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A FRAMEWORK FOR ANALYSIS OF TEACHERS' GEOMETRIC CONTENT KNOWLEDGE AND GEOMETRIC KNOWLEDGE FOR TEACHING

ABSTRACT. Current reform-driven mathematics documents stress the need for teachers to provide learning environments in which students will be challenged to engage with mathematics concepts and extend their understandings in meaningful ways (e.g., National Council of Teachers of Mathematics, 2000, *Curriculum and evaluation standards for school mathematics*. Reston, VA: The Council). The type of rich learning contexts that are envisaged by such reforms are predicated on a number of factors, not the least of which is the quality of teachers' experience and knowledge in the domain of mathematics. Although the study of teacher knowledge has received considerable attention, there is less information about the teachers' content knowledge that impacts on classroom practice. Ball (2000, *Journal of Teacher Education*, 51(3), 241–247) suggested that teachers' need to 'deconstruct' their content knowledge into more visible forms that would help children make connections with their previous understandings and experiences. The documenting of teachers' content knowledge for teaching has received little attention in debates about teacher knowledge. In particular, there is limited information about how we might go about systematically characterising the key dimensions of quality of teachers' mathematics knowledge for teaching and connections among these dimensions. In this paper we describe a framework for describing and analysing the quality of teachers' content knowledge for teaching in one area within the domain of geometry. An example of use of this framework is then developed for the case of two teachers' knowledge of the concept 'square'.

KEY WORDS: framework for geometric knowledge mapping, geometric knowledge, mathematics teacher knowledge for teaching, teacher education, teachers' knowledge deconstruction

INTRODUCTION

A principal theme in current reform-driven mathematics documents is the need for teachers to provide learning environments in which students will be challenged to engage with mathematics concepts and extend their understandings in meaningful ways. Recent discussions of the role of the mathematics teacher emphasise the importance of teachers helping students to develop knowledge structures that will allow the student productively to explore a suitable range of mathematical problems. The sense of this perspective is clearly articulated in the recommendations, such as that of the National Council of Teachers of Mathematics (1989, p. 128), that there is a need for teachers to 'shift from dispensing information to facilitating learning'. Knapp (1997) views the adoption by teachers of this shift in conceptualisation of teaching as one of the central planks of the broad reform movement in mathematics and science teaching. One part of this movement is the specification of the crucial role that teachers and their knowledge play in influencing the knowledge and understandings constructed by students. A critical assumption made in development of the agenda for reform is that the teacher should have access to a well-developed, good quality, body of mathematical content knowledge. In this study we address the issue of how to characterise the quality of a teacher's mathematical content knowledge. The purpose of this study was twofold. First, we develop a framework for identifying dimensions of quality in teachers' content knowledge for geometry and teaching of a concept in geometry. Second, we use this framework to describe the knowledge provided by two teachers who completed a number of tasks that were designed to access their knowledge of this concept.

Teacher Knowledge and Mathematics Teaching

Recent research about development of teachers' competence in mathematics has identified three major components of teachers' knowledge base which permit them to perform their role effectively: Mathematics content knowledge, pedagogical knowledge, and the blend of knowledge of content and pedagogy. *Mathematical content knowledge* includes information such as mathematical concepts, rules and associated procedures for problem solving. *Pedagogical knowledge* refers to teachers' understanding of their students, and the processes involved in teaching. The *blend of content and pedagogical knowledge* includes understandings about why some children experience difficulties when learning a particular concept while others find it easy to assimilate, knowledge about useful ways to conceptualise and represent the chosen concept (Feiman-Nemser, 1990), the quality of explanations that teachers generate prior to and during instruction (Leinhardt, 1987), and perceptions about the nature of mathematics. This blend has also been labelled as *pedagogical content knowledge* (Shulman, 1986). In recent years, researchers interested in improving children's mathematical performance have argued that the quality of teachers' own knowledge has a strong influence on how that knowledge is accessed and exploited during planning for a lesson and instruction (Clark & Peterson, 1986; Lawson & Chinnappan, 1994; Schoenfeld, 1992).

While the study of teacher knowledge has received considerable attention, there is less information about teachers' content knowledge that impacts on classroom practice. In highlighting this issue, Ball (2000) made the distinction between 'knowing how to do mathematics and knowing it in ways that enable its use in practice' (p. 243). Ball suggested that teachers' need to 'deconstruct' their content knowledge into more visible forms that would help children make connections with their previous understandings and experiences. The following quote adumbrates the issue.

> 'Understanding the use of mathematics in the work of teaching is critical area ripe for further examination. It is not only what mathematics teachers know but also *how they know it*, and what they are able to mobilise mathematically in the course of teaching' (Ball, Lubienski & Mewborn, 2001, p. 451).

The documenting of teachers' content knowledge for teaching has received little attention in debates about teacher knowledge. In particular, there is limited information about how we might go about characterising the qualities of this knowledge in a systematic manner. In this paper, we describe a framework designed to allow us to undertake such a characterisation. In so doing we examine the issue of 'how they know it' that Ball et al. (2001) drew attention to.

Geometric Knowledge

Students' understandings of geometry have received considerable attention in most curriculum documents, and this area is regarded as providing important foundations for appreciation of other mathematics topics such as algebra. K-12 mathematics curriculum documents (Board of Studies, 2002; National Council of Teachers of Mathematics, 2000) have identified geometric knowledge under various themes such as spatial concepts, attributes of 2-dimensional (2-D) and 3-dimensional (3-D) shapes, plane geometry, deductive geometry and coordinate geometry. All these areas involve spatial thinking and use of conventions in geometry diagrammatic representations. Vinner and Dreyfus (1989) suggested that formal *concept definitions* and *images* as provided in the curriculum may be different from images of the concept that individual students develop. In a traditional classroom, teachers may focus on the former at the expense of the latter, which could include individual students' idiosyncratic ways of understanding geometry concepts.

In their analysis of perception of geometric figures, Gray, Pinto, Pitta and Tall (1999) highlighted the power of language in helping learners to make hierarchical classifications. They suggested that 'through verbal discussion, instruction and construction, the child may begin to see hierarchies with one idea classified within another, so that a square is a rectangle is a quadrilateral' (p. 112). Thus, the language used by a teacher plays a central role in the development of understandings about 2-D shapes and their relations to other shapes.

From the teaching point of view, current models of embedded learning emphasise not only properties of shapes but also the identification of these shapes in an array of real-life contexts. For example, teachers' content knowledge about 2-D shapes would involve not only descriptions about features of parallelograms but also the construction of parallelograms using pattern blocks, and the awareness of use of these shapes in real life activities. Teachers must also be able to articulate the relationships between parallelograms and other 2 and 3-D figures, such as rectangles, triangles, rhombus and quadrilaterals in general. That is, teachers need to bring a level of representational fluency to the teaching of 2-D shapes. This fluency, which should include language that is associated with 2-D shapes, can be argued to reflect teachers' geometric knowledge for teaching. In order to improve our sense of what content knowledge matters in teaching geometry, we would need to identify the 'critical components' of the deconstructed knowledge referred to by Ball (2000). In the first instance, this requires a fine-grained analysis of not only the content of geometric knowledge but also its deconstruction and reorganisation which is important for accessing and making concepts visible to students.

Geometric Knowledge Connectedness

It is generally accepted that, all other things being equal, a teacher with a better quality knowledge base will be more able to assist students than one with lesser quality knowledge (Grossman, 1995; Munby, Russell & Martin, 2001). Researchers have emphasised the importance of recognising the connected nature of the teacher's knowledge base. Robinson, Even and Tirosh (1992) suggested that in order to understand the depth of teachers' knowledge and understanding it was necessary to examine the network of interconnected schemas and procedures that form the knowledge base. Schoenfeld (1988) observed that development of mathematical thinking requires not only mastering various facts and procedures, but also understanding connections among them, and suggested that there is value in providing detailed descriptions of the structures that support such thinking. As yet, we have very few detailed discussions of the ways in which the quality of connectedness of elements of teacher knowledge can be investigated and represented.

The starting point for our attempt to represent the quality of mathematics teachers' knowledge of content was Mayer's (1975, p. 529) notion of knowledge connectedness. Mayer described the accumulation of new information in long-term memory as adding new nodes to memory and connecting the new nodes with components of the existing network. *Internal connectedness* refers to the degree to which new nodes of information are connected with one another to form a single well-defined structure or schema. This sense of connectedness refers both to the presence of nodes related to a schema and to the quality of the relationships established among those nodes. The broad notion of quality here can be related, in part, to what Anderson (2000) refers to as the strength of a memory trace. Seen in this way, the stronger the connections among the nodes in a particular schema, the better is the quality of that structure. Mayer (1975) referred to *external connectedness* as the degree to which newly established knowledge structures are connected with structures already existing in the learner's knowledge base. For example a teacher might be expected to relate a schema for proportion with schemas for ratio or fraction.

One important dimension related to the quality of that structure. Is the identification of what connections are present in a knowledge structure. Other things being equal, the more comprehensive the connections in a knowledge structure are, the more 'rich' or more elaborated is the structure, the more useful it will be in problem solving (Anderson, 2000). However, it is also apparent that the nature of the connections within a knowledge structure, not just the number of connections, is also important. Some time ago Bruner (1966) referred to knowledge representations as having degrees of 'power', and Wittrock (1990) has more recently described both student and teacher understandings as having 'generative' capacity. Both power and generative capacity draw attention to the quality of the connections in a knowledge structure. The more powerful and more generative a structure, the more widely it can be applied in problem solving (Bruner, 1966). So we might expect different individuals to have connections between proportion and ratio or fraction that differ in power. In similar vein, we might expect a student's new schema for proportion to have both a certain quality in its internal structure (internal connectedness) and a certain quality in its connections to related schemas (external connectedness). This analysis of connectedness was used by Chinnappan (1998a, b) who argued that the linking of the different pieces of knowledge of geometry and trigonometry reflect deeper and richer understandings.

The above discussion relating to the quality of connections in a knowledge representation omits the question of how to describe such quality. In developing the framework discussed below, we have set out one way in which we think the quality of a knowledge structure related to mathematical content knowledge can be described. We have used the notions of internal (within-schema) and external (between-schema) connectedness for representing the structural dimensions of teachers' knowledge and have defined specific features of those structures as indices of quality. These notions provide a way to represent the complexity of geometric knowledge base in a manner that focuses on the state of organisation of that knowledge.

Representation of Geometric Knowledge Structure in Maps

As the study of dimensions of teachers' geometric knowledge for teaching involves the examination and specification of schemas and relationships, we need tools that will help us to represent the organisational features of that knowledge. An intuitively appealing and effective procedure for representing knowledge structure is that known as concept mapping. The *concept map* has emerged in a number of forms in the literature of educational research, though the term is most commonly associated with the work of Joseph Novak and his colleagues in the science education program at Cornell University (e.g., Novak, 1990; Novak & Gowin, 1984). In establishing this representational format, Novak drew extensively on the descriptions of learning that had been developed by Ausubel (1968). The establishment of meaningful relationships among concepts was contrasted by Ausubel with rote learning in which concepts were not embedded in rich conceptual networks, but were left relatively unelaborated and conceptually isolated within the broad conceptual structure. In the Ausubelian view, the growth of knowledge was characterised by the gradual development of more complex and more differentiated structures organised in a hierarchical pattern. The different parts of this structure could be related, or integrated, through the establishment of propositional links. The hierarchical structure of a concept map was seen by Novak as instantiating the process of knowledge growth that Ausubel termed subsumption.

Concept maps, in general, are graphs, or networks, consisting of nodes and labelled lines (Lawson, 1994). The nodes are used to indicate concepts, or categories, while the lines correspond to a relation between pairs of concepts. The label on the line tells how the two concepts are related. Shavelson, Lang and Lewin (1993) referred to this relation as a proposition and argued that concept maps represent important aspect of learner's propositional knowledge in a domain.

A major difference among the various types of concept maps is their basic structure and the degree of control one wishes to impose in constructing the maps. A hierarchical concept map is better suited to assessing concepts that are organised in a top-down fashion, where the top-level of the map shows the most inclusive and subsumptive concept. However, if one is interested in elucidating multiple links among concepts or among concept clusters, and the integration of this information in the generation of explanations involving analogies and/or metaphors, then a web-like structure will be more appropriate (Beissner, Jonassen & Grabowski, 1993). Furthermore, the structural analysis of forms of spatial representation carried out by Novick and Hurley (2001) suggests that a network structure can provide flexibility in representation of direction of relationships and of linking of units that are not available with matrix or hierarchy formats. It is for this reason that we have used a network structure for the concept maps in our framework.

Concept mapping techniques have also been argued to be appropriate for representing complex interrelationships among schematised knowledge within and between domains (Jonassen, Beissner & Yacci, 1993). A number of recent studies have used concept maps to assess conceptual understanding in mathematics and science (Coleman, 1993; Laturno, 1994; Markman, Mintzes & Jones, 1994; Williams, 1998). In this project, we have attempted to realise more of the potential of mapping representations to provide indices of the quality of teacher's geometric knowledge and, to some extent, the transformation of this knowledge for practice. In doing this we are not intending to suggest that we have captured an enduring representation of an individual teacher's knowledge base. We assume that all knowledge bases are in a constant state of evolution, so that what we are representing is conceptual space that has been activated across the times of our interaction with the participants. In addition we have chosen to use the term 'schema' as the basic organisational unit within the map rather than 'concept'. In doing this we are following the use of schema proposed by Anderson (2000), that we see as allowing for representation of declarative, procedural and conditional characteristics of a category than does the label 'concept'. The schema is seen to be composed of relationships among a number of features.

METHOD

Participants

The teachers whose geometric knowledge is the subject of this report were two of five experienced teachers recruited for a larger project, along with five novice teachers. The experienced teachers were selected using two criteria: first, they had at least 15 years of mathematics teaching experience at the high school level; and second, they were all recommended by their peers or professional subject associations as exemplary teachers. The maps developed for this report were based on the knowledge exhibited by two teachers, Gary and Sue. In addition to having 20 years of teaching experience, Gary was the head of the mathematics department at his high school in an Australian capital city and was also involved in the writing of mathematics textbooks for high school students. Sue was one of the senior mathematics teachers in a private girls' school in a different capital city.

The teachers were interviewed individually during school hours and their responses were audio-taped and videotaped for later transcription. They were told that the purpose of the study was to find out what teachers know about topics in geometry and about the teaching of these topics.

Procedure

Three interviews were conducted with each participant, each lasting about 1 h. The interview schedule allowed each participant multiple opportunities to access knowledge and provided a range of activities for prompting such access. Throughout the interviews, the teachers were reminded also to consider geometry knowledge that was relevant to their teaching. During the first interview the teacher was asked to talk about a list of focus schemas in the areas of geometry, trigonometry, and coordinate geometry that were relevant to the school curriculum and their teaching. In the first instance teachers were asked to talk about the concept of square (focus schema 1) and their understandings about the teaching and learning a square. As this was a free recall session we asked questions such as: 'Tell me what you know about square' and 'Tell me how you would teach square to your students'. The teachers were invited to use diagrams to explain their thoughts if they wished to do so. Following teachers' responses to square, the teachers were asked similar questions about 12 other focus schemas (squares, rectangles, lines, similar triangles, congruent triangles, parallel lines, area, coordinates, triangles, right-angled triangles, regular hexagons, regular octagons, circles). All 13 focus schemas were required to solve the four problems given to the teachers during the second interview.

During the second interview, the teachers were asked to think aloud as they solved four problems. The problem-solving activity was included because we anticipated that the application of knowledge to a problem might lead to activation of knowledge additional to that accessed in the first free-recall interview. Two of the problems involved the use of knowledge related to the focus schema square. When the teachers indicated that they had completed a solution they were asked if the problem could be solved in any other ways. They were also asked to comment on any feature of their solution that could be related to the way their students would solve these problems. For example, teachers were asked, 'How would you expect your students to tackle this problem', 'What type of difficulties would you expect your students to experience if they are given these problem, Why?' The above prompts were expected to elicit teachers' further understandings about how their students would approach the problem thus providing data about how teachers integrate knowledge about focus schemas (from Interview 1) and the use of these schemas in a problem situation.

The format for the final interview was a series of probing questions designed to give the teacher the opportunity to access relevant knowledge that had not been activated in the previous two interviews. For example, in a discussion with a teacher who had not yet mentioned symmetry, we might have commented; 'You haven't said anything about symmetry of a square yet'. Or if a particular relationship had not been discussed we asked: 'How are these two concepts related?'

The activities in the three interviews were designed to provide good estimates of the teachers' knowledge in the area of geometry and knowledge for teaching geometry. We can never be certain that we have tapped all that a teacher might have constructed and deconstructed about a specific topic. However, we argue that the use of the free-recall, problem solving and detailed probing activities did provide a good estimate of the functionally available knowledge of geometry and for teaching geometry.

In addition to the experienced teachers recruited for the project, information was also sought from an academic mathematician. This mathematician was a member of a university mathematics department whose area of professional expertise was geometry. The focus of our interview with this mathematician was generation of a list of features and relationships for each of the focus schemas that could be used as a guide in evaluating the degree of completeness and accuracy of a teacher's geometry knowledge base. It is important to note that this information provided by this academic mathematician was not used as a template or a scoring rubric.

Map Structure

We adopted a simple form of representation for the node-link structure for the maps that can be used to identify teachers' knowledge of geometry (KG) and knowledge of geometry for teaching (KGT). The boxes and ovals, or nodes, in the maps indicated schemas and features, and the lines joining the boxes/ovals showed that a relationship was expressed between a schema and features, or between schemas. We identified four areas of KG and KGT about the focus schema and the relations that had been built around that schema.

1. The defining features of the focus schema (Defining Features)
   The term 'defining features' is used here to refer to necessary properties of the focus schema. For example, in the case of square, these features are that the sides are equal, all the interior angles are equal, and the opposite sides are parallel.
2. The related features of the focus schema (Related Features)
   Related features of the focus schema include information that one could derive by going beyond the basic defining properties of the schema. For instance, information about the formula for determining the area of a square would fall into this category of related features.
3. Relationships between the focus schema and other schemas (Other Schemas)
   In addition to information that was activated about a particular focus schema, teachers would also make links between the focus schema and other geometric figures, such as the connection between a square and a rhombus.
4. Other representations of the focus schema (Applications)
   In this section of the map we considered the different representations of the focus schema, such as analogies, metaphors, illustrations or

real life examples. For example, the idea that some floor tiles are square in shape, or that tessellation of tiles is used in tiling an area, falls into this category of information.

The above-mentioned four areas reflect the complex structure of focus concepts such as that of square that evolves from single words that Gray et al. (1999) refer to in their analysis of role of language in perceptions about shapes. Defining and related features are related to the notion of concept images (Vinner & Dreyfus, 1989). The framework shows emergence of hierarchies that emanate from use of key words such as square. Gray et al. (1999) suggest that the mental images and physical objects that are supported by the linguistic descriptions could evolve in a 'more pure and imaginative way'.

Structure of Mapping Template

The template for our map is shown in Figure 1. Information related to the defining features of the focus schema was recorded in the bottom left corner, and elaborations on these features (related features) were recorded in the lower right section of the map. D1 refers to defining feature 1 of the schema. D11 shows features arising from D1 and so on. In the bottom right section, R11 shows schemas emanating from related feature, R1. Links to other schemas were recorded on the upper right part of the map and are indicated by nodes labelled as S1, S21 and so on. Information relating to applications and alternative representations was included in the top left part of the map, so that A1 is used to depict instances of teachers using other representations for, or applications of, the focus schema. Where relationships were described by the teachers, labels were included on the line joining a pair of schema, or on the line between a schema and feature. Arrows on lines were used to show the direction of relationship of one node to another noted by the participant. If there were instances of links between nodes in related and other schemas, these were indicated by lines with appropriate arrows. Such links are not shown in Figure 1.

Analysing the Maps

We have proposed that the connectedness of knowledge can be described both in terms of the number of knowledge components present, as well as in terms of the qualitative relations that exist among the knowledge components. The scope, or range of knowledge, can be seen as a quantitative feature of a knowledge base that reflects

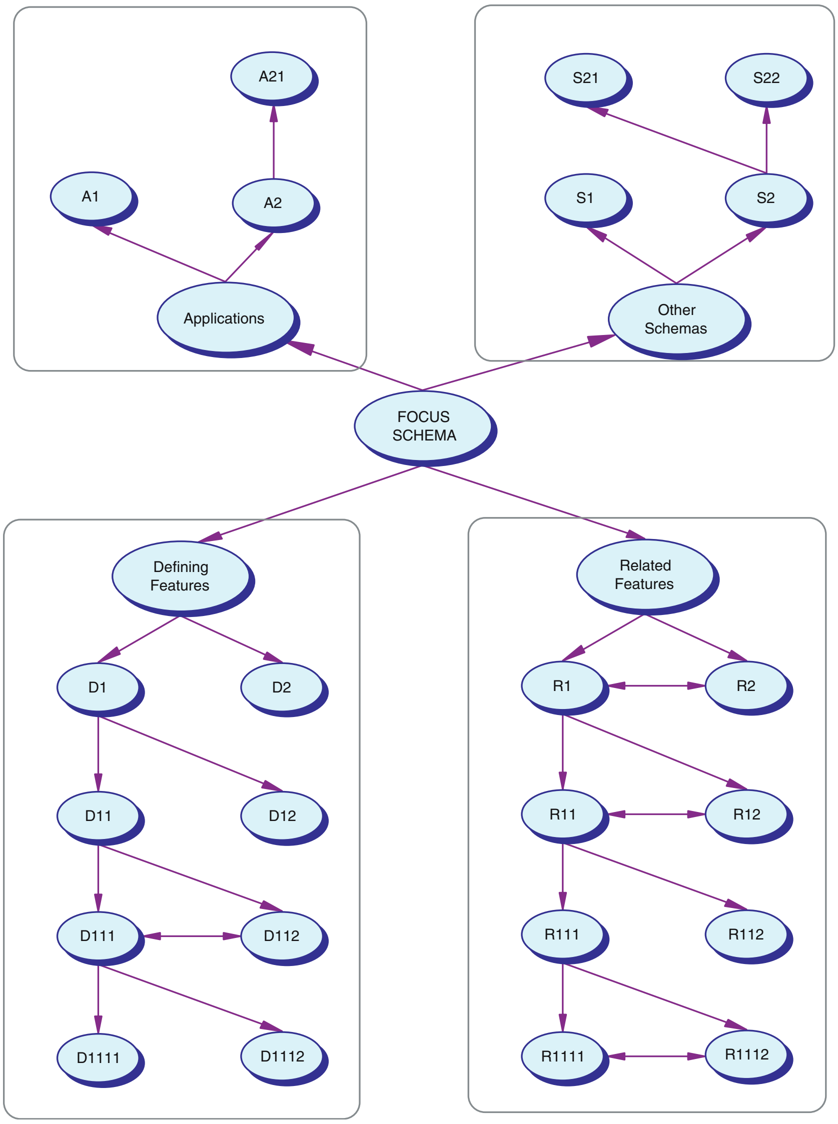

Figure 1. Map template for representing connectedness.

teachers' knowledge of geometry (KG). The characteristics of the organisation of the knowledge base, which require more qualitative judgements, are chosen to represent the depth of elaboration of the knowledge structure. The depth of elaboration is used as a measure of teachers' knowledge of geometry for teaching (KGT) because we contend that the elaborations constrain teachers to reflect on their content knowledge and deconstruct them in ways that their students could relate to.

The basic units of analysis used to construct the maps were nodes and links. Nodes were established for all schemas associated with geometric terms and for all the features associated with these terms that were mentioned by the participant during the interviews. The participating teachers did not play any role, either in the construction or the revision of the maps. That is, the construction of the concept maps for each of the participating teachers was based solely on data generated during the three interviews. Data from each of the teacher interviews were compared in order to ensure that what was said in the nodes and links were correct. The reliability of this process was established by consensus using two coders.

In Figure 1, the within-schema links refer to connections among the *Defining features* and those among the *Related features* of square. The between-schema links refer to links between the focus schema for square and *Other* schemas. The following measures were derived for each map.

Quantity

We interpreted quantity as having two sub-categories: (a) Number of nodes, and (b) Number of links. The concept maps were analysed in two ways in order to generate values for 'Number of nodes' and 'Number of links' listed in Table I. First, we considered nodes that appeared on the first layer away from the focus concept. For example, in the case of Gary's concept map (Figure 2), nodes with labels 'Angles' and 'Sides' were deemed to be located in the first layer, as these had the first direct link to our focus concept, 'Square'. We then counted the number of nodes that evolved from each of these Layer 1 nodes, and added one more for the starting node. This procedure yielded the result 5 for both 'Angles' and 'Sides' for Gary. Only nodes at the first layer were considered in this analysis. A similar procedure was followed for the counting of links, which included the link from the focus concept to the layer one node in question. Again, for Gary we obtained 5 links for 'Angles' and 'Sides'.

Cross-links made between *Defining* and *Related* features were also identified. In Gary's map there were three cross-links between *Related* features and the Sides node and the score of 3 is shown under Cross-linking in Table I. This analysis of the concept map in order to generate values for quantitative indicators of knowledge connectedness was

TABLE I
Within-Schema Indicators of Knowledge Connectedness

	Quantitative indicators				Qualititative indicators															
					Integrity				Connectedness											
	Number of nodes		Number of links		Complete-ness[b]		Accu-racy[b]		Depth[c]		Branc-hing[b]		Cross-linking		Complexity of relationships					
															Simple		Moderate		Complex	
Teacher[a]	G	S	G	S	G	S	G	S	G	S	G	S	G	S	G	S	G	S	G	S
Defining features																				
Angles	5	4	5	4					L3	L3	M	M	0	0	0		5	4		
Sides	5	5	5	5					L3	L3	M	M	3	1	0		5	5		
Subtotal	10	9	10	9	H	H	H	H							0	0	10	9	0	0
Related features																				
Area	2	2	2	2					L2	L2	L	L	0	0	0	0	2	2		
Perimeter	2		2						L2		L				0	0	2			
Symmetry	10	3	10	3					L3	L3	H	L	0	0	0	0	9	3	1	
Tessellation	1		1						L1		L				0	0	1			
Diagonals		1		2						L1		L	0	0	0	0		1		1
Subtotal	15	6	15	7			H	H									14	6	1	1
Applications	0	0	0	0																
Total	25	15	25	16									3	1	0	0	24	15	1	1

[a]G = teacher Gary; S = teacher Sue.
[b]For Integrity and branching H = High rating.
[c]For depth L1, L2, L3 = Level 1 ratings, Level 2 ratings, Level 3 ratings.

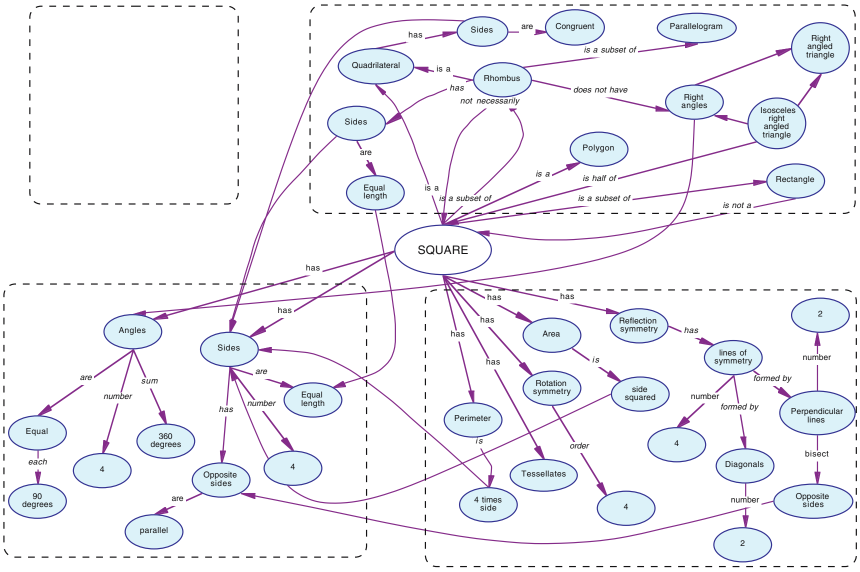

Figure 2. Connectedness map for teacher Gary.

also applied to between-schema knowledge components and the results of this are shown in Table II.

Quality

The quality of the concept maps was analysed for the two broad dimensions of *integrity* and *connectedness*. Integrity was analysed in terms of (a) completeness and (b) accuracy. A judgement of integrity here is not an absolute judgement about what a teacher knows, but was seen more functionally as a rating of the knowledge that the teacher was prompted to access by the range of specific research procedures used here. A rating of 'High' for completeness indicated the presence of all defining features. A rating of 'Moderate' for completeness was assigned if one defining feature was missing and a rating of 'Low' was given if more than one defining feature was missing.

Accuracy refers to the degree of correctness of the information provided by teachers. Information that is not correct may be manifested in various forms. An incorrect piece of knowledge could be a misconception (McKeown & Beck, 1990) or it could be 'garbled' (Perkins & Simmons, 1988). For example, if a teacher could not differentiate between the lines of symmetry in a square and a rhombus this would be a case of a misconception. If there was no evidence of garbled knowledge or

TABLE II
Between-Schema Indicators of Knowledge Connectedness

	Quantitative indicators				Qualititative indicators													
					Integrity		Connectedness											
	Number of nodes		Number of links		Accuracy[b]		Depth[c]		Branc-hing[b]		Cross-linking		Complexity of relationships					
													Simple		Moderate		Complex	
Teacher[a]	G	S	G	S	G	S	G	S	G	S	G	S	G	S	G	S	G	S
Layer 1 Schema																		
Quadrilateral	3		3				L3				1				3			
Rhombus	4	1	7	2			L3	L1	L		1				3	1	4	1
Polygon	1		1				L1								1			
Isoc rt triangle	3		4				L2		L		1		3				1	
Right angle																		
Rectangle	1	1	2	2			L1	L1								1	2	1
Right triangle		3		3				L2		L		1		2		1		
Face		2		2				L2								2		
Total	12	7	17	9	H	H					3	1	3	2	7	5	7	2

[a]G = teacher Gary; S = teacher Sue.
[b]For Integrity and branching H = High rating.
[c]For depth L1, L2, L3 = Level 1 ratings, Level 2 ratings, Level 3 ratings.

misconceptions a rating of 'High' was given for accuracy. 'Moderate' and 'Low' ratings were assigned for one, or more than one instance, of misconceptions or garbled knowledge respectively.

Connectedness was represented by four sub-categories: (a) depth, (b) branching, (c) cross-linking and (d) complexity of relationships. Depth refers here to the extension of connections in a concept map in vertical directions along a single path. Within-schema depth is a measure of the degree of vertical connection in a schema. We refer to this spread as occurring over different vertical layers of the nodes in the concept map. Between-schema depth is a measure of vertical connections among schema. Links made over more than two layers were assigned a rating of 'Level 3' for depth; those across two layers and at a single layer being scored as 'Level 2' and 'Level 1' respectively on this measure.

Branching is a measure of the number of paths associated with a Layer 1 node. If there was evidence of branching from a node at a single layer a rating of 'Low' was assigned. Branching at two layers and at more than two vertical layers was assigned ratings of 'Medium' and 'High' respectively. Cross-linking is a measure of horizontal linking between branches from a node or between sections of the map. The scores for cross-linking represent the number of cross-links among features for the within-schema scoring or from other schemas to features of the focus schema in the between-schema ratings.

The complexity of a stated label for a relationship was also rated. If the description of the label showed evidence of elaboration, or of bi-directionality, it was rated as 'Complex'. Elaboration here refers to any specification of the nature of a relationship between nodes that went beyond definition or allocation of membership. If there was no evidence of elaboration of the relationship, a rating of 'Moderate' was given for complexity. A 'Simple' rating indicated presence of a link that was implied. Parallel scoring systems were used for all between-schema measures, except completeness because the extent of between-schema links is open ended.

RESULTS AND DISCUSSION

The Teachers' Concept Maps

As shown above, all four dimensions of our framework describe aspects of KG but it is the quality of elaboration within each component that qualifies as KGT. Figures 2 and 3 show the maps that were constructed for the focus schema of square for the two teachers, Gary and

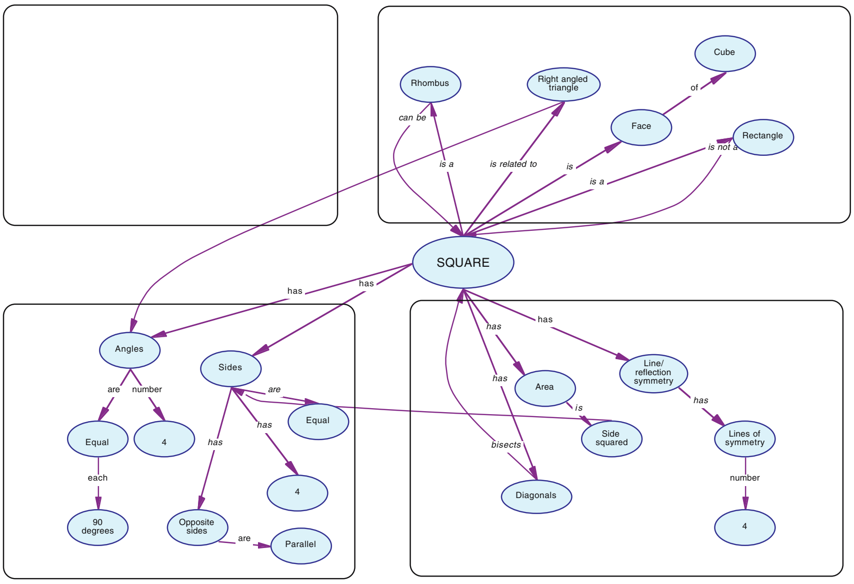

Figure 3. Connectedness map for teacher Sue.

Sue. A first impression is that Gary accessed a larger number of connections involving squares, as shown by comparison of the number of arrows emanating to and from the focus schema. Comparison of the lower halves of the two figures shows that the difference between the teachers was associated with the accessing of related features of square. Gary's output contained more nodes and links for the related features. The same pattern was apparent in the upper right section of the figures. Again Gary made more connections to other schemas. As shown by the blank section in the upper left corner of both figures, in the activities engaged in here, neither teacher made any statements that referred to applications or examples.

Gary's map suggests that he not only has a well-developed understanding about the properties of square and also has an extensive network of connections between squares and other geometrical figures. He was able to highlight differences and similarities that exist between square and related figures. For example, he represented a square as a polygon, as a quadrilateral and made explicit its relationship to an isosceles right-angled triangle. He identified bi-directional relationships between square and rhombus and square and rectangle. In contrast, Sue's map contained fewer such connections to other schemas. Nevertheless both teachers have built up a considerable amount of KG.

The same general comment applies to the related features section of the two maps. Gary showed that the symmetry of a square could be identified in more than one way. This understanding of symmetry is extended to his observations that squares tessellate. Again this section of Sue's map was less developed. Scoring of the two maps provided a more detailed account of the differences in the output of the two teachers.

Within-Schema Ratings

Results of the analysis for the focus schema of square in terms of quantity and quality are provided in Tables I and II. Table I contains data on indicators that were internal to the concept of square, links that indicated within-schema organisation. Gary's concept map contained a total of 10 nodes related to the defining features of the focus schema, and 15 nodes associated with the related features. Analysis of the number of links among these nodes also showed similar scores for defining and related features. The number of nodes and links in Sue's map was similar for the defining features, but lower for the related features. In terms of the estimates provided by our analysis Gary's knowledge of square was more extensive than Sue's. However, both teachers have constructed KG that shows the links between language and images that children need to learn about 2-D shapes.

In the qualitative analysis there were high ratings for integrity in the within-schema sections of both teachers' maps and the ratings for depth were similar for the two maps. In Gary's representation of symmetry there was a greater extent of branching than that produced by Sue. This suggests that this is a more elaborated chunk of knowledge for Gary indicating a higher level of deconstruction of content knowledge to KGT. However, the lack of cross-linking between these two branches of symmetry indicates that there seems to be little integration between knowledge about line symmetry and rotational symmetry. Overall, Gary's map showed more cross-linking, though none of the cross-links was explicitly labelled by either teacher. The complexity of the labels used by the teachers to describe within-schema relationships was also similar, with most relationships being rated as moderate, indicating lack of elaboration of these relationships. Here one can detect room for further developments of Gary's KGT.

Between-Schema Ratings

Table II shows the ratings given to the teachers' knowledge about relationships between square and other schemas. Scores were derived for each of the schemas in the first layer of nodes extending from the focus schema (Layer 1 Schemas). The quantitative analysis showed that Gary again accessed a larger number of nodes than did Sue and established a greater number of links among these nodes. Of these, the highest number of links and nodes accessed by Gary involved the rhombus schema, suggesting the development of mature KGT in this area. For both teachers the relationships established were accurate. With regard to between-schema connectedness, with one exception, the ratings for both teachers were generally lower than was the case for their within-schema representations. For both, the depth ratings in the between-schema analysis were lower on average and there was less evidence of branching. In Gary's map the links with greatest extent of depth and branching were those between square and rhombus, rhombus and parallelogram, and rhombus and quadrilateral, thus indicating the clustering of these schemas.

Gary provided more instances of complex labelling of relationships than did Sue, though there was no instance where such descriptions involved extensive elaboration. For example, Gary mentioned that an isosceles right-angled triangle is half of a square. However, he did not go further and discuss the implications of such relationships in using or deriving, say, Pythagoras' theorem, or trigonometric ratios. These extended connections and elaborations constitute features of well-developed KGT. Both teachers made explicit links between features of square and other schemas, with Gary making more of these links than Sue. In addition, Gary explicitly linked the square with quadrilateral and rhombus schemas, and linked the latter to the isosceles right-angled triangle schema.

The scoring of the maps provided the detail supporting the interpretation derived from visual inspection of the maps. The specification of qualities of the maps allowed us to make judgements that were not just quantitative ones. Thus, through use of the scoring system, we were able to make clear that not only did Gary's output show a more extensive network of linked nodes, but that more of the nodes in his map were linked by complex relationships. This relational information contained in related features and related schemas might be called upon during teaching for problem solving. Our analysis suggests that in a problem where relationships between, say a square and isosceles triangles, or between a square and a quadrilateral, needed to be ascertained, Gary would more readily access such knowledge than would Sue because of the better state of his KGT.

CONCLUSION

We contend that the design of the analytical procedure for making judgements about the quality of teachers' knowledge is soundly based, and that the indicators we have developed represent important features of KG and KGT in a small area of geometry.

The analysis shows that Gary has a rich set of connections with evidence of complex differentiation among schemas, and within the focus schema in some instances. Overall, his knowledge has high integrity and shows evidence of substantial branching in certain areas. In the between-schema analysis, about half of the relations that were discussed by Gary were complex in nature. The within-schema analysis provided less evidence of cross-linking between branches than might have been expected for an experienced teacher, though there was such cross-linking in other parts of his map. For instance, as shown in Figure 1, Gary discussed area, perimeter, rotational symmetry and reflection symmetry. He did not, however, discuss relationships between the two types of symmetry, area and perimeter. The effects of different transformations on attributes of square could be expected to be a topic of questions from some students. Further, concepts of symmetry and transformations could be discussed in terms of matrix representations and coordinates. This is an important area of learning mathematics that would help students draw more complex links among 2-D figures and other topics in school curriculum. Thus, Gary could deconstruct his content knowledge to a higher degree than was evident in this analysis.

We suggest that this mapping procedure is useful for purposes where the systematic characterisation of teachers' KG and KGT is required. There are other implications of this analysis. The comparison of the output of the two teachers in this study does suggest that the knowledge bases they could call upon in their teaching are quantitatively and qualitatively different. We have, for purposes of illustration, focussed only on representation of one focus schema. However, if the patterns reported here were confirmed for other schemas in this part of mathematics, it seems clear that the knowledge resources available to Gary, and so to his students, would be richer than those in Sue's case. Clearly, we do not want to extend the province of this claim unreasonably. There are many other factors associated with effective teaching than the quality of the teacher's content knowledge base and we make no claims in this regard about the two teachers discussed here. However, we do suggest that the lack of integration between the different branches of schema knowledge of teachers could impact on their teaching, in that they and their students might not draw out important distinctions and similarities between key knowledge schemas in classroom interchanges.

Although the focus of the analysis in this paper has been on one simple concept we do not think that the analytical procedure we have described is limited to use with such concepts, nor is it limited to the field of geometry. It is important to note that even when the concept of square was the focus of discussion, the teachers also accessed knowledge of more complex geometric schemas, such as symmetry and congruence, and that these could be represented in the analysis. The reason for our focus on relatively simple figures arose from the objectives of our larger project in which we have also examined the understandings of the students being taught by the teacher participants.

In the present study, we attempted to generate estimates of quality of teachers' knowledge of this part of geometry and for teaching geometry by inviting the participants to discuss their understandings in three different but related contexts. However, observing the same teachers in action might enrich further these data about knowledge for teaching geometry. In that case, it should not be assumed that observation of teachers in action would necessarily result in accessing by the teachers of different sets of knowledge from that accessed using tasks such as the ones involved in this study. The knowledge access process depends in part on the cues provided by the situation and it could be the case that the teaching situation is less rich in cues than say, the problems and question probes used in this project. For students, it is clear that they can leave much of their available knowledge inert, during problem solving (Watson & Lawson, 1995), and the same may happen with teachers if lessons did not involve a degree of problematising of the content. So there is some interesting further research to be undertaken in observing teachers' knowledge access while they are teaching.

We contend that the system of analysis used in this study takes us beyond that used in other studies that have employed concept mapping or similar graphical systems as a means of representing knowledge states. The current system takes us beyond the point reached by Williams' (1998) study in mathematics, enabling us to present a more detailed and differentiated description of the dimensions of a teacher's knowledge base for subject matter content and the teaching of this content. The indices used in the framework provide a way to make more specific statements about a teacher's 'depth of understanding'. Overall, our experience with this procedure suggests that there is value in pursuing Mayer's (1975) distinction between internal and external connectedness in order to make judgements about the qualitative features of a knowledge network.

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NOTES

In the top right-hand corner of Figure 2, two attributes of *Other Schemas* appear in circles, while the remaining ones are in ovals. This is due to the technical feature of the software which was used to construct the map. We do not attribute any significance to the ovals and circles.

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