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Mathematical sophistication among preservice elementary teachers

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Abstract This study explores the ways in which eleven preservice elementary teachers used a web-based teacher resource to apply a mathematical definition, to correct a procedural error in arithmetic, and to make sense of a story requiring the multiplication of fractions. In our analysis we propose a framework to compare the behaviors and values expressed by our participants with the values and norms of the mathematical community. This analysis suggests that many preservice elementary teachers are profoundly mathematically unsophisticated. In other words, they displayed a set of values and avenues for doing mathematics so different from that of the mathematical community, and so impoverished, that they found it difficult to create fundamental mathematical understandings.

Keywords Autonomy · Mathematical sophistication · Mathematics · Preservice elementary teachers · Teacher education

> *The only way to learn mathematics is to do mathematics.*
>
> Paul Halmos, Hilbert Space Problem
> (as found in Gallian, 1998, p. 51)

As mathematicians at a large, comprehensive university, we are faced with the challenge of preparing future generations of elementary teachers to teach mathematics. This is a daunting task indeed, given that many elementary teacher education students and practicing elementary teachers reveal a paucity of mathematics content knowledge (e.g., Ball, 1990; Ma, 1999) and overtly authoritarian beliefs about the nature of mathematical

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behavior (Schuck, 1996; Seaman, Szydlik, Szydlik, & Beam, 2005). We are left to wonder how best to design our programs for prospective elementary teachers. Should we emphasize mathematics content or stress mathematics methods and connections to the elementary curriculum? What forms should these courses take and which faculty members are best qualified to teach them?

At the center of this debate lies a persistent tension between the education and the mathematics communities over the types of mathematical knowledge elementary teachers need. The research literature distinguishes several components of teacher knowledge including content knowledge, knowledge of learning and the learner, general pedagogical knowledge, and pedagogical content knowledge. Fennema and Franke (1992) provide a synthesis of this literature and then propose that teacher knowledge is both situated and changeable. ''Within a given context, teachers' knowledge of content interacts with knowledge of pedagogy and students' cognitions and combines with beliefs to create a unique set of knowledge that drives classroom behavior'' (p. 162). We now describe recent work (Ball, 1993; Ball, 2000; Hill, Rowan, & Ball, 2005; Ma, 1999) that is aligned with this model and points to the need for teachers who can mediate between the understanding and ideas brought by the students and the mathematical demands of the content. Shulman (1986) writes, ''The key to distinguishing the knowledge base of teaching lies at the intersection of content and pedagogy, in the capacity of a teacher to transform the content knowledge he or she possesses into forms that are pedagogically powerful and yet adaptive to the variations in ability and background presented by the students'' (p.15). In the end, we will propose an emerging point of consensus between the mathematics education community and the community of mathematicians.

Ma (1999), based on her study of 23 United States and 72 Chinese elementary teachers, observed that the effective teachers were those who demonstrated profound understandings of *fundamental mathematics*. Such teachers exhibited deep understanding of basic mathematical ideas (e.g., number sense, operations, algorithms, number systems), the ability to make connections among those ideas, the use of multiple representations of those ideas, and an understanding of the curriculum that allowed for longitudinal coherence in their teaching. In the forward to Ma's book, Shulman asserted that Ma's conception of what it means to understand mathematics ''…emphasizes those aspects of knowledge most likely to contribute to a teacher's ability to explain important ideas to students'' (in Ma, 1999, p. xi). Thus, her conception of mathematical knowledge is profoundly pedagogical; it is tied to the specific tasks required of teachers as they assess, guide, and nurture the mathematical thinking of children.

Through study of their own work as elementary teachers, both Ball (1993) and Lampert (2000) have rendered detailed accounts of the complex and demanding mathematical work of teaching. The teacher must understand the rich connections among mathematical ideas. She must bridge gaps between students' use and standard mathematical use of notation and language. At the same time she must model and request the mathematical behaviors of sense making, conjecturing, and reasoning. For example, Lampert (2000), in her analysis of her work with fifth-grade students graphing the distance a car travels in 15 minutes, asserts that teaching occurs:

> in the terrain of division, in the relationship between division and fractions, and in a larger domain of what mathematicians might call ''multiplicative structures'' where rate, ratio, functions, and other big related ideas reside.… It occurs in my using words that keep the concrete context of time, speed, and distance in the conversation and in connecting this context to the mathematical terrain of rate and ratio. It occurs

in rehearsing familiar number facts and in encouraging reasoning about when assertions make sense. (p. 27)

More recently, Ball and her colleagues (Ball & Bass, 2003; Hill, Schilling, & Ball, 2004) have set out to identify and measure a *mathematical knowledge for teaching* needed to carry out the mathematical work of teaching. The following summary prepared by a cadre of mathematicians and mathematics educators captures the flavor of this knowledge.

> Effective teaching requires an understanding of the underlying meaning and justifications for the ideas and procedures to be taught, and the ability to make connections among topics. Fluency, accuracy, and precision in the use of mathematical terms and symbolic notation are also crucial. Teaching demands knowing appropriate representations for a particular mathematical idea, deploying these with precision, and bridging between teachers' and students' understanding. It requires judgment about how to reduce mathematical complexity and manage precision in ways that make the mathematics accessible to students while preserving its integrity. (Ball, et al., 2005, p. 4)

Effective teachers of mathematics should understand mathematical definitions, representations, examples and notations, and recognize those which are most powerful in supporting children's understanding; they should hear the mathematical thinking of children and guide and extend that thinking; they should recognize the nature of children's errors and alternate conceptions, and help them to create counterexamples and arguments. In their study of first and third grade classrooms, Hill, Rowan, and Ball (2005) demonstrated that teachers' mathematical knowledge for teaching positively predicted gains in mathematical achievement of their students. Their work suggests that we must improve ''…not just what mathematics teachers know, but how they know it and what they are able to mobilize mathematically in the course of teaching'' (Ball, 2000, p. 243).

The purpose of this paper is to elucidate a related facet of teacher knowledge that we term *mathematical sophistication*, and to demonstrate that this construct is helpful in explaining why some preservice teachers fail to make sense of mathematics. Specifically, we show that preservice elementary teachers in our study could not use a teacher resource to refresh fundamental mathematical ideas, and we demonstrate that this failure was due, in large part, to their lack of mathematical sophistication. While this construct is informed by our own work as mathematicians and by our observations and reading of the work of practicing research mathematicians, it seems logically 'prerequisite' and intimately intertwined with the mathematical work of teachers as described by Ma and Ball. If children construct knowledge, then the pedagogically powerful forms of mathematical content knowledge suggested by Shulman might intersect those same elements that allow mathematicians to create new mathematics. In other words, aspects of doing mathematics that make the subject accessible to the mathematical community might also make it accessible to students.

Our research perspective is one of social constructivism. We assert that cultural and social processes are fundamental to mathematical activity and that the culture of the mathematics classroom plays an essential role in developing ''mathematical disposition'' among students (Yackel & Cobb, 1996, p. 458). This perspective is described as follows by Bauersfeld (1993):

> The understanding of learning and teaching mathematics … support[s] a model of participating in a culture rather than a model of transmitting knowledge. Participating in the processes of a mathematics classroom is participating in a culture of

> using mathematics, or better: a culture of mathematizing as a practice. The many skills, which an observer can identify and will take as the main performance of a culture, form the procedural surface only. These are the bricks of the building, but the design for the house of mathematizing is processed on another level. (p. 4)

Within this paradigm, student understanding is tied to the mathematical norms of the school mathematics classroom, a culture that may be at odds with the culture of the mathematical community. It is this social constructivist perspective that suggests we compare students' mathematical behaviors and beliefs with those of mathematicians.

We use *mathematical sophistication* to describe the result of enculturation into the community of practicing mathematicians. In other words, a mathematically sophisticated individual has taken *as her own* the values and ways of knowing of the mathematical community. We note that the construct of ''mathematical sophistication'' is related to beliefs about the nature of mathematical behavior. An abundance of research suggests that preservice elementary teachers are ''unsophisticated;'' for example, they often believe that doing mathematics means memorizing and applying formulas to contrived textbook exercises (Ball, 1990; Carpenter, Lindquist, Mattews & Silver, 1983; Schuck, 1996). In this paper, we seek explicitly to identify some fundamental norms of the community of mathematicians and to demonstrate how these norms can help us to understand why many preservice teachers find mathematics difficult. In the following framework we attempt to list and clarify these norms.

Framework

We assert that the difference between a sophisticated mathematics student and a naive one lies in her beliefs about the nature of mathematical behavior, her values concerning what it means to know mathematics, and particularly in her avenues of experiencing mathematical objects and her distinctions about language. Specifically, we propose the following list of norms that indicate mathematical sophistication and form the ''design for the house of mathematizing.'' Both authors have doctorates in mathematics, are faculty members of the mathematics department of a midsize comprehensive university, and teach mathematics to preservice elementary teachers, to mathematics majors, and to graduate students in mathematics education. The content of the courses we teach includes topics in number theory, geometry, data analysis, calculus, and abstract algebra. We generated this list as members of the community of mathematicians and based on writings by prominent mathematicians as indicated. We acknowledge that there is no single view of 'what mathematics is,' yet we assert that the following list is a relevant subset of that which mathematicians value.

1) Mathematicians seek to understand patterns based on underlying structure. ''Seeing and revealing hidden patterns is what mathematicians do best'' (Steen, 1990, p. 1). ''The mathematician's patterns, like the painter's or the poet's must be beautiful; the ideas like the colours or the words, must fit together in a harmonious way. Beauty is the first test: there is no permanent place in the world for ugly mathematics'' (Hardy, 1941, p. 14).
2) Mathematicians make analogies by finding the same essential structure in seemingly different mathematical objects. Poincaré asserted, ''Mathematics is the art of giving the same name to different things'' (in O'Connor & Robertson, 2003, Quotations section, ¶ 7). ''A mathematician is a person who can find analogies between theorems; a better mathematician is one who can see analogies between proofs and the best

mathematician can notice analogies between theories. One can imagine that the ultimate mathematician is one who can see analogies between analogies'' (Banach in Ulam, 1976, p. 203).

3) Mathematicians make and test conjectures about mathematical objects and structures. ''When you try to prove a theorem, you don't just list the hypothesis, and then start to reason. What you do is trial and error, experimentation, guesswork'' (Halmos, 1985, p. 321).
4) Mathematicians create mental (and physical) models for and examples and non-examples of mathematical objects. This is the way we come to create and understand our definitions, and thus understand our mathematical objects. ''A good stock of examples, as large as possible, is indispensable for a thorough understanding of any concept, and when I want to learn something new, I make it my first job to build one'' (Halmos, in Gallian, 1998, p. 40). Models give us intuition about the behavior of mathematical objects and their relationships to other objects and allow us to check results for reasonableness.
5) Mathematicians value precise mathematical definitions of objects (Tall, 1992). Definitions provide us both necessary and sufficient criteria for classifying objects, creating taken-as-shared meanings, and making arguments. ''What is a good definition? For the philosopher or the scientist it is a definition which applies to the objects defined, and only those; it is the one satisfying the rules of logic'' (Poincaré, 1946, p. 430). ''The mathematician is not concerned with the current meaning of his technical term…. The mathematical definition *creates* the mathematical meaning'' (Polya, 1957, p. 86).
6) Mathematicians value an understanding of why relationships make sense. Poincaré claimed, ''Mathematicians do not study objects, but relations among objects; they are indifferent to the replacement of objects by others as long as relations do not change. Matter is not important, only form interests them'' (in Gallian, 1998, p. 115).
7) Mathematicians value logical arguments and counterexamples as our sources of conviction (Tall, 1992). These help us to understand relationships among mathematical objects and provide us autonomy. ''Proof is the idol before whom the pure mathematician tortures himself'' (Eddington, 1928, p. 337).
8) Mathematicians value precise language and have fine distinctions about language. We need these distinctions to communicate assertions and to make and evaluate arguments. For example, we carefully distinguish between ''and'' and ''or,'' ''there is something, such that for all'' and ''for all, there is something such that,'' ''at most'' and ''at least,'' necessary and sufficient conditions, and converse and contrapositive forms, to name just a few. ''Ordinary language is totally unsuited for expressing what physics really asserts, since the words of everyday life are not sufficiently abstract. Only mathematics and mathematical logic can say as little as the physicist means to say'' (Russell, 1931, p. 82). Laplace asserted, ''Such is the advantage of a well-constructed language that its simplified notation often becomes the source of profound theories'' (in O'Connor and Robertson, 2004, Quotations section, ¶ 5).
9) Mathematicians value symbolic representations of, and notation for, objects and ideas. Powerful notation helps us to organize our own thinking and to communicate meaning to others. Leibniz claimed, ''In symbols one observes an advantage in discovery which is greatest when they express the exact nature of a thing briefly and, as it were, picture it; then indeed the labor of thought is wonderfully diminished'' (in Simmons, 1992, p.156). ''A good notation has a subtlety and suggestiveness which at times makes it almost seem like a live teacher'' (Russell in Newman, 1956, Vol. 3, p. 1856).

The National Council of Teachers of Mathematics (NCTM) recognizes the above norms of the mathematical community and promotes them. For example, NCTM (2000) asserted that ‘‘understanding of patterns’’ is a fundamental aspect of doing mathematics; that ‘‘…all students must make and investigate mathematical conjecture’’ (p. 122); and that students ‘‘…[u]se the language of mathematics to express mathematical ideas precisely’’ (p. 128). Furthermore, NCTM recommends that all students ‘‘…recognize reasoning and proof as fundamental aspects of mathematics,’’ (p. 56), and ‘‘…use representations to model and interpret … mathematical phenomena’’ (p. 67).

We stress that mathematical sophistication does not imply an understanding of any specific definition, mathematical object, or procedure. Rather, having mathematical sophistication means possessing the *avenues of knowing* of the mathematical community that allow one to construct mathematics for oneself. For example, a mathematician may have no understanding of knot theory. But because she is mathematically sophisticated, she can read a book on the subject and teach a course. (As university mathematicians, we often teach content courses without any prior knowledge of the standard definitions or theorems. We learn the ideas by studying the definitions and reconstructing and inventing the relevant arguments and procedures during the term.) We assert that she has *access* to understanding the mathematical ideas of knot theory because she has internalized the above values and behaviors. We now show how lack of mathematical sophistication denied this access to the students in our study.

Research design and methods

Participants and setting

Elementary teacher education students at our university are required to complete three mathematics content courses. At the start of the first course (Number Systems), each student completes a 30-item multiple-choice *Inventory* designed to assess conceptual and procedural knowledge of fundamental mathematics. The examination requires students to apply definitions, to complete calculations, to make sense of stories involving geometric shapes, decimals, or fractions, and to interpret graphs and tables. We identified three target problem-types commonly answered incorrectly on the examination to use in this study. (Each of the three problems was answered incorrectly by at least two-thirds of the 58 students who took the examination in the semester the study was conducted.) The problems were also selected in a way that we might observe several indicators of mathematical sophistication; that is, we chose problems in which understanding definitions (#5 on our list of values), interpreting examples (part of #4 on our list of values), and using context to create models (another part of #4 on our list of values) might be important components to finding a solution.

We wanted to determine if student difficulty with these problems stemmed from a lack of recall of algorithms and terms (the common student complaint, ‘‘I know this stuff, but it’s been so long, I just don’t remember how to do it…’’) or if their lack of procedural proficiency with elementary mathematics such as division of decimals or finding a GCF reflects a deeper lack of understanding of the ways of learning and knowing mathematics.

We used student work on the Inventory to identify a pool of students who had answered incorrectly all three of the target problems. Eleven participants were chosen from this pool to represent several demographic factors including gender, academic history in college mathematics, age, major, and class. The mean score (number of items correct out of 30) on the initial taking of the Inventory for the interview participants was 14.45 (s.d. = 4.1, n = 11); 6 scored between 25% and 50% correct and the remaining 5 scored between 50% and 75% correct. We chose not to select as an interview participant any student who scored above 75% correct, as we were interested in studying those students with apparent lack of proficiency in basic mathematical skills. Of the 58 students who took the Inventory, all but 10 scored in this range (25%–75% correct), and so our participants are not atypical of our population of students in the course. In addition, all but one of the interview students earned a C or better in the course.

Data collection

As the emphasis of this study was on interpreting students' understandings of basic mathematical tasks with a view to observing their use or non-use of various indicators of mathematical sophistication, qualitative research methods were used in this study. These methods are described next.

Interviews with students

During the first four weeks of the semester, each of the eleven participants completed a semi-structured, one-on-one interview. In the interview, the participant was first asked to solve three problems isomorphic to the target problems and to discuss her thinking about each problem and its solution. The student was not told whether her solutions were correct. Then she was allowed to study independently (in any manner she chose) and alone with a resource similar to one to which she might have access as a practicing teacher. After twenty minutes, the interviewer returned and asked the student if she was ready to discuss the problems again or if she would like additional time at the site. All eleven students indicated that they were ready to resume the interview. The participant was now asked to revisit each interview problem, to make any changes she felt appropriate, and to explain again her thinking about the problem. If a student changed her thinking in any way, she was asked to explain why, and to discuss how the resource contributed to her revised thinking. The interviews were audiotaped and the written work was videotaped.

Interview tasks

The interview tasks are presented in Table 1. In finding the solution to Task 1, the students must distinguish between a ''factor'' and a ''multiple'' and compute a greatest common factor (GCF) and least common multiple (LCM). Thus, students might find it helpful to focus on understanding the relevant definitions and considering examples and

Table 1 Mathematical tasks used in interviews

Task Number	Mathematical Problem
One	What is the greatest common factor of 60 and 105?
	What is the least common multiple of 60 and 105?
Two	Calculate $0.4 \div 0.05$.
Three	Brooke has a $\frac{3}{4}$ pound (12 oz.) bag of M&M's. If she gives $\frac{1}{3}$ of the bag to Taylor, what fraction of a pound does Taylor receive?

non-examples (#4 and #5 on our list of values). Task 2 assessed the division of decimals. A student could solve this problem either by mimicking an example of the standard procedure or by appealing to a model of division (e.g., ''How many times does 0.05 fit into 0.4?'' Or ''How many nickels are in 40 cents?''). It was also important to attend to the meaning of the notation and to use it with precision (#9 on our list of values). Task 3 required students to make sense of a word problem involving multiplication of fractions. In order to understand this problem, it was essential for students to attend carefully to the meaning of the language in the problem (#8 on our list of values), and helpful to use a model for fraction multiplication (#4 on our list of values).

The teacher resource

The study resource is a website coordinated with the *Everyday Math* (University of Chicago School Mathematics Project, 2004) curriculum, found at http://www.math.com/homeworkhelp/EverydayMath.html. It provides teachers and parents with statements of mathematical definitions, explanations, examples, and practice problems for a variety of topics, including each mathematical idea that was needed for an interview problem. Although the pedagogy of the website is not always consistent with a constructivist theory of learning and the explanations are not always the most elegant or intuitive, we chose it for this study because *Everyday Math* is the curriculum used in the elementary schools in our district, and it is typical of the type of resource practicing teachers might have available. Participants were shown how to access the information they might need to assist them, all of which was available on a single menu. All eleven students reported that the website was easy to use.

To give the reader a sense of the site, we demonstrate the resource material relevant to the first part of the first interview problem: determine the GCF of 60 and 105. At the website, the student finds the appropriate definition, a review of the procedure for finding the GCF of two numbers, and several examples (see Figure 1). Definitions and examples of ''factor,'' ''multiple,'' and ''prime factor'' are accessible by a direct link from this page. Similarly, the website lists the steps in the common algorithms for ''moving'' the decimal

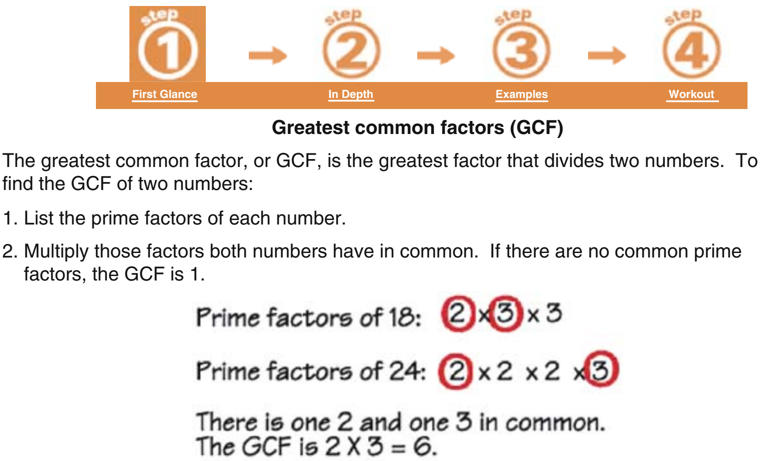

Fig. 1 Everyday Math webpage for greatest common factor

point in a long division problem and for multiplying fractions, together with examples illustrating the use of each. In our opinion, the pedagogy of the website would not be appropriate for students learning these concepts for the first time, but could be useful as a concise review of previously learned ideas. It might seem that the students who would most benefit from the short time at the site would be those who had known the definitions of terms and the procedures at one stage of their schooling and had simply forgotten them. Indeed, we used the website intervention in our interview protocol precisely to discover how often this was the case and to observe the manner in which students did or did not use the avenues of knowing of the mathematics community. Our premise is that the true underlying cause of students' lack of skill is not merely a need for knowledge refreshment, but rather is a paucity of ''accessing skills,'' a profound lack of mathematical sophistication.

Data analysis

To prepare the data for analysis, the videotapes of student written work on each problem were viewed to annotate the written work using students' comments during interview questioning. Careful attention was paid to distinguishing work and comments made prior to use of the teacher resource from work and comments make after such use. This annotated written work was then analyzed as to the correctness of the solutions prior to and after use of the teacher resource in order to prepare Table 2. After the written data were thus clarified, the researchers listened to each audiotape many times in order to code student actions and comments. This coding of the data took several iterations, with early analysis suggesting the need for a framework by which we could describe the concept of mathematical sophistication. In the final analysis, we used the items in our framework as categories. As a further check on the reliability and validity of the coding of the data, we asked a colleague who was not a part of the research team to listen to the tapes and independently to assess the mathematical sophistication displayed by the students' work and words. He concurred with our assessment of the correctness of students' solutions and with our coding of their explanations in all cases.

Results

The presentation of our analysis of the data is twofold. First, we explore participants' understandings of three mathematical tasks (see Table 1) before and after their independent

Table 2 Number of interview students with correct problem solution

Interview Tasks	On Initial Skills Inventory (in class)	In Interview before Time at Website	In Interview after Time at Website	On Final Skills Inventory (in class)
#1a – GCF of 60 & 105	0	2	3	7
#1b – LCM of 60 & 105	0	1	1	4
#2 – 0.4 / 0.05	0	0	5	2
#3 – multiplication of fractions word problem	0	3	4	7

Note: n = 11

work with the teacher resource. Second, we apply the proposed framework and use it to illuminate reasons for the students' successes and failures in working on these tasks.

Student work on the mathematical tasks

Recall that the interview tasks consisted of three problems that were essentially a subset of those on the Inventory. All the interview participants indicated that these were mathematical tasks that they were able to solve at some point in their schooling. In the words of Kay, ''A lot of this stuff I'm looking at – I know how to do it. I've done it before. But I haven't done it in so long, I've forgotten how.'' They also expressed a need to ''refresh the procedures to use to get the answers.'' Table 2 shows the number of interview participants who solved each problem in the interview correctly both before and after ''refreshment'' time with the teacher resource. The number with correct solutions after time at the website resource includes those who had correct solutions prior to spending time there.

Comparing Columns 2 and 3 on Table 2, we see that none of the 11 interview participants gained the ability to find a Least Common Multiple (Task 1b), and only one gained the ability to find a Greatest Common Factor (Task 1a) after using the teacher resource. Meg, a freshman dual elementary and special education major, shared her frustration, ''Even after looking at that website for awhile, the GCF and LCM… I couldn't figure it out.'' Likewise no participant who had been unable to solve Task 3 prior to using the website resource could interpret the one-step word problem correctly after using it. The only participant who did correct her solution after using the site said she did so because she had recalled that the word ''of'' in the problem statement meant ''multiply.'' We acknowledge that the learning process is complex, and it is possible that students learned something more from the website than they revealed in the interview. Thus, in our analysis, we will focus on what our participants *did* at the site in addition to what they appeared to have learned from their time there.

While, as indicated above, no participant appeared to gain understanding of terms, examples, notation or models from the resource, almost half were able to use the resource to correct procedural errors in applying the division algorithm for decimals (Task 2). Successful participants were those who initially made a correct interpretation of the division notation (that is, they set up the ''long division box'' with 0.05 on the outside and 0.4 on the inside) and performed a long division, but were unsure of how to ''move'' the decimal point in the answer.

Thus, even though these students expected to be able ''to pick it up quick,'' they were generally unsuccessful in ''refreshing'' their ''know how,'' despite claims that the website was easy to navigate and to read. Kara reported, ''I think it is a very good website for people who are trying to help kids with math. [It has] good explanations and the definitions were helpful.''

From Column 4 on Table 2 we see that at the end of the semester (after the first content course), 7 of the 11 interview participants were able to find the correct GCF (of 24 and 300), 4 correctly determined the LCM, and 7 were able to solve a one-step word problem involving multiplication of fractions. Thus there appears to have been some increase in understanding and skill that may be based on the participants' experiences in this content course. While gratifying, the nature of this increase is not the focus of this study, but is noted for information. We also observe that, although 5 participants corrected their procedural knowledge (on how to ''move'' decimals) in the interview, only 2 of these were still able to answer this type of problem correctly at the end of the semester. In other words, the improvement in procedural knowledge gained at the site did not appear to last.

Mathematical sophistication

In order to make sense of the participants' inabilities to use the teacher resource to assist them in solving the three interview tasks, we now contrast the approaches students said they had used while studying at the site with the ways of knowing of the mathematical community as presented in the framework. Only one student in the study distinguished between a factor and a multiple prior to work at the site, and all eleven were unsure of the meaning of a least common multiple (LCM). As Sue explained, ''For me I feel they [the words 'factor' and 'multiple'] are interchangeable,'' and Emmy stated, ''Multiples are numbers that multiply together to give the number and a factor is a number that goes into the number.'' Eventually, most students decided that the greatest common factor (GCF) was the *largest* number that went into both (and then typically found the largest *prime* factor) and the LCM was the *smallest* number that went into both (and then generally found the smallest *prime* factor). Hence, attending to the definitions while at the site would have been beneficial for all participants.

We assert that a mathematician, when faced with using an unfamiliar definition, will persist in understanding the precise statement of that definition (#5 on our list of values). She will create examples and non-examples and explore why each criterion given in a definition is important (#4 on our list of values). The students did not do this. Typically they ignored the definitions and spent their time at the site trying to follow the procedures for computing a GCF and an LCM. They focused almost exclusively on *what to do*, and not on sense making. After time at the site, six of the ten students who initially could not find an LCM now attempted to mimic the procedure they had studied (building the LCM from the prime factorizations of two numbers) to find the LCM of 60 and 105. None was successful. None could describe what it was they were finding or why the procedure made sense (#6 on our list of values). None could provide an intuitive definition of LCM. Of the remaining four students who were unable to compute correctly the LCM of 60 and 105 initially, two students stated that the site confirmed their initial (incorrect) understanding, and the remaining two claimed to have studied the definitions, but then said they were unable to understand them or to use them in the context of the problem.

Initially, no student could find the correct quotient of 0.4 and 0.05 (Task 2). Two students attempted to compute instead the quotient of 0.05 and 0.4 (reversing the divisor and the dividend); two unsuccessfully used a model of repeated additions (How many times does 0.05 fit into 0.4?); two participants tried to think about what needed to be multiplied by 0.05 to get 0.4; and three students attempted to perform the long division but were unsure how to ''move'' the decimal point. The remaining two students did not attempt the problem, but one stated that the answer must be bigger than 0.4 because ''…dividing by a point-something, I'm pretty sure makes it greater.'' In all cases, the students would have benefited from the following avenues of knowing of the mathematical community: careful attention to the notation (so they were performing the correct division or so that they could be more effective in their use of their models) (#9 on our list of values); rounding and estimation of the size of the answer; and the creation of a model for division (such as thinking about how many nickels fit into 40 cents) (#4 on our list of values) so that they could make sense of the problem and assess the reasonableness of their answers (#6 on our list of values).

The resource site shows the standard long division procedure along with explanation and examples; its only model is ''division as the inverse of multiplication.'' The site advocates rounding decimal numbers in order to estimate the reasonableness of answers, and it encourages the reader to multiply the answer and divisor to check that the product is the dividend. This ''check'' is included as part of each example. The students focused exclusively on reproducing the procedure. In fact, after time at the site even the students who initially had models for the problem abandoned them. Recall that they had not been told that their initial answers were incorrect. Typically, they did not estimate the size of an answer beforehand. Five students used the standard procedure to obtain the correct answer. Meg, however, was not convinced that the result was correct. ''I know 8 isn't the answer. That doesn't make sense. You have to get a decimal.'' After time at the site two students ''moved the decimals'' and then, in the end, *moved them back*, getting 0.08 as their (incorrect) answer. Kara explained, ''So I would get the same thing (0.08), I just had to do it the right way.'' None of the students who had reversed the quotient noticed this after time at the site. All students indicated that this was a type of problem they always did with a calculator. Ann stated, ''We learned how to do it [divide decimal numbers] on the calculator. As far as doing it in my head, oh no, I don't remember how to do this. And without a calculator I felt lost [when taking the inventory test in class].''

Students were uniformly uncertain how to approach the story problem that required multiplication of fractions (Task 3). Typically, they responded to the presence of fractions in the problem by determining immediately a common denominator. ''First, I'd attempt to get a least common denominator…'' (Sue on first reading). And then a majority attempted a subtraction because the M&Ms were ''given away.'' They appeared to focus on key words from the problem rather than making sense of the situation in the story. Prior to spending time on the website, three of the 11 students correctly determined that Taylor received ¼ pound of M&Ms. One of these (Emmy) used her contextual knowledge of cooking to model the problem. The other two students drew pictures to represent the fractions and decided that the answer was ¼ (although one of these incorrectly said '' ¼ of the bag'' at one point). Of those students who did not get the correct result, six attempted subtraction (two of whom could not correctly determine the difference) and two used division (both of whom could not correctly determine the quotient) to solve Task 3.

A mathematician faced with an unfamiliar or confusing story would attend very carefully to the language in the problem (#8 on our list of values), making certain that she was able to make sense of the situation. She might draw a picture or create another model for the problem (#4 on our list of values), or, in the interview task, make a temporary conversion to ounces. Finally, she would consider whether her answer made sense in the context of her model (#6 on our list of values). Indeed, the three students who did these things were successful and the remaining students were not. The website provides fraction definitions, fraction algorithms, sample story problems, the assertion that 'of' typically means multiply, and an example to make sense of this assertion. However, in order to find this explanation, the student would have needed to click on the section on ''multiplying fractions,'' and most did not know that this operation was required for the problem. Instead, the interview students typically used the site to refresh their memories of the procedure for the operation they had initially chosen without further scrutiny of the problem's meaning or the reasonableness of answers. Thus, the information at the site did not prove useful for them in solving the interview problems.

Discussion and conclusions

Many preservice elementary teachers begin their education programs unable to perform basic computations or to explain fundamental mathematical ideas. Typically, they attribute these difficulties to the length of time that has passed since they last considered such problems or to their dependence on the calculator. University mathematics instructors, on the other hand, have been heard to blame their students' school preparation, their inattentiveness in class, or their unwillingness to study. Our data, however, suggest yet another explanation: preservice elementary teachers are profoundly mathematically unsophisticated. In other words, they display a set of values and avenues for learning mathematics that is so different from that of the mathematical community and so impoverished, that their attempts to create fundamental mathematical understandings often meet with little success. While it may be that this lack of mathematical sophistication is a result of students' prior experiences in school mathematics, we claim that the weakness in their preparation goes well beyond a simple deficiency in procedural knowledge.

The participants of this study were not just unable to find a least common multiple of two numbers; they did not even attempt to make sense of the relevant definitions provided by a teacher resource. They were not just unable to recall the procedure for dividing 0.4 by 0.05; they were sometimes unable to set up the correct quotient (even as they followed an isomorphic example), and they often did not focus on giving meaning to the problem or the answer. They were not just unable to operate with fractions; they did not attend carefully to language in a story problem, and they did not attempt to use relevant explanations. What they did instead was to mimic the procedural examples provided by the resource - even at the expense of their own tentative models and understandings. What educators would call poor procedural knowledge was an indicator of poor conceptual understanding, and worse, was an indicator of an *inability to use the avenues of knowing of the mathematical community* to gain either conceptual or procedural understanding. We acknowledge that we studied only eleven preservice elementary teachers at one stage of their program, and that we did not select as participants high-achieving students. However our participants did earn fairly good grades (grades B and C) in their content course and are likely to graduate and become elementary teachers.

Our data support the assertion that mathematics courses for teachers should focus on their enculturation (in the sense of Bauersfeld, 1993) into the mathematical community. We advocate not necessarily an authentic replica of the mathematical community in classrooms for preservice teachers but rather an increased focus on the relevant and powerful avenues of knowing used by mathematicians. Elementary teachers must come to value precise definitions, develop habits of classifying objects based on those definitions, and invent examples and non-examples of mathematical objects as ways of enabling them to learn mathematics. They must learn to appreciate the power of mathematical language and notation. They need to create models that have meaning. Acquiring these aspects of mathematical sophistication is essential if preservice elementary teachers are to learn the mathematics they need to understand for teaching.

This enculturation into the mathematical community's avenues of knowing is as important as a deep understanding of any specific content knowledge. In their 2001 report on the mathematical education of teachers, the Conference Board of the Mathematical Sciences recommended, ''along with building mathematical knowledge, mathematics courses for perspective teachers should develop the habits of mind of a mathematical thinker…. Most of all, prospective teachers need to learn how to learn mathematics'' (p. 8). This study supports this recommendation and indeed, we claim that developing ''the habits of mind of a mathematical thinker'' should be the crucial role that mathematics content courses play in the preparation of teachers.

Our assertions are also consistent with the current reform efforts in mathematics education and the process recommendations for school mathematics made by the National

Council of Teachers of Mathematics (NCTM, 2000). Indeed, Ball (2000) and her colleagues have drawn similar conclusions.

> In our recent work on mathematics teaching, we expected to see that concepts such as place value and decimal notation would be central, and they have been, as have operations and methods of reasoning. However, beyond that, we have been struck by the unanticipated by recurrent prominence of certain mathematical notions. For instance, we have found that ideas about equivalence, similarity, and even isomorphism emerge across many instances of ordinary and extraordinary teaching and learning. We have also uncovered salient issues involving mathematical language: symbolic notation, mapping among representations, and definitions of terms. (p. 244)

These activities are not just crucial facets of mathematical knowledge, but they are part of the mathematical work of teaching. Elementary teachers should define terms in ways that will help children classify objects, solve problems, and make arguments. They should create examples and non-examples that enable their students to clarify definitions and concepts. Elementary teachers should use mathematical symbolism and notation to capture the mathematical ideas that their students express. They should choose and use many different models of mathematical ideas to provide multiple learning opportunities for their students. While Ball reached these conclusions by studying the mathematical work of teaching from the perspective of a mathematics educator, this study supports similar conclusions from the perspective of the mathematics community. In other words, mathematical sophistication and mathematical knowledge for teaching are inextricably linked, and should represent a common goal for both mathematics content and methods courses.

If, then, such courses for preservice teachers should focus on the mathematical work of teaching using the avenues of knowing of the mathematical community, we need faculty instructors who are able to teach these courses from both perspectives; that is, they, like their students, must be acculturated in both communities. This presents a serious dilemma. Mathematicians are mathematically sophisticated, but they are often unable to make that sophistication transparent to students, and most are unfamiliar with the mathematical knowledge for teaching elementary school mathematics. For example, they are unfamiliar with intuitive mathematical ideas held by children, with the sequencing of the school curriculum, and with the representations, examples and tasks that are most effective at bridging children's mathematical models with more sophisticated ideas. Many mathematics educators, on the other hand, are familiar with the mathematical knowledge for teaching but lack mathematical sophistication. And neither mathematical sophistication nor the mathematical knowledge for teaching is easily attained. Research is needed (in the vein of Chazen & Ball, 1999; Goos, 2004; Yackel, 2001; Yackel & Cobb, 1996) to document processes of this enculturation of faculty in both communities as well as to describe classroom and other experiences that increase mathematical sophistication among students.

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