J Math Teacher Educ (2007) 10:415–432
DOI 10.1007/s10857-007-9054-8

The design of tasks in support of teachers' development of coherent mathematical meanings

Patrick W. Thompson · Marilyn P. Carlson · Jason Silverman

Published online: 27 November 2007

Abstract We examine the role of tasks that have the intended effect of teachers re-conceiving the mathematics they teach as comprising a coherent body of meaningful ideas. We ground our discussion in ideas of trigonometry and modular functions and draw from a professional development research project to illustrate our approach. In this project, many teachers experienced dissonance that was rooted in their commitments to their curricular knowledge of trigonometry. Teachers who built new meanings into a coherent whole were those who coordinated them at a micro level. Teachers who saw implications of their own reasoning for student learning were also successful at expressing that reasoning in natural language. We saw a similar pattern in the case of teachers' creation of meanings for action and process conceptions of mod(f(x),g(x)). Teachers who gained insight into implications of their own activities for student learning were the teachers who reasoned at a micro level in regard to the meaning of mod, who coordinated that meaning with a covariational perspective on the behavior of functions, and who expressed that coordination in natural language. We conclude that a primary feature of tasks that promote teachers' construction of coherent mathematical meanings is that they support an overall effort to have teachers engage in the coordination of meanings in the context of explaining significant ideas and relationships.

Keywords Tasks · Coherence · Meaning · Reflection · Trigonometry · Covariation · Teachers · Professional development

Educational tasks, by definition, are designed by someone to be performed by someone else; they are designed for a purpose and with an intended effect. To consider the role of particular tasks in mathematics education we must therefore consider the designers' intentions and their theory of how the tasks will have their intended effect.

P. W. Thompson (✉) · M. P. Carlson
Mathematics and Statistics, Arizona State University, P.O. Box 873604, Tempe, AZ 85287, USA
e-mail: pat.thompson@asu.edu

J. Silverman
Drexel University, Philadelphia, PA, USA

To us, a task has one of three purposes: To have learners engage in repetitive activity (sometimes known as *practice*), to have learners engage in reflective abstraction, and to support instructors' intentions to engender discussions in which learners and teachers take their ongoing activity as the object of discourse. Though these three purposes become intertwined over time, they are distinct at any moment.

While it may seem obvious, it is worth noting nevertheless that tasks do not have agency. Tasks do not elicit behavior any more than a hammer elicits hammering. They do not stimulate thinking any more than does a paragraph of text. Tasks affect learners, or not, because the learner accepts what is offered, or not, in the context of his or her own meanings, goals, interests, and commitments. It is in this sense that one must design tasks with the learner in mind. An "informed" design is then one in which the designer is guided by a model of the learner that includes the learner's understanding of the context in which he receives the task, which includes the learner's model of the person assigning the task. A designer might offer an initial task knowing that it will be interpreted by learners in ways that differ predictably from what he intends they eventually understand, but which will provide springboards for moving discussions in directions not possible without the initial (mis)interpretations. This approach draws from a model of teaching in radical constructivism – informed interventionists (i.e., folks with models of what they hope learners will learn) place themselves in positions to be interpreted in ways they intend by the persons they wish to affect (Thompson, 2000).

It is also worthwhile to expand on our meaning of repetitive activity (practice). First, we start with Piaget's position that assimilation is the source of schemes.

> Assimilation thus understood is a very general function presenting itself in three nondissociable forms: (1) functional or reproductive assimilation, consisting of repeating an action and of consolidating it by this repetition; (2) recognitive assimilation, consisting of discriminating the assimilable objects in a given scheme; and (3) generalizing assimilation, consisting of extending the field of this scheme .... It is therefore assimilation which is the source of schemes ... assimilation is the operation of integration of which the scheme is the result. (Piaget, 1977, pp. 70–71)

To assimilate a task means to understand it and its entailments. What you want learners to repeat is understanding the task and the pattern of reasoning that leads to a resolution of a perplexity that the task introduces into the learner's thinking. As Cooper (1991) makes clear, we must be explicit about the activity we intend that learners repeat. It is too common that designers think of practice as being repeated behavior and not repetitive activity. Cooper points out that repetitive behavior leads to habits. Repeated reasoning leads to schemes of thought.

This article deals with the role of tasks in two related aspects of the mathematical professional development of teachers: (a) helping future and current mathematics teachers develop coherent mathematical meanings and (b) serving as a context within which discussions of the implications and utility of someone's having coherent meanings can be held. The development of coherent meanings is nontrivial. To introduce coherence into one's meanings necessarily requires a learner to reflect on the meanings she holds and to adjust them so that they are compatible in overlapping domains. It requires taking one's thinking and meaning as objects of thought.

The reason for our focus on teachers' coherent meanings is pragmatic: If a teacher's conceptual structures comprise disconnected facts and procedures, their instruction is likely to focus on disconnected facts and procedures. In contrast, if a teacher's conceptual structures comprise a web of mathematical ideas and compatible ways of thinking, it will at least be possible that she attempts to develop these same conceptual structures in her students. We believe that it is mathematical understandings of the latter type that serve as a necessary condition for teachers to teach for students' high-quality understanding.

We introduce the theme of coherence with a question: *Suppose we stipulate that all angles are measured in degrees. What, then, is the value of cos(sin35°), in degrees?* The answer, as will be explained, is 48°.[1] But we do not ask this question merely to get an answer. The primary reason we ask it is that, to answer it, one must have highly coherent meanings of angle measure and trigonometric functions.

Common meanings of angle measure and trig function often are not coherent. For example, if by "degree" one means 1/360 of a complete rotation, by "sin" we mean "opposite over hypotenuse", and by "cos" we mean adjacent over hypotenuse, then sin(35°) having a value of 0.5736 means that in a right triangle having one angle measuring 35/360 of one rotation, the ratio of the side opposite that angle to the hypotenuse is 0.5736. But this presents a clash of conceptual categories when thinking about sin(35°) as an argument to cosine: How can a ratio of two lengths be an amount of rotation?

The solution to this conundrum is to have meanings for angle measure and trigonometric functions that differ from what we described. Suppose we have an angle. Assume an arbitrary circle centered at the angle's vertex. By "degree" we mean an arc on the circle whose length is 1/360 the circle's circumference, by "angle measure in degrees" we mean the length of the arc subtended by the angle, measured in arcs of length 1/360 the circle's circumference. Imagine an embedded right triangle made so that its hypotenuse is the circle's radius and one angle is formed by the angle in question. By "sine of an angle" we mean the percent of the radius' length made by the length of the side "opposite" the origin in the embedded right triangle. By "cosine of angle" we mean the percent of the radius' length made by the length of the side "adjacent" the origin. Now it all fits together conceptually:

- 35° is an arc having a length that is 35/360 times as long as the circle's circumference.
- sin(35°) is a length that is 0.5736 times as long as the circle's radius.
- A length that is 0.5736 times as long as the circle's radius produces an arc on the circle whose length is $0.5736/2\pi$ times as long as the circle's circumference, or 32.864°.
- cos(32.864°) is a length that is 0.834 times as long as the circle's radius
- A length that is 0.834 times as long as the circle's radius produces an arc on the circle whose length is $0.834/2\pi$ times as long as the circle's circumference, or 48.125°.

Several points are worth mentioning. First, this body of meanings makes it evident that the unit for the value of sine and cosine is one radius, no matter how angles are measured. Second, this body of meanings keeps *length* as a core concept.[2] All numbers, in this system of meanings, are lengths.[3] Third, angles can be measured in any unit that is proportional to a circle's circumference.[4]

To return to the theme of this article, the design of tasks in support of teachers' development of coherent mathematical meanings, we point out several features of the

[1] If you put your calculator in degree mode, then sin(35) produces 0.5736 and cos(0.5736) produces 0.9999. However, if you enter cos(sin(35)), you will get 0.8399. Neither result is in degrees, but it is noteworthy that the stepwise process produces a different result than does the composition.

[2] It would be more accurate to say that *magnitude* is the core concept. See Thompson & Saldanha (2003) for a discussion of magnitude versus measure.

[3] Tangent might seem an exception, but one can standardize tangent in a triangle that has the circle's radius as its horizontal leg. In that case, all values of tangent can be thought of as numbers of radii.

[4] The *grad* (or *grade*, or *gon*), introduced in France circa 1900 and still used in engineering, is an arc whose length is 1/400th a circle's circumference.

question that initiated the above analysis. First, we anticipated that the question itself, which asks for a value of cosine to be expressed in degrees, would not make sense to someone having a common way of thinking about trigonometry.[5] Second, as an instructional question, we anticipated a conversation of a general form surrounding the questions, "What are you measuring when you measure an angle?", "How do you measure an angle given what you imagine you are measuring?", and so on. Third, we anticipated raising the issue of coherence as a criterion by which we judge the adequacy of meanings. This all is in line with using a task as a didactic object (Carlson & Thompson, 2005; Silverman, 2004; Thompson, 2002), an object designed with the intent that it support an instructor's goal of generating reflective conversations around particular ideas and issues.

A focus on meaning

We illustrate the issues surrounding the design of tasks to support teachers' construction of coherent mathematical meanings by drawing on a current research project. We worked with 14 secondary mathematics teachers over the 2005–2006 school year. The teachers taught in an affluent school district located in the suburbs of a major metropolitan area within the southwestern United States. They had volunteered to participate for three years in a project that offered them the opportunity to improve their instruction by improving their mathematical knowledge and their knowledge about student learning.

In Fall 2005 they participated in a course, meeting for 3 hours once a week for 15 weeks at one of their schools. The course was designed with the intent that teachers re-conceptualize much of the secondary mathematics curriculum as being grounded in ideas of covariation and functional relationships.[6] Each teacher had a project-provided laptop with a graphing program (*Graphing Calculator*, by PacificTech), *Geometers Sketchpad*, and *Microsoft Office*.

The course

We will focus on the parts of the course that dealt with trigonometry and with process conceptions of function (Carlson, Oehrtman, & Thompson, in press; Dubinsky & Harel, 1992). Prior to the trigonometry segment, teachers engaged in specially designed activities and assignments to build their ability to think covariationally. By "think covariationally" we mean to imagine two quantities whose magnitudes vary simultaneously and to devise methods to record their simultaneous variation. From this perspective, graphs emerge as records of covariation, where each point on a graph is located by a convention that allows the coordinates of its position to represent the states of each quantity simultaneously. A graph in its entirety emerges by way of tracking quantities' varying magnitudes using this convention (Carlson, Jacobs, Coe, Larsen, & Hsu, 2002; Carlson et al., in press; Saldanha & Thompson, 1998; Thompson, 1994a, 1994b, 2002). Covariational reasoning plays a prominent role in our treatment of trigonometric functions.

We began the unit on trigonometry by asking, "What is one measuring when measuring an angle? After some debate, teachers settled that we are measuring "portion of a full turn", and also suggested that an angle's actual measure is that part of 360°, or any other

[5] In the US, this way of thinking is captured by the mnemonic "SOH-CAH-TOA", which stands for "Sine is Opposite over Hypotenuse", "Cosine is …", and so on.

[6] The course website is at http://tpc2.net/Courses/Func1F05/.

number that is assigned to one full turn (e.g., 2π), that is proportional to the fraction of one turn when rotating from a reference side to the other side. The instructor also asked for definitions of sine and cosine. Teachers, unanimously mentioning SOH-CAH-TOA, said that sine and cosine were ratios of a side in a right triangle to the triangle's hypotenuse. With these meanings as background, the instructor asked a series of leading question, "With these meanings, what is the meaning of sin(90°)? cos(100°)? Can you have a 100° angle in a right triangle? What does x represent when you graph $\sin(x)$ in Cartesian coordinates? What will you vary in a triangle to show how $\sin(x)$ varies as x varies?" Teachers' answers were ad hoc, confirming that their meanings for angle measure, sine, and cosine that they had given did not form a coherent system.

The instructor finally asked, "What meaning can we give to $\frac{\sin(x)}{x}$?" When no one could give a meaning other than it is the ratio of two numbers, the instructor pointed out that being able to give a meaning to $\frac{\sin(x)}{x}$ that is coherent with meanings of angle measure and trigonometric function is essential to understanding large parts of the calculus, and that the class would start from scratch to build these meanings. The instructor also pointed out that rebuilding these meanings was not simply for their personal gratification, but that if their students are to learn calculus of trigonometric functions meaningfully, their pre-calculus teachers must build appropriate meanings from the beginning.

In the above account, we dwelt on the opening discussion of the first lesson to convey the nature of interactions between the teachers and us, and to convey the course's persistent focus on creating meanings that are coherent across settings. The remaining account will be less detailed, and will highlight selected tasks, their intended effect, and provide brief accounts of teachers' struggles to produce meanings that supported flexible reasoning about trigonometric functions and flexible thinking in regard to teaching them.

Angle measure and trigonometric functions

At the outset, teachers accepted our suggestion to measure angles by measuring a circle's subtended arc in units that are proportional to the circumference. We introduced 'arc length as angle measure' as another convention, intending that teachers eventually would see the coherence that this meaning, used consistently, introduces into all of trigonometry. Teachers agreed, but were unsure why they should use this convention. They also agreed that sine and cosine of an angle could be thought of as the percent of a circle's radius constituted by the length of the arc terminus' ordinate or abscissa.[7,8] An arc of length 1/360th of a circle's circumference, by convention, is one degree, but using a circle's radius as a unit of length for measuring arcs produces a considerable benefit: The arguments of trig functions and the values of trig functions are measured in the same unit.[9]

[7] El'konin and Davydov (Davydov, 1975; El'konin & Davydov, 1975) speak of "a/b" as meaning "the measure of a in units of b".

[8] We agree with one reviewer who noted that this percent is not a length, but we disagree that it is dimensionless. The unit of this percent is "radii per radius."

[9] Having the argument to sine and the value of sine measured in the same unit is why $\lim_{x \to 0} \frac{\sin x}{x} = \frac{\pi}{180}$. When x is in degrees, $\lim_{x \to 0} \frac{\sin x}{x} = \frac{\pi}{180}$. The reason is that when comparing the length of a very small arc to the y-coordinate of the arcs endpoint, they have about the same length when both are measured in units of one radius. The measure of the arc is about $\pi/180$ times the length of the y-coordinate when the arc is measured in units of 1/360th the circle's circumference and the y-coordinate is measured in units of one radius. This is like saying the ratio of two equal lengths is 1 when we measure both in feet, while it is 12 when we measure one in feet and the other in inches.

Teachers became accustomed to thinking of arc length as angle measure through practicing measuring angles by drawing circles of a convenient radius and then using a string having the same length as the radius as the unit of length by which to measure arcs. They then used the same string to estimate sine and cosine of these angles by measuring the ordinates and abscissas obtained by placing a coordinate system appropriately (centered at angle's vertex, horizontal axis along one side). Many teachers were quite surprised at the accuracy they could achieve with a simple string segmented into 16ths.

We were initially unsuccessful in communicating to teachers that our primary aim was to build a coherent system of meanings, one that rested on ideas of proportionality and arc length as angle measure. Marcy's written note at the bottom of a homework assignment from this activity expressed a common sentiment.

> **Marcy**: I do get what you are trying to get us to do here, but I am still unable to see the true value of it as an alternative method for what seems easier using degrees to start with. I am at times enjoying the **struggle** [sic] that we are going through and I do finally understand where the concept of radian comes from, but again, this deep understanding may be important to us—I still question its necessity to the understanding of pre-calc and calculus problems in general.

A fundamental conflict in imagery was at the root of Marcy's concern with the use of arc length as angle measure (which she equated with radian measure). Teachers' root image of angle measure was amount of turn. When one imagines a partial turn, that image need not entail an arc length. We suspect that Marcy's (and teachers') common experience with angles and triangles is that one *sees* an amount of turn in a diagram and one *sees* a number that is given as its measure. In these cases one need not imagine a circle in which an angle is embedded. Thinking of angle measure as an amount of turn is sufficient. However, when one is given an angle and asked to measure it, one must impose an arc of a circle on it, which is what one does even with a protractor, and the angle's measure will be in units that are proportional to the circle's circumference, because the size of the protractor is immaterial. Teachers did not see protractors, which in their experience are in degrees, as imposing a circle on an angle. Even when using a protractor, they saw an angle measure as indicating some fraction of one full turn that is then scaled by whatever number one uses to indicate a full turn.

In session 3 of the unit we asked teachers about the legitimacy of the following way of reasoning:

> Sin() and cos() have periods of 2π with respect to their arguments, meaning whenever their arguments vary by 2π, their values will repeat. So, $\sin(3x + 5)$ will repeat whenever $3x + 5$ varies by 2π, so $\sin(3x + 5)$ will repeat whenever x varies by $2\pi/3$.

Their initial reaction was that this is a complicated way to think about something they already knew—"Why talk about arguments? Why not just talk about shifts and dilations?" They began to see the power of this way of thinking when we asked them, "Use this line of reasoning to explain why the graph of $y = \sin(x^2)$, $x \geq 0$, behaves as it does."[10]

We designed another set of questions about trig functions that, we hoped, would require teachers to coordinate basic meanings of angle measure, sine, and cosine in order to

[10] As x gets larger, it has to vary less for x^2 to vary by 2π, and thus as x gets larger, $\sin(x^2)$ will go through more complete cycles for each increase in x of a given amount. Put another way, the graph generated parametrically as $(x, y) = (t^2, \cos(t^2))$, $0 \leq t \leq 2\pi$ will generate a standard cosine graph over the interval $[0,4\pi^2]$.

explain why the functions behave as they do. One question was particularly fruitful: *Use meanings of angle measure, sine, and cosine to explain the behaviors of g(x) = cos(sin(x)) and of h(x) = cos(10sin(x)).*

Figure 1 shows the graph of $g(x) = \cos(\sin(x))$; Figure 2 shows the graph of $h(x) = \cos(10\sin(x))$. Most teachers did not know where to start. The remaining teachers began their explanations by showing graphs of $\cos(x)$, $\sin(x)$, and $10\sin(x)$. We pointed out to both groups that they were not incorporating the *meanings* of angle measure, sine, and cosine into their attempts to understand and explain the functions' behaviors, because they were not saying what x stood for nor what $\sin(x)$ and $\cos(x)$ meant. To incorporate meanings, one must refer to arc lengths, ordinates, and abscissas in a unit circle.

Figure 3 shows that for an arc of length β, the value of $\sin(\beta)$ becomes the arc length at which $\cos(\beta)$ is determined. Thus, as β wraps around the circle, $\sin(\beta)$ varies from 0 to 1 to 0 to -1 and back to 0. Thus, as β wraps around the circle, the argument to cosine varies from 0 to 1 to 0 to -1 and back to 0, thus explaining the graph in Figure 1. In Figure 4, we see that as β wraps around the circle, $10\sin(\beta)$ varies from 0 to 10 to 0 to -10 and back to 0. Thus, as β wraps around the circle, the argument to cosine varies from 0 to 10 (wrapping around the circle 1.6 times), to 0 (unwrapping back to 0), to -10 (wrapping 1.6 times around the circle negatively), and back to 0.

In class, we suggested to teachers that they work in pairs, to use diagrams of circles and angle measures, and to vary the value of β slowly as they tracked the value of $10\sin(\beta)$ and of $\cos(10\sin(\beta))$. After about five minutes, one of us asked, "How many of you are placing $10\sin(\beta)$ back onto a circle?" Only one pair of teachers did this. The others tried to track the covariation using just the variation of β on one circle, and thus forgot that the argument to cosine must be an arc length.

We note that teachers were severely challenged to create explanations of these functions' behaviors that were based in meanings of angle measure, sine, and cosine. Most spent 50 minutes in class on this activity and then had to revise their explanations outside of class. Their explanations suffered a number of defects, the most common being an

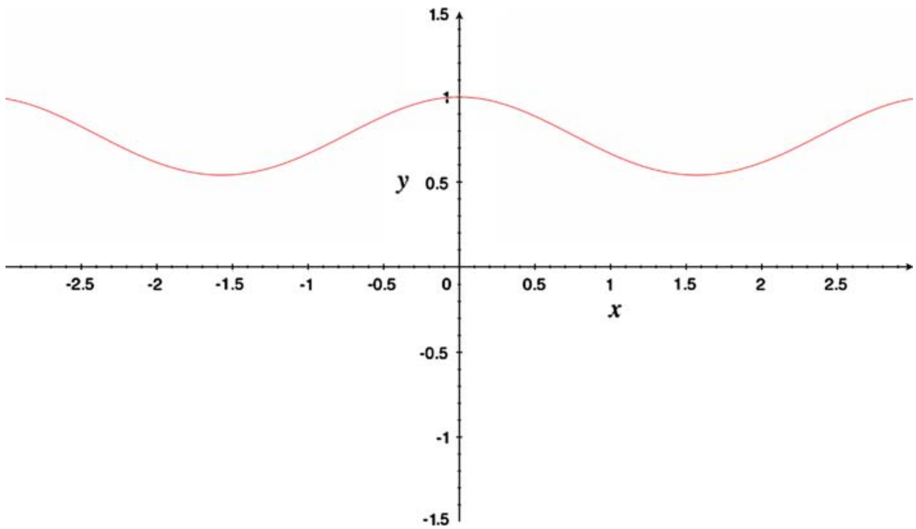

Fig. 1 Graph of $g(x) = \cos(\sin(x))$

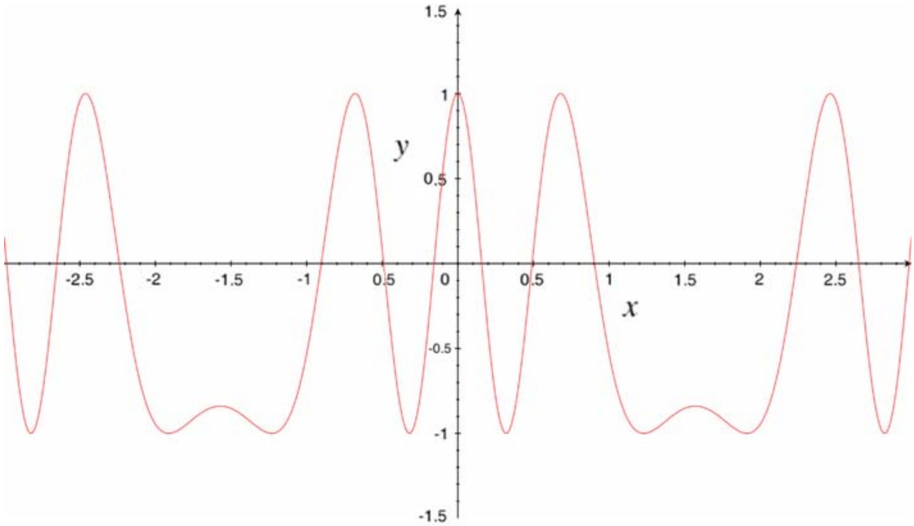

Fig. 2 Graph of h(x) = cos(10sin(x))

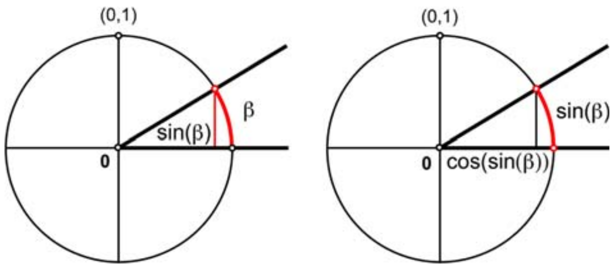

Fig. 3 Value of sin(β) becomes argument to cos

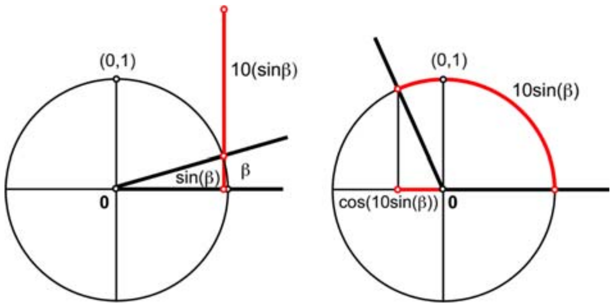

Fig. 4 Value of 10sin(β) becomes argument to cos

attempt to explain the behavior of cos(10sin(β)) in one fell swoop—without imagining small changes in the angle's measure. For example, Kristine, a teacher of calculus, wrote (using x in place of β):

> As x varies from 0 to 2π, $\sin(x)$ varies from -1 to 1, $10\sin(x)$ varies from -10 to 10, and $\cos(10\sin(x))$ varies from $\cos(-10)$ to $\cos(10)$. (Kristine, submitted Oct 3, 2005)

While Kristine did attempt to incorporate aspects of covariation into her explanation, she did not realize that her explanation conveyed little about the covariational nature of this function, nor did she realize that her explanation did not incorporate meanings of angle measure, sine, and cosine. Kristine did not talk about the fine-grained behavior of the function, which she would have obtained had she focused on varying the value of the argument in smaller bits. In fact, Kristine's explanation prompted the instructor to drive home this point by asking teachers to draw a graph that fits this description: *As x varies from 2 to 5, f(x) varies from 0 to 2.2 Sketch what the graph of f might look like.* All teachers sketched a graph that was either linear or had modest variations. He then revealed the graph in Figure 5.

Quinton (an Algebra II teacher) handily coordinated meanings of angle measure, sine, cosine, and their representations in graphs. In his response to, "Explain the behavior of $f(x) = \cos(8\sin(5x))$," he explained the graph in Figure 6 in terms of what x represented (an arc length) and in terms of any value of $8\sin(5x)$ itself becoming an arc length when used as an argument to cosine.

> In some cases, like the example shown above, the cosine graph changes from decreasing to increasing before completing an entire cycle (such as, near $x = 0.3$). The reason for this is that as the arc length x increases from 0 to $\pi/10$, $8\sin(5x)$ will increase from 0 to 8. ... As x increases past $\pi/10$, $8\sin(5x)$ passes a local maximum, so $8\sin(5x)$, which now is an arc length for the cosine function, will begin to decrease. As the arc length decreases, the horizontal coordinate of its endpoint, which is cosine of $8\sin(5x)$, will repeat its values in reverse order and thus the cosine function will

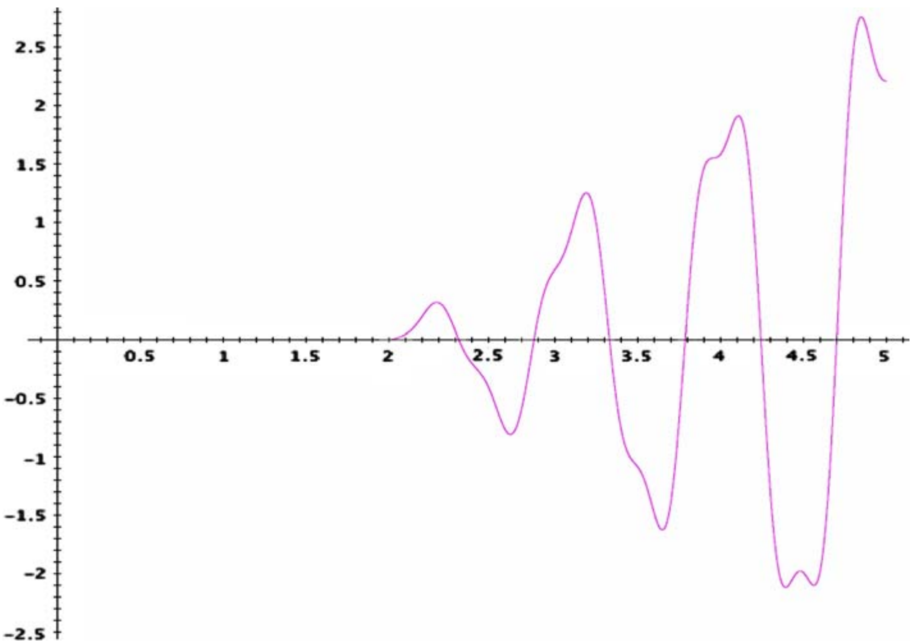

Fig. 5 As x varies from 2 to 6, $f(x)$ varies from 0 to 2.2

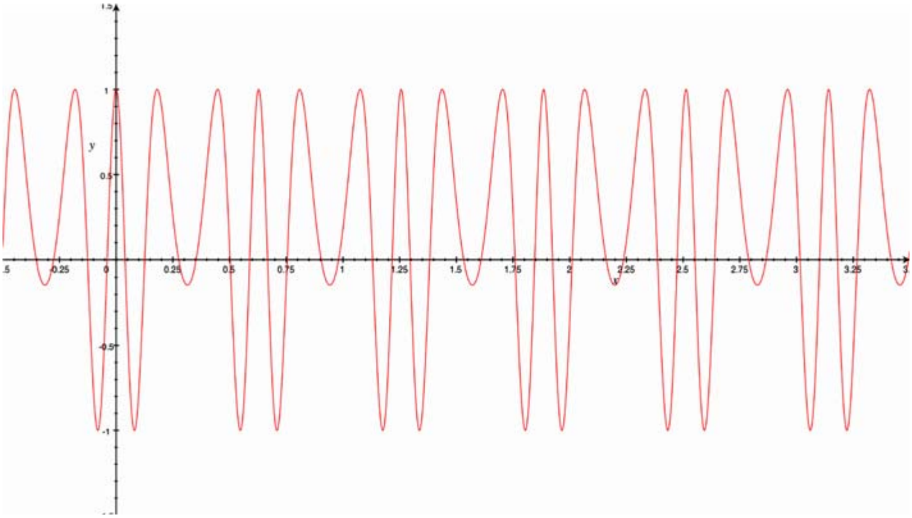

Fig. 6 Quinton's graph of cos(8sin(5x))

> reverse its direction despite not completing a full cycle. This will occur similarly at every minimum and maximum of 8sin(5*x*). (Quinton, submitted October 2, 2005)

Though Quinton's submitted work did not include illustrations of circles or arc lengths, videotapes of his work in class while he thought about the problem clarify his word usage. In class, he drew two circles similar to Figure 3 and Figure 4, but with less detail. He used the first circle to track the values of *x* (an arc length), 5*x* (an arc length), and 8sin(5*x*). He used the second circle to simultaneously track the value of 8sin(5*x*) as an argument to cos. This suggests that, in the above text, he drew meaning for *x*, *sin*, and *cos* from those images. We thus interpret Quinton's reasoning to be an explanation of how to *anticipate* the graph in Figure 6 from the function's definition, not how to interpret the graph after seeing it.

Quinton said	Quinton imagined
... as the arc length x increases from 0 to π/10, 8sin(5x) will increase from 0 to 8	Stretching the arc x from 0 to π/10 on the first circle will cause the arc of length 5x on the first circle to stretch from 0 to π/2, whence sin(5x) in the first circle will vary from 0 to 1, and thus 8sin(5x) will vary from 0 to 8.
As x increases past π/10, 8sin(5x) passes a local maximum, so 8sin(5x), which now is an arc length for the cosine function, will begin to decrease.	The value of sin(5x) in the first circle decreases from 1, so 8sin(5x), which is now an arc length on the second circle, decreases from 8, so as the value of x increases past π/10, the values of the cosine function (the x-coordinate of the terminus of arc length 8sin(5x)), will begin to retrace themselves.

Kristine's earlier statement, in which she gave a "fell swoop" explanation of a function's behavior, illustrates a challenge teachers faced throughout the course—to develop personal images of explanations that are capable of conveying what one has in mind to someone who does not already understand what one intends to convey. Teachers often described a graph's appearance instead of using the graph as data in explaining why the function behaved as it did. The reason for this, we believe, is that, until this course, it was not in their experience to make meanings "do work" for them in their teaching or in their reasoning. Their curricular knowledge was about the procedures that they intended students learn, not about ideas they intended students to have. One of us once asked, "What is the big idea here?" about a sophisticated chain of meaning-based reasoning. Nell, who taught 9th-grade algebra, replied, "To follow a procedure carefully?"[11]

While this article is about the design of tasks and is not a report of this intervention's results, we feel compelled to note the persistent difficulty that many of these teachers experienced in trying to coordinate meanings of angle measure, sine, and cosine—notwithstanding the considerable coaching we gave in class. We suspect that this difficulty has to do with the nature of reflective abstraction, which by definition entails one's coordination of meanings to form a scheme at a level of thought which thematizes concrete actions (Dewey, 1910; Piaget, 2001). We will return to this point in the final section.

Later in the course we addressed the distinction between "action" and "process" conceptions of function definitions. This is the introduction:

> Dubinsky and Harel (1992) pointed out two very different conceptions of function definitions held by people. One, an action conception, is typified by a person looking at a formula as a prescription for calculating, a "command to calculate", so to speak. Students holding this view imagine function values being calculated, laboriously, one at a time. Students holding an action conception of a formula cannot envision a graph as emerging from two quantities covarying simultaneously, nor can they see it as a mapping from one set of values to another set of values.
>
> A person looking at a formula as "self evaluating" typifies another conception of formulae, a process conception. It evaluates itself. We give it a number and, bang, we get a number back. This does not mean that the person can, in fact, perform calculations instantaneously. Rather, he or she *envisions* the formula as giving results instantaneously.
>
> How do students develop process conceptions of formulae and function definitions? By experiencing supportive instruction that persistently employs the mantra, "variables vary, go slow!". (Functions 1 course, November 14, 2005).

Our goal in this section of the course was to have teachers come to understand the last paragraph above by way of personally experiencing the need to go through an action phase of understanding a function definition before being able to reason with a process conception of that function. As such, we needed to design a function for which they would not have ready-made ways of thinking about it. We settled on the *mod* function.[12]

> We normally think that b and a in "a mod b" stand for whole numbers. 27 mod 3 is 0, because $27 \div 3$ has remainder 0. 27 mod 5 is 2, because $27 \div 5$ has remainder 2. But we can generalize this idea to fractions and irrational numbers, too. The definition of "a mod b" that does this is:

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[11] We are reminded of Stigler et al.'s report of the TIMSS video study that while 38% of Japanese lessons and 28% of German lessons addressed a significant mathematical idea, 0% of US lessons did so (Stigler, Gonzales, Kawanaka, Knoll, & Serrano, 1999).

[12] The selection of the *mod* function was motivated by one of us recalling a proof in real analysis that begins with the statement, "Let I be $\mathbb{R}/1$, the real numbers mod 1."

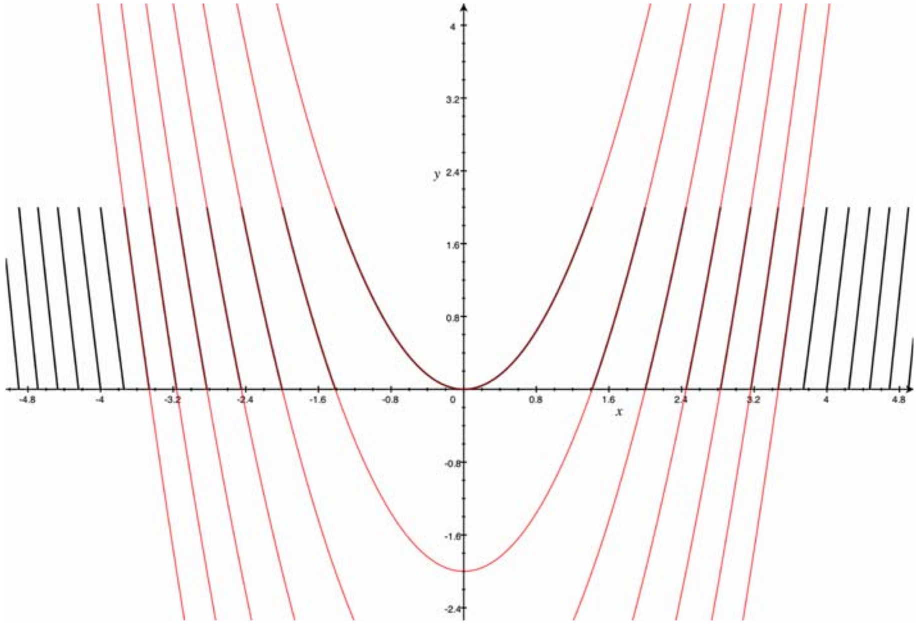

Fig. 7 Graphs of $y = \text{mod}(x^2, 2)$ and $y = x^2 - a$, $a = 0, 2, 4, \ldots$

> (a mod b) is the remainder obtained when subtracting mb from a, where m is the largest integer less than or equal to a/b.
> By this definition, (6.5 mod 2.1) = 0.2, since 3 is the greatest integer less than or equal to 6.5/2.1, and 6.5 − (3)(2.1) = 0.2. Similarly, (6.5 mod −2.1) = −1.9 because −4 is the largest integer less than or equal to 6.5/(−2.1) and 6.5 − (−4)(−2.1) = −1.9. (Functions 1, November 14, 2005)

In class, we practiced how to calculate (a mod b) for various values of a and b. Teachers used 17 minutes of practice with various values before they had a mental image of what one is doing when one calculates a mod b by the process of calculating the greatest integer less than or equal to a/b (call it g), and then calculating $a - gb$. During this 17 minutes they first internalized the definition of mod for non-integral values of a and b by going through elaborated steps of calculating specific values, and at the end they developed justifications for the generalizations that (a mod b) ≤ 0 when $b < 0$ and (a mod b) ≥ 0 when $b > 0$.

We followed this initial activity with the question, "With this definition of mod in mind, predict the graph of $y = \text{mod}(x^2, 2)$.[13] While the task appears to be about the function defined by the formula mod(x^2, 2), it really was about coming to build a process conception of *mod* so that one can then imagine the simultaneous variation of x and mod(x^2, 2).

We discussed what the graph of $y = \text{mod}(a^2, 2)$ would look like for incremental values of a. Teachers had to adjust to the new complication of squaring mod's first argument before calculating the value of mod(x^2, 2). We also discussed why and where the function's graph would "break". After having to remind them of the mantra, "variables vary, go slow!", they reasoned that the function would be 0 every time x^2 is an even number, so the graph's overall behavior would be like $y = x^2$ for values of x being between 0 and $\sqrt{2}$, like

[13] The expression "mod(x^2, 2)" for "x^2 mod 2" is due to *Graphing Calculator*'s syntax.

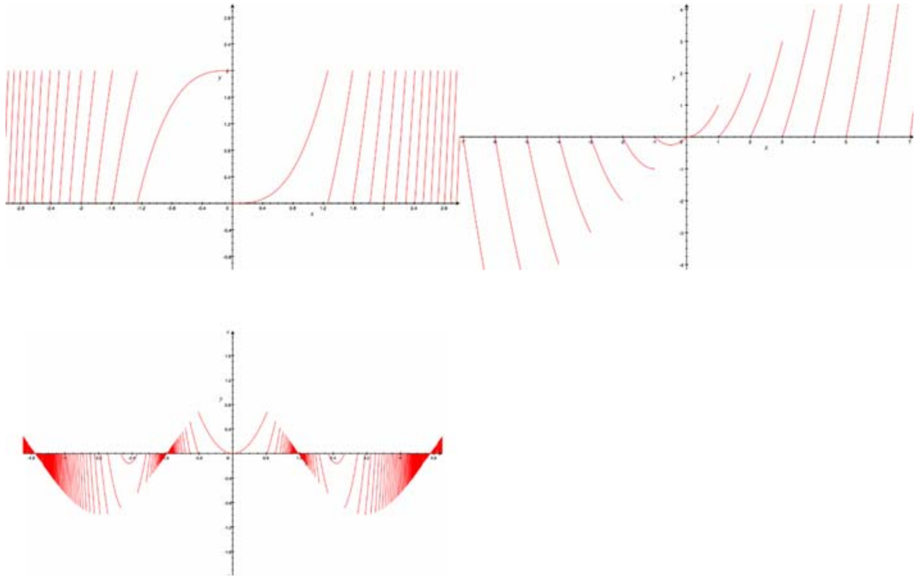

Fig. 8 Graphs of mod(x^3, 2) , mod(x^2, x), and mod(x^2, cos(x))

$y = x^2 - 2$ for values of x between $\sqrt{2}$ and 2, like $y = x^2 - 4$ for values of x between 2 and $\sqrt{6}$, etc. We checked this by graphing each of these functions on top of the graph of $y =$ mod(x^2, 2), getting Figure 7.

At each stage, we requested that teachers explain *why* the graphs behave as they do, and insisted that their explanations be rooted in their meaning of mod(a, b) and in the idea that the value of mod(x^2, 2) varies as the value of x varies.

So that teachers would have opportunities to solidify and extend their conception of the *mod* function, we asked them, as homework, to explain the behaviors of $y =$ mod(x^3, 2), $y =$ mod(x^2, x), and $y =$ mod(x^2, cos(x)), first by predicting the appearance of each graph and then by refining their explanation in light of seeing the graph. We insisted that their explanations give insight into why the graphs appear as they do, including why they break where they do (the graphs are shown in Figure 8).[14]

All teachers but one gave satisfactory explanations of the first two; only Quinton gave a satisfactory explanation of the third. Explanations employed the meaning of *mod* and showed that they reasoned covariationally, although not all included covariation explicitly. An explanation of these graphs, however, was only the end of the preparatory stage of the question we really wanted teaches to consider, which was:

> What have you learned from this assignment about how you might help students develop process conceptions of function definitions and covariational understandings of functions?

[14] The graph of mod(x^2, cos(x)) is especially interesting. It becomes much more comprehensible when you superimpose the graphs of x^2 and $\left(\text{floor}\left(\frac{x^2}{\cos(x)}\right)\right)\cos(x)$, where floor is ‘the greatest integer less than or equal to’ function.

We of course hoped that teachers would generalize from their personal experience with these tasks to realize that for students to construct a process conception of a function definition they must conceive it first at an action level and internalize it through repeated applications of it, first for individual values, then over small intervals, then over larger intervals. Responses from teachers who returned the assignment were:

Teacher	*Summary of teacher's statement about pedagogical point of the task*
Adrian	You can become better at predicting the behavior of a function by looking at the behavior over smaller intervals rather than trying to predict it all at once
Augusta	Look at easier functions before more complex functions
Bernardo	Students have to see a pattern in the way you get a value from a function before they can see a function as a process. The role of the teacher is to help them see those patterns
Carin	Students will start with an action view of a function; process views of a function come with practice and time
Earline	Students need a process conception of function to answer questions that request a rule or formula. To have them understand domain and range, have students go through calculating a function's value for successive values of x that are very close together
Estella	We should teach mod functions so that students understand covariation
Jim	Explore patterns from simple to complex
Kristine	Students need time to struggle
Liz	(Did not answer)
Nell	I see now that I've always taught a process conception because I've always emphasized rules
Quinton	I can see that when I've talked about functions with my students, I was looking at them from a process view and they were looking at them from an action view. So we probably miscommunicated a lot. A major implication is that I need to let students "play" with functions, getting lots of values for various values of x, before we start talking about general properties
Sheila	Have students make predictions before having GC draw a graph

Bernardo, Carin, and Quinton seemed to learn about their students' learning by attending to their own experiences in conceptualizing mod as a function. Earline drew an implication from her activity for moving from action to process conceptions—that a process conception of function is essential for thinking about a function's domain and range. Adrian reiterated, in her own words, the mantra, "Variables vary, go slow!" Augusta and Jim saw the lesson of these questions to be their scaffolding—one should move from simple to complex examples. Estella, Kristine, and Sheila saw things that were largely unrelated to developing a process conception of function.

It is worth restating that all teachers except Liz were at least modestly successful at explaining why these mod functions behave as they do. Several of them saw in their experiences a lesson for kinds of support students would need. The question, then, is why the other teachers did not see a lesson in their own conceptualization of mod-as-a-process about students making the same conceptualizations of functions in their curriculum. When we looked back at the collection of assignments, we noticed several differences between those who learned lessons about students' development of process conceptions of functions and those who did not. Bernardo, Carin, Quinton, and Earline incorporated covariation explicitly into their explanations of why the functions behaved as they do, and this showed most vividly in their explanations of mod(x^2, cos(x)). Augusta and Jim employed covariation explicitly in their explanation of mod(x^2, 2), less so for mod(x^2, x), and not at all for mod(x^2, cos(x)), instead relying on structural properties, such as looking for solutions to the equation $x^2 = n\cos(x)$ as places where the graph would break. Adrian, Estella, Kristine, Sheila, and Nell's explanations broke the function's domains into intervals, but referred to what the function would "look like" over those intervals. They employed covariation in their reasoning about the functions' behaviors, but they did not employ covariation in their explanations. Liz's explanations on the first two functions suggested she did not even employ covariation in her own reasoning.

The discernable relationship between the degree to which teachers employed covariation in their explanations of the functions' behaviors and the degree to which they saw implications for student learning tells us about conditions under which teachers might use their own activity to learn about student learning. In this set of tasks, successful teachers' activities appeared to go through these phases:

- Internalize the definition of mod by using it repeatedly to calculate values of mod(m, n) for various and then critical values of m and n. That is, form an initial image of mod as a process that produces a number when given two numbers.
- Refine their initial image in the context of envisioning mod($f(x)$, n), holding n constant while covarying x and $f(x)$. This requires that they interiorize[15] the definition of mod so that, from their perspective, it becomes self-evaluating. They can then focus on imagining x and $f(x)$ varying simultaneously.
- Express their reasoning about functions' behaviors veridically, in natural language. To do this successfully requires that teachers become self-aware of their reasoning processes.
- (Advanced) Refine their process-image of mod($f(x)$, n) so that it can accommodate mod($f(x)$, $g(x)$). This involves coordinating coordinations as they anticipate x, $f(x)$, $g(x)$, and mod($f(x)$, $g(x)$) varying simultaneously.

In brief, teachers needed to develop a process conception of mod to reason covariationally about it, they needed to reason covariationally to successfully predict the function's behavior, and they needed to become aware of that reasoning to explain why the functions behave as they do over every part of their domain. Those teachers who became self-aware of the reasoning behind their successful predictions and explanations were then positioned to project that reasoning and its development onto images of their students' thinking.

Discussion

We began this article by illustrating what we mean by having a coherent body of powerful meanings in trigonometry (arc length as angle measure, sine and cosine as percents of a radius) and how teachers' initial trigonometric meanings departed from those. We then described selected tasks that were designed to provide occasions for teachers to develop the meanings we saw as essential to understanding trigonometric functions coherently. We then described how some teachers experienced dissonance that was rooted in their commitments to their curricular knowledge of trigonometry (trigonometry starts with triangles,

[15] Internalize and interiorize are Piagetian terms that distinguish between phases of idea development. At first one can carry out actions in thought, but later the person can anticipate the outcome of acting and take that anticipated outcome in place of having acted.

not angles; angles are measured in fractions of a rotation; trigonometry is about solving triangles, etc.). Teachers who built new meanings into a coherent whole were those who coordinated them at a micro level. Teachers who saw implications of their own reasoning for student learning were also successful at expressing that reasoning in natural language.

We saw a similar pattern in the case of teachers' creation of meanings for action and process conceptions of mod($f(x)$, $g(x)$). Teachers who gained insight into implications of their own activities for student learning were the teachers who reasoned at a micro level in regard to the meaning of mod, who coordinated that meaning with a covariational perspective on the behavior of functions, and who expressed that coordination in natural language.

To bring this back to the nature of tasks, we must remind ourselves of our earlier statement that tasks affect learners according to what the learner makes of them. Some teachers accepted our tasks the way we had hoped. Those who did were affected more or less in the way we intended. A primary consideration here is the other teachers—those who were not affected in the way we intended. How did their context (commitments, meanings, and intentions) bar them from advancing in the same way as the others? We have several hypotheses about this, based on a diffuse corpus of interactions with them over the semester.

- In the case of trigonometry, many teachers held a strong commitment to their knowledge of their curriculum—that students would be tested on SOH-CAH-TOA, that they must teach SOH-CAH-TOA—and to the body of meanings entailed by that approach, despite having to create local patches to overcome incoherence of those meanings.
- Some teachers thought the tasks were about how to answer questions, while we intended that they be about building ever more coherent meanings. As such, they did not attend to making meanings work for them, and thus found more sophisticated tasks to be unrelated to what they had already done.
- Some teachers were only partially successful at holding basic meanings in mind while attempting to coordinate them in the context of tasks. As such, they felt confused about the task and what we hoped they would accomplish.
- Some teachers short-circuited their reasoning process by attending to figural patterns that emerged from their activity, and then attempted to base further reasoning on those patterns. We say that their reasoning was "short-circuited" because by basing further reasoning on figural patterns, they did not achieve the coordination of meanings that would sustain the kind of reflection that yields thematic, summative images of that reasoning.

With regard to trigonometry, our next round of modifications will use these same tasks, but we will attempt to preface them with situations that will, we hope, bring teachers' commitments into the open. We now know that it is insufficient to demonstrate incoherence of a body of meanings if those meanings "work" for what teachers imagine themselves teaching. We also learned that when teachers are unaware of for what they are preparing students (e.g., trigonometric functions in analysis and calculus), they cannot appreciate the inadequacy of what they currently teach.

With regard to understanding the implications for student learning of trigonometric functions and action/process conceptions of functions (as opposed to simply having them), we also anticipate that teachers *must* engage in the coordination of meanings that is necessary for productive reflection. Otherwise, teachers see these meanings as external to themselves and not as what they want their students to understand. Our challenge is to find a way to preface our tasks so that they understand this necessity.

Acknowledgements Research reported in this article was supported by National Science Foundation Grant No. EHR-0353470 and EHR-0412537. Any conclusions or recommendations stated here are those of the authors and do not necessarily reflect official positions of NSF.

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