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FACILITATING STUDENTS' PROBLEM SOLVING IN A TECHNOLOGICAL CONTEXT: PROSPECTIVE TEACHERS' LEARNING TRAJECTORY

ABSTRACT. In the past decade, there has been an increased emphasis on the preparation of teachers who can effectively engage students in meaningful mathematics with technology tools. This study presents a closer look at how three prospective teachers interpreted and developed in their role of facilitating students' mathematical problem solving with a technology tool. A cycle of planning–experience–reflection was repeated twice during an undergraduate course to allow the prospective teachers to change their strategies when working with two different groups of students. Case study methods were used to identify and analyze critical events that occurred throughout the different phases of the study and how these events may have influenced the prospective teachers' work with students. Looking across the cases, several themes emerged. The prospective teachers (1) used their problem solving approaches to influence their pedagogical decisions; (2) desired to ask questions that would guide students in their solution strategies; (3) recognized their own struggle in facilitating students' problem solving and focused on improving their interactions with students; (4) assumed the role of an explainer for some portion of their work with students; (5) used technological representations to promote students' mathematical thinking or focus their attention; and (6) used the technology tools in ways consistent with the nature of their interactions and perceived role with students. The implications inform the development of an expanded learning trajectory for what we might expect as prospective teachers develop an understanding of how to teach mathematics in technology-rich environments.

KEY WORDS: case study, mathematics teacher development, pre-service teacher education, problem solving, prospective teachers, teacher roles, technology

Preparing prospective mathematics teachers for classrooms in the 21st century is a complex task. Many of these prospective teachers were taught school mathematics during the 1990s when reform documents such as the *Curriculum and Evaluation Standards for School Mathematics* (National Council of Teachers of Mathematics, NCTM, 1989) were only beginning to affect state and local curriculum. More importantly, the influences of reform ideas were not consistently implemented in mathematics classrooms (e.g., Ferrini-Mundy & Schram, 1997) resulting in the current generation of prospective teachers having only a modicum of learning experiences that captured the spirit and vision of the NCTM (1989, 2000).

Two aspects of reform in mathematics education are of interest for this paper: the importance of problem solving, and the increased availability and use of a variety of learning tools, especially technology. Prospective teachers need experience as learners of mathematics that emphasizes problem solving and use of learning tools. However, in the context of using technology tools, Olive and Leatham (2000) found that prospective teachers' use of such tools in their own mathematical learning is often insufficient to understand how to help *students* learn mathematics with these tools. Prospective teachers also need to understand how students use a variety of tools to solve problems and what this implies about their use to facilitate students' problem solving.

Most problem-solving literature uses frameworks from Polya (1957) and Schoenfeld (1985) to guide instruction and research. Of the four aspects of students' problem solving emphasized by Schoenfeld – resources, heuristics, control, and beliefs – teachers play a critical role in helping students choose resources, implement heuristics, and control their problem solving actions. The use of physical resources (learning tools) in a problem-solving context can afford or constrain students' ability to use certain heuristics and to actively control their problem solving (Healy & Hoyles, 2001). In such contexts, teachers must continually make pedagogical decisions concerning how much input and the nature of that input in order to facilitate students' use of resources, heuristics, and control.

Early experiences with students can help prepare prospective teachers for the pedagogical challenges of effectively engaging students in problem solving. These experiences, when coupled with reflection, can enhance prospective teachers' understanding of the complexity of facilitating different aspects of students' problem solving. This study investigates two research questions about prospective teachers' engagement in an iterative cycle of *planning–experience–reflection*.

1. How do prospective mathematics teachers make sense of their interactions with students when facilitating problem solving with a technology tool?
2. What is the nature of prospective mathematics teachers' perturbations resulting from interactions with students and available tools, and do reflections on these perturbations influence their pedagogical decisions?

CONCEPTUAL FRAMEWORK AND BACKGROUND

A coordinated perspective of learning provides the underlying conceptual framework for this study. The perspective involves individuals' constructive process of resolving perturbations through reflecting on their actions (and subsequent effects). This allows for an abstraction of ideas with a lens on the context in which meanings are socially negotiated through interactions (Tzur & Simon, 1999; Voigt, 1996; von Glasersfeld, 1995). Additionally, this perspective is augmented by the view of available tools and mathematical tasks as both enabling and constraining learning (Wertsch, 1991; Graue & Walsh, 1998). Thus, the available tools, the mathematical tasks, and the social interaction among students and between students and teacher, all operate interactively as potential meaning-making agents for students' learning and a teacher's pedagogy (Figure 1).

In addition to making sense of students' meaning-making interactions, prospective teachers must develop an understanding of their role in facilitating students' problem solving. The perturbations occur for prospective teachers as they observe and interact with students and reflect on their own and students' interactions with each other, the mathematics, and the available tools (Figure 2). Additionally, prospective teachers are influenced by their beliefs, knowledge, and social interactions with students, peers and faculty. It is within these social contexts that teachers make sense of their role as instructional designer, teacher, and evaluator of students' understanding in an environment rich with a variety of tools for learning.

In the context of this study, prospective teachers and students have access to a specific software application as well as to paper and pencil. A brief review of literature of learning to teach with technology tools

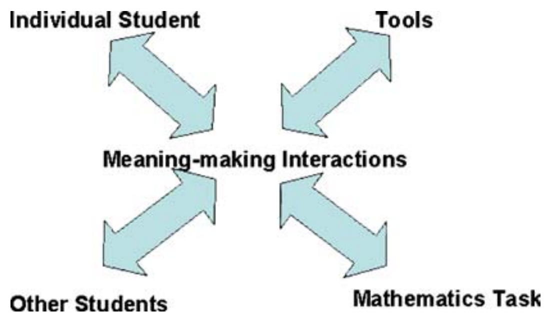


Figure 1. A coordinated perspective on contributions to meaning-making interactions.

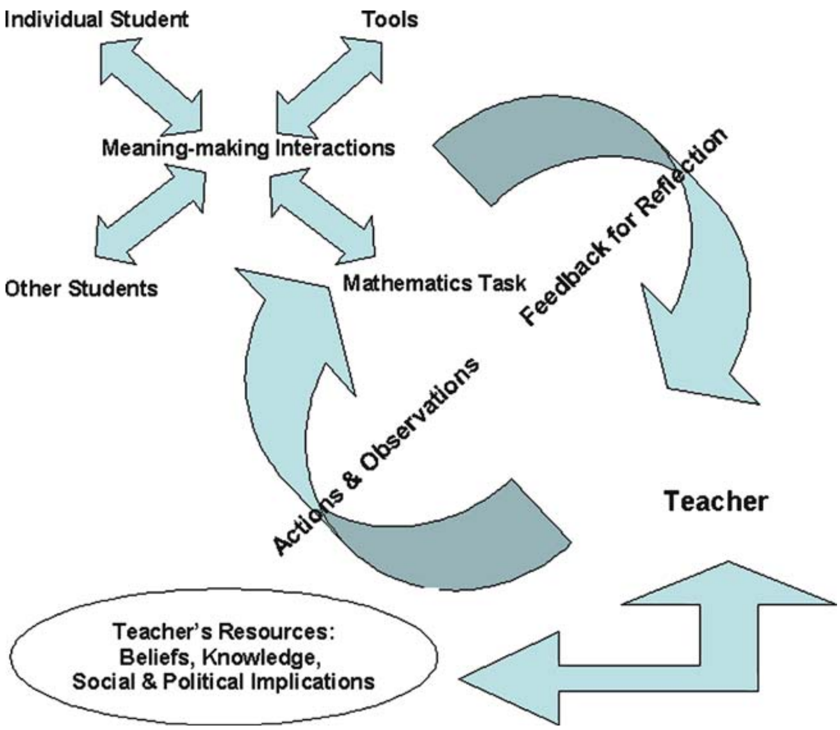


Figure 2. Teacher's contributions to and sense-making of interactions.

can provide insight into some of the difficulties prospective teachers may have facilitating students' problem solving in such an environment.

Learning to Teach with Technology

Computer technology can enhance students' problem solving by providing an environment that allows them to engage in playful exploration, test ideas, receive feedback, and make their understanding public and visible (Clements, 2000). Several researchers have studied prospective and practicing teachers learning to use computer and calculator technologies in teaching mathematics. When beginning to use technology, teachers often focus on classroom management, use highly structured lessons, or use technology only for remediation or practice (Drier, 1998; Heid, Blume, Zbeik, & Edwards, 1999; Manoucherhi, 1999; Tharp, Fitzsimmons, & Ayers, 1997; Thomas, Tyrell, & Bullock, 1996). Such uses of technology and a personal need for structure and authority in a classroom may be related to teachers' beliefs about mathematics learning and teaching in general, and appropriate uses of technology in particular (Drier, 2001). Many teachers have a computational orientation towards teaching mathematics that can be seen to stem from an underlying belief that doing mathematics is a rule--driven, right or wrong endeavor (Thompson, 1984; Thompson, Philip, Thompson, & Boyd, 1994).

Prospective teachers tend to have similar computational and authoritative approaches to teaching and hold beliefs that technology should be used after prerequisite mathematics knowledge and skills have been mastered (Drier, 2001; Turner & Chauvot, 1995). After extensive experiences engaging in technology-based mathematics activities and discussing pedagogical issues with a teacher educator, prospective teachers' beliefs about the use of technology can shift to include use of technology for conceptual understanding (Drier, 2001).

Teachers' beliefs about teaching, learning and the use of technology influence the pedagogical goals they make and carry out. However, teachers' "professed" beliefs do not always match the beliefs as attributed by researchers when analyzing teachers' interactions with students (Aguirre & Speer, 2000). It is difficult to ascertain how prospective teachers' professed beliefs influence their interactions with students without observing them and the ways in which they use available tools. Beliefs are enacted as prospective teachers make sense of their role in the interaction system as depicted in Figure 2.

Heid, Sheets and Matras (1990) and Farrell (1996) identify roles that practicing teachers assume when teaching in technology-rich environments. Heid et al. (1990) claim that teachers may assume three different types of roles: (1) a *technical assistant* to help students trouble shoot issues with using technology tools, (2) a *collaborator* with students in problem solving, and (3) a *facilitator and catalyst* to monitor and help students proceed in problem solving, often by asking prompting questions. In Farrell's (1996) research, she identifies six types of roles that teachers assume: (1) a *manager* of the classroom, often as an authoritarian, (2) the *task setter* who often questions students but still decides most strategies for doing mathematics, (3) an *explainer* who gives rules and sets the focus of any problem solving, (4) a *counselor* who advises, encourages, and stimulates students' problem solving (similar to *facilitator*), (5) a *fellow investigator* who participates in problem solving along side students (similar to *collaborator*), and (6) as a *resource* of information, both mathematical and technical (similar to *technical assistant*).

In observations of many 5-minute classroom episodes from a pre-calculus course, Farrell found a high number of incidences where teachers act as *managers* and as a *task setter* and *explainer*. In contrast, there are a low number of incidences where the teacher acts in a role as a *resource* and *fellow investigator*. When teachers in Farrell's study are not using technology, they assume the role of an *explainer* more often than when technology is being used. Thus, it appears that the use of technology tools can promote less reliance on the teacher as an explainer, although teachers still tend to set the task and manage students' interactions.

By working with students solving a problem with technology, prospective teachers can make sense of their role in the teaching and learning process. The work of Bowers and Doerr (2001) illustrates how teachers experience perturbations as learners and teachers, reflect on activities and effects of those activities, and think critically about pedagogical decisions for using technology to promote students' understanding. The prospective and practicing teachers in Bowers' and Doerr's study first use a microworld as learners of mathematics and then as a teacher with a small group of students. Through extensive class discussions in the role of learners, the teachers recognize the potential value of conceptual explanations and debate over the ordering of such exploratory activities in relation to learning formal symbolism. The insights from their role as a teacher include the value of capitalizing on students' incorrect explanations or misconceptions and the affordances and constraints of technology tools on students' mathematical activities. This type of dual activity places teachers in contexts where they may experience a perturbation as both learners and teachers and must make sense of, and possibly reorganize, their understandings of learning and teaching mathematics.

The design of this research study is informed by the prior research on teachers' beliefs about teaching and learning with technology, and an understanding of the context within which prospective teachers can develop pedagogical insights. In addition, the various roles as described by Heid et al. (1990) and Farrell (1996) allow the researcher, as a teacher educator, to anticipate how prospective teachers may enact their beliefs and knowledge as roles for interacting with students.

LEARNING AND RESEARCH CONTEXT

Course Setting

This research study was conducted in the context of a "Teaching Mathematics with Technology" course for prospective middle and secondary school mathematics teachers. The course content focused on exploring mathematics tasks using various technology tools (e.g., dynamic geometry software, spreadsheets, probability simulators) and discussing teaching strategies for using such tools with students. In this course, the teacher educator often engaged prospective teachers in a brief technology-based activity as if she was working with students, and then discussed the pedagogical aspects of the activity explicitly with the prospective teachers. This type of teacher education pedagogy allowed prospective teachers opportunities to learn mathematics, technology, and pedagogical methods-type skills (e.g., planning, hypotheses about students' learning, questioning techniques, reactive "on-the-fly" decision making). This design was informed by knowing that prospective teachers can have pedagogical insights in the role of learners of mathematics and by discussing pedagogical issues and decisions with peers and an "expert" (Bowers & Doerr, 2001; Drier, 2001). The teacher educator/researcher decided to enhance these course experiences by implementing an intervention to engage prospective teachers in opportunities to promote pedagogical perturbations and growth.

Cycle for Enacting Teacher Education Goals

The cycle in this research project is similar to Simon's (1995) Mathematics Teaching Cycle and a model described by Artzt (1999) that emphasizes teachers' cognitive processes in pre-active (planning), interactive (monitoring and regulating), and postactive (evaluating and revising) phases of instructional practices. Artzt uses this model as a basis for enabling prospective teachers to reflect on their teaching. Simon's (1995) model addresses the "inherent challenge to integrate the teacher's goal and direction for learning with the trajectory of students' mathematical thinking and learning" (p. 121). As such, he recognizes that a teacher's beliefs and knowledge affect the development of a hypothetical learning trajectory (HLT) for students that includes a teacher's goals for the learner, plan for activities and questions, and an initial understanding of students' learning process with these activities and possible cognitive obstacles. The heart of Simon's model is analyzing students' understandings, reflecting, and making subsequent instructional decisions. This is consistent with the ‘‘actions and observation’’ that feed into the meaning-making interactions (Figure 2) and the ‘‘feedback for reflection’’ stemming from those interactions that help a teacher adjust instructional goals or activities.

The project I report involves the development of a 6-phase cycle in a deliberate attempt to enact the conceptual framework by creating a situation for prospective teachers to experience perturbations while students solve a problem with technology. My intent was that by repeating the experience with different students, prospective teachers could use their reflections to inform their HLT for students both ‘‘on the fly’’ and for the subsequent experience. The prospective teachers enact the six phases[1] by

(1) Individually solving the Fish Farm problem (Figure 3) using the java applet, submitting a solution to the MathForum, receiving feedback from a mentor at the MathForum, and discussing the problem with peers and the teacher educator/researcher.
(2) Developing anticipatory ideas and planning a HLT for students.
(3) Interacting with two students as they solve the Fish Farm problem with a java applet.
(4) Discussing the experience with peers, reflecting, and planning of revised HLT for different students.
(5) Interacting with two different students as they solve the Fish Farm problem with a java applet.
(6) Reflecting on their role in facilitating students’ problem solving with technology and their understanding of what the students understood about the problem.

In the reflection phases of the project (2, 4, 6), the prospective teachers were given guiding prompts to help focus their reflective activity. Several prompts in Phase 2 help prospective teachers think about students’ hypothetical learning trajectory by considering possible solution strategies, difficulties students may have, and questions that might be asked to help students overcome those difficulties. In Phase 4, the prospective teachers were asked to reflect on their interactions with students, students’ understanding and problem solving, and changes or improvements desirable for the next group of students. In Phase 6, the prospective teachers were prompted to compare the two experiences and to reflect on what may have caused any similarities or differences in their interactions with students and how students solved the problem.

A Fishy Family:

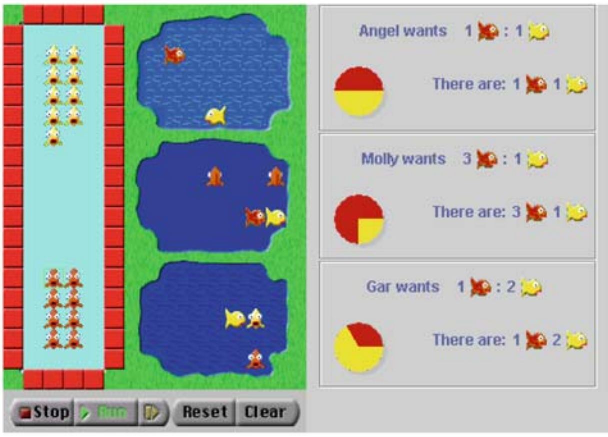
For their birthday, the Carp triplets received 26 tropical fish: 13 females and 13 males. They discussed ways to divide the fish among their three tiny backyard ponds.

Angel said, "I want the same number of male and female fish in my pond."

"Okay," said Molly. "I want three times as many males as females in my pond."

"Then I want twice as many females as males in my pond," Gar replied.

Is there a way to put all 26 fish into those three ponds, while giving each triplet what he or she wants? Use the applet to explore this question.



Questions:

1. How many male fish and female fish does each triplet get in his or her pond? Describe the work you did to find the solution. Sample questions you can answer: Into which pond did you put fish first? How many fish of each kind went into that pond? Why? What was your next step? How were you sure a pond had the correct ratio?
2. Given the 13 males and 13 females, what are ALL the possible numbers of male and female fish that would satisfy the ratio of 1 male to 2 female fish in Gar's pond? Explain why these different amounts are equivalent to the ratio 1:2.

Bonus: Explain why all possible answers in question 2 result in the same pie graph for Gar's pond.

Figure 3. Fish Farm problem and java applet used in study. [Available online at http://www.escot.org/resources/applets/fish1].

The Problem Task and Technology Tool

The problem (Figure 3) used in this study is designed with an accompanying java applet to give students access to different possible solution strategies and representations to help them make sense of the problem. The problem and applet are part of the MathForum's "Problem of the Week" and are designed to allow for a variety of possible classroom uses, either directed or non-directed by a teacher.[2] The applet contains three linked representations such that as a user drags and drops a fish in and out of ponds, the ratio and pie graph displays update accordingly.

This problem is open-ended since there are multiple strategies to use and two possible correct solutions to question #1 (see Figure 3). Although students can solve this problem using manipulatives or paper-and-pencil, the intention of using the problem situation and applet is to allow students to enact different strategies and solution paths, use part–part and part–whole reasoning about equivalent ratios, and to promote reasoning about the dynamic pie graph representation of the ratios. The bonus question (see Figure 3) is designed to induce a perturbation for students about the relationship between a part–part and a part–whole representation of a ratio. Many students intuitively think about a 1 male to 2 female ratio as representing a one-half situation and do not easily make the transition to a "1/3 males" representation. I hypothesized that such a perturbation for students may also cause a perturbation for the prospective teachers as they make decisions about how to help the students.

This problem and applet have the potential for a variety of different student approaches that may challenge pedagogically the prospective teachers without necessarily causing an extra challenge mathematically or technologically. The technology tool requires students only to drag and drop icons rather than needing to know specific technology skills in more generic software tools (e.g., graphing calculator, spreadsheets). Pedagogical perturbations may occur when a prospective teacher needs to make an instructional decision to respond to students' interactions with each other, the mathematics, or the technology tool.

METHODS

Case Study Participants

A focus on three prospective teachers as case studies (Stake, 1995) allows for a critical analysis of their development as they enact the teacher educator's planned hypothetical learning trajectory. The prospective teachers were third year secondary mathematics education majors enrolled in the "Teaching Mathematics with Technology" course taught by the researcher. All prospective teachers participated in project activities with the option of allowing their work to be used for research purposes. Out of 13 prospective teachers, all agreed for written work to be used, 11 agreed to videotaping, and 7 volunteered as a possible case study participant. The three case studies were deliberately chosen from these 7 volunteers. These three prospective teachers had participated in a research study (Cavey, Berenson, Clark, & Staley, 2001) related to designing lesson plans on concepts of ratio during an introductory mathematics education course the previous semester. In addition, they represented a range of achievement in mathematics courses and in their mid-semester grades in the current course. Thus, these students had common prior experiences thinking about how to teach the concept of ratio and represent a range of achievement in their college courses.

Brandi is a female with high achievement in her college-level mathematics and excellent grades in the current course during the first eight weeks. Chandler is a male with above average achievement in his college-level mathematics and above average grades in the current course. Griffin is a male with average achievement in college-level mathematics and average grades in the course at mid-semester.

Data Collection

Each case study prospective teacher worked with eighth grade students at a specially-equipped computer. These computers had a webcam and microphone that captured video/audio of interactions between the students and prospective teacher. The computers were also equipped with an additional microphone and a PC-to-TV converter to capture the monitor display and to record voices as students used the applet. The complete data corpus for the study included 12 videos (6 internal & 6 external videos) and all written work from prospective teachers and 8th grade students. Whole class discussions of the prospective teachers were not recorded since several had not agreed to video/audio recordings. The teacher educator's notes about these discussions were the only data available from these phases of the project. Although these discussions probably introduced insights and perturbations for the participants, there is no direct evidence for analysis.

Methods of Analysis

The methods for analysis included use of an analytical model for analyzing video data to identify critical events (Powell, 2001), a constant comparative method (Strauss & Corbin, 1990) to look for patterns, and an interpretation cycle within a case and across cases (Lesh & Lehrer, 2000). The analysis was initially linear through the six phases for each case study to study the prospective teachers' interactions with students (e.g., questions they asked students, how they used the tools in the java applet, and the pedagogical decisions they made ''on the fly''). Critical events were marked for two reasons: (1) when an important pedagogical decision was made in anticipation of or in reaction to interaction with students or technology; and (2) when there appeared to be a significant event that may cause a pedagogical perturbation. Constant comparative techniques were used to look across all six phases to identify related critical events and to hypothesize the nature of the relationship between these events (e.g., inconsistent, directionally influential). Finally, an interpretation cycle was used to analyze within and across cases to note similarities and differences in the prospective teachers' pedagogical learning trajectories.

ANALYSIS OF PROSPECTIVE TEACHERS' ENACTED LEARNING TRAJECTORY

The learning trajectories of each prospective teacher provide rich examples of the complexities and intricacies of learning to facilitate students' problem solving. The three enacted learning trajectories provide examples and analysis of critical events from each phase that appear to cause perturbations and influence their pedagogical decisions. Figures 4–6 provide a synopsis of the critical events for Brandi, Chandler, and Griffin, respectively. The analysis of these critical events refers directly to those listed in each figure.

Brandi's Trajectory

There are four critical events for Brandi in Phase 1 (see Figure 4). First, her solution process for question #1 is well written and involves the coordination of several aspects in the problem (CE1). She has an error in her solution to question #2 since she notes that ''1:2, 2:4, 3:6, 4:8, 5:10, 6:12, 7:14, and 8:16'' will satisfy Gar's ratio without realizing 7:14 and 8:16 do not meet the constraints of the problem (CE2). During the whole class discussion, several classmates share their solution processes and solutions, including Brandi. Several classmates comment on the efficiency and ''elegance'' of her method (CE3). The number of equivalent ratios for Gar's pond (question #2) is discussed in class, and Brandi is privy to the justification for why 6:12 is the greatest ratio allowed (CE4).

Phase 1: Solving Problem & Discussion with Peers

CE1: Her solution to Q1 coordinates equivalent ratios in pond and remaining fish in tank and she provides detailed justification.
CE2: She has error in Q2 stating 7:14 and 8:16 are viable ratios without considering constraint of 13 males.
CE3: She shares solution process for Q1 and classmates comment on her efficiency and elegance
CE4: She listens to (does not participate in) class discussion about mistake with 7:14 and 8:16

Phase 2: Planning

CE5: Anticipates having to explain aspects of problem or concepts
CE6: Hypothesizes 3 questions she could use to help students justify work, but questions actually ask for clarification.
CE7: Does minimal written planning and shows lack of careful anticipation of students' actions

Phase 3: Experience with Students

CE8: While solving Q1, tells students when ratios are "satisfied"
CE9: Rewards/encourages processes similar to hers
CE10: Confirms solution to Q1 is correct without asking for justification
CE11: Has students trade control of mouse and asks them to find another way to solve Q1
CE12: Leads students with her strategy and solution
CE13: Leads students in Q2 with "next one" strategy
CE14: Corrects student's suggestion of 7:14 as ratio in Q2
CE15: Explains pie graph
CE16: Encourages students to explore other tasks in time remaining, asks questions based on status of applet

Phase 4: Reflection/Planning

CE17: Praises self on "trading mouse" decision
CE18: Perceives students explained solutions
CE19: Praises self on ratio and pie graph explanations
CE20: Surprised by students use of multiplicative reasoning in finding solution to Q1
CE21: Admits she gave too many hints
CE22: Wants to avoid giving too much information and add wait time

Phase 5: Experience with Students

CE23: Allows students more freedom and asks open-ended questions when they solve Q1
CE24: Tells students when ratios are "satisfied"
CE25: Encourages multiplicative reasoning
CE26: Does not encourage pursuit of second solution
CE27: Leads students in Q2 with "next one" strategy
CE28: Switches students focus in Q2 between additive and multiplicative strategies
CE29: Students confused, she is visibly frustrated
CE30: Asks students to reason why 7:14 is not possible when they suggest it
CE31: Uses paper and gives lengthy explanation of pie graph and equivalent ratios, in terms of part-part and part-whole

Phase 6: Final Reflection

CE32: Recognizes she prompted too much but links this with her frustration in engaging them (does not link to their confusion)
CE33: Thinks paper-based explanation works better than applet
CE34: Thought applet was good to give feedback and help for explanations
CE35: Recognizes struggle with posing non-leading questions and thought project helped her develop better questioning techniques.

Figure 4. Brandi's critical events throughout each phase.

In Phase 2, Brandi anticipates she may need to explain aspects of the problem or concepts to the students (CE5) and hypothesizes several questions that would prompt students to justify their work (CE6). These questions, however, are actually clarification-type questions and would not necessarily lead to justification (e.g., "So, is Molly and Gar's ratio satisfied?"). Overall, she has not spent much time on anticipating actions or questions (that are expressed in written form) to help facilitate students' problem solving and use of the applet (CE7). This lack of careful planning may be related to a high confidence in her own problem solving methods (CE1 and CE3) and seems critical when considering her learning trajectory.

As she works with the students in Phase 3, several critical events appear to be influenced from earlier events in Phases 1 and 2. Her interactions with students and the control she exerts to guide their problem solving (CE9, CE11, CE12, CE13) are highly leading and can be traced back to her own problem solving (CE1 and CE3) and lack of planning (CE7, which was probably influenced by CE1 and CE3). Her lack of preparation may have left her without a choice to implement anything other than her own strategy. Brandi's planned use of clarification questions in CE6 to help students justify their work demonstrates that, at least in her written plan, she was not prepared to ask for justification or reasoning from the students (CE8 and CE10).Considering her anticipation in CE5 that she would need to explain aspects of the problem to students, there are several events that follow (CE8, CE10, CE14, CE15) where she does, in fact, do this. Brandi's use of open-ended questions in CE16 demonstrate a very different approach from that which she took in helping students complete the questions in the problem. This event demonstrates at least one instance when she is capable of asking such questions and responding to students' actions and applet status in a more constructive manner. The fact that the more open-ended questions appear later in the sequence of interactions with students could indicate she believes (researcher's hypothesis[3]) that structured teacher-led interactions should be used to complete instructional goals (questions posed in the Fish Farm task) and that student-guided explorations are auxiliary to instructional goals.

Her reflection and planning in Phase 4 are related to the critical events in Phase 3. Two critical events in this phase are noteworthy. She believes, incorrectly, that students explained their solutions (CE18) and she is surprised by their use of multiplicative reasoning (CE20). Although she praises many of her pedagogical decisions, CE18 seems to indicate that she also perceives that the students were cognitively involved in explanations, which is contradictory to what actually happened. This is also not completely aligned with her recognition that her interactions with students may have been too overbearing and that she wants to change this approach (CE21 and CE22). Overall, her reflection and planning are focused on *her actions* and indicate an underdeveloped awareness of how her actions are actually affecting students' meaning-making interactions (see Figure 2).

During her work with the second group of students (Phase 5), Brandi allows the students much more self-directed control when solving question 1 (CE23) and asks questions such as "what do you think we should do now?" She seems more willing to allow them to pursue different solution paths and comfortable asking questions that do not necessarily lead students down a particular solution path. Her actions follow from her plans from Phase 4 (CE22) to avoid giving too much information and add wait time. Although she did not point directly back to her experience of asking students more open-ended questions in CE16, this shift in her need for structured teacher guidance may be related to her success with students in CE16 (researcher's hypothesis). She also encourages students to use multiplicative reasoning, as a result of her reflection in CE20, and improves her response in CE14 by asking them in CE30 to justify why 7:14 is not a possible solution. However, she still tells the students when ratios are satisfied and leads students in a solution for question 2 (CE24 and CE27). Her switching of strategies and resulting frustration and students' confusion (CE28 and CE29) led her to fall back into the role of an explainer (CE31) and could be influenced (researcher's hypothesis) by her lack of anticipation of students' difficulties (CE7).

Brandi's final reflection in Phase 6 reveals that she recognizes her difficulties (CE32 and CE35), but also demonstrates that she still is focused on her actions and the use of the technology tool and paper to aid her actions (CE33 and CE34), rather than its effect on students' actions and problem solving. It is clear that Brandi's own solution processes and mathematical understandings have a direct impact on her interactions with students. What is not clear, is if she is aware of this phenomenon. She does seem aware of her struggle to ask appropriate questions and to not lead the students, but does not express a realization that her understandings of the mathematics influenced that struggle and her tendency to lead the students and assume a role of an expert.

Chandler's Trajectory

The critical events described in Figure 5 highlight Chandler's trajectory through the six phases. During Phase 1, Chandler solves the problem in question #1 with a focus on the constraints in Molly's pond (CE1) and the use of the pie graph to aid and justify his reasoning (CE2). He does not present a solution to question #2 but hears the solution presented by others in class (CE3). Chandler shares his two different methods for finding a solution with his classmates, seems very interested in contributions from his peers about how they solved the problem, and asks questions and makes comments on their solution methods (CE4). This curiosity about different solution methods seems to affect his work in subsequent phases.

Chandler seems to plan carefully in Phase 2 and maintains a focus on students' actions in CE5, CE6 and CE8 and how he must ask questions and interact with students CE5, CE7 and CE8. His interest and curiosity about students' solution processes (CE6) stems from CE4 in Phase 1. He believes "my biggest problem will be asking good questions so as to make them think and head in the right direction, but without giving away the answer or leading in the direction they need to go" (CE7). At this phase, he seems open to students' ideas and aware that he may have difficulty in asking appropriate questions but wants to maintain student-controlled solution methods and explanations. He also expresses concern that one student may dominate the conversation and that he may need to intervene to ensure the students work collaboratively (CE8).

In his work with students in Phase 3, several of the critical events (CE9, CE10, CE12, CE13, CE14) seem directly influenced by his forethought on how he wanted to interact with the students (CE5, CE7, CE8). His pedagogical decision in CE12 to allow students to explore a large number of fish in Molly's pond is related to his own solution strategy and reflection on the constraints of the problem (CE1). In addition, his use of the pie graph as a focus for questions (CE11, CE15, CE16) is also similar to his own use of this representation (CE2).

Chandler's interactions with the students and his pedagogical decisions in CE15 and CE16 are of particular importance. While the students' second solution is shown in the applet, he engages the students in a conversation about equivalent ratios using the numerical and pie graph displays. He asks the students to justify Angel's pie graph (8:8), and encourages the students to explain further why they know it is "half." He asks the students to look at Molly's pie graph (1/4 males) as a percentage to think about a part–whole relationship. The students are confused when they look at Gar's pond and Chandler asks, "why are his males less than half?" After a brief pause, Chandler uses fractions to explain the pie graph and why 2:4 also works in Gar's pond. It is interesting that he uses the concept of percentages to anchor the discussion about Molly's ratio (75% and 25%)

Phase 1: Solving Problem & Discussion with Peers
CE1: His solution process for Q1 begins with Molly's pond and is focused on equivalent ratios
CE2: He misunderstands Q2 and gives another solution to Q1 using pie graphs as visualization and part of his reasoning for finding second solution
CE3: No evidence he has considered solution to Q2 on his own but is privy to class discussion
CE4: In class, he shares his two different methods for finding a solution and asks questions and comments on peer's solution methods

Phase 2: Planning
CE5: Anticipates students may have questions about problem constraints and equivalent ratios but that they should work together to solve the problem
CE6: Interested in students' strategies to distribute fish.
CE7: Believes challenge will be asking questions "without giving it all away"
CE8: Anticipates encouraging students to collaborate

Phase 3: Experience with Students
CE9: Allows students to choose their own strategy and waits patiently as they make decisions
CE10: Asks questions to aid student' control
CE11: Struggles to ask students question about Gar's pie graph, does not correct student misconception of ½
CE12: In finding second solution, he allows student to put 9:3 in Molly's pond and asks students about constraint.
CE13: Students work for 4 minutes with minimal intervention from Chandler
CE14: At impasse, he focuses students on Gar' pond.
CE15: Uses combination of pie graph, prompting questions and explanation to discuss justification of solution.
CE16: Uses percents to discuss Molly's ratio (25% 75%) but fractions to discuss Gar's pond (1/3 2/3)
CE17: Does not directly address Q2 or Bonus

Phase 4: Reflection/Planning
CE18: Believes students understood part-part and part-whole reasoning and equivalent ratios
CE19: Reflects on students' initially misunderstand Gar's pie graph and later realize their mistake
CE20: Plans to create a situation for students to think about pie graph
CE21: Desires to have students explain more

Phase 5: Experience with Students
CE22: Asks students to reflect on Angel's pie graph
CE23: Gives students hints while solving Q1
CE24: Technology glitch causes student confusion
CE25: Appears frustrated and gives students a lot of hints to compensate for technology error.
CE26: Asks students to justify solution to Q1
CE27: Has students draw pie graphs for Molly's and Angel's pond on paper and asks questions related to percents.
CE28: Asks students to use pie graphs for reasoning when finding second solution
CE29: Allows students to place large ratios in Molly's pond asks them to consider constraints
CE30: He uses pie graph to explain Gar's ratio as 1/3 males

Phase 6: Final Reflection
CE31: Learned what "provoking" questions to ask from Phase 3 and used these in Phase 5
CE32: Believes using the java applet promotes students working together and gives them access to different problem solving strategies
CE33: Appreciates difficultly of allowing students to use a tool and pursue a solution path without directly intervening

Figure 5. Chandler's critical events throughout each phase.

but uses fractions (1/3 and 2/3) to discuss Gar's ratio (CE16). He may be trying to use a familiar numerical representation for each ratio and to avoid using repeating decimals (researcher's hypothesis). Even though he used a combination of student questions and his own explanations, his insight on how best to use the representations in the applet demonstrates his careful attention to his role in students' meaning-making interactions with a tool (as in Figure 2).

In Phase 4, Chandler reflects on students' understandings (CE18 and CE19). The episodes with the pie graph seem to have caused a pedagogical perturbation for him as he remembers the perturbation for students when they initially suggest Gar's pie graph is "half" and later realize it is not (CE19). Upon reflection of this interaction with students (CE19), he plans to use the tools in the applet to create a situation that may cause a perturbation for the next students (CE20). Chandler also recognizes that he often tells students whether they are correct and does not always ask students to justify their answers (CE21). Although he is obviously struggling with asking questions, he is attuned to his own actions in relation to students' meaning-making interactions with each other, the tool, and himself, and he wants to improve.

In Phase 5, Chandler implements his plan to have students predict the pie graphs in CE22 and CE27. In CE27, he asks them questions about the pie graphs for Angel's and Molly's ratios and has them draw an image of the graphs on paper. He also asks them to think about the percentages for each pie graph and how they relate to the ratios. He does not ask them to predict Gar's pie graph. However, he spends time making sure they understand why Angel's ratio of 1:1 gives a 50% male pie graph. One possible interpretation of this action is that he is trying to provide a scaffold for students to reason later about Gar's pie graph and the common misconception of 1:2 representing a "half," which his students in Phase 3 demonstrated (CE11).

Although Chandler asks the students significantly more questions (or gives hints) in this phase, he bases the majority of his questions on students' current solution status (CE23, CE25, CE28). An unforeseen technology glitch occurs in this episode when one male fish suddenly loses its ability to be recognized by the applet, and thus is not counted in the ratio and pie graph displays. The students are frustrated and spend the vast majority of their time dealing with the technology glitch and struggling to find and justify a solution (CE24, CE25, CE26). As in CE12, he again allows students to pursue solution paths with Molly's pond that he knows do not lead directly to a solution (CE29). This pedagogical decision contributes to the amount of time devoted to students' finding a second solution and lack of time left to explore the ratio in Gar's pond with the pie graph, which, instead, he explained in CE30. However, since he discusses the pie graphs and equivalent ratios throughout the episode, the students are engaged in thinking about the intended mathematics in question #2 and the bonus question. It seems Chandler has become flexible enough to respond to students and pose questions that reflect the intention of the three questions in the problem without feeling constrained to investigating the problems in the order in which they are written.

His final reflection in Phase 6 illustrates his focus on his questioning skills throughout the project (CE31). He appears to believe that his role as a teacher is to facilitate students' work by guiding them with thought-provoking questions and to minimize teacher-led explanations. Chandler often struggles with this role. He is aware of how he interacts with students and seems to analyze carefully that interaction to make future decisions and plans. His reflections about the use of the tool (CE32 and CE33) are aligned with how he perceives his role as a guide and desires to maximize student-directed activity.

Griffin's Trajectory

The critical events for Griffin in Phase 1 (Figure 6) set the stage for related events in later phases. His approach to solving the problem includes an overall recognition and use of ratios that are equivalent to Angel's pond (CE1 and CE3). His strategy of finding a second solution by starting with the first solution and rearranging fish to maintain a sum of fish in Molly and Gar's pond of 1:1 demonstrates an approach which is different from the other prospective teachers in the class (CE3). Since he does not justify his solution in (CE1 and CE2) or attend to question #2 (CE3), there is not much evidence that he has carefully worked through that aspect of the problem. In addition, his error in interpreting the pie graph (CE4) leads to a heightened awareness of difficulty students may have with interpretation of a 1:2 ratio as representing 1/3 males (CE5).

In his initial planning phase, Griffin expresses his anxiety (CE6 and CE8) that may stem from a lack of confidence or from the public mistake he made in CE4. He is curious about how students will approach the problem and wants to give them freedom to explore their ideas (CE7). He appears to value a teacher acting as an encourager and providing additional contexts to which students can relate (CE9). His planning seems focused on how his interactions may affect students' problem solving.

Griffin's interactions with students in Phase 3 demonstrate a few ways in which his prior critical events may be affecting his actions. His

Phase 1: Solving Problem & Discussion with Peers
CE1: Solution to Q1 focuses on distributing fish in Molly and Gar's ponds so that sum is in a 1:1 ratio, allowing remaining fish in tank to be placed in Angel's pond. Explains process well.
CE2: Does not justify why solution is correct
CE3: Misunderstands Q2 and gives another solution for Q1 that starts from first solution
CE4: Volunteers in class that Gar's ratio represents one half
CE5: Admits he has to think twice about Gar's ratio as 1/3 males and imagines students will too

Phase 2: Planning
CE6: Comfortable with problem content but worried his own solution methods might confuse the students
CE7: Curious about students solution methods
CE8: Intimidated by using technology and afraid of doing "something stupid"
CE9: Anticipates a need to engage students and hypothesizes using real world examples of ratios to stimulate interest

Phase 3: Experience with Students
CE10: When student asks about "circle", he uses percents to connect fish in pond to pie graph representation
CE11: Does not interfere with students' control of problem solving, but often makes encouraging statements
CE12: When students pause briefly, he directs attention to fish remaining in pond and leads them to a solution
CE13: Asks students to explain process to a "friend"
CE14: Connects solution to selling fish to a customer
CE15: When students start to find second solution, he intervenes and suggests starting with their first solution
CE16: Students are confused, he provides minimal guidance
CE17: Asks students to explore constraints in Molly's pond but then tells students the constraints
CE18: Leads students with a hint for Q2
CE19: Has students predict Gar's pie graph on paper but when they predict a "half", he explains why it is incorrect
CE20: Uses scenario of bikes to explain ratio of 1:2 as 1/3

Phase 4: Reflection/Planning
CE21: Recognizes he struggled to have students justify answers or provide wait time and plans to improve
CE22: Thinks he often talked to much and needs to guide students more with appropriate questions
CE23: Comments on how quick students found both solutions
CE24: Concerned with students' understanding of constraints of problem

Phase 5: Experience with Students
CE25: Asks students which pond's ratio is similar to initial ratio in tank (13:13)
CE26: Students want to discuss solution to Q1 without applet, but he asks them to use applet to show steps
CE27: Has students record solution to Q1 on paper
CE28: Focuses students on Molly's pond when finding second solution
CE29: Disregards student's suggestion of 9:3 and suggests starting with 3:1 for Molly's pond
CE30: Seems fidgety and distracted
CE31: Discusses constraints of problem with one student
CE32: Skips Q2, has students place 2:4 in Gar's pond and explains connection between pie graph and ratio with paper
CE33: Repeat pie graph explanation so student can record

CE34: Chats with students for remaining 6 minutes

Phase 6: Final Reflection
CE35: Feels he did excellent job in guiding students to make discoveries
CE36: Notes that he had to suggest students to use applet because they seemed hesitant
CE37: Notes that one girl may have a lack of understanding of the problem and the pie graphs
CE38: Realizes he did not take second session as seriously as first
CE39: Believes value of using applet is the freedom it affords students in solving problem

Figure 6. Griffin's critical events throughout each phase.

suggestions for strategies (CE12 and CE15) are based on his own solution to the problem (CE1 and CE3), even though he was worried that students may be confused with his methods (CE6), and they actually did become confused (CE16). He attempts to allow students to explore the problem and control their own actions as he gives encouraging statements (CE11 and CE16). However, this seems to be a struggle for him as he tries to control which strategies they employ, but then does not ask questions that would help students examine their implementation of these strategies (CE16). He sometimes intervenes and leads students to a solution (CE12, CE17, CE18, CE19). Griffin implements his plan (CE9) and makes careful pedagogical uses of real world situations to help students explain their work (CE13) or for him to explain a concept (CE14 and CE20). His inconsistency in his interactions with the students (e.g., CE11 and CE15) is an indication of his struggle to make sense of his role and to find a balance between "free exploration" and structured teacher guidance or explanations.

Griffin's reflection and planning in Phase 4 demonstrates that he is attuned to some of his own struggles in Phase 3 (CE11, CE15, CE16, CE17, CE18) to be a guide and encourager (CE22) and balancing that with the need for students to justify their reasoning and solutions (CE21). He attends carefully to his students' actions and the effect of his actions, but there are some inconsistencies. He notes, "they were successful rather quickly in finding the two possible solutions" (CE23), although it actually takes a total of 29 minutes for the students to find both solutions. He is also concerned that they "couldn't understand why there weren't more than two solutions" (CE24) even though he has attempted to cause this perturbation for students (CE17). He is not reflecting critically (at least in his written reflection) on the students' confusion when finding the second solution (CE16), yet he attends to their difficulty in the exploration of the constraints in Molly's pond (CE17). Perhaps because the students eventually find two solutions he considers that aspect of his interactions with them a success (CE23), and is only concerned with the aspect where the students do not completely make sense of the situation (CE24).

During Phase 5, Griffin begins with a focusing question (CE25) that is related to his own problem solving strategy (CE1). Instead of implementing his strategy, the students embark on a discussion among themselves about what ratio to put in Gar's pond and seem hesitant to use the applet until they know the total number of fish to place in a pond (CE26). Although the students seem more comfortable only using the applet to implement their solution, Griffin wants them to utilize the applet as a tool for finding the solution (CE26). Two possible interpretations of this action are (1) he wants an external representation to be able to follow their thinking process, or (2) he is not prepared to engage in a mathematical discussion with the students without the aid of the applet. Griffin's interactions concerning Molly's pond and the constraints of the problem (CE28 and CE31) can be traced back to his concern about students' confusion (CE24) and his prior interactions in Phase 3 (CE17).

Griffin has implemented his plan (CE22) to guide the students more in their problem solving (CE25 and CE28). Yet, as the session progresses, he becomes more teacher-centered and directed in his interactions (CE29, CE32, CE33). His demeanor (CE30 and CE34), however, suggests a much more relaxed and casual approach – one where he is not attending carefully and making decisions based on students' reasoning. He seems to have a goal of completion without honoring students' abilities to reason through the task and has several minutes to spare at the end of the episode (CE34).

In each of the five critical events in his final reflection (CE35–CE39), Griffin exhibits an acute awareness of his interactions with the students in Phase 5. He acknowledges that, "the second session was definitely more laid back in a negative way.... I wanted to serve more as a guide does the second time around, however except for a few instances; I did not let them struggle enough" (CE38). He feels that what he learned from the interactions in the first session with students has prepared him for whatever could arise in the second session. His comments here seem inconsistent with his self praise at the beginning of this reflection that he did "an excellent job in guiding them in discoveries" (CE35). This inconsistency may be an indication of his struggle to make sense of what his role should be in interacting with students. Griffin also believes the main purpose of allowing students to use the java applet is to help them visualize and freely manipulate, struggle and explore (CE39); and he believes that he tries to allow students to have this freedom when they are solving the problem (CE35). Although he may have a vision of what it means to facilitate students' problem solving, his reflections in Phases 4 and 6 indicate he is aware of his struggle to enact that vision.

DISCUSSION AND IMPLICATIONS

The *planning–experience–reflection* cycle provides opportunities for prospective teachers to begin to struggle with issues of facilitating students' problem solving. The prospective teachers in this study were able to make their struggle an open and reflective activity and used it as an opportunity to improve their practice. There are both similarities and differences between how these prospective teachers make sense of their role in facilitating students' problem solving. Looking across the cases, six themes emerge that will be used to anchor the discussion and implications from this study. The prospective teachers:

(1) Use their own mathematical problem solving approaches to influence their pedagogical decisions.
(2) Desire to ask questions that can guide students in their solution strategies without "giving it all away."
(3) Recognize their own struggle in facilitating students' problem solving and seem focused on improving aspects of their interactions with students.
(4) Assume the role of an explainer for part of each facilitation phase.
(5) Make pedagogical decisions to use representations in the java applet to promote students' mathematical thinking or focus their attention on specific aspects of the problem.
(6) Use the technology tools in ways consistent with the nature of their interactions and perceived role with students.

The first theme applied across all three cases as the tracing of critical events demonstrates the influence of a prospective teacher's own problem solving on their pedagogical decisions for facilitating students' use of resources, heuristics, and control. In a research study on teachers' problem solving preferences and characterizations of particular problem solving strategies, Leikin (2003) found that teachers prefer to explain problems to others using strategies that are "easier to explain" and preferred strategies for their own problem solving that are "more beautiful" or elegant in their opinion. With regard to teaching others, Leikin found that teachers prefer to use strategies they characterize as "more convincing." However, the teachers in Leiken's study were not observed actually explaining or teaching problem solving to students. In the context of this current study, the prospective teachers certainly seemed attracted to elegant solution strategies in their own problem solving, but their strategies had a direct impact on their pedagogical decisions while they were facilitating students' problem solving. The results from Leikin's study may be yet an another example where teachers' professed beliefs, or problem solving preferences, are not aligned with their actual practices (Aguirre & Speer, 2000). The results from this current study demonstrate that it may not only be the solution method teachers find "most compelling" that influences their teaching practices.

Within a technological context, Zbiek (1995) found that although a teacher may desire to have an exploratory approach to using technology tools, a need for structure can influence them to fall back on using their own mathematics explorations as a way to guide students through tasks to reach their mathematical conclusions. These prospective teachers are no exception. As seen in the tracing of her critical events, Brandi's "elegant" solution methods and her high success level as a college mathematics student were consistent with her belief about and tendency towards teacher-centered interactions. Chandler's curiosity about multiple solution strategies guides his focus on a student-centered approach; however there is evidence that his solution process and use of the pie graph as a helpful representation influences his decisions, questions, and comments. Griffin's solution strategy also influences his interactions with students. These prospective teachers may have benefited from a broader experiential base from which to develop an understanding of how learners solve problems with various tools. If we desire teachers to become comfortable with methods and solutions that are different from their own, we need to provide opportunities for them to analyze and discuss students' mathematical problem solving.

The second and third themes are related to the prospective teachers' sense-making process in learning to facilitate students' problem solving. The results from the analysis of each trajectory show that critical events often serve as pedagogical perturbations that affect prospective teachers' future reflections, plans, and interactions with students. From their initial planning through their final reflections, each prospective teacher is aware of their personal struggle and focuses on improving aspects of their interactions with students. They recognize they *should* ask non-leading questions and that they have difficulties posing such questions. Chandler is the most consistent in his efforts to improve his use of questions. Both Brandi and Griffin ask a few of these types of questions, but are not able to sustain their effort. Thus, they do not gather much evidence of students' understanding. Chandler, however, seems to gather more evidence and actually makes decisions based on his assessment of students' understandings. This phenomenon has implications for teacher education, which typically involves prospective teachers learning a variety of teaching methods, writing lesson plans out of context, and learning to use different teaching tools. A shift in teacher education practices that provide opportunities for prospective teachers to analyze students' mathematical work and make lesson plans that are situated within a context and grounded in students' current mathematical understandings may be in order.

The fourth theme indicates that each prospective teacher took the lead at some point to explain a concept or solution to students. This typically occurs when they are trying to help students understand concepts of equivalent ratios and part–part and part–whole interpretations of a ratio. Although Brandi assumes the role of an explainer much more frequently and consistently, Chandler and Griffin both have instances when they explain concepts to students. The role of an explainer seems to be comfortable for practicing and prospective teachers, and is influenced by their beliefs about teaching and learning mathematics and their struggle to make sense of how to facilitate students' problem solving. This is aligned with Farrell's (1996) findings of a high level of incidences of the role of *explainer*, with and without the use technology.

The fifth and sixth themes are focused on the prospective teachers' use of tools in the environment, including paper and pencil, to promote students' problem solving. Similar to the findings of Bowers and Doerr (2001), these prospective teachers develop pedagogical strategies for taking advantage of the affordances of the technology tool. At some point, each prospective teacher makes use of the representations in the java applet. The representations appear to have become "didactic objects" (Thompson, 2002) for the prospective teachers since they used the representations didactically to focus students' attention, generate conversations, or pose additional tasks. However, they each also chose *not* to use the applet and to use paper and pencil (e.g., drawing a pie graph, writing ratio and fraction representations) or imagery strategies (e.g., imagine changes in the pie graph if more female fish are added) as didactic objects for at least one discussion or explanation. This indicates that the prospective teachers are able to hypothesize how various tools might enable or constrain the learning process and make their pedagogical decisions based on this hypothesis. These prospective teachers have had many opportunities within the current course and the lesson-planning research project within the introductory course (Cavey et al., 2001) to experience, discuss, and reflect on how different tools and representations can be used to help students learn mathematics. These opportunities seem to contribute to their sense-making process and suggest that such opportunities have an important place in mathematics teacher education programs.

The prospective teachers' beliefs about and particular uses of the applet seem consistent with how they facilitate students' problem solving. Brandi believes the technology tools are good for giving "immediate feedback" and confirming correct answers. She sometimes uses the tools as part of her explanations (e.g., moving fish into Gar's pond to discuss equivalent ratios). Chandler uses technology as a way to involve students working together and gain access to different problem solving strategies (e.g., encouraging them to move the fish between the ponds and tank). He also uses the tools purposely to engage students in thinking and justifying a solution (e.g., predicting and then viewing pie graphs). Griffin believes the main purpose of using technology tools is to allow students to manipulate freely and to explore a problem. This aligns with his role as an encourager and with the few instances when he uses the representations to have them explore a situation (e.g., asking students to consider problem constraints if Molly's pond has 9 males and 3 females). The nature of prospective teachers' interactions with students and their subsequent use of the technology tool is probably influenced by their beliefs about mathematics, teaching, and learning and is supportive of similar findings from prior research (e.g., Drier, 1998; Manoucherhi, 1999; Tharp et al., 1997; Turner & Chauvot, 1995).

Given the similar nature of prospective teachers' interactions with students and technology, placing teachers in technology-rich learning environments is not a sole catalyst for changing their understanding of and beliefs about learning and teaching mathematics. Perhaps opportunities like the one described in this research can help prospective teachers begin to make sense of this complex process. However, extended opportunities for reflection, support, and professional development opportunities must be sustained as prospective teachers continue into the practice of teaching in their own classrooms.

AN EXTENDED HYPOTHETICAL LEARNING TRAJECTORY

The results of this research can contribute to the development of a learning trajectory for prospective teachers that blends theoretical constructs with results from this and others' research. There were several key components of this study's results that are pertinent for developing an extended learning trajectory. First, it seems that engaging in an iterative *planning–experience–reflection* cycle allows prospective teachers to critically reflect upon and improve their practice. Secondly, technology is another tool, albeit a more complex one than other hands-on materials, that teachers may use in their lessons to enact their framework of teaching and learning. Merely having prospective teachers use a technology tool with students will not suffice as a change agent for beliefs about teaching and learning mathematics. Finally, it is not surprising that prospective teachers struggle to pose questions and critically to assess students' problem solving while using a technology tool. Because these prospective teachers are placed in a situation where they are concerned with their role as a teacher, they do not yet effectively coordinate the dual activity of informing their actions and making decisions while gathering evidence and formulating a hypothesis about students' understanding (see Figure 2).

As content knowledge is different from, but an essential aspect of, pedagogical content knowledge (Shulman, 1986), so knowledge of how to solve problems with various tools is different from, but an essential part of, pedagogical knowledge of facilitating students' problem solving with such tools. With respect to technology tools, Olive and Leatham (2000) are correct in their claim that teachers' use of technology in their own learning of mathematics is insufficient to foster an understanding of how to teach students to learn mathematics with technology tools. The work of Bowers and Doerr (2001) and the results of this study indicate that prospective teachers can develop and refine their pedagogical understandings of using technology tools with students. However, the results of this study also suggest that placing prospective teachers in the role of a teacher, without first giving them the opportunity to understand how students learn mathematics and solve problems, may lead to frustration and result in an egocentric focus on one's own actions.

What follows are my suggestions for an extended hypothetical learning trajectory for prospective teachers as they learn how to use technology to engage students in mathematical tasks. In the first phase, I suggest that teacher educators should engage prospective teachers in exploring mathematics with various tools and acquire a solid understanding of the affordances and constraints of using these tools in their own learning. In the next phase, prospective teachers can gain an understanding of how various tools offer both opportunities and constraints in the learning of others, particularly for the students in the grade range they wish to teach. This allows prospective teachers to critically analyze students' work with technology tools as non-participant observers. This analysis can help them develop an understanding of how other learners, and possibly other teachers, interact with tools while doing mathematics.

Teacher educators can then engage prospective teachers in an iterative planning–experience–reflection cycle with a small number of students *with an increased focus on designing and using higher-level questions and making pedagogical decisions based on an assessment of students' understanding*. The purpose of this cycle is to shift the prospective teachers from an egocentric focus on their actions to a focus on understanding students' work. The reflection that follows such an experience can lead prospective teachers to examine critically students' work and their interactions via videotape. The fourth phase in the trajectory is to engage in a similar iterative planning–experience–reflection cycle with a classroom of students, perhaps during student teaching.

The culminating aspect of my proposed hypothetical learning trajectory is helping teachers develop a robust understanding of pedagogical issues of using learning tools with students in the mathematics classroom and being able to engage in critical analysis of appropriate uses of such tools, especially technology. This last phase is essential within the realities of a rapidly changing technological landscape of the 21st century, where more and more tools for learning will be technology-based. Teachers will continually be faced with learning how to use emerging technologies in new classroom contexts and cannot solely rely on understandings that are situated within familiar contexts and tools. They must be able to abstract their pedagogical understanding of teaching and learning mathematics with current tools into a cohesive vision and personal theoretical framework that moves beyond the boundaries of current classroom contexts. This abstraction and robust understanding takes time to develop. As Hiebert, Morris, and Glass (2003) have noted, "it is both more realistic and more powerful to help prospective teachers learn how to learn to teach mathematics effectively when they begin teaching" (p. 202). If prospective teachers enter the profession with reflective habits and beginning the shift from a focus on their actions to students' understanding, their classroom and professional development experiences will propel them forward in developing such a robust theoretical framework.

As mathematics teacher educators, we need to engage in research on this proposed trajectory and continue collaborative work and discussions about the development of appropriate pedagogy in technology-rich mathematics classrooms. Future research should focus on how prospective teachers make sense of students' understanding while working with technology tools. Gaining a deeper understanding of this aspect of the trajectory can allow mathematics teacher educators to understand prospective teachers' growth and further refine the key aspects for developing appropriate pedagogical content knowledge for the classrooms of tomorrow.

NOTES

[1] The complete project cycle for prospective teachers includes two additional phases. After phase 6, the prospective teachers assess each of the students' written work in a small group session with peers and respond to students via email. In addition, the prospective teachers work with their four students together for a follow-up problem using a different technology tool. Due to loss of some data in each of these situations, their work in the later phases of the project is not included in this analysis.

[2] The Math Forum (http://mathforum.org) is a large mathematics education resource portal and the Problem of the Week (PoW, http://mathforum.org/pow/) is one of the most popular services they provide. During 1999–2001, middle school mathematics problems and accompanying java applets were designed and implemented as part of the NSF-funded grant (REC #9804930) *Educational software components of tomorrow: A Testbed for sustainable development of interoperable objects for middle school mathematics* (ESCOT).

[3] "Researcher's hypothesis" indicates that this is a plausible interpretation made under the circumstances and constraints of the project, type of data collected, and researcher's experience as the teacher educator in the study.

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