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Partial Trade Credit Policy of Retailer in Economic Order Quantity Models for Deteriorating Items with Expiration Dates and Price Sensitive Demand

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Abstract In a supplier-retailer-customer supply chain, a credit-worthy retailer frequently receives a permissible delay on the entire purchase amount without collateral deposits from his/her supplier (i.e., an up-stream full trade credit). By contrast, a retailer usually requests his/her credit-risk customers to pay a fraction of the purchase amount at the time of placing an order, and then grants a permissible delay on the remaining balance (i.e., a down-stream partial trade credit). Also, in selecting an item for use, the selling price of that item is one of the decisive factors to the customers. It is well known that the higher selling price of item decreases the demand rate of that item where the lesser price has the reverse effect. Hence, the demand rate of an item is dependent on the selling price of that item. In addition, many products such as fruits, vegetables, high-tech products, pharmaceuticals, and volatile liquids not only deteriorate continuously due to evaporation, obsolescence and spoilage but also have their expiration dates. However, only a few researchers take the expiration date of a deteriorating item into consideration. This paper proposes an economic order quantity model to allow for: (a) the strategy that supplier offers retailer a full trade credit policy whereas the retailer offers their customers a partial trade credit policy, (b) selling price dependent demand rate, (c) a profit maximization objective and (d) deteriorating items not only deteriorate continuously but also have their expiration dates. For the objective function sufficient conditions for the existence and uniqueness of the optimal solution are provided. An efficient algorithm is designed to determine the optimal pricing and inventory policies for the retailer. Finally, numerical examples are presented to illustrate the proposed model and the effect of key parameters on optimal solution is examined.

Keywords EOQ model · Deterioration · Partial trade credit · Expiration dates · Pricing

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1 Introduction

Today's research is interested in focusing on supply chain models which have real life applications. In real life business via share marketing, trade credit financing becomes a powerful tool to improve sales and profits in an industry. Bhigham [5] gave financial management term "net credit". "Net credit" means a supplier/retailer offers the retailer/customer a fixed time period, say 30 days, to settle the total amount against the purchases made without penalty for his retailer/customer to increase sales and reduce on-hand stock. However, if the payment is not settled within the allowable trade credit period, the interest is charged on the unsold stock under the agreed terms and conditions. During the delay period (i.e. credit period) the retailer can accumulate revenue on sales and earn interest on that revenue via share market investment or banking business and delay the payment up to the last day of the delay period offered by the supplier. This permissible delay in payments reduces the cost of holding stock because it reduces the amount of capital investment in stock for the duration of the offered trade credit. Teng [42] illustrated two more benefits of trade credit policy: (1) it attracts new customers who consider trade credit policy to be a type of price reduction; and (2) it should cause a reduction in sales outstanding, since some established customer will pay more promptly in order to take advantage of trade credit more frequently. However, the strategy of granting credit terms adds an additional dimension of default risk to the supplier [45].

According to Teng et al. [45], the strategy of granting credit terms adds an additional dimension of default risk to the supplier and/or the retailer. In commercial practice, to reduce non-payment risks, a retailer frequently offers a partial trade credit to its credit risk customer who must make partial payment of the purchase amount at the time of placing an order and settle the unpaid balance at the end of credit period.

In supply chain management, it is too difficult to preserve deteriorating items for all business sectors. Many products such as fruits, vegetables, medicines, high-tech products, pharmaceuticals, and volatile liquids not only deteriorate continuously due to evaporation, obsolescence and spoilage but also have their expiration dates, i.e., the product will have a maximum lifetime which is time bound. However, only a few researchers take the expiration date of a deteriorating item into consideration. In the present article, we consider the replenishment policies for inventory which are subject to deteriorate continuously and also have their expiration dates.

On the other hand, to take the decision about procuring of items, inventory management is generally influenced by pricing of that item. Again, in selecting an item for use, the selling price of that item is one of the decisive factors to the customers. It is well known that the higher selling price of item decreases the demand rate of that item where the lesser price has the reverse effect. Hence, the demand rate of an item is dependent on the selling price of that item. Incorporating this effect, we investigate the dependency on pricing for deteriorating items with their expiration dates.

In this study, we propose an EOQ model for deteriorating items with expiration dates was developed in a supply chain with up-stream full trade credit and down-stream partial trade credit financing. As demand for products can evidently be affected by sale price, we assume that the demand rate is linked to selling price. Under these conditions, we model the retailer's inventory system as profit maximization problem to determine the retailer's optimal replenishment and pricing policy. We then construct and prove several theoretical results to characterize the optimal solution. An easy-to-use algorithm is designed to determine the optimal pricing and inventory policies for the proposed model. Finally, numerical examples are presented to illustrate the proposed model and the effect of key parameters on optimal solution is studied.

2 Literature Review

In practice, the seller usually provides to her/his buyer a permissible delay in payments to stimulate sales and reduce inventory. During the credit period, the buyer can accumulate the revenue and earn interest on the accumulative revenue. However, if the buyer cannot pay off the purchase amount during the credit period then the seller charges to the buyer interest on the unpaid balance. Due to significant practical relevance of trade credit policy, numerous inventory models under the condition of delay in payments have been discussed. The models of Goyal [16], Aggarwal and Jaggi [1], Shinn [38], Liao et al. [28], Teng [42], Chung and Huang [12], Ouyang et al. [35], Mahata and Goswami [29], Jaber [23], Sana and Chaudhuri [36], Chang et al. [6], Balkhi [3], Teng et al. [47], Min et al. [34], Zhou and Zhou [54] are worth mentioning in this direction.

Due to instability of financial market it is crucial for enterprises to receive trade credit from their suppliers in order to pre-finance production, but it is also important to extend trade credit in order to sell goods to their constrained customers. This succession of new credit relationship is usually referred to by the generic term "two-level trade credit". Two-level of trade credit refers that the supplier provides the permissible delay period M (i.e. the supplier trade credit), to the retailer, and the retailer in turn offers the trade credit period N, to its customers (i.e. the retailer trade credit). Huang [18] and Biskup et al. [4] have studied an inventory model under two-level trade credit policy assuming that the credit period offered by the retailer is shorter than the credit period offered by the supplier. Huang [19, 20] extended Huang [18] by incorporating the limited storage space and finite replenishment rate, respectively. Later, Teng and Goyal [46] proposed a generalized formulation of Huang, 2003, 2006 Huang's models [18, 19] and Teng and Chang [44] modified Huang [20] wherein the assumption that the trade credit offered by the supplier is longer than trade credit offered by the retailer is relaxed.

In daily life, the deteriorating of goods is a frequent and common phenomenon. Incorporating this feature in model formulation, Chung and Huang [11] amended Huang [18] by developing two-warehouse inventory model for deteriorating items under trade credit financing. Min et al. [33] formulated an inventory model for deteriorating items under stock-dependent demand and two-level trade credit to study the retailer's optimal ordering policy. Soni [39] extended the work of Min et al. [33] by incorporating a constraint on the maximum inventory level. Again, it is usually observed that customers pay reasonable prices of commodity on the basis of its quality and longevity. Hence, pricing strategy becomes one of the most important aspect for business organizations to sell deteriorating inventory and enhance revenues. In this context, Thangam and Uthayakumar [48] presented two-echelon trade credit financing model for perishable items to derive optimal credit period, selling price and replenishment time with price and credit linked demand, wherein the authors extended the work of Jaggi et al. [24] by relaxing the assumption that the retailer's trade credit period (M) is not necessarily longer than the customer's trade credit period (N). Dye and Ouyang [14] established EOQ model for deteriorating items to determine optimal selling price, replenishment number and replenishment schedule with time and price demand under two levels of trade credit policy. Other interesting articles can be found in Mahata and Goswami [30], Mahata and Mahata [31], Tsao [49], Chang et al. [6], Kreng and Tan [26, 27], Ho [17], Chung [13], and their references. All the above mentioned papers did not consider the fact that deteriorating items have their expiration dates. In fact, the study of deteriorating items with expiration dates has received a relatively little attention in the literature. Currently, Bakker et al. [2] provided an excellent review of inventory systems with deterioration since 2001.

According to Teng et al. [45], the strategy of granting credit terms adds an additional dimension of default risk to the supplier and/or the retailer. In commercial practice, to reduce non-payment risks, a retailer frequently offers a partial trade credit to its credit risk customer who must make partial payment of the purchase amount at the time of placing an order and settle the unpaid balance at the end of credit period. Huang and Hsu [22] developed an economic order quantity (EOQ) model in which retailer gets full trade credit but offers partial trade credit to the customer. Teng [43] established optimal ordering policies for a retailer who offers distinct (i.e. full or partial trade credit) trade credits to its good and bad customers. Jaggi and Verma [25] formulated an EOQ model under partial trade credit financing for a two level of supply chain i.e. the retailer as well as the customer must pay a portion of the purchase amount at the time of placing an order. Soni and Patel [40] developed an integrated inventory system involving variable production and defective items under retailer partial trade credit policy to establish best policy for retail price, the replenishment cycle and the number of shipment from the supplier to the retailer. Mahata [32] developed an inventory model within economic production quantity (EPQ) framework for exponentially deteriorating items under retailer partial trade credit policy. Feng et al. [15] developed the retailer's inventory system within the EPQ framework to determine the retailer's optimal inventory cycle time and optimal payment time under cash discount and partial trade credit. Chen et al. [9] proposed an economic production quantity (EPQ) model for deteriorating items in a supply chain with both up-stream and down-stream trade credit financing. Wu and Chan [50] established optimal lot-sizing policies for a retailer who sells a deteriorating item to credit-risk customers by offering partial trade credit to reduce his/her risk. Chen and Teng [8] proposed an EOQ model for a retailer to obtain its optimal ordering policy when his/her product not only deteriorates continuously but also has a maximum lifetime and his/her supplier offers a permissible delay in payments. Wu et al. [51] built an EOQ model for the retailer to obtain its optimal credit period and cycle time in a supplier-retailer-buyer supply chain in which the retailer receives an up-stream trade credit from the supplier while offers a down-stream trade credit to the buyer with deteriorating items not only deteriorate continuously but also have their expiration dates and down-stream credit period increases not only demand but also opportunity cost and default risk. Recently, Seifert et al. [37] presented an excellent review of trade credit financing. Some relevantly recent articles in trade credit financing were developed by Chern et al. [10], Taleizadeh [41], and Yang et al. [53].

3 Notations and Assumptions

The following notations and assumptions are used throughout.

3.1 Notations

A The ordering cost per order in dollars.

h	The inventory holding cost per dollar per unit per year excluding interest charges.
c	The unit purchasing cost in dollars.
s	The unit selling price in dollars with $(s > c)$.
M	The retailer's trade credit period offered by the supplier in years.
N	The customer's trade credit period offered by the retailer in years.
α	The fraction of the purchase cost in which the customer must pay the retailer at the time of placing an order with $0 \leq \alpha \leq 1$.
$1-\alpha$	The portion of the purchase cost for which the retailer offers its customer a permissible delay of N periods.
I_e	The interest earned per dollar per year.
I_c	The interest charged per dollar in stocks per year.
t	The time in years.
$I(t)$	Inventory level in units at time t.
$\theta(t)$	The time-varying deterioration rate at time t, where $0 \leq \theta(t) < 1$.
m	The expiration date or maximum lifetime in years of the deteriorating item.
T	Replenishment cycle time in years (a decision variable).
Q	The order quantity.
$TP(s,T)$	The annual total profit in dollars of inventory system, which is a function of s and T.

3.2 Assumptions

Next, the following assumptions are made to establish the mathematical inventory model.

1. The market demand for the item is assumed to be sensitive to the customer's retail prices and is defined as $D(s) = as^{-b}$, which is a decreasing function of the retail price s; where $a(> 0)$ is a scaling factor and $b(> 1)$ is a price elasticity coefficient. For notational simplicity, $D(s)$ and D will be used interchangeably in this paper.
2. All deteriorating items have their expiration dates. The physical significance of the deterioration rate is the rate to be closed to 1 when time is approaching to the maximum lifetime m. The items deteriorates at a rate $\theta(t)$ which depends on time as follow:

$$\theta(t) = \frac{1}{1+m-t}, \quad 0 \leq t \leq T \leq m. \tag{1}$$

   Note that it is clear from Eq. 1 that the replenishment cycle time T must be less than or equal to m, and the proposed deterioration rate is a general case for non-deteriorating items, in which $m \to \infty$ and $\theta(t) \to 0$.
3. During the credit period offered by the supplier, the retailer uses the sales revenue to earn interest at a rate I_e. At the end of the permissible delay period, the retailer pays the purchasing cost to the supplier and pays interest charges at a rate of I_c for the items in stock or the items already sold but have not been paid for yet.
4. Replenishment rate is instantaneous.
5. In today's time-based competition, we may assume that shortages are not allowed to occur.
6. Time horizon is infinite.

4 Model Formulation

The retailer receives Q units at $t = 0$. Hence, the inventory starts with Q units at $t = 0$, and then gradually depletes to zero at $t = T$ due to the combination effect of demand and deterioration. Hence, the inventory level is governed by the following differential equation:

$$\frac{dI(t)}{dt} + \theta(t)I(t) = -D; \quad 0 \le t \le T, \tag{2}$$

with the boundary condition $I(T) = 0$. Solving the differential equation (Eq. 2), we obtain obtain the inventory level at time t as

$$I(t) = D(1+m-t)\ln\left(\frac{1+m-t}{1+m-T}\right), \; 0 \le t \le T. \tag{3}$$

As a result, the retailer's order quantity is

$$Q = I(0) = D(1+m)\ln\left(\frac{1+m}{1+m-T}\right). \tag{4}$$

The annual total relevant cost consists of the following elements:

1. Annual ordering cost is $\frac{A}{T}$.
2. Annual purchase cost per cycle is $\frac{c}{T}I(0) = \frac{cD(1+m)}{T}\ln\left(\frac{1+m}{1+m-T}\right)$
3. Annual stock holding cost (excluding interest charges)

$$= \frac{h}{T}\int_0^T I(t)dt$$
$$= \frac{hD}{T}\left[\frac{(1+m)^2}{2}\ln\left(\frac{1+m}{1+m-T}\right) + \frac{T^2}{4} - \frac{(1+m)T}{2}\right].$$

4. Based on the values of M (i.e., the time at which the retailer must pay the supplier to avoid interest charge), T (i.e., the replenishment cycle time), and $T + N$ (i.e., the time at which the retailer receives the payment from the last customer), we have to examine following three situations: (1) $0 < T + N \le M$, (2) $T \le M \le T + N$, and (3) $M \le T$. Note that different approaches are available in existing literature to calculate the interest earned and interest charged. In this paper, we have employed Teng [43] approach throughout in this article.

Situation 1 $0 < T + N \le M$ (i.e., $0 < T < M - N$, see Fig. 1)

In this case, the retailer receives all returns from the customers before paying the purchase amount to the supplier. Consequently, the retailer does not incur interest charges in this case. On the other hand, interest earned per cycle is

$$\frac{I_e}{T}\left[\left(\int_0^T \alpha sDtdt + \alpha sDT(M-T)\right)\right.$$
$$\left.+\left(\int_N^{T+N}(1-\alpha)sD(t-N)dt + (1-\alpha)sDT(M-T-N)\right)\right]$$
$$= \frac{sI_eD}{2}\left[T + 2\alpha(M-T) + 2(1-\alpha)(M-T-N)\right].$$

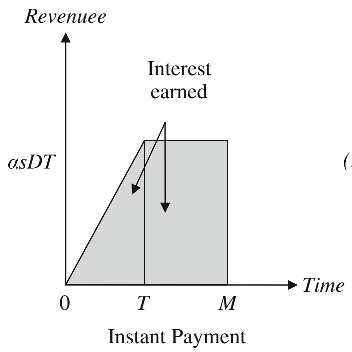

Fig. 1 $N \leq M$ and $T \leq T + N \leq M$

Situation 2 $T \leq M \leq T + N$ (i.e., $M - N \leq T \leq M$, see Fig. 2)

By the time M, the retailer has two sources to accumulate revenue in an account that earns I_e per dollar per year: (1) from the portion of partial payment (starting 0 through M) and (2) from the portion of delayed payment (starting N through M). Therefore, the interest earned per cycle is

$$\frac{I_e}{T}\left[\int_0^T \alpha s D t dt + \alpha s D T (M - T) + \int_N^M (1-\alpha) s D (t - N) dt\right]$$
$$= \frac{s I_e D}{2T}\left[\alpha T^2 + 2\alpha T (M - T) + (1-\alpha)(M - N)^2\right].$$

Since $M \leq T + N$, the retailer has to finance all the items sold after $M - N$ at an interest charged I_c per dollar per year. Consequently, interest charged per cycle is

$$\frac{cI_c}{sT}\left[\int_M^{T+N} (1-\alpha) s D (t - M) dt\right] = \frac{(1-\alpha) c I_c D}{2T}(T + N - M)^2.$$

Situation 3 $M \leq T$ (see Fig. 3)

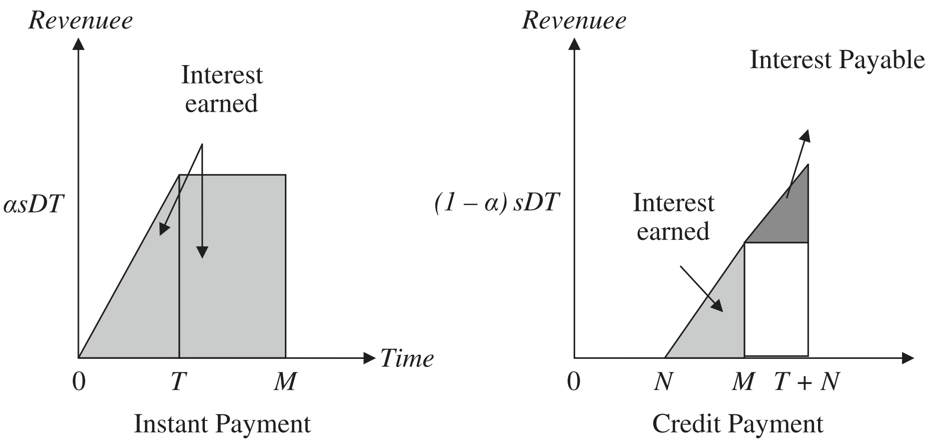

Fig. 2 $N \leq M$ and $T \leq M \leq T + N$

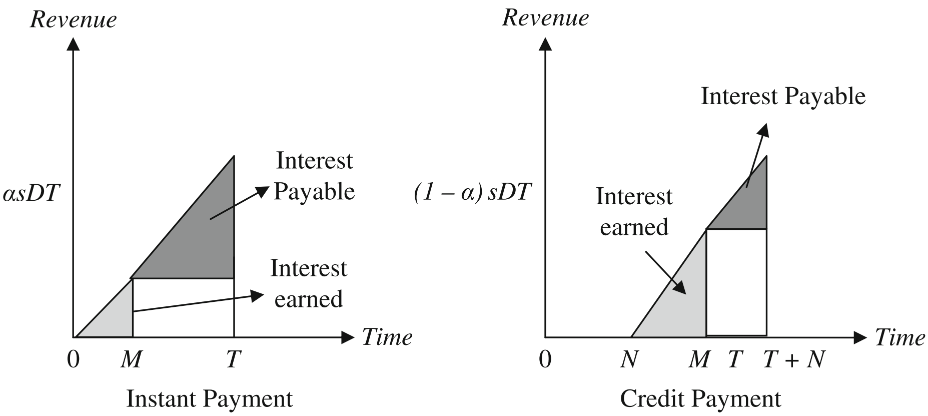

Fig. 3 $N \leq M$ and $M \leq T$

Again, the retailer can accumulate revenue from two sources: (1) from the portion of partial payment (starting 0 through M) and (2) from the portion of delayed payment (starting N through M). Therefore, the interest earned per cycle is

$$\frac{I_e}{T}\left[\int_0^M \alpha s D t dt + \int_N^M (1-\alpha)sD(t-N)dt\right] = \frac{sI_eD}{2T}\left[\alpha M^2 + (1-\alpha)(M-N)^2\right].$$

As $M \leq T$, the retailer needs to finance (1) all items sold after M for the portion of instant payment and (2) all items sold after $M-N$ for the portion of credit payment at an interest charged I_c per dollar per year. Hence, the interest charged per cycle is

$$\frac{cI_c}{sT}\left[\int_M^T \alpha s D(1+m-t)\ln\left(\frac{1+m-t}{1+m-T}\right)dt + \int_M^{T+N}(1-\alpha)sD(t-N)dt\right]$$
$$= \frac{cI_cD}{T}\left[\alpha\left\{\frac{(1+m-M)^2}{2}\ln\left(\frac{1+m-M}{1+m-T}\right) + \frac{T^2-M^2}{2} - \frac{(1+m)(T-M)}{2}\right\}\right.$$
$$\left. + \frac{(1-\alpha)}{2}(T+N-M)^2\right].$$

Therefore, the total profit per unit time for the retailer when $N \leq M$ is given by,

$$TP_1(s,T) = \begin{cases} TP_{11}(s,T), & \text{if } 0 < T \leq M-N \\ TP_{12}(s,T), & \text{if } M-N < T \leq M \\ TP_{13}(s,T), & \text{if } T \geq M \end{cases} \tag{5}$$

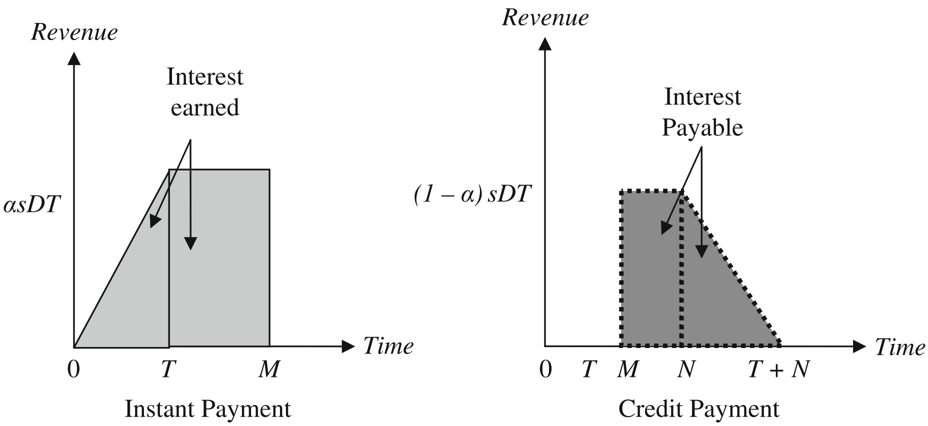

Fig. 4 $N \geq M$ and $T \leq M$

where

$$TP_{11}(s,T) = sD - \frac{cD(1+m)}{T}\ln\left(\frac{1+m}{1+m-T}\right) - \frac{A}{T} - \frac{hD}{T}\left[\frac{(1+m)^2}{2}\ln\left(\frac{1+m}{1+m-T}\right) + \frac{T^2}{4} - \frac{(1+m)T}{2}\right] + \frac{sDI_e}{2}\left[T + 2\alpha(M-T) + 2(1-\alpha)(M-T-N)\right], \quad (6)$$

$$TP_{12}(s,T) = sD - \frac{cD(1+m)}{T}\ln\left(\frac{1+m}{1+m-T}\right) - \frac{A}{T} - \frac{hD}{T}\left[\frac{(1+m)^2}{2}\ln\left(\frac{1+m}{1+m-T}\right) + \frac{T^2}{4} - \frac{(1+m)T}{2}\right] + \frac{sDI_e}{2T}\left[\alpha T^2 + 2\alpha T(M-T) + (1-\alpha)(M-N)^2\right] - \frac{(1-\alpha)cI_cD}{2T}(T+N-M)^2, \quad (7)$$

$$TP_{13}(s,T) = sD - \frac{cD(1+m)}{T}\ln\left(\frac{1+m}{1+m-T}\right) - \frac{A}{T} - \frac{hD}{T}\left[\frac{(1+m)^2}{2}\ln\left(\frac{1+m}{1+m-T}\right) + \frac{T^2}{4} - \frac{(1+m)T}{2}\right] + \frac{sI_eD}{2T}\left[\alpha M^2 + (1-\alpha)(M-N)^2\right] - \frac{cI_cD}{T}\left[\alpha\left\{\frac{(1+m-M)^2}{2}\ln\left(\frac{1+m-M}{1+m-T}\right) + \frac{T^2-M^2}{2} - \frac{(1+m)(T-M)}{2}\right\} + \frac{(1-\alpha)}{2}(T+N-M)^2\right]. \quad (8)$$

Case 2 $N \geq M$

Based on the values of M and T, we have to explore following two situations: (a) Situation 1: $T \leq M$ and (b) Situation 2: $T \geq M$.

(a) Situation 1. $T \leq M$ (see Fig. 4)

During $[0, M]$, the retailer accumulates his revenue in interest bearing account at the rate I_e per dollar per year. As a result, the interest earned by the retailer is

$$\frac{I_e}{T}\left[\int_0^T \alpha s D t dt + \alpha s D T(M-T)\right] = \frac{\alpha s I_e D}{2}\left[T + 2(M-T)\right].$$

The retailer must arrange the finance for (1) paying the supplier at the end of trade credit M and (2) the items already sold but not paid for till $T+N$. The resultant interest charged will be

$$\frac{cI_c}{sT}\left[(1-\alpha)sDT(N-M) + (1-\alpha)sD\int_N^{T+N}(T+N-t)dt\right]$$
$$= \frac{(1-\alpha)cI_cD}{2}\left[T + 2(N-M)\right].$$

(b) Situation 2: $T \geq M$ (see Fig. 5)

In this case, the retailer can accumulate interest for portion of the partial payment till M at the rate I_e per dollar per year. Therefore the interest earned per cycle will be

$$\frac{I_e}{T}\int_0^M \alpha s D T dt = \frac{\alpha s I_e D M^2}{2T}.$$

The retailer must finance for (1) the items sold after M, (2) the entire amount of the delayed payment at the end of the trade credit M, and (3) the items already sold but not yet paid for till $T+N$. So, the interest charged per cycle is

$$\frac{cI_c}{sT}\left[\int_M^T \alpha s D(1+m-t)\ln\left(\frac{1+m-t}{1+m-T}\right)dt\right.$$
$$\left.+(1-\alpha)sDT(N-M) + (1-\alpha)sD\int_N^{T+N}((T+N)-t)dt\right]$$
$$= \frac{cI_cD}{T}\left[\alpha\left\{\frac{(1+m-M)^2}{2}\ln\left(\frac{1+m-M}{1+m-T}\right) + \frac{T^2-M^2}{2} - \frac{(1+m)(T-M)}{2}\right\}\right.$$
$$\left.+(1-\alpha)\left\{T(N-M) + \frac{T^2}{2}\right\}\right].$$

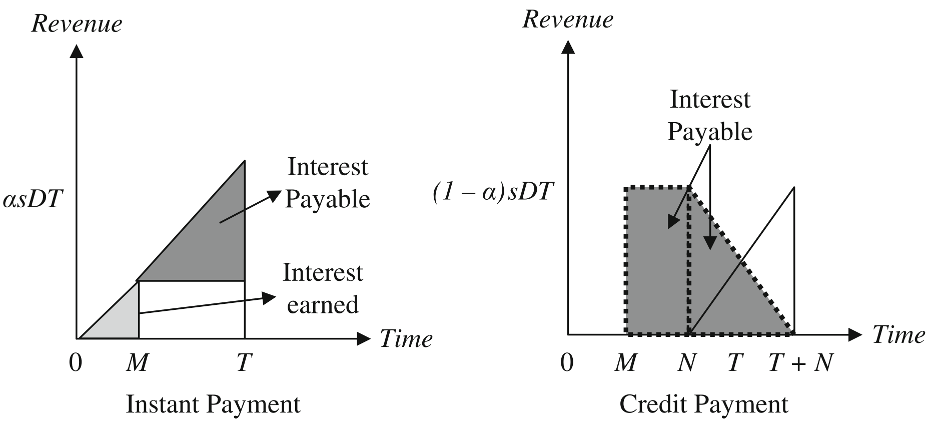

Fig. 5 $N \geq M$ and $T \geq M$

Consequently, the total profit per unit time for the retailer when $N \geq M$ is

$$TP_2(s,T) = \begin{cases} TP_{21}(s,T), & \text{if } T \leq M \\ TP_{22}(s,T), & \text{if } T \geq M \end{cases} \tag{9}$$

where

$$\begin{aligned} TP_{21}(s,T) &= sD - \frac{cD(1+m)}{T}\ln\left(\frac{1+m}{1+m-T}\right) - \frac{A}{T} - \frac{hD}{T}\left[\frac{(1+m)^2}{2}\ln\left(\frac{1+m}{1+m-T}\right)\right. \\ &\quad \left. + \frac{T^2}{4} - \frac{(1+m)T}{2}\right] + \frac{\alpha s I_e D}{2}\left[T + 2(M-T)\right] - \frac{(1-\alpha)cDI_c}{2}\left[T + 2(N-M)\right], \end{aligned} \tag{10}$$

$$\begin{aligned} TP_{22}(s,T) &= sD - \frac{cD(1+m)}{T}\ln\left(\frac{1+m}{1+m-T}\right) - \frac{A}{T} - \frac{hD}{T}\left[\frac{(1+m)^2}{2}\ln\left(\frac{1+m}{1+m-T}\right)\right. \\ &\quad \left. + \frac{T^2}{4} - \frac{(1+m)T}{2}\right] + \frac{\alpha s D I_e M^2}{2T} - \frac{cDI_c}{T}\left[\alpha\left\{\frac{(1+m-M)^2}{2}\ln\left(\frac{1+m-M}{1+m-T}\right)\right.\right. \\ &\quad \left.\left. + \frac{T^2-M^2}{2} - \frac{(1+m)(T-M)}{2}\right\} + (1-\alpha)\left\{T(N-M) + \frac{T^2}{2}\right\}\right]. \end{aligned} \tag{11}$$

Hence our problem is,

$$maximize\ TP(s,T) = \begin{cases} TP_1(s,T), & \text{if } N \leq M \\ TP_2(s,T), & \text{if } N \geq M \end{cases} \tag{12}$$

where $TP_i(s,T)$, for $i = 1, 2$ is defined in Eqs. 5 and 9 respectively.

It is to be noted that, for fixed s, $TP_{11}(s, M-N) = TP_{12}(s, M-N)$, $TP_{12}(s,M) = TP_{13}(s,M)$ and $TP_{21}(s,M) = TP_{22}(s,M)$. Hence, for fixed s, $TP_i(s,T)$ is a continuous function on $T > 0$, for $i = 1, 2$.

5 Theoretical Results and Optimal Solution

In this section, we discuss how to obtain the optimal ordering cycle length T^*, as well as the optimal selling price s^*, in the following two cases.

5.1 Optimal Solution for the Case of $N \leq M$

For fixed s, the first order partial derivative of $TP_{11}(s,T)$ with respect to T is

$$\begin{aligned} \frac{\partial TP_{11}(s,T)}{\partial T} &= \frac{1}{T^2}\left[A + (1+m)D\left\{c + \frac{h(1+m)}{2}\right\}\left\{\ln\left(\frac{1+m}{1+m-T}\right) - \frac{T}{1+m-T}\right\}\right] \\ &\quad - \frac{1}{4}(h + 2sI_e)D. \end{aligned} \tag{13}$$

Motivated by Eq. 13, we assume an auxiliary function, say $F_{11}(T)$, $T \in (0, M-N]$, where

$$\begin{aligned} F_{11}(T) &= A + (1+m)D\left\{c + \frac{h(1+m)}{2}\right\}\left\{\ln\left(\frac{1+m}{1+m-T}\right) - \frac{T}{1+m-T}\right\}\Bigg] \\ &\quad - \frac{1}{4}(h + 2sI_e)DT^2. \end{aligned} \tag{14}$$

Differentiating $F_{11}(T)$ with respect to $T \in (0, M-N]$, we have

$$\frac{dF_{11}(T)}{dT} = -\left[\frac{(1+m)DT}{(1+m-T)^2}\left\{c+\frac{h(1+m)}{2}\right\}+\frac{1}{2}(h+2sI_e)DT\right] < 0. \tag{15}$$

Thus, $F_{11}(T)$ is strictly decreasing function with respect to $T \in (0, M-N]$. Moreover, $\lim_{T\to\infty} F_{11}(T) = -\infty$, $F_{11}(0) = A > 0$ and

$$\begin{aligned} F_{11}(M-N) &= A+(1+m)D\left[c+\frac{h(1+m)}{2}\right]\left[\ln\left(\frac{1+m}{1+m-M+N}\right)-\frac{M-N}{1+m-M+N}\right] \\ &\quad -\frac{1}{4}(h+2sI_e)D(M-N)^2. \end{aligned} \tag{16}$$

If $F_{11}(M-N) \leq 0$ then by Intermediate value Theorem there exists unique value of T (say $T_{11} \in (0, M-N]$) such that $F_{11}(T_{11}) = 0$.

Conversely, if $F_{11}(M-N) > 0$, we have $F_{11}(T) > 0$, for all $T \in (0, M-N]$ which implies $TP_{11}(s, T)$ is strictly increasing function of $T \in (0, M-N]$. Hence, $F_{11}(T)$ has a maximum value at the boundary point $T = M-N$. For convenience, let

$$\begin{aligned} \Delta_1 &= A+(1+m)D\left[c+\frac{h(1+m)}{2}\right]\left[\ln\left(\frac{1+m}{1+m-M+N}\right)-\frac{M-N}{1+m-M+N}\right] \\ &\quad -\frac{1}{4}(h+2sI_e)D(M-N)^2. \end{aligned} \tag{17}$$

Based on above arguments, we obtain the following lemma.

Lemma 1 *Let T_{11}^* denotes the optimal value of $T \in (0, M-N]$. For fixed s, the profit function $TP_{11}(s, T)$ is concave and reaches its global maximum at point $T = T_{11}^*$.*

Proof From above discussion, T_{11}^* which maximizes profit function $TP_{11}(s, T)$ for fixed s, is given by

$$T_{11}^* = \begin{cases} T_{11}, & \text{if } \Delta_1 \leq 0 \\ M-N, & \text{if } \Delta_1 > 0 \end{cases} \tag{18}$$

At point $T = T_{11}^*$

$$\left[\frac{\partial^2 TP_{11}(s,T)}{\partial T^2}\right]_{T=T_{11}^*} = -\frac{1}{T_{11}^*}\left[\frac{(1+m)D}{(1+m-T_{11}^*)^2}\left\{c+\frac{h(1+m)}{2}\right\}+\frac{1}{2}(h+2sI_e)D\right] < 0.$$

Thus, T_{11}^* gives global maximum for the profit function $TP_{11}(s, T)$. This completes the proof. □

On the other hand, for fixed T_{11}^* defined in Eq. 6 consider the first order partial derivative of $TP_{11}(s, T_{11}^*)$ with respect to s which gives

$$\frac{\partial TP_{11}(s, T_{11}^*)}{\partial s} = -\frac{(b-1)a}{s^b}+\frac{b\delta_1}{s^{b+1}}+\frac{b\delta_2}{s^{b+1}}-\frac{(b-1)\delta_3}{s^b}, \tag{19}$$

where

$$\delta_1 = \frac{ca(1+m)}{T_{11}^*} \ln\left(\frac{1+m}{1+m-T_{11}^*}\right),$$

$$\delta_2 = \frac{ha}{T_{11}^*}\left[\frac{(1+m)^2}{2}\ln\left(\frac{1+m}{1+m-T_{11}^*}\right) + \frac{(T_{11}^*)^2}{4} - \frac{(1+m)T_{11}^*}{2}\right]$$

$$\delta_3 = \frac{aI_e}{2}\left[T_{11}^* + 2\alpha(M-T_{11}^*) + 2(1-\alpha)(M-T_{11}^*-N)\right].$$

Equating Eq. 19 with zero and solving for s (denoted by s_{11}^*) we obtain

$$s_{11}^* = \frac{b}{b-1}\left(\frac{\delta_1+\delta_2}{a+\delta_3}\right). \tag{20}$$

Furthermore, at point $s = s_{11}^*$,

$$\left[\frac{\partial^2 TP_{11}(s, T_{11}^*)}{\partial s^2}\right]_{s=s_{11}^*} = \frac{1-b}{(s_{11}^*)^{b+1}}(a+\delta_3) < 0. \tag{21}$$

Thus, s_{11}^* is the global optimal which maximizes profit function $TP_{11}(s, T_{11}^*)$ for fixed T_{11}^*. That is, we have following result.

Lemma 2 *For fixed $T_{11}^* \in (0, M-N]$ the profit per unit time $TP_{11}(s, T_{11}^*)$ has a unique global maximum value at the point $s = s_{11}^*$, which is shown as in* Eq. 20.

Next for fixed s, the first order partial derivative of $TP_{12}(s, T)$ with respect to T is

$$\frac{\partial TP_{12}(s,T)}{\partial T} = \frac{1}{T^2}\left[A + (1+m)D\left\{c + \frac{h(1+m)}{2}\right\}\left\{\ln\left(\frac{1+m}{1+m-T}\right) - \frac{T}{1+m-T}\right\} - \frac{hDT^2}{4}\right.$$
$$\left. - \frac{sDI_e}{2}\left\{\alpha T^2 + (1-\alpha)(M-N)^2\right\} - \frac{(1-\alpha)cDI_c}{2}\left\{T^2 - (M-N)^2\right\}\right]. \tag{22}$$

Motivated by Eq. 22, we assume an auxiliary function, say $F_{12}(T)$, $T \in [M-N, M]$, we have

$$F_{12}(T) = A + (1+m)D\left\{c + \frac{h(1+m)}{2}\right\}\left\{\ln\left(\frac{1+m}{1+m-T}\right) - \frac{T}{1+m-T}\right\} - \frac{hDT^2}{4}$$
$$- \frac{sDI_e}{2}\left\{\alpha T^2 + (1-\alpha)(M-N)^2\right\} - \frac{(1-\alpha)cDI_c}{2}\left\{T^2 - (M-N)^2\right\}. \tag{23}$$

Differentiating $F_{12}(T)$ with respect to $T \in [M-N, M]$, we have

$$\frac{dF_{12}(T)}{dT} = -\left[\frac{(1+m)DT}{(1+m-T)^2}\left\{c + \frac{h(1+m)}{2}\right\} + \frac{1}{2}\left\{h + 2\alpha sI_e + 2(1-\alpha)cI_c\right\}DT\right] < 0. \tag{24}$$

Thus, $F_{12}(T)$ is strictly decreasing function with respect to $T \in [M-N, M]$. Moreover, $\lim_{T\to\infty} F_{12}(T) = -\infty$,

$$F_{12}(M-N) = A + (1+m)D\left[c + \frac{h(1+m)}{2}\right]\left[\ln\left(\frac{1+m}{1+m-M+N}\right) - \frac{M-N}{1+m-M+N}\right]$$
$$-\frac{1}{4}(h+2sI_e)D(M-N)^2. \tag{25}$$

and

$$F_{12}(M) = A + (1+m)D\left[c + \frac{h(1+m)}{2}\right]\left[\ln\left(\frac{1+m}{1+m-M}\right) - \frac{M}{1+m-M}\right] - \frac{1}{4}hDM^2$$
$$-\frac{sDI_e}{2}\left\{\alpha M^2 + (1-\alpha)(M-N)^2\right\} - \frac{(1-\alpha)cDI_c}{2}\left\{M^2 - (N-M)^2\right\}. \quad (26)$$

If

$$(1+m)D\left[c + \frac{h(1+m)}{2}\right]\left[\frac{M-N}{1+m-M+N} - \ln\left(\frac{1+m}{1+m-M+N}\right)\right] + \frac{1}{4}(h+2sI_e)D(M-N)^2$$
$$\leq A \leq (1+m)D\left[c + \frac{h(1+m)}{2}\right]\left[\frac{M}{1+m-M} - \ln\left(\frac{1+m}{1+m-M}\right)\right] + \frac{1}{4}hDM^2$$
$$+\frac{sDI_e}{2}\left\{\alpha M^2 + (1-\alpha)(M-N)^2\right\} + \frac{(1-\alpha)cDI_c}{2}\left\{M^2 - (N-M)^2\right\},$$

then by Intermediate Value Theorem there exists unique value of T (say $T_{12} \in [M-N, M]$) such that $F_{12}(T_{12}) = 0$. Conversely, if

$$A < (1+m)D\left[c + \frac{h(1+m)}{2}\right]\left[\frac{M-N}{1+m-M+N} - \ln\left(\frac{1+m}{1+m-M+N}\right)\right] + \frac{1}{4}(h+2sI_e)D(M-N)^2,$$

we have $F_{12}(T) < 0$, for all $T \in [M-N, M]$ which implies that $TP_{12}(s, T)$ is strictly decreasing function of $T \in [M-N, M]$. Hence, $TP_{12}(s, T)$ has a maximum value at the boundary point $T = M - N$. Again, if

$$A > (1+m)D\left[c + \frac{h(1+m)}{2}\right]\left[\frac{M}{1+m-M} - \ln\left(\frac{1+m}{1+m-M}\right)\right] + \frac{1}{4}hDM^2$$
$$+\frac{sDI_e}{2}\left\{\alpha M^2 + (1-\alpha)(M-N)^2\right\} + \frac{(1-\alpha)cDI_c}{2}\left\{M^2 - (N-M)^2\right\},$$

we have $F_{12}(T) > 0$, for all $T \in [M-N, M]$ which implies $TP_{12}(s, T)$ is strictly increasing function of $T \in [M-N, M]$. Hence, $TP_{12}(s, T)$ has a maximum value at the boundary point $T = M$.

For convenience, let Δ_1 be defined as in Eq. 17, and

$$\Delta_2 = A + (1+m)D\left[c + \frac{h(1+m)}{2}\right]\left[\ln\left(\frac{1+m}{1+m-M}\right) - \frac{M}{1+m-M}\right] - \frac{1}{4}hDM^2$$
$$-\frac{sDI_e}{2}\left\{\alpha M^2 + (1-\alpha)(M-N)^2\right\} - \frac{(1-\alpha)cDI_c}{2}\left\{M^2 - (N-M)^2\right\}. \quad (27)$$

Based on above arguments and fact that $\Delta_1 > \Delta_2$, we obtain the following lemma.

Lemma 3 *Let T_{12}^* denotes the optimal value of $T \in [M-N, M]$. For fixed s, the profit function $TP_{12}(s, T)$ is concave and reaches its global maximum at point $T = T_{12}^*$.*

Proof It follows from above discussion that T_{12}^* which maximizes profit function $TP_{12}(s, T)$ for fixed s, is given by

$$T_{12}^* = \begin{cases} M-N, & \text{if } \Delta_1 < 0 \\ T_{12}, & \text{if } \Delta_2 \leq 0 \leq \Delta_1 \\ M, & \text{if } \Delta_2 > 0 \end{cases} \quad (28)$$

At point $T = T_{12}^*$

$$\left[\frac{\partial^2 TP_{12}(s, T)}{\partial T^2}\right]_{T=T_{12}^*} = -\frac{1}{T_{12}^*}\left[\frac{(1+m)D}{(1+m-T_{12}^*)^2}\left\{c + \frac{h(1+m)}{2}\right\} + \frac{1}{2}\{h + 2\alpha sI_e + 2(1-\alpha)cI_c\}D\right] < 0.$$

Thus, T_{12}^* gives global maximum for the profit function $TP_{12}(s,T)$. This completes the proof. □

On the other hand, for fixed T_{12}^* defined in Eq. 7 consider the first order partial derivative of $TP_{12}(s,T_{12}^*)$ with respect to s which gives

$$\frac{\partial TP_{12}(s,T_{12}^*)}{\partial s} = -\frac{(b-1)a}{s^b} + \frac{b\xi_1}{s^{b+1}} + \frac{b\xi_2}{s^{b+1}} - \frac{(b-1)\xi_3}{s^b} + \frac{b\xi_4}{s^{b+1}}. \tag{29}$$

where

$$\begin{aligned}
\xi_1 &= \frac{ca(1+m)}{T_{12}^*}\ln\left(\frac{1+m}{1+m-T_{12}^*}\right),\\
\xi_2 &= \frac{ha}{T_{12}^*}\left[\frac{(1+m)^2}{2}\ln\left(\frac{1+m}{1+m-T_{12}^*}\right) + \frac{(T_{12}^*)^2}{4} - \frac{(1+m)T_{12}^*}{2}\right]\\
\xi_3 &= \frac{aI_e}{2T_{12}^*}\left[\alpha(T_{12}^*)^2 + 2\alpha T_{12}^*(M-T_{12}^*) + (1-\alpha)(M-N)^2\right].\\
\xi_4 &= \frac{(1-\alpha)caI_c}{2T_{12}^*}(T_{12}^*+N-M)^2
\end{aligned} \tag{30}$$

Equating Eq. 29 with zero and solving for s (denoted by s_{12}^*) we obtain

$$s_{12}^* = \frac{b}{b-1}\left(\frac{\xi_1+\xi_2+\xi_4}{a+\xi_3}\right). \tag{31}$$

Furthermore, at point $s = s_{12}^*$,

$$\left[\frac{\partial^2 TP_{12}(s,T_{12}^*)}{\partial s^2}\right]_{s=s_{12}^*} = \frac{1-b}{(s_{12}^*)^{b+1}}(a+G_1) < 0. \tag{32}$$

Thus, s_{12}^* is the global optimal which maximizes profit function $TP_{12}(s,T_{12}^*)$ for fixed T_{12}^*. That is, we have following result.

Lemma 4 *For fixed $T_{12}^* \in [M-N, M]$ the profit per unit time $TP_{12}(s,T_{12}^*)$ has a unique global maximum value at the point $s = s_{12}^*$, which is shown as in* Eq. 31.

Likewise, for fixed s, the first order partial derivative of $TP_{13}(s,T)$ with respect to T is

$$\begin{aligned}
\frac{\partial TP_{13}(s,T)}{\partial T} &= \frac{1}{T^2}\left[A + (1+m)D\left\{c + \frac{h(1+m)}{2}\right\}\left\{\ln\left(\frac{1+m}{1+m-T}\right) - \frac{T}{1+m-T}\right\} - \frac{hDT^2}{4}\right.\\
&\quad -\frac{sDI_e}{2}\left\{\alpha M^2 + (1-\alpha)(M-N)^2\right\} - cDI_c\left[\alpha\left\{\frac{(1+m-M)^2}{2}\left(\frac{T}{1+m-T}\right.\right.\right.\\
&\quad \left.\left.\left.\left. -\ln\left(\frac{1+m-M}{1+m-T}\right)\right) + \frac{T^2+M^2}{2} - \frac{(1+m)M}{2}\right\} + \frac{(1-\alpha)}{2}\left\{T^2-(M-N)^2\right\}\right]\right].
\end{aligned} \tag{33}$$

Using the arguments similar to those above, there exists unique value of T (say $T_{13} \in [M,\infty)$) such that $\frac{\partial TP_{13}(s,T)}{\partial T} = 0$. Hence, we can easily obtain the following lemma.

Lemma 5 *Let T_{13}^* denotes the optimal value of $T \in [M,\infty)$. For fixed s, the profit function $TP_{13}(s,T)$ is concave and reaches its global maximum at point $T = T_{13}^*$.*

Proof The optimal value T_{13}^* which maximizes profit function $TP_{13}(s, T)$ for fixed s, is given by

$$T_{13}^* = \begin{cases} T_{13}, & \text{if} \quad \Delta_2 \geq 0 \\ M, & \text{if} \quad \Delta_2 < 0 \end{cases} \tag{34}$$

At point $T = T_{13}^*$

$$\left[\frac{\partial^2 TP_{13}(s, T)}{\partial T^2}\right]_{T=T_{13}^*} = -\frac{1}{T_{13}^*}\left[\frac{(1+m)D}{(1+m-T_{13}^*)^2}\left\{c + \frac{h(1+m)}{2}\right.\right.$$
$$\left.\left. + \frac{hD}{2} + \alpha c D I_c \left\{\frac{(1+m-M)^2}{2(1+m-T)^2} + 1\right\}\right\}\right] < 0.$$

Thus, T_{13}^* gives global maximum for the profit function $TP_{13}(s, T)$. This completes the proof. □

On the other hand, for fixed T_{13}^* defined in (34) the first order partial derivative of $TP_{13}(s, T_{13}^*)$ with respect to s is

$$\frac{\partial TP_{13}(s, T_{13}^*)}{\partial s} = -\frac{(b-1)a}{s^b} + \frac{b\zeta_1}{s^{b+1}} + \frac{b\zeta_2}{s^{b+1}} - \frac{(b-1)\zeta_3}{s^b} + \frac{b\zeta_4}{s^{b+1}}. \tag{35}$$

where

$$\zeta_1 = \frac{ca(1+m)}{T_{13}^*} \ln\left(\frac{1+m}{1+m-T_{13}^*}\right),$$

$$\zeta_2 = \frac{ha}{T_{13}^*}\left[\frac{(1+m)^2}{2}\ln\left(\frac{1+m}{1+m-T_{13}^*}\right) + \frac{(T_{13}^*)^2}{4} - \frac{(1+m)T_{13}^*}{2}\right]$$

$$\zeta_3 = \frac{aI_e}{2T_{13}^*}\left[\alpha M^2 + (1-\alpha)(M-N)^2\right]$$

$$\zeta_4 = \frac{caI_c}{T_{13}^*}\left[\alpha\left\{\frac{(1+m-M)^2}{2}\ln\left(\frac{1+m-M}{1+m-T_{13}^*}\right) + \frac{(T_{13}^*)^2 - M^2}{2} - \frac{(1+m)(T_{13}^* - M)}{2}\right\}\right.$$
$$\left. + \frac{(1-\alpha)}{2}(T_{13}^* + N - M)^2\right].$$

Equating Eq. 35 with zero and solving for s (denoted by s_{13}^*) we obtain

$$s_{13}^* = \frac{b}{b-1}\left(\frac{\zeta_1 + \zeta_2 + \zeta_4}{a + \zeta_3}\right). \tag{36}$$

Furthermore, at point $s = s_{13}^*$,

$$\left[\frac{\partial^2 TP_{12}(s, T_{13}^*)}{\partial s^2}\right]_{s=s_{13}^*} = \frac{1-b}{(s_{13}^*)^{b+1}}(a + \zeta_3) < 0. \tag{37}$$

Thus, s_{13}^* is the global optimal which maximizes profit function $TP_{13}(s, T_{12}^*)$ for fixed T_{13}^*. That is, we have following result.

Lemma 6 *For fixed $T_{13}^* \in [M, \infty)$ the profit per unit time $TP_{13}(s, T_{12}^*)$ has a unique global maximum value at the point $s = s_{13}^*$, which is shown as in* Eq. 36.

Combining the above Lemmas 1, 3 and 5, we obtain the following result.

Theorem 1 *For any s,*

(a) If $\Delta_1 \leq 0$, the retailer's optimal replenishment cycle length is $T = T_{11}$.
(b) If $\Delta_2 \leq 0 \leq \Delta_1$, the retailer's optimal replenishment cycle length is $T = T_{12}$.
(c) (a) If $\Delta_2 \geq 0$, the retailer's optimal replenishment cycle length is $T = T_{13}$.

Proof It immediately follows from the facts that $TP_{11}(s, M - N) = TP_{12}(s, M - N)$, $TP_{12}(s, M) = TP_{13}(s, M)$, Lemma 1, 3 and 5. □

5.2 Optimal Solution for the Case of $N \geq M$

For fixed s, the first order partial derivative of $TP_{21}(s, T)$ with respect to T is

$$\frac{\partial TP_{21}(s,T)}{\partial T} = \frac{1}{T^2}\left[A + (1+m)D\left\{c + \frac{h(1+m)}{2}\right\}\left\{\ln\left(\frac{1+m}{1+m-T} - \frac{T}{1+m-T}\right\}\right.\right.$$
$$\left. -\frac{1}{4}\left\{h + 2\alpha s I_e + 2(1-\alpha)cI_c\right\}DT^2\right]. \tag{38}$$

Using the similar arguments as in Case 1, there exists unique value of T (say $T_{21} \in (0, M]$ such that $\frac{\partial TP_{21}(s,T)}{\partial T} = 0$. Hence, we can easily obtain the following lemma.

For convenience let

$$\Delta_3 = A + (1+m)D\left\{c + \frac{h(1+m)}{2}\right\}\left\{\ln\left(\frac{1+m}{1+m-M}\right) - \frac{M}{1+m-M}\right\}$$
$$-\frac{1}{4}\left\{h + 2\alpha s I_e + 2(1-\alpha)cI_c\right\}DM^2. \tag{39}$$

Lemma 7 *Let T_{21}^* denotes the optimal value of $T \in (0, M]$. For fixed s, the profit function $TP_{21}(s, T)$ is concave and reaches its global maximum at point $T = T_{21}^*$.*

Proof The optimal value T_{21}^* which maximizes profit function $TP_{21}(s, T)$ for fixed s, is given by

$$T_{21}^* = \begin{cases} T_{21}, & \text{if} \quad \Delta_3 \leq 0 \\ M, & \text{if} \quad \Delta_3 > 0 \end{cases} \tag{40}$$

At point $T = T_{21}^*$

$$\left[\frac{\partial^2 TP_{21}(s,T)}{\partial T^2}\right]_{T=T_{21}^*} = -\frac{1}{T_{21}^*}\left[\frac{(1+m)D}{(1+m-T_{21}^*)^2}\left\{c + \frac{h(1+m)}{2}\right\}\right.$$
$$\left. + \frac{1}{2}\left\{h + 2\alpha s I_e + 2(1-\alpha)cI_c\right\}\right] < 0.$$

Thus, T_{21}^* gives global maximum for the profit function $TP_{21}(s, T)$. This completes the proof. □

On the other hand, for fixed T_{21}^* defined in Eq. 40 the first order partial derivative of $TP_{21}(s, T_{21}^*)$ with respect to s is

$$\frac{\partial TP_{21}(s, T_{21}^*)}{\partial s} = -\frac{(b-1)a}{s^b} + \frac{b\eta_1}{s^{b+1}} + \frac{b\eta_2}{s^{b+1}} - \frac{(b-1)\eta_3}{s^b} + \frac{b\eta_4}{s^{b+1}} \tag{41}$$

where

$$\begin{aligned}
\eta_1 &= \frac{ca(1+m)}{T_{21}^*}\ln\left(\frac{1+m}{1+m-T_{21}^*}\right),\\
\eta_2 &= \frac{ha}{T_{21}^*}\left[\frac{(1+m)^2}{2}\ln\left(\frac{1+m}{1+m-T_{21}^*}\right) + \frac{(T_{21}^*)^2}{4} - \frac{(1+m)T_{21}^*}{2}\right]\\
\eta_3 &= \frac{\alpha a I_e}{2}\left[T_{21}^* + 2(M - T_{21}^*)\right]\\
\eta_4 &= \frac{(1-\alpha)caI_c}{2}\left[T_{21}^* + 2(N-M)\right].
\end{aligned}$$

Equating Eq. 41 with zero and solving for s (denoted by s_{21}^*) we obtain

$$s_{21}^* = \frac{b}{b-1}\left(\frac{\eta_1+\eta_2+\eta_4}{a+\eta_3}\right). \tag{42}$$

Furthermore, at point $s = s_{21}^*$,

$$\left[\frac{\partial^2 TP_{21}(s, T_{21}^*)}{\partial s^2}\right]_{s=s_{21}^*} = \frac{1-b}{(s_{21}^*)^{b+1}}(a+\eta_3) < 0. \tag{43}$$

Thus, s_{21}^* is the global optimal which maximizes profit function $TP_{21}(s, T_{11}^*)$ for fixed T_{21}^*. That is, we have following result.

Lemma 8 *For fixed $T_{21}^* \in (0, M]$ the profit per unit time $TP_{21}(s, T_{21}^*)$ has a unique global maximum value at the point $s = s_{21}^*$, which is shown as in* Eq. 42.

Analogously, for fixed s, the first order partial derivative of $TP_{22}(s, T)$ with respect to T is

$$\begin{aligned}
\frac{\partial TP_{22}(s,T)}{\partial T} &= \frac{1}{T^2}\Bigg[A + (1+m)D\left\{c + \frac{h(1+m)}{2}\right\}\left\{\ln\left(\frac{1+m}{1+m-T}\right) - \frac{T}{1+m-T}\right\} - \frac{hDT^2}{4}\\
&\quad -\frac{\alpha s D I_e M^2}{2} - cDI_c\Bigg[\alpha\Bigg\{\frac{(1+m-M)^2}{2}\left(\frac{T}{1+m-T} - \ln\left(\frac{1+m-M}{1+m-T}\right)\right)\\
&\quad +\frac{T^2+M^2}{2} - \frac{(1+m)M}{2}\Bigg\} + (1-\alpha)\frac{T^2}{2}\Bigg]\Bigg].
\end{aligned} \tag{44}$$

Using the similar arguments as in Case 1, there exists unique value of T (say $T_{22}^* \in [M, \infty)$) such that $\frac{\partial TP_{22}(s,T)}{\partial T} = 0$. Hence, we can easily obtain the following lemma.

Lemma 9 *Let T_{22}^* denotes the optimal value of $T \in [M, \infty)$. For fixed s, the profit function $TP_{22}(s, T)$ is concave and reaches its global maximum at point $T = T_{22}^*$.*

Proof The optimal value T_{22}^* which maximizes profit function $TP_{22}(s, T)$ for fixed s, is given by

$$T_{22}^* = \begin{cases} T_{22}, & \text{if} \quad \Delta_3 \geq 0 \\ M, & \text{if} \quad \Delta_3 < 0 \end{cases} \tag{45}$$

At point $T = T_{22}^*$

$$\left[\frac{\partial^2 TP_{22}(s, T)}{\partial T^2}\right]_{T=T_{22}^*} = -\frac{1}{T_{22}^*}\left[\frac{(1+m)D}{(1+m-T_{22}^*)^2}\left\{c + \frac{h(1+m)}{2}\right\}\frac{hD}{2} + cDI_c\left\{\frac{\alpha(1+m-M)^2}{2(1+m-T)^2} + 1\right\}\right] < 0.$$

Thus, T_{22}^* gives global maximum for the profit function $TP_{22}(s, T)$. This completes the proof. □

On the other hand, for fixed T_{22}^* defined in Eq. 45 the first order partial derivative of $TP_{22}(s, T_{22}^*)$ with respect to s is

$$\frac{\partial TP_{22}(s, T_{22}^*)}{\partial s} = -\frac{(b-1)a}{s^b} + \frac{b\lambda_1}{s^{b+1}} + \frac{b\lambda_2}{s^{b+1}} - \frac{(b-1)\lambda_3}{s^b} + \frac{b\lambda_4}{s^{b+1}} \tag{46}$$

where

$$\lambda_1 = \frac{ca(1+m)}{T_{22}^*}\ln\left(\frac{1+m}{1+m-T_{22}^*}\right),$$

$$\lambda_2 = \frac{ha}{T_{22}^*}\left[\frac{(1+m)^2}{2}\ln\left(\frac{1+m}{1+m-T_{22}^*}\right) + \frac{(T_{22}^*)^2}{4} - \frac{(1+m)T_{22}^*}{2}\right]$$

$$\lambda_3 = \frac{\alpha a I_e M^2}{2T_{22}^*}$$

$$\lambda_4 = \frac{caI_c}{T_{22}^*}\left[\alpha\left\{\frac{(1+m-M)^2}{2}\ln\left(\frac{1+m-M}{1+m-T_{22}^*}\right) + \frac{(T_{22}^*)^2 - M^2}{2} - \frac{(1+m)(T_{22}^* - M)}{2}\right\} + (1-\alpha)\left\{T_{22}^*(N-M) + \frac{(T_{22}^*)^2}{2}\right\}\right].$$

Equating Eq. 46 with zero and solving for s (denoted by s_{22}^*) we obtain

$$s_{22}^* = \frac{b}{b-1}\left(\frac{\lambda_1 + \lambda_2 + \lambda_4}{a + \lambda_3}\right). \tag{47}$$

Furthermore, at point $s = s_{22}^*$,

$$\left[\frac{\partial^2 TP_{21}(s, T_{22}^*)}{\partial s^2}\right]_{s=s_{22}^*} = \frac{1-b}{(s_{22}^*)^{b+1}}(a + \lambda_3) < 0. \tag{48}$$

Thus, s_{22}^* is the global optimal which maximizes profit function $TP_{21}(s, T_{22}^*)$ for fixed T_{22}^*. That is, we have following result.

Lemma 10 *For fixed $T_{22}^* \in [M, \infty)$ the profit per unit time $TP_{22}(s, T_{22}^*)$ has a unique global maximum value at the point $s = s_{22}^*$, which is shown as in* Eq. 47.

Combining the above Lemmas 7 and 9, we obtain the following result.

Theorem 2 *For any s*

(a) If $\Delta_3 \leq 0$, the retailer's optimal replenishment cycle length is $T = T_{21}$.
(b) If $\Delta_3 \geq 0$, the retailer's optimal replenishment cycle length is $T = T_{22}$.

Proof It immediately follows from the facts that $TP_{21}(s, M) = TP_{22}(s, M)$, Lemmas 7 and 9. □

Based on the concavity behavior of the objective function with respect to the decision variables the following algorithmic procedure was developed to identify global optimal solution for (s, T).

6 Numerical Examples

In this section, we use above algorithm to run several numerical examples in order to illustrate theoretical results as well as to gain some managerial insights.

Example 1 An inventory system with the following data is considered.

$D = as^{-b}$ units/year where $a = 5 \times 10^6$ and $b = 2.3$, $A = \$120$/order, $h = \$/7$/unit/year, $c = \$15/unit$, $I_c = \$0.15/\$$/year, $I_e = \$0.12/\$/$ year, $m = 1$ year.

Applying the procedure of the proposed algorithm for $\alpha = 0$, 0.5 and 0.9, we summarize the computational results for different values of M (in year) by varying N (in year) in Tables 1, 2 and 3 respectively.

Based on the computational results, we can obtain the following managerial insights:

(a) It is observed that, for fixed α and M, when customer credit period N increases, the optimal retail price (s^*) and the optimal cycle time (T^*) increases, whereas the optimal total profit per unit time $TP(s^*, T^*)$ decreases. These results imply that a longer delay payment period provided by the retailer leads to lower demand rate and a higher retail price. It may be interesting to observe that when $\alpha = 0.9$, the optimal replenishment cycle time (T^*) increases as long as $M \leq N$ and then decreases subsequently.
(b) For fixed values of N and α, it can be noted that the optimal retail price (s^*) and the optimal length of replenishment cycle (T^*) decreases, whereas retailer's profit per unit time $TP(s^*, T^*)$ increases with an increase in retailer's credit period M. These results imply that a longer credit period provided by the supplier may ultimately cause the retailer to shorten the replenishment cycle length to take advantage of trade credit frequently.
(c) With an increase in the value of the parameter α, the optimal retail price (s^*) and the optimal length of replenishment cycle (T^*) decreases, whereas retailer's profit per unit time $TP(s^*, T^*)$ increases. These results indicate that if the amount of the part payment of purchase cost is more, then the demand can be stimulated by reducing the retail price and thereby the retailer can raise the profit.

Example 2 In this example, we study the effect of expiration date of the deteriorating item, m. All the parameters are identical to those in Example 1 except $\alpha = 0.5$, $M = 45/365$

Algorithm 1

Step 1: Set $j = 1$, $s^{(j)} = c$.

Step 2: Compare the values of M and N. If $N \leq M$, then go to Step 3 otherwise, go to Step 4.

Step 3: Determine Δ_1 and Δ_2 from Eqs. 17 and 27 respectively. Execute any one of the following cases (3.1), (3.2), (3.3), (3.4), (3.5).

(3.1) If $\Delta_1 < 0$, then determine the value $T_{11}^{(j)}$ by solving Eq. 13. Then, substitute $T_{11}^{(j)}$ into Eq. 20 to obtain the corresponding value $s_{11}^{(j+1)}$. Let $s^{(j+1)} = s_{11}^{(j+1)}$ and $T^{(j)} = T_{11}^{(j)}$, go to Step 5.

(3.2) If $\Delta_2 < 0 < \Delta_1$, then determine the value $T_{12}^{(j)}$ by solving Eq. 22. Then, substitute $T_{12}^{(j)}$ into Eq. 31 to obtain the corresponding value $s_{12}^{(j+1)}$. Let $s^{(j+1)} = s_{12}^{(j+1)}$ and $T^{(j)} = T_{12}^{(j)}$, go to Step 5.

(3.3) If $\Delta_2 > 0$, then determine the value $T_{13}^{(j)}$ by solving Eq. 33. Then, substitute $T_{13}^{(j)}$ into Eq. 36 to obtain the corresponding value $s_{13}^{(j+1)}$. Let $s^{(j+1)} = s_{13}^{(j+1)}$ and $T^{(j)} = T_{13}^{(j)}$, go to Step 5.

(3.4) If $\Delta_1 = 0$, then set $T_{11}^{(j)}$ (or $T_{12}^{(j)}$)$= M - N$ and then, substitute $T_{11}^{(j)}$ (or $T_{12}^{(j)}$) into Eq. 20 or 31 to obtain the corresponding value $s_{11}^{(j+1)}$ (or $s_{12}^{(j+1)}$). Let $s^{(j+1)} = s_{11}^{(j+1)}$ and $T^{(j)} = T_{11}^{(j)}$, go to Step 5.

(3.5) If $\Delta_2 = 0$, then set $T_{12}^{(j)}$ (or $T_{13}^{(j)}$)$= M$ and then, substitute $T_{12}^{(j)}$ (or $T_{13}^{(j)}$) into Eq. 31 or 36 to obtain the corresponding value $s_{12}^{(j+1)}$ (or $s_{13}^{(j+1)}$). Let $s^{(j+1)} = s_{12}^{(j+1)}$ and $T^{(j)} = T_{12}^{(j)}$, go to Step 5.

Step 4: Determine Δ_3 from Eq. 39. Execute any one of the following cases (4.1), (4.2), (4.3).

(4.1) If $\Delta_3 < 0$, then determine the value $T_{21}^{(j)}$ by solving Eq. 38. Then, substitute $T_{21}^{(j)}$ into Eq. 42 to obtain the corresponding value $s_{21}^{(j+1)}$. Let $s^{(j+1)} = s_{21}^{(j+1)}$ and $T^{(j)} = T_{21}^{(j)}$, go to Step 5.

(4.2) If $\Delta_3 > 0$, then determine the value $T_{22}^{(j)}$ by solving Eq. 44. Then, substitute $T_{22}^{(j)}$ into Eq. 47 to obtain the corresponding value $s_{22}^{(j+1)}$. Let $s^{(j+1)} = s_{22}^{(j+1)}$ and $T^{(j)} = T_{22}^{(j)}$, go to Step 5.

(4.3) If $\Delta_3 = 0$, then set $T_{21}^{(j)}$ (or $T_{22}^{(j)}$)$= M$ and then, substitute $T_{21}^{(j)}$ (or $T_{22}^{(j)}$) into Eq. 42 or 47 to obtain the corresponding value $s_{21}^{(j+1)}$ (or $s_{22}^{(j+1)}$). Let $s^{(j+1)} = s_{21}^{(j+1)}$ and $T^{(j)} = T_{21}^{(j)}$, go to Step 5.

Step 5: If the difference between $s^{(j)}$ and $s^{(j+1)}$ is small enough (i.e. $|s^{(j)} - s^{(j+1)}| < \epsilon$), then set $s^* = s^{(j)}$ and $T^* = T^{(j)}$. Thus, (s^*, T^*) is the optimal solution. Otherwise go back to Step 2.

Step 6: Compute corresponding $TP(s, T)$ and Q^* from Eq. 4.

year and $N = 30/365$ year. Computational results are summarized in Table 4 for $m \in \{0.6, 0.8, 1.0, 1.2, 1.4\}$.

Table 1 Computational results for different values of M and N when $\alpha = 0.0$

M	N	s^*	T^*	Q^*	$TP^*(s^*, T^*)$
10/365	10/365	32.7299	0.1046	232.89	27354.12
	15/365	32.8001	0.1048	232.31	27281.21
	30/365	32.9811	0.1056	230.82	27071.79
	45/365	33.1611	0.1063	229.15	26860.12
	60/365	33.3299	0.1071	281.61	26647.41
15/365	10/365	32.6811	0.1043	233.30	27431.07
	15/365	32.7339	0.1046	230.89	27354.11
	30/365	32.9150	0.1053	231.31	27138.23
	45/365	33.0899	0.1061	229.71	26930.31
	60/365	33.2759	0.1068	228.15	26719.98
30/365	10/365	32.5001	0.1022	231.89	27679.13
	15/365	32.5499	0.1031	232.81	27601.12
	30/365	32.7401	0.1046	232.98	27353.31
	45/365	32.8999	0.1053	231.31	27138.11
	60/365	33.1001	0.1061	229.71	26928.21
45/365	10/365	32.3063	0.0985	226.79	27979.98
	15/365	32.3674	0.0999	228.97	27875.43
	30/365	32.5519	0.1031	232.81	27592.14
	45/365	32.7339	0.1046	232.98	27353.31
	60/365	32.9151	0.1053	231.31	27138.23
60/365	10/365	32.1619	0.0980	228.27	28304.49
	15/365	32.2099	0.0982	227.78	28196.19
	30/365	32.3673	0.0999	228.97	27875.39
	45/365	32.5515	0.1031	232.79	27591.89
	60/365	32.7339	0.1046	232.92	27353.28

Based on the computational results we can obtain the following managerial insights: From Table 4, it can be observed that as m increases the optimal length of replenishment cycle (T^*), the retailer's profit per unit time $TP^*(s^*, T^*)$ as well as the optimal order quantity Q^* increases whereas the optimal price s^* decreases. This observation reveals that if the expiration date of the deteriorating item m is longer, then it is worth to increase the length of replenishment cycle T in order to increase the sales and the annual total profit $TP^*(s^*, T^*)$.

Example 3 In this example we shall assess the impact of the systems parameters over the decision variables. For this, let us consider the inventory model with the following parametric values.

$D = as^{-b}$ units/year where $a = 5 \times 10^6$ and $b = 2.3$, $A = \$130$/order, $h = \$7$/unit/year, $c = \$15$/unit, $I_c = 0.15/\$/\text{year}$, $I_e = 0.12/\$/\text{year}$, $m = 1$ year and $\alpha = 0.5$.

For the above input data, the model gives the optimal result as $TP^* = \$28526.19$, $s^* = \$27.1489$, $T^* = 0.0951$ year and $Q^* = 240.84$ units.

Table 2 Computational results for different values of M and N when $\alpha = 0.5$

M	N	s^*	T^*	Q^*	$TP^*(s^*, T^*)$
10/365	10/365	32.6733	0.1041	233.04	27430.02
	15/365	32.7516	0.1040	231.43	27330.89
	30/365	32.8419	0.1050	230.63	27223.12
	45/365	32.9322	0.1048	229.83	27115.79
	60/365	33.0227	0.1052	227.02	27009.51
15/365	10/365	32.6122	0.1036	233.01	27510.13
	15/365	32.6425	0.1038	232.84	27472.11
	30/365	32.7961	0.1035	229.42	27276.71
	45/365	32.8864	0.1039	228.65	27169.12
	60/365	32.9767	0.1043	227.84	27062.41
30/365	10/365	32.4287	0.1010	230.48	27778.11
	15/365	32.4589	0.1015	230.91	27733.11
	30/365	32.5499	0.1022	230.97	27612.36
	45/365	32.4637	0.0847	192.56	27455.21
	60/365	32.5512	0.0847	191.21	27344.71
45/365	10/365	32.2581	0.0983	227.32	28088.12
	15/365	32.2878	0.0990	228.27	28035.51
	30/365	32.3784	0.1005	230.11	27892.21
	45/365	32.4681	0.1012	230.10	27770.53
	60/365	32.5579	0.1015	229.26	27660.24
60/365	10/365	32.1141	0.0978	228.76	28413.02
	15/365	32.1379	0.0979	228.53	28358.68
	30/365	32.2157	0.0987	229.01	28197.41
	45/365	32.3056	0.1002	230.81	28053.12
	60/365	32.3949	0.1009	230.78	27932.27

The sensitivity analysis of the model is carried out by changing the each parameter at a time and keeping other parameters as fixed whose values are defined in above model. Effects of these changes on the optimal solution are examined using the measurement of $\Delta s/s^*\%$, $\Delta T/T^*\%$, $\Delta Q/Q^*\%$, and $\Delta TP/TP^*\%$. The measurement can be explained as follows. For the measure of $\Delta s/s^*\%$, where $\Delta s = s^{**} - s^*$, s^{**} is the optimal value of selling price for the model when one of the parameters increases or decreases by 20% and 40% while all other parameters remain unchanged. Hence, $\Delta s/s^*\%$ denotes the change in selling price and can be used to measure this parameter's sensitivity on the selling price. Analogously, the measures of $\Delta T/T^*\%$, $\Delta Q/Q^*\%$ and $\Delta TP/TP^*\%$ indicate each parameter's sensitivity on cycle time T, ordering quantity Q and average profit TP, respectively. The sensitivity analysis results are presented in Table 5.

Based on the computational results shown in Table 5, the following features can be observed.

(1) The optimal cycle time and the optimal order quantity is more sensitive whereas the optimal selling price and average profit is less sensitive with respect to change of parameters A and h. The reasons for these phenomena are apparent.

Table 3 Computational results for different values of M and N when $\alpha = 0.9$

M	N	s^*	T^*	Q^*	$TP^*(s^*, T^*)$
10/365	10/365	32.6247	0.1038	233.11	27491.72
	15/365	32.7171	0.1034	230.63	27371.12
	30/365	32.7354	0.1035	230.47	27349.15
	45/365	32.7533	0.1036	230.31	27327.52
	60/365	32.7721	0.1037	230.15	27305.86
15/365	10/365	32.5635	0.1031	232.78	27575.29
	15/365	32.5696	0.1032	232.74	27567.79
	30/365	32.7001	0.1001	227.89	27388.91
	45/365	32.7183	0.1021	227.73	27367.18
	60/365	32.7365	0.1022	227.59	27345.46
30/365	10/365	32.3796	0.1001	229.27	27855.89
	15/365	32.3857	0.1002	229.36	27846.86
	30/365	32.4037	0.1004	229.37	27822.39
	45/365	32.2697	0.0847	214.48	27762.69
	60/365	32.2871	0.0846	215.21	27740.23
45/365	10/365	32.2193	0.0982	227.69	28174.59
	15/365	32.2251	0.0983	227.88	28164.11
	30/365	32.2432	0.0986	228.23	28135.12
	45/365	32.2610	0.0988	228.24	28110.43
	60/365	32.2788	0.0988	228.03	28087.91
60/365	10/365	32.0761	0.0977	229.15	28500.05
	15/365	32.0803	0.0978	229.09	28489.14
	30/365	32.0961	0.0979	226.17	28456.73
	45/365	32.1137	0.0982	229.53	28427.61
	60/365	32.1315	0.0983	229.54	28402.72

(2) As the purchasing cost varies, the optimal selling price, the optimal cycle time, the optimal ordering quantity and the average profit change significantly. It may be interesting to observe that the percentage changes in $\Delta s/s^*\%$ and $\Delta T/T^*\%$ are nearly equal to the percentage changes in the purchase cost at various level. It is rational

Table 4 Computational results for m

m	s^*	T^*	Q^*	$TP^*(s^*, T^*)$
0.6	32.3932	0.0992	227.11	27856.71
0.8	32.3783	0.1005	230.09	27892.17
1.0	32.3632	0.1018	233.18	27928.10
1.2	32.3552	0.1028	235.85	27954.38
1.4	32.3401	0.1039	238.07	27982.96

Table 5 Sensitivity analysis

Parameter		% change	$\Delta s/s^*(\%)$	$\Delta T/T^*(\%)$	$\Delta Q/Q^*(\%)$	$\Delta TP/TP^*(\%)$
A	78	−40	−0.75	−23.32	−22.05	2.18
	104	−20	−0.37	−10.98	−10.31	1.03
	130	0	0	0	0	0
	156	+20	0.35	9.97	9.22	−0.92
	182	+40	0.64	19.23	17.62	−1.76
h	4.2	−40	−0.53	14.76	16.21	1.28
	5.6	−20	−0.26	6.59	7.24	0.63
	7	0	0	0	0	0
	8.4	+20	0.24	−5.47	−5.99	−0.59
	9.8	+40	0.47	−10.09	−11.05	−1.12
c	9	−40	−40.26	−40.15	95.37	97.05
	12	−20	−20.17	−19.8	34.52	34.56
	15	0	0	0	0	0
	18	+20	20.24	19.23	−21.89	−21.6
	21	+40	40.55	38.39	−36.64	−36.23
a	3×10^6	1	30.66	−23.27	−41.66	
	4×10^6	−20	0.4	12.35	−10.9	−20.9
	5×10^6	0	0	0	0	0
	6×10^6	+20	−0.29	−9.06	9.83	21
	7×10^6	+40	−0.52	−16.07	18.84	42.08
I_e	0.072	−40	0.33	3.92	3.17	−0.48
	0.096	−20	0.16	1.92	1.55	−0.24
	0.120	0	0	0	0	0
	0.144	+20	−0.16	−1.84	−1.48	0.25
	0.168	+40	−0.32	−3.61	−2.9	0.51
I_c	0.09	−40	0.01	1.68	1.67	0.07
	0.12	−20	0.004	0.82	0.82	0.04
	0.15	0	0	0	0	0
	0.18	+20	−0.004	−0.80	−0.79	−0.04
	0.21	+40	−0.01	−1.57	−1.56	−0.07

that higher purchasing cost of a product decreases the average profit and increase the selling price and order cycle time such that the average profit is maximized.

(3) The variations in parameters I_e and I_c show minor deviation in decision variables and associated average profit. This shows that in estimating I_e and I_c may not result in much more deviation from the optimal results.

(4) The percentage changes in the demand rate parameter (a) causes significant changes in the optimal cycle time, the optimal ordering quantity and the average profit whereas the selling price is comparatively insensitive to changes with increasing percentage changes in the level of fuzziness of scaling factor (a). This indicates that the retailer

should carefully estimate the value of demand rate parameter (a) and then make ordering decisions.

7 Some Special Cases for Non-deteriorating Items

Firstly, if there is no expiration date i.e., if the maximum lifetime of items is approaching to infinity, then our proposed model is a generalized model for non-deteriorating items, in which m is approaching infinity. Using Calculus, L'Hospital's Rule, and simplifying terms, we can simplify the problem for non-deteriorating items as shown below.

$$\lim_{m\to\infty}(1+m)\ln\left(\frac{1+m}{1+m-T}\right) = \lim_{m\to\infty}\frac{\frac{d}{dm}\ln\left(\frac{1+m}{1+m-T}\right)}{\frac{d}{dm}\frac{1}{1+m}} = \lim_{m\to\infty}\frac{\frac{-T}{(1+m)(1+m-T)}}{\frac{-1}{(1+m)^2}} = \lim_{m\to\infty}\frac{T(1+m)}{1+m-T} = T. \tag{49}$$

Consequently, the retailer's order quantity per cycle in Eq. 4 becomes

$$Q = I(0) = D(1+m)\ln\left(\frac{1+m}{1+m-T}\right) = DT \text{ when } m\to\infty. \tag{50}$$

Similarly, we can get the following results:

$$\lim_{m\to\infty}\left[\frac{(1+m)^2}{2}\ln\left(\frac{1+m}{1+m-T}\right) - \frac{(1+m)T}{2}\right] = \frac{1}{2}\lim_{m\to\infty}\left[\frac{\ln\left(\frac{1+m}{1+m-T}\right) - \frac{T}{1+m}}{\frac{1}{(1+m)^2}}\right]$$
$$= \lim_{m\to\infty}\left[\frac{\frac{-T}{(1+m-T)(1+m)} + \frac{1}{(1+m)^2}}{\frac{-2}{(1+m)^2}}\right] = \frac{1}{2}\lim_{m\to\infty}\left[\frac{T^2(1+m)}{2(1+m-T)}\right] = \frac{1}{4}\lim_{m\to\infty}T^2 = \frac{T^2}{4}. \tag{51}$$

As a result, we know that the retailer's holding cost excluding interest charge per cycle is simplified to

$$\lim_{m\to\infty} hD\left[\frac{(1+m)^2}{2}\ln\left(\frac{1+m}{1+m-T}\right) + \frac{T^2}{4} - \frac{(1+m)T}{2}\right] = \frac{hDT^2}{2}. \tag{52}$$

Likewise, we have the following results:

$$
\begin{aligned}
&\lim_{m\to\infty}\left[\frac{(1+m-M)^2}{2}\ln\left(\frac{1+m-M}{1+m-T}\right)-\frac{(1+m)(T-M)}{2}\right]\\
&=\frac{1}{2}\lim_{m\to\infty}\left[\frac{\ln\left(\frac{1+m-M}{1+m-T}\right)-\frac{(1+m)(T-M)}{(1+m-M)^2}}{\frac{1}{(1+m-M)^2}}\right]\\
&=\frac{1}{2}\lim_{m\to\infty}\left[\frac{\frac{-(T-M)}{(1+m-M)(1+m-T)}+\frac{(T-M)(1+m+M)}{(1+m-M)^2}}{\frac{-2}{(1+m-M)^2}}\right]\\
&=\frac{1}{2}\lim_{m\to\infty}\left[\frac{(T-M)-3(1+m)M+M^2+(1+m+M)T}{2(1+m-T)}\right]=\frac{1}{4}(T-M)(T-3M). \qquad (53)
\end{aligned}
$$

Therefore, for non-deteriorating items, the retailer's annual total profit in Eq. 6 is reduced to

$$
TP_{11}(s,T)=(s-c)D-\frac{A}{T}-\frac{hDT}{2}+\frac{sDI_e}{2}\left[T+2\alpha(M-T)+2(1-\alpha)(M-T-N)\right]. \qquad (54)
$$

Similarly, if there is no expiration date, then we get

$$
\begin{aligned}
TP_{12}(s,T)\;=\;&(s-c)D-\frac{A}{T}-\frac{hDT}{2}+\frac{sDI_e}{2T}\left[\alpha T^2+2\alpha T(M-T)+(1-\alpha)(M-N)^2\right]\\
&-\frac{(1-\alpha)cI_cD}{2T}(T+N-M)^2, \qquad (55)\\
TP_{13}(s,T)\;=\;&(s-c)D-\frac{A}{T}-\frac{hDT}{2}+\frac{sI_eD}{2T}\left[\alpha M^2+(1-\alpha)(M-N)^2\right]\\
&-\frac{cI_cD}{T}\left[\alpha\left\{\frac{1}{4}(T-M)(T-3M)+\frac{T^2-M^2}{2}\right\}+\frac{(1-\alpha)}{2}(T+N-M)^2\right]. \qquad (56)
\end{aligned}
$$

This simplified problem with $b = 0$ and $\alpha = 0$ has been solved by Teng and Goyal [46].

In fact, several previous models are indeed special cases of the proposed inventory model here.

- (i) When $m \to \infty$, $\alpha = 0$, $b = 0$, then the proposed model is simplified to that in Teng and Goyal [46].
- (ii) When $m \to \infty$, $\alpha = 0$, $N = 0$, $b = 0$, then the proposed model is similar to that in Teng [42].
- (iii) When $m \to \infty$, $\alpha = 0$, $N = 0$, $s = c$, $b = 0$, then the proposed model is reduced to that in Goyal [16].

8 Conclusion

The use of a down-stream partial trade credit to reduce default risks with credit-risk customers has received a very little attention by the researchers. In this paper, we have formulated an EOQ model to allow for: (1) selling price dependent demand rate, (2) a profit maximization objective, and (3) deteriorating items with maximum life time in a supply chain in which the retailer receives an up-stream full trade credit from his/her supplier while offers a down-stream partial trade credit to his/her credit-risk customers. By analyzing the profit function, we developed theoretical results and algorithm to obtain optimal solutions. Moreover, we have shown that the proposed model is a generalized case for non-deteriorating items and several previous EOQ models. Finally, we provided numerical examples to illustrate the proposed model, and examined the effect of key parameters on the optimal solution. The results in numerical examples suggest that the retailer should encourage the customer to raise the amount of part payment which in turn reduces the selling price and generate more profit to the retailer. The results also indicate that the retailer should grant the credit period to its customer by taking into account the credit period offered from the supplier and initial payment made by the customer to earn more profit.

In practice, the contributions of this paper and the approach we considered to solve the problem are significant because the retailer has to decide whether it is worthwhile to alter the regular ordering pattern to exploit other opportunities and assess their monetary impact to find the optimal ordering policy under realistic conditions linking marketing as well as operations management concerns. Finally, this paper brings attention into the trade credit that is of major importance in the operations of enterprises in many economics. In future research, one can extend this model for two-part trade credit term i.e., supplier offers two payment options: trade credit and early-payments with discount price to the retailer. One can also extend this model for more general supply chain networks, for example, multi-echelon or assembly supply chains.

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