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Multi-objective Compromise Allocation in Multivariate Stratified Sampling Using Extended Lexicographic Goal Programming with Gamma Cost Function

Yousaf Shad Muhammad · Javid Shabbir · Ijaz Husain · Mitwali Abd-el.Moemen

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Abstract In the present paper, a new Gamma cost function is proposed for an optimum allocation in multivariate stratified random sampling with linear regression estimator. Extended lexicographic goal programming is used for solution of multi-objective non-linear integer allocation problem. A real data set is used to illustrate the application.

Keywords Multivariate stratified sampling · Compromise allocation · Multi-objective programming · Nonlinear cost function · Lexicographic goal programming

1 Introduction

A good sampling plan plays a significant role in a statistical study and provides a close approximation to the population parameters. The selection of appropriate sampling plan and samples can produce more reliable parameters. The important consideration in a stratified random sampling design is the sample size allocation. The sample is allocated in each stratum with the criterion either to minimize variance of stratified sample mean for a fixed cost or to minimize cost for the specified variance.

Stratified random sampling is used to increase precision following some cost mechanism. Allocation of sample size n_h to individual stratum becomes more complicated in a study or survey while using stratified random sampling scheme. Mostly, the sampling efficiency depends on how the sample size is allocated. In multivariate stratified sampling, individual optimum allocation can be used when the characteristics are correlated but in case when the characteristics are uncorrelated, a suitable criterion is needed for allocation of sample sizes which should be optimum for all the characteristics in some sense.

Y. S. Muhammad (✉) · J. Shabbir · I. Husain
Department of Statistics, Quaid-i-Azam University, Islamabad, Pakistan
e-mail: yousuf@qau.edu.pk

M. Abd-el.Moemen
College of Law and Political Science, King Saud University, Riyadh, Saudi Arabia

Cochran [6] highlighted that it is difficult to work out an allocation which is optimum for all characteristics unless the characteristics are highly correlated and the deviation between strata variances is very small. Compromised allocation may be suitable in such situation. Holmberg [13] addressed the problem of compromised allocation in multivariate stratified sampling by taking into consideration the minimization of sum of variances or coefficients of variation of population parameters and minimization of sum of efficiency loss which may result due to increase in variance because of using the compromise allocation.

The solution of a problem needs some compromise allocation criteria which makes the allocation optimum for all characteristics. For example, an allocation which minimizes the trace of variance-covariance matrix of the estimator of population mean or the weighted average of variances or that maximizes the total relative efficiency of the estimators as compared to corresponding individual optimum allocation is discussed in [28]. Many authors include [1, 2, 5, 7–10, 14–19] and [29] used different compromise criterion to solve allocation problem in stratified sampling.

In the existing literature described above, different estimators of population parameters are used for same purpose but have less efficiency and precision. In the present paper, regression estimator is used because of its efficiency and precision over other estimators [24]. The rest of the paper is organized as follows: Section 2 explains the model for the cost function. Problem formulation is discussed in Section 3. Section 4 explains lexicographic programming. The results and discussions are presented in Section 5.

2 Gamma Cost Function

The cost of a survey is the major factor of sample allocation to various strata. Tschuprow [27] and Neyman [21] proposed an allocation procedure that minimizes the variance of sample mean under a linear cost function $n = \Sigma_{h=1}^{L} n_h$ in a stratified random sampling. Neyman [21] used lagrange multiplier optimization technique to get optimum sample size for a study of a single variable. The linear cost function used in stratified sampling is given as follows:

$$C = c_0 + \sum_{h=1}^{L} c_h n_h, \tag{1}$$

where C denotes total budget available for a survey and $c_h (h = 1, 2, ..., L)$ represent measurement per unit cost in the hth stratum, c_0 represents fixed cost of survey, and n_h is number of sample units selected from the hth stratum.

Considering a quadratic cost function, including unit cost and traveling cost within strata as Bearwood et al. [3] proposed, the shortest route among k randomly allocated sampling units in the region is asymptotically proportional to $\sqrt{k}$ for a large k. Varshney et al. [29] used a quadratic cost function for large sample size given as follows:

$$C = c_0 + \sum_{h=1}^{L} c_h n_h + \sum_{h=1}^{L} \tau_h \sqrt{n_h}, \tag{2}$$

where τ_h is travel cost for a unit within the hth stratum.

It is important to determine a true functional form of the cost function so that the appropriate form should be considered. In most of the situations, the per unit measurement cost,travel cost within strata, reward to respondent, and labor cost are important factors in a survey. A polynomial cost function accounting per unit measurement cost, traveling cost within strata, reward, and labor cost may be a good approximation to actual cost of a survey. Reward given to a respondent may reflect the preciousness of the respondent's view point, availability, approachability, time, etc., where labor cost may be a multiple of time units consumed to collect the data from the respondents.

Now, if time taken to collect the data from a sampling unit follows an exponential distribution ([12, 23], and [20]) with rate λ then probability distribution function for time used is approximately $f(x) = \lambda \exp^{-\lambda x} \quad \forall x > 0$ and integer where $\lambda = 1/(averge\ time)$. Sum of independently identically exponentially distributed random variables follows a Gamma distribution with parameters n_h and λ [12, 23]. Moreover, this idea can be extended to the whole sample from all strata, and the Gamma function will have the parameters $\sum_{h=1}^{L} n_h$ and λ. Considering this, we propose the following cost function:

$$\acute{C} = \sum_{h=1}^{L} \acute{c_h} n_h + \sum_{h=1}^{L} \tau_h n_h^{\delta} + \omega \int_0^{\infty} \lambda exp^{-\lambda t} \frac{(\lambda t)^{\sum_{h=1}^{L} n_h - 1}}{(\sum_{h=1}^{L} n_h - 1)!} d_t, \tag{3}$$

where $\acute{C} = C - c_0$, $\acute{c_h}$ unit cost along with reward paid to respondents in strata h, t is time taken by the interviewer, δ represents the effect of travel within strata, and Gamma function represents the effect of labor cost. $\acute{c_h} = c_h + r_h$, where r_h is reward paid within the strata h, equally to all n_h units. If different rewards are paid to n_h units within strata, we can use average reward paid $\bar{r}_h$. Here, ω is the cost of unit time. The value of δ is determined by solving problems using methods discussed by Winston [30], Taha [25], and Hiller and Lieberman [11].

Cost occurs on labor time is replaced by aggregate expected cost using function $\omega \sum_{h=1}^{L} E(T_h) = \omega \sum_{h=1}^{L} \left(\int_0^{\infty} \lambda exp^{-\lambda t} \frac{(\lambda t)^{\sum_{h=1}^{L} n_h - 1}}{(n_h - 1)!} d_t \right) = \omega \sum_{h=1}^{L} \frac{n_h}{\lambda}$ for different choices of λ estimated from methods discussed in [23] and [12].

3 Extended Lexicographic Goal Programming

Mathematically speaking, we allow our generic goal program to have Q goals, which may be $j = 1, ..., Q$. We also define n_{jh} decision variables. These are the factors over which the decision maker(s) have control and define the decision to be made. Each goal has an achieved value, Z_j, on its underlying criterion. Z_j is a function of the decision variables. The whole situation may be expressed below:

$$\left.\begin{array}{r} \text{Minimize } Z_j = f(n_{jh}) \\ \text{Subject to} \\ \sum_{h=1}^{L} \acute{c_h} n_{jh} + \sum_{h=1}^{L} \tau_h n_{jh}^{\delta} + \omega \int_0^{\infty} \lambda exp^{-\lambda t} \frac{(\lambda t)^{\sum_{h=1}^{L} n_{jh} - 1}}{(\sum_{h=1}^{L} n_{jh} - 1)!} d_t \le \acute{C} \\ 2 \le n_{jh} \le N_h \\ n_{jh} \text{ are integers } \forall\, h = 1, 2, ..., L \text{ and } j = 1, 2, ..., Q. \end{array}\right\} \tag{4}$$

Note that in this generic form, no assumptions have yet been made about the nature of the decision variables of goals. The decision maker(s) sets a real target level for each goal which is denoted by Z_j^* (generally an individual optimal of the jth objective). This then leads to the basic formulation of the jth goal:

$$\hat{Z}_j + d_j^- - d_j^+ = Z_j^*,$$

where d_j^- and d_j^+ are negative and positive deviational variables. They are also called goal variables. $\hat{Z}$ compromised value of jth goal.

Sometimes, the set of goals are termed as soft constraints. That is, the decision maker(s) desires to optimize each goal but if the goal is not achieved, then this does not imply that the solution is infeasible. Goal programming also allows for an addition of a set of linear programming style hard constraints whose violation will make the problem infeasible. These are modeled by adding the condition

$$\underline{\hat{n}}_j \epsilon \mathcal{F},$$

where $\mathcal{F}$ is feasible region established by points in decision space and $\underline{\hat{n}}_j = [n_{j1}, n_{j2}, ..., n_{jL}]$.

Finally, the unwanted deviational variables are put into an achievement function whose purpose is to minimize them and ensure that solution is "as close as possible" to the set of desired goals.

Lexicographic goal programming is termed as preemptive goal programming. The distinct feature of lexicographic goal programming is the existence of priority levels for objectives. The objectives are prioritized in order of their importance. All unwanted deviations are minimized at each priority level. The generic form the programme of compromise allocation can be written as follows:

$$\left.\begin{array}{r} \text{Minimize } [f_1(\underline{d}_j^-, \underline{d}_j^+), f_2(\underline{d}_j^-, \underline{d}_j^+), \cdots, f_Q(\underline{d}_j^-, \underline{d}_j^+)] \\ \text{Subject to} \\ \hat{Z}_j + d_j^- - d_j^+ (\leq \ or \ \geq) Z_j^* \\ \underline{\hat{n}}_j \epsilon \mathcal{F}, \\ n_{jh} \text{ are integers } \forall\, h = 1, 2, ..., L \text{ and } j = 1, 2, ..., Q. \end{array}\right\} \quad (5)$$

where $f_1, f_2, \cdots, f_Q$ represent priority-wise functions and $\underline{d}_j^-, \underline{d}_j^+$ are vectors of unwanted deviations in the respective priority.

The other techniques are weighted goal programming (WGP), which formulate to minimize a composite objective function formed by a weighted sum of unwanted deviational variables. The third is MINMAX (Chebyshev) goal programming, which attempts to minimize the maximum deviation from the desired goals.

In most of the cases, the goal programming variant is chosen without justifying the reason for the selection. It then appears as the choice of the goal programming variant is related to the analyst's taste or to the capability of getting solution. However, the selection of the right goal programming variant or mix of variants is a crucial matter if we want the goal programming model to capture the essential features of the reality modeled [22].

Goal programming can be analyzed in terms of utility theory which always maximize the utility. The utility function described from the given situation may be of any form, i.e., linear, non-linear, etc., and a certain satisfaction level of aspiration for a particular goal can be set within a feasible space [22]. Using the programming techniques discussed in [4] and [22], a goal program becomes equivalent to minimize the weighted discrepancy for a certain aspiration level $\forall\ j = 1, 2, ..., Q$ goals within a feasible space. Now, if we consider that negative deviational variable and positive deviational variable have different impact on achievement function is a particular preference sequence. Let W_{1j} and W_{2j} represent the weights of normalizing parameter and preferential of negative deviation variable $(\underline{d}_j^-)$ and positive deviation variable ($\underline{d}_j^+$) on the jth goal, respectively, then following formulation is discussed in [22]:

$$\left.\begin{array}{r} \text{Minimize } \sum_{j=1}^{Q}[f_j(W_{1j}\underline{d}_j^-, W_{2j}\underline{d}_j^+)] \\ \text{Subject to} \\ \hat{Z}_j + d_j^- - d_j^+ (\leq \ or \ \geq) Z_j^* \\ \underline{\hat{n}}_j \epsilon \mathcal{F}, \\ n_{jh} \text{ are integers } \forall\, h = 1, 2, ..., L \text{ and } j = 1, 2, ..., Q. \end{array}\right\} \tag{6}$$

The maximum utility function may subject to deviate from its desired aspiration level. An Archimedean goal programming model has a clear utility interpretation, "it implies the maximization of a separable and additive utility function in the Q attributes considered" [22]. The MINMAX (Chebyshev) structure corresponds to a utility function where the maximum deviation is minimized. This structure is discussed in [26] and [22] as follows:

$$\left.\begin{array}{r} \text{Minimize } D \\ \text{Subject to} \\ [\underline{d}_j^-, \underline{d}_j^+] \leq D \\ \hat{Z}_j + d_j^- - d_j^+ (\leq \ or \ \geq) Z_j^* \\ \underline{\hat{n}}_j \epsilon \mathcal{F}, \\ n_{jh} \text{ are integers } \forall\, h = 1, 2, ..., L \text{ and } j = 1, 2, ..., Q \end{array}\right\} \tag{7}$$

where D is maximum deviation from utility.

The concept of extended goal programming, the utility maximization of the Archimedean and MINMAX (Chebyshev) goal programming models, can be generalized as follows:

$$\left.\begin{array}{r} \text{Minimize } (1-\rho)D + \rho \sum_{j=1}^{Q}[f_j(W_{1j}\underline{d}_j^-, W_{2j}\underline{d}_j^+)] \\ \text{Subject to} \\ [W_{1j}\underline{d}_j^-, W_{2j}\underline{d}_j^+)] \leq D \\ \hat{Z}_j + d_j^- - d_j^+ (\leq \ or \ \geq) Z_j^* \\ \underline{\hat{n}}_j \epsilon \mathcal{F}, \\ n_{jh} \text{ are integers } \forall\, h = 1, 2, ..., L \text{ and } j = 1, 2, ..., Q \end{array}\right\} \tag{8}$$

Parameter ρ assigns the importance attached to the minimization of the weighted sum of unwanted deviation variables. Above formulation increase the feasible region by relaxing the constraint $(1-\rho)[W_{1j}\underline{d}_j^-, W_{2j}\underline{d}_j^+)] \leq D$ imposed in [22] into $[W_{1j}\underline{d}_j^-, W_{2j}\underline{d}_j^+)] \leq D$ as $0 \leq \rho \leq 1$. Integer nonlinear programming problems have a small feasible solution grid and we are already compromising on allocating sample size. This will help us to find feasible and optimal solution considering larger grid using this relaxation.

4 Application in Stratified Random Sampling

Consider a finite population of size N which is divided into L mutually exclusive strata such that $N = \sum_{h=1}^{L} N_h$. We draw a simple random sample of size n_h independently from each stratum such that $\sum_{h=1}^{L} n_h = n$.

Let, we have a data Y_{jhi} for $j = 1, 2, ..., Q$ characteristics, $h = 1, 2, ..., L$ strata with $i = 1, 2, 3, ...N_h$ sampling units in the h^{th} stratum. Let $\bar{y}_{jh}$ and $\bar{x}_{jh}$ are the sample means, $\bar{Y}_{jh}$ and $\bar{X}_{jh}$ are the strata mean of the study variable Y_{jhi} and the auxiliary variable X_{jhi}, respectively, of the jth characteristics in the hth stratum. Let S^2_{yjh} and S^2_{xjh} are strata variances and S_{yxjh} is strata covariance between the study and the auxiliary variables for the jth characteristic in the hth stratum.

Let $b_{jh} = \frac{s_{yxjh}}{s^2_{xih}}$ and $\beta_{jh} = \frac{S_{yxjh}}{S^2_{xjh}}$ are sample and strata regression coefficients, respectively, and $W_h = \frac{N_h}{N}$ is the known stratum weight. The traditional regression estimator is $\bar{y}_{j,lr} = \sum_{h=1}^{L} W_h \bar{y}_{j,lrh}$, where $\bar{y}_{j,lrh} = \bar{y}_{jh} + b_{jh}(\bar{X}_{jh} - \bar{x}_{jh})$.

The (MSE) of $\bar{y}_{j,lr}$ is given by

$$MSE(\bar{y}_{j,lr}) = \sum_{h=1}^{L} W_h^2 \left(\frac{1}{n_h} - \frac{1}{N_h}\right)\left(S^2_{yjh} - 2\beta_{jh} S_{yxjh} + \beta^2_{jh} S^2_{xjh}\right), \tag{9}$$

Ignoring the finite population correction, we have

$$MSE\left(\bar{y}_{j,lr}\right) = \sum_{h=1}^{L} \frac{W_h^2 \Delta'_{jh}}{n_h},$$

where

$$\Delta'_{jh} = S^2_{yjh} - 2\beta_{jh} S_{yxjh} + \beta^2_{jh} S^2_{xjh}.$$

We use coefficient of variation instead of mean square error as it is free from units:

$$C.V\left(\bar{y}_{j,lr}\right) = \sqrt{\frac{MSE\left(\bar{y}_{j,lr}\right)}{\bar{Y}_j^2}}$$

or

$$C.V\left(\bar{y}_{j,lr}\right) = Z_j = \sqrt{\sum_{h=1}^{L} \frac{\delta'_{jh}}{n_h}}, \tag{10}$$

where

$$\delta'_{jh} = \frac{W_h^2 \Delta'_{jh}}{\bar{Y}_j^2}.$$

A sample size $n = \sum_{h=1}^{L} n_h (\forall h = 1, 2, ..., L)$ is determined by using proposed Gamma cost function in Eq. 3 as constraint by minimizing vector of coefficients of variation of the estimators of population mean for each characteristic $Y_j (j = 1, 2, ..., Q)$. This problem is formulated in multi-objective multivariate integer non-linear programming as discussed by many authors, e.g., [5, 9, 10, 16, 17, 19] and [28].

$$\left.\begin{array}{r} \text{Minimize } \left(Z_1, Z_2, ..., Z_Q\right) \\ \text{Subject to} \\ \sum_{h=1}^{L} \acute{c_h} n_h + \sum_{h=1}^{L} \tau_h n_h^{\delta} + \omega \int_0^{\infty} \lambda exp^{-\lambda t} \frac{(\lambda t)^{\sum_{h=1}^{L} n_h - 1}}{(\sum_{h=1}^{L} n_h - 1)!} d_t \leq \acute{C} \\ 2 \leq n_h \leq N_h \\ n_h \text{ are integers and } n_h \in \mathbf{F}\ \forall\ h = 1, 2, ..., L. \end{array}\right\} \tag{11}$$

5 Numerical Illustration

For numerical illustration, the data from agricultural census in Iowa state 1997 and 2002 conducted by the National Agricultural Statistics Service, USDA, Washington D.C. as reported by [19] is used. Let we assume

Y_1 denote the quantity of corn harvested in 2002,
Y_2 denote the quantity of oats harvested in 2002,
X_1 denote the quantity of corn harvested in 1997,
X_1 denote the quantity of oats harvested in 1997.

The data summary is given as: $\bar{Y_1} = 474973.90$, $\bar{X_1} = 405654.19$, $\bar{Y_2} = 1576.25$, $\bar{X_2} = 2116.70$

A complete formulation of the problem is shown in Appendix.

6 Results and Discussion

GAMS optimization software is used to solve these programs. The $\acute{\delta_{jh}} = \frac{W_h^2 \acute{\Delta_{jh}}}{\bar{Y}_j^2}$ is re-scaled to

$$\acute{\delta_{jh}} = 1000 \times \frac{W_h^2 \acute{\Delta_{jh}}}{\bar{Y}_j^2}$$

to make possible optimum allocation; otherwise, program (GAMS) does not consider changes in decision variables n_{jh} due to a very small fractional optimal solution.

The optimum allocation in four strata for first characteristic are 8, 28, 13, and 12, respectively, with an aggregate cost of 19934 when an upper limit of 20,000 is fixed. Optimum value of coefficient of variation (CV) is 0.271 which devalues (marginal value, i.e., $\partial Z/\partial x$) −0.002 for a unit rise in sample size from strata one and two, −0.001 and −0.004 for a unit in strata three and four, respectively. Higher devaluation in CV was observed in stratum four which indicates its preference of increased sample size if it is to be increased. But strata one and two are fully part of our sample so stratum two is most favorable at the current status. However, situation may be different if we change the cost of the survey.

When we run optimum allocation model for second characteristic, it allocates 4, 32, 45, and 10 units in four strata, respectively. Now, stratum three is fully part of our sample so devaluation of −0.006 related to this stratum get no attention. The priority sequence considering marginal values is established, which is strata one, two, and four with marginal values −0.006, −0.005, and −0.003, respectively. The optimum values of CV is 0.959 with cost level at 35472. These optimal results are obtained at cost less then 35500.

These nonlinear integer programming problems have limited feasible solution points. If we discuss arbitrary cost structure, it is observed that optimal solution with reasonable allocation in all strata starts from arbitrary cost between 19489 and 39056 for first characteristic and 35044 to 39056 for second characteristics. Below these limits, programs return lower bound solutions of decision variables and above these limits, a complete selection of strata. Obviously, a lowest cost is needed for minimum integer allocation.

In multi-objective non-linear integer program, we set two additional constraints which bound CVs below to their individual optimum values. Using $\rho = 0.4$ for unwanted sum of deviations from individual optimum values and $1 - \rho = 0.6$ for maximum deviation from utility, we minimize the goal objective or achievement function under originally defined cost and decision space (decision variables) constraints. The compromise optimum allocation for four strata are 5, 33, 45, and 12, respectively. CVs are further reduced to 0.261 and 0.957 for characteristics one and two, respectively, with a cost of 35902 (higher from both individual optimum levels). This is because we have applied an upper bound on CVs. Changing the ρ, different results are expected.

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Appendix

From the data given in numerical illustration and the method discussed in application in stratified random sampling, we formulate the following model.

Individual Model for each Characteristic (j=1, 2, . . . , Q).

Our decision variables are n_h ($\forall\ h = 1, 2, ..., L$). Values of $\acute{c_h}$ and τ_h are replaced from Table 1 $\forall\ h = 1, 2, ..., L$. δ replaced with arbitrary values 0.5 and 2.0 (i.e., square-root and square, as both the references in literature), ω is the cost for a unit time of labor (say 100, 150, etc. per hour per individual only for solution purpose. One can change all these values with their actual values in his/her study). Estimates of Gamma function $\int_0^\infty \lambda exp^{-\lambda t} \frac{(\lambda t)^{\sum_{h=1}^{L} n_{jh}-1}}{(\sum_{h=1}^{L} n_{jh}-1)!} d_t$ replaced with its expected time used $\sum_{h=1}^{L} E(T_h) = \sum_{h=1}^{L} \left(\int_0^\infty \lambda exp^{-\lambda t} \frac{(\lambda t)^{\sum_{h=1}^{L} n_{jh}-1}}{(n_{jh}-1)!} d_t \right) = \sum_{h=1}^{L} \frac{n_{jh}}{\lambda}$ for different choices of λ where $\lambda = \frac{1}{avg\ time}$ taken to collect data for response Y_{jhi} (say 15 min, 20 min, etc. on the average from an individual). In the above formulation, cost constraint becomes,

$$\sum_{h=1}^{L} \acute{c_h} n_{jh} + \sum_{h=1}^{L} \tau_h n_{jh}^{\delta} + \omega \sum_{h=1}^{L} \frac{n_{jh}}{\lambda} \leq \acute{C}$$

Table 1 Data summery

h	N_h	W_h	S^2_{y1h}	S^2_{x1h}	S^2_{y2h}	S^2_{x2h}	S_{x1y1h}	S_{x2y2h}	c_h	r_h	$\acute{c}_h$	τ_h	β_{1h}	β_{2h}	$\acute{\delta}_{1h}$	$\acute{\delta}_{2h}$
1	8	0.0808	29267524195.5	21601503189.8	777174.1	1154134.2	24360422802.3	902170.6	20	100	120	8	1.1249	0.7834	0.000066	0.000181
2	34	0.3434	26079256582.8	19734615816.7	4987812.9	7056074.8	22003466630.3	5813439.5	8	60	68	6	1.1150	0.8239	0.000809	0.009411
3	45	0.4545	42362842460.8	27129658750.0	1074510.6	2082871.3	33367597192.0	1285355.6	6	50	56	5	1.2300	0.6171	0.001212	0.023390
4	12	0.1212	30728265336.9	17258237358.5	388378.5	732004.9	21033769867.3	456991.5	15	80	95	7	1.2188	0.4243	0.000332	0.000610

c_h and τ_h are given in previous reference paper. r_h, a collective labor cost and reward for a unit time, we are proposing according to approximate local rates. Obviously, results will be different if some one choose different costs

$$\left.\begin{aligned}
\text{Min } \left(Z_j\right) = C.V\left(\bar{y}_{j,lrs}\right) = \sqrt{\frac{MSE\left(\bar{y}_{j,lrs}\right)}{\bar{Y}_j^2}} = \sqrt{\sum_{h=1}^{L}\frac{W_h^2\acute{\Delta}_{jh}}{\bar{Y}_j^2 n_{jh}}} = \sqrt{\sum_{h=1}^{L}\frac{\acute{\delta}_{jh}}{n_{jh}}}\\
\text{Min } (Z_1) = \sqrt{[0.000066\ 0.000809\ \ 0.001212\ 0.000332] * [1/n_{11}\ 1/n_{12}\ 1/n_{13}\ 1/n_{14}]^T}\\
\text{Min } (Z_2) = \sqrt{[0.000181\ 0.009411\ \ 0.02339\ 0.000610] * [1/n_{21}\ 1/n_{22}\ 1/n_{23}\ 1/n_{24}]^T}\\
\text{Subject to}\\
[120\ 68\ \ 56\ 95] * [n_{11}\ n_{12}\ n_{13}\ n_{14}]^T + [8\ 6\ \ 5\ 7] * \left[n_{11}^{0.5}\ n_{12}^{0.5}\ n_{13}^{0.5}\ n_{14}^{0.5}\right]^T + 100\sum_{h=1}^{L}\frac{n_{jh}}{4} \le \acute{C}\\
2 \le n_{jh} \le N_h\\
n_{jh} \text{ are integers } \ \forall\, h = 1, 2, ..., L \text{ and } j = 1, 2, ..., Q
\end{aligned}\right\}$$

Multi-Objective Model for Characteristics ($j = 1, 2, ..., Q$).

From the above models, we obtain optimal values Z_1^* and Z_2^* for two characteristics $j = 1$ and $j = 2$. Selecting an arbitrary value(s) of ρ (say 0.1,0.2, 0.3,.....,1.0), we establish the following model ($\rho = 0.4$),

$$\left.\begin{aligned}
\text{Minimize } 0.6D + 0.4(d_1^+ + d_2^+)\\
\text{Subject to}\\
d_1^+ \le D\\
d_2^+ \le D\\
\sqrt{[0.000066\ 0.000809\ 0.001212\ 0.000332] * [1/n_{11}\ 1/n_{12}\ 1/n_{13}\ 1/n_{14}]^T} - d_1^+ + d_2^+ \le Z_1^*\\
\sqrt{[0.000181\ 0.009411\ \ 0.02339\ 0.000610] * [1/n_{11}\ 1/n_{12}\ 1/n_{13}\ 1/n_{14}]^T} - d_1^+ + d_2^+ \le Z_2^*\\
[120\ 68\ \ 56\ 95] * [n_{11}\ n_{12}\ n_{13}\ n_{14}]^T + [8\ 6\ \ 5\ 7] * \left[n_{11}^{0.5}\ n_{12}^{0.5}\ n_{13}^{0.5}\ n_{14}^{0.5}\right]^T + 100\sum_{h=1}^{L}\frac{n_{jh}}{4} \le \acute{C}\\
n_{jh} \text{ are integers } \ \forall\, h = 1, 2, ..., L \text{ and } j = 1, 2, ..., Q.
\end{aligned}\right\}$$

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