.

Similarly, another teacher, David, a mathematics coach, noticed that one student divided her towers in two groups. Then, the student called the towers in one group "originals" or "mothers," and the towers in the other group "children." Interestingly, more than just labels, the words "mother" and "children" reveal a particular strategy for thinking about the Tower Problem. IML researchers have come to call it, "The doubling strategy." In the strategy, students first build an initial set of towers by some criteria. Then, they build another set of towers from the first group by making opposites of the initial towers. As a result, the original number of towers doubles, which explains the name of the strategy. Now, because the second group of towers is built from the first group of towers, one can understand why the student referred to by the teacher called the towers in first group of towers "originals" or "mothers," and called the second group of towers "children:"

David: I think it [the research session] went very well. Looking at the [tower] combinations together and how they [students] are rationalizing, why they are coming up with different combinations has been very interesting. One young lady, Kori [student], was asked by John [researcher]: "Why are these [towers] called 'originals'?" She said: "Well this is the 'original'; this is like the 'mother' and then she went on and explained what the rest of the combinations were. She used the blue, which she considered the mother. [Pointing to a group of towers] she said, "These are the children". And she made different combinations with the blue, like on top. Then she followed through and [pointing to a tower] she said, "This is the red mother, and these are the different combinations." She utilized that thinking [mother-children] to try to develop different combinations of towers.

Regarding insights into students' reasoning, the statements, "They did a good job explaining in their groups" by Ms Wilson, and "It has been very interesting," by David suggest that, once again, the teachers were impressed by the students' reasoning abilities.

This time, the teachers were impressed at the students' ability to come up with their own language to describe or represent their thinking. Ms Wilson also noticed that, "He [the student] wasn't able to explain what he was doing, even though he could work with those multi-cubes very well." This is another insight into students' reasoning, namely that students are capable of engaging in valid thinking even when they cannot explain or articulate their thinking.

Mathematical justification

The teachers paid particular attention to the ways in which the students tried to justify their reasoning to peers and researchers, particularly when working on the Tower Problem, where they had to answer the question: "How do you know that you have found all towers?" One elementary school teacher, Kwame, noticed that one student "wasn't able to rationalize" his solution to the problem, not because the student did not have a valid argument, but because "She [the student] wasn't quite sure" of how to articulate it:

Kwame: I observed the same table [as another teacher] and the portion where she [a student] had a three-color combination. I think she eventually figured it out, but she wasn't quite sure. Her thinking was something like, "It is three towers high and I have a color at each level. So, that is how I know I have it." She wasn't able to rationalize it. [But] <u>that</u> was basically the response we [researchers] were searching for.

A middle school mathematics teacher, William, described a debate among a group of students during a student session in which the students were discussing whether diagrams or text alone were enough for a mathematical justification to be considered valid. The teacher viewed students' debate as an example of the students starting to develop an "unofficial rubric" of what counts as a convincing justification in mathematics:

William: From listening to [the students,], basically posting their work [on the walls], they developed an unspoken rubric to determine what is needed to make this a convincing piece of work. One group stated that the diagrams [on one poster] are fine, but there was no text to support the diagram. And then [on another poster] there's text without support for a diagram. So, it seems as if like the students have developed upon themselves the specific rubric to determine what is convincing or not convincing.

A mathematics teacher coach, Kathleen admitted that she was "impressed" that the students were starting to engage in "non-frivolous" discussions and each other questions like, "Are you convinced?" or "Why and why not?" The teacher saw this as evidence of students "Acting as if they were real mathematicians:"

Kathleen: Some of the discussion was very detailed. It wasn't frivolous discussion. It wasn't just like "I don't know this or that." They were, <u>really</u>, acting as if they were real mathematicians. Walking around, I was impressed to see the interactions and even the communication among the students. Within the groups, they conversed about what was missing [in their arguments]. They asked each other "Are you convinced?", "Where are you convinced?" or "Why or why not?" And they kept on trying to be more specific. So, I thought the conversations were really good, on point.

An elementary school teacher, Jennifer, made interesting observations about the role of mathematical justification in school curricula. She pointed out that justification was rare in the curricula of lower grades, and this explained why some IML students often looked confused when the researchers asked them to justify their thinking. She suggested introducing mathematical justification in lower grades. However, she also added that if students were to be successful in learning how to justify their thinking, the emphasis should be on students' explanations of their thinking rather than on formal written mathematical proofs:

Jennifer: I think, it is really, important for them [students] to see what it [justification] looks like. Because we have this issue in the elementary school where you give kids an open-ended question and they'll give me an answer and I will write on the paper, "Can you prove it to me?" and fourth-graders will say, "Well, what do you mean by that? They really don't understand that. So, I'm sure kids who are coming into sixth grade have really not had that many experiences. So, if we show them some models of what justification looks like, they'll have a sense of "Oh, this is what you mean." You know, it's not just drawing [on paper] what I've built; it's also trying to talk about what my thought process is in how I'm determining that I have everything.

The reports suggest teachers' awareness of several insights into students' ability to engage in mathematical justification, namely that (1) students are capable of having a valid mathematical argument even when they cannot articulate it, and, when given the opportunity, students can naturally (2) develop ideas about what counts as a valid mathematical justification, and (3) adopt justification as part of their mathematical practice. The reports show teachers realizing that (4) emphasizing students' explanations rather just formal written mathematical proofs is strong way of promoting students understanding of mathematical justification.

Conditions that support growth of mathematical reasoning

Finally, several teacher interns contributed reports that showed attention to conditions that promote the students' mathematical reasoning. One teacher, Daniel, described how two students discovered improper fractions while working together on a fraction-as-number activity. According to the teacher, the students decided to pose themselves an extension of a problem that had been assigned to all students. As a result of solving it, they came up with the number name "ten-ninths:" The teacher attributed the discovery to allowing students' opportunities to "explore" ideas or engage in what she called "nonsensical play:"

Daniel: He [the student] asked her [another student], "What would you call the orange rod?" even before it [the question] was proposed [by the researchers later in the day]. Then, she put it [the rods] together and said, "See, it would be called ten-ninths! You didn't even know that!" And then they kind of played with each other. Chanel [student] said, "I explained it to him how to get the answer." He was like "No you didn't!" And he got a paper and he drew all of the different shapes because he had been playing with them all along and moving the one [white] block. So, when the question was asked: "What is the name of each of the fractions?" He knew right away that if this is one-ninth, then the others are two-ninths, three-ninths, four-ninths, and so on. Right away he made that connection. And it came together for me that you [need to] allow them to explore a bit with the blocks because really – while it seems like nonsensical play – something registered for him that right away because as soon as the question was asked, boy, he had the right answer. For me that was priceless.

Another teacher, Kathleen, reflected on the researchers' deliberate decision to avoid telling the students if their mathematical arguments or solutions were correct or false, and, instead, encourage the students to collectively be arbiters of the validity of their arguments. The teacher thought that researchers' decision was having an empowering to the students, turning them into increasingly more autonomous and more confident mathematical learners:

Kathleen: One of the things I like about your idea [the researchers' decision to avoid being the ones to decide if a claim is correct or false], by the way, is that you are saying that you don't need an authority to figure out if what you've done is right or wrong. It can come from you, and boy, that's a lot of empowerment. That's very special.

The reports suggest that their experience in IML project allowed the teachers to realize that allowing students to explore mathematical ideas and to decide for themselves about validity of their argument can have the positive effect of turning them into more powerful and autonomous learners.

Discussion

The previous section described insights into students' mathematical reasoning that a group of mathematics teachers were able to gain from their year-long observations of students' mathematical activity in the IML project. The teachers were able to see that students can naturally come up with problem-solving strategies, create a language to describe their mathematical thinking, adopt justification as part of their mathematical practice, develop valid ideas about what counts as an acceptable mathematical justification, and come up with mathematical discoveries. The observations also helped the teachers realize that giving students the opportunity to explore ideas and decide on the validity of their mathematical arguments can help students develop into powerful and autonomous mathematical thinkers, and encouraging students to explain their mathematical thinking is a powerful way of promoting students' ability to engage in mathematical justification. One observational report even created an opportunity for teachers to deepen their understanding of fractions. All this supports the claim in this article that teachers' observations of students' mathematical activity in research projects on students' development of mathematical ideas can provide much-needed opportunities for teachers to learn about students' mathematical reasoning.

Our experience shows that several instances of rich mathematical reasoning form students often go unnoticed to teachers in mathematics classrooms. Our analysis suggests that three main factors enhanced the teachers' learning of students' mathematical reasoning in the IML project. First, the IML was a *research* project on students' development of mathematical ideas. In particular, the IML project was different from other teaching experiments where researchers have particular concepts that they want students to construct, often by traversing an anticipated learning trajectory (e.g., Simon 1995). In the IML project, there were no specific ideas that students were expected to learn, nor were there particular stages that students were supposed to traverse. The students' ways of reasoning and how they constructed their knowledge were the results of the research, not of pre-conceived goals. For this reason, IML researchers called the project a learning rather than teaching experiment. The result were several opportunities for teachers to observe *reasoning-rich* students' mathematical activities some of which have been documented elsewhere (Maher et al. 2007; Weber et al. 2008). This certainly enhanced the teachers' learning about students' reasoning. Second, the IML was *an after-school* project. This means that there was not a fixed curriculum and there were no typical pressures of regular classrooms such as having to cover large amounts of material and preparing students for standardized tests. Not only did this allow the project to have its research focus, but also made teachers more willing to participate in the project, which often enhances motivation for learning. Finally, as mentioned above, in their role as interns, the teachers worked closely with *researchers* in the IML project. In particular, during the debriefing, meetings, researchers contributed observations of students' mathematical reasoning that helped the teachers attend to particular aspects of students' mathematical reasoning. Researchers also promoted teachers' collective discussion of their observations, which helped the teachers articulate their insights into students' mathematical reasoning. Such actions from researchers certainly helped promote teachers' learning in the project.

The results of this study are significant, particularly with regard to future research projects. It is particularly significant that the teachers were impressed by the students' mathematical reasoning. Cohen and Ball (1999) argue that "Instructional capacity is partly a function of what *teachers know students are capable of doing* and what they think they are capable of achieving with students" (Emphasis added, p. 7). They claim that teachers who believe that students are capable of doing well tend to be more successful teachers than teachers who believe that students are capable only of modest work. They claim that this is often the case in high-poverty minority schools. Our experience suggests that this is the case also in affluent, mainstream culture schools. In any case, the fact that the teachers were impressed by the students' mathematical reasoning in the IML project suggests that teachers' observations of students' mathematical activity in IML-type projects may help promote teachers' positive beliefs and dispositions about students' mathematical abilities. This is an issue that requires further research with carefully designed instruments to measure possible changes in teachers' beliefs as a result of their participation in IML-type studies.

As indicated above, researchers built on Gilberto's observational report on the students' understanding of fractions to introduce the distinction between fractions as operators and fractions as numbers to all teachers. The distinction was new to the teachers, which generated an opportunity for the teachers to expand their mathematical knowledge. This suggests the possibility that teachers' observations of students' mathematical activity in IML-type project may promote growth of teachers' mathematical knowledge. This issue may be worth investigating further. There is a consensus that there is a way of knowing mathematics that is effective teaching, and it involves understanding mathematics in flexible ways, including the ways in which learners might understand it (Adler and Davis 2006; Ball 2000; Hill and Ball 2004; Ma 1999). In particular, Ball (2000) argues that studying situations of mathematical knowledge in use [by students, for example] helps teachers develop such understanding of mathematics. Therefore, this study suggests that teachers' observations of students' mathematical activity in IML-type settings might help teachers develop an understanding of mathematics that is effective for teaching.

Finally, another area for future research regards ways of incorporating the benefits of IML-type after-school programs in teacher education and professional development. One way is to run such programs and encourage teacher participation in them. Another way is to involve teachers in the study of videotapes of students' mathematical activities from such programs. This strategy has claimed success in promoting teachers' knowledge of students' reasoning (Maher 2007; Sherin 2007; Kazemi and Franke 2004; Koellner et al. 2006). However, both options also raise the complex question regarding the extent to which after-school programs can have positive impact in promoting effective teaching practices in regular mathematics classrooms. After-school programs and regular mathematics classrooms may be related, but are also different environments. This makes it difficult to explain how the former can have a direct impact on classroom teaching. More research is needed on this issue. Our experience working with teachers in after-school programs and showing them videos from such settings has at least given us insights into how teachers respond to such experiences. In general, the teachers express amazement at the students' mathematical abilities. Then, they split up into two opposing groups: the group of those who decide to find ways to incorporate the "after-school approach" in their own classrooms, and the group of those who dismiss any impact of the approach on grounds that after-school programs are different from regular classroom environments. One pattern seems to be emerging. Teachers in the first group tend to be among those who stay long enough in after-school programs to gain deep exposure to the complexity of promoting students' mathematical reasoning. These teachers become reluctant to engage in traditional teaching approaches based on showing and telling. That, we find, is the good news!

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