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A new Hybrid Projection Algorithm for Solving the Split Generalized Equilibrium Problems and the System of Variational Inequality Problems

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Abstract In this paper, we introduced modified Mann iterative algorithms by the new hybrid projection method for finding a common element of the set of fixed points of a countable family of nonexpansive mappings, the set of the split generalized equilibrium problem and the set of solutions of the general system of the variational inequality problem for two-inverse strongly monotone mappings in real Hilbert spaces. The strong convergence theorem of the iterative algorithm in Hilbert spaces under certain mild conditions are provided.

Keywords Split generalized equilibrium problem · Nonexpansive mapping · Inverse-strongly monotone mapping · General system of the variational inequality problem

1 Introduction

The *split feasibility problem* (SFP) in Hilbert spaces for modeling inverse problems which arise from phase retrievals and in medical image reconstruction was first introduced by

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Censor and Elfving [11, 12]. It has been found that the SFP can also used to mode the intensity modulate radiation therapy [13, 14].

The SFP is formulated as finding a point $\hat{x}$ with the property

$$\hat{x} \in C, \quad A\hat{x} \in Q, \tag{1}$$

where C and Q are the nonempty closed convex subsets of the infinite-dimensional real Hilbert spaces H_1 and H_2, respectively, and $A \in B(H_1, H_2)$ (i.e., A is a bounded linear operator from H_1 to H_2).

A special case of the SFP is called the *convex constrained linear inverse problem* [15], that is, the problem to finding an element $\hat{x}$ such that

$$\hat{x} \in C, \quad A\hat{x} = b \in Q. \tag{2}$$

In fact, it has been extensively investigated in the literature using the projected Landweber iterative method [15, 16].

Recently, Moudafi [17] introduced the following split equilibrium problem (SEP):

Let $F_1 : C \times C \to \mathbf{R}$ and $F_2 : Q \times Q \to \mathbf{R}$ be nonlinear bifunctions and $A : H_1 \to H_2$ be a bounded linear operator, then the *split equilibrium problem* (SEP) is to find $\hat{x} \in C$ such that

$$F_1(\hat{x}, x) \geq 0, \quad \forall x \in C, \tag{3}$$

and such that

$$\hat{y} = A\hat{x} \in Q \text{ solves } F_2(\hat{y}, y) \geq 0, \quad \forall y \in Q. \tag{4}$$

When looked separately, (3) is the classical equilibrium problem (EP) and we denoted its solution set by $EP(F_1)$. The SEP Equations (3) and (4) constitutes a pair of equilibrium problems which have to be solved so that the image $\hat{y} = A\hat{x}$, under a given bounded linear operator A, of the solution $\hat{x}$ of the EP (3) in H_1 is the solution of another EP (4) by $EP(F_2)$.

The solution set SEP Equations (3) and (4) is denoted by $\Xi = \{p \in EP(F_1) : Ap \in EP(F_2)\}$.

In 2013, Kazmi and Rivi [18] consider a *split generalized equilibrium problem* (SGEP): Find $\hat{x} \in C$ such that

$$F_1(\hat{x}, x) + h_1(\hat{x}, x) \geq 0, \quad \forall x \in C, \tag{5}$$

and such that

$$\hat{y} = A\hat{x} \in Q \text{ solves } F_2(\hat{y}, y) + h_2(\hat{y}, y) \geq 0, \quad \forall y \in Q, \tag{6}$$

where $F_1, h_1 : C \times C \to \mathbf{R}$ and $F_2, h_2 : Q \times Q \to \mathbf{R}$ are nonlinear bifunctions and $A : H_1 \to H_2$ is a bounded linear operator.

They denoted the solution set of generalized equilibrium problem (GEP) Equations (5) and GEP (6) by $GEP(F_1, h_1)$ and $GEP(F_2, h_2)$, respectively. The solution set of SGEP Equations (5)-(6) is denoted by $\Gamma = \{p \in GEP(F_1, h_1) : Ap \in GEP(F_2, h_2)\}$.

If $h_1 = 0$ and $h_2 = 0$, then SGEP Equations (5)-(6) reduces to SEP Equations (3)-(4). If $h_2 = 0$ and $F_2 = 0$, then SGEP Equations (5)-(6) reduces to the equilibrium problem considered by Cianciaruso et al. [3].

Recall, a mapping $S : C \to C$ is said to be *nonexpansive* if

$$\|Sx - Sy\| \leq \|x - y\|, \tag{7}$$

for all $x, y \in C$. We denote the set of fixed point of S by $Fix(S)$. If C is bounded closed convex and S is a nonexpansive mapping of C into itself, then $Fix(S)$ is nonempty (see [19]). We denote weak convergence and strong convergence by notations $\rightharpoonup$ and $\to$, respectively. A mapping A of C into H_1 is called *monotone* if

$$\langle Au - Av, u - v \rangle \geq 0, \quad \forall u, v \in C. \tag{8}$$

A mapping A of C into H_1 is called *α-inverse-strongly monotone mapping* if there exists a positive real number α such that

$$\langle Au - Av, u - v\rangle \geq \alpha\|Au - Av\|^2, \;\; \forall u, v \in C. \tag{9}$$

It is obvious that any α-inverse-strongly monotone mappings A is monotone and Lipschitz continuous.

In 1953, Mann [27] introduced the iteration as follows: a sequence $\{x_n\}$ defined by

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)Sx_n, \tag{10}$$

where the initial guess element $x_0 \in C$ is arbitrary and $\{\alpha_n\}$ is a real sequence in [0, 1]. The Mann iteration has been extensively investigated for nonexpansive mappings. One of the fundamental convergence results is proved by Riech [28]. In an infinite-dimensional Hilbert space, the Mann iteration can conclude only weak convergence [29]. Attempts to modify the Mann iteration method (10) so that strong convergence is guaranteed have recently been made. Generally speaking, the algorithm suggested by Takahashi and Toyoda [20] is based on two well-known types of methods, namely, on the projection-type methods for solving variational inequality problems and so-called hybrid or outer-approximation methods for solving fixed point problems. The idea of "hybrid" or "outer-approximation" types of methods was originally introduced by Haugazeau in 1968 (see [30]).

On the other hand, for finding an element of $Fix(S) \cap VI(C, A)$ under the assumption that a set $C \subset H_1$ is closed and convex, a mapping S of C into itself is nonexpansive and a mapping A of C into H_1 is a α-inverse-strongly monotone mapping. Takahashi and Toyoda [20] introduced the following iterative scheme:

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)SP_C(x_n - \lambda_n Ax_n), \;\; n \geq 0, \tag{11}$$

where $x_0 = x \in C$, $\{\alpha_n\}$ is a sequence in (0,1) and $\{\lambda_n\}$ is a sequence in $(0, 2\alpha)$. They shown that if $Fix(S) \cap VI(C, A) \neq \emptyset$, then the sequence $\{x_n\}$ converges weakly to some $z = P_{Fix(S)\cap VI(C,A)}x_0$.

Let C be a closed convex subset of real Hilbert space H_1. Let $A, B : C \to H_1$ be two mappings. We consider the following problem of finding $(x^*, y^*) \in C \times C$ such that

$$\begin{aligned} \langle \lambda Ay^* + x^* - y^*, x - x^*\rangle &\geq 0, \;\; \forall x \in C, \\ \langle \mu Bx^* + y^* - x^*, x - y^*\rangle &\geq 0, \;\; \forall x \in C, \end{aligned} \tag{12}$$

which is called a *general system of variational inequalities*, where $\lambda \geq 0$ and $\mu \geq 0$ are two constants. This problems has been studied by Kumam [21] and Kumam and Kumam [22]. The solution set of (12) is denoted by $GVI(C, A, B)$. In particular, if $A = B$, then problem (12) reduced to finding $(x^*, y^*) \in C \times C$ such that

$$\begin{aligned} \langle \lambda Ay^* + x^* - y^*, x - x^*\rangle &\geq 0, \;\; \forall x \in C, \\ \langle \mu Ax^* + y^* - x^*, x - y^*\rangle &\geq 0, \;\; \forall x \in C, \end{aligned} \tag{13}$$

which is defined by Verma [23, 24], and is called the *new system of variational inequalities*. Further, if we add up the requirement that $x^* = y^*$, then problem (13) reduces to the classical variational inequality $(VI(C, A))$ which was originally introduced and studied by Stampacchina [25] in 1964.

Very recently, Ceng et al. [26] introduce and study a relaxed extragradient method for finding a common of the solution set of (12) for α and β-inverse-strongly monotone mappings and the set of fixed points of a nonexpansive mapping in a real Hilbert space. Let $x_1 = u \in C$ and $\{x_n\}$ are given by

$$y_n = P_C(x_n - \mu B x_n)$$
$$x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n S P_C(y_n - \lambda A y_n), \quad n \in \mathbf{N}. \tag{14}$$

Then, they proved that the iterative sequence $\{x_n\}$ converges strongly to a solution of the problem (12).

In this paper, we motivated and inspired by above results, we introduce the following modified Mann iterative scheme defined in Theorem 1 for finding a common element of the set of fixed points of a countable family of nonexpansive mappings, the set of split generalized equilibrium problem and the set of solutions of the general system of the variational inequality for β_1, β_2-inverse strongly monotone mappings in Hilbert spaces. Consequently, we prove a strong convergence theorem by modify Mann hybrid iterative algorithm which solves some fixed point problems, a split generalized equilibrium problem and a general system of the variational inequality.

2 Preliminaries

Let H_1 be a real Hilber space. Then

$$\|x - y\|^2 = \|x\|^2 - \|y\|^2 - 2\langle x - y, y\rangle, \tag{15}$$

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y\rangle, \tag{16}$$

and

$$\|\lambda x + (1 - \lambda)y\|^2 = \lambda\|x\|^2 + (1 - \lambda)\|y\|^2 - \lambda(1 - \lambda)\|x - y\|^2, \tag{17}$$

for all $x, y \in H_1$ and $y \in [0, 1]$. It is also known that H_1 satisfies the *Opial's condition* [6], i.e., for any sequence $\{x_n\} \subset H_1$ with $x_n \rightharpoonup x$, the inequality

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\| \tag{18}$$

holds for every $y \in H_1$ with $x \neq y$. Hilbert space H_1 satisfies the *Kadee-Klee property* [8] that is, for any sequence $\{x_n\}$ with $x_n \rightharpoonup x$ and $\|x_n\| \to \|x\|$ together imply $\|x_n - x\| \to 0$.

We recall some concepts and results which are needed in sequel. A mapping P_C is said to be *metric projection* of H_1 onto C if for every point $x \in H_1$, there exists a unique nearest point in C denoted by $P_C x$ such that

$$\|x - P_C x\| \leq \|x - y\|, \quad \forall y \in C. \tag{19}$$

It is well known that P_C is a nonexpansive mapping and is characterized by the following property:

$$\|P_C x - P_C y\|^2 \leq \langle x - y, P_C x - P_C y\rangle, \quad \forall x, y \in H_1. \tag{20}$$

Moreover, $P_C x$ is characterized by the following properties:

$$\langle x - P_C x, y - P_C x\rangle \leq 0, \tag{21}$$

$$\|x - y\|^2 \geq \|x - P_C x\|^2 + \|y - P_C x\|^2, \quad \forall x \in H_1, y \in C, \tag{22}$$

and

$$\|(x - y) - (P_C x - P_C y)\|^2 \geq \|x - y\|^2 - \|P_C x - P_C y\|^2, \quad \forall x, y \in H_1. \tag{23}$$

It is known that every nonexpansive operator $T : H_1 \to H_1$ satisfies, for all $(x, y) \in H_1 \times H_1$, the inequality

$$\langle (x - T(x)) - (y - T(y)), T(y) - T(x)\rangle \leq \frac{1}{2}\|(T(x) - x) - (T(y) - y)\|^2 \tag{24}$$

and therefore, we get, for all $(x, y) \in H_1 \times Fix(T)$,

$$\langle x - T(x), y - T(x)\rangle \leq \frac{1}{2}\|T(x) - x\|^2, \tag{25}$$

(see, e.g., Theorem 3 in [4] and Theorem 1 in [5]).

Let A be a monotone mapping of C into H_1, it is easy to observe that,

$$u \in VI(C, A) \Leftrightarrow u = P_C(u - \lambda Au), \quad \forall\lambda > 0. \tag{26}$$

We also have that, for all $u, v \in C$ and $\lambda > 0$,

$$\begin{aligned}
&\|(I - \lambda A)u - (I - \lambda A)v\|^2 \\
&= \|(u - v) - \lambda(Au - Av)\|^2 \\
&= \langle (u - v) - \lambda(Au - Av), (u - v) - \lambda(Au - Av)\rangle \\
&= \langle u - v, u - v\rangle - \lambda\langle u - v, Au - Av\rangle - \lambda\langle Au - Av, u - v\rangle \\
&\quad + \lambda^2\langle Au - Av, Au - Av\rangle \\
&= \|u - v\|^2 - 2\lambda\langle u - v, Au - Av\rangle + \lambda^2\|Au - Av\|^2 \\
&\leq \|u - v\|^2 - 2\alpha\|Au - Av\|^2 + \lambda^2\|Au - Av\|^2 \\
&\leq \|u - v\|^2 + \lambda(\lambda - 2\alpha)\|Au - Av\|^2.
\end{aligned}$$

So, if $\lambda \leq 2\alpha$, then $I - \lambda A$ is a nonexpansive mapping from C to H.

Lemma 1 [1] *Let $F : C \times C \to \mathbf{R}$ be a bifunction satisfying the following assumptions:*

(i) $F(x, x) \geq 0$ *for all* $x \in C$*;*
(ii) F *is monotone, i.e.,* $F(x, y) + F(y, x) \leq 0$ *for all* $x \in C$*;*
(iii) F *is upper hemicontinuous, i.e., for each* $x, y, z \in C$,

$$\limsup_{t\to 0} F(tz + (1 - t)x, y) \leq F(x, y); \tag{27}$$

(iv) *For each* $x \in C$ *fixed, the function* $y \mapsto F(x, y)$ *is convex and lower semicontinuous;*

let $h : C \times C \to \mathbf{R}$ such that

(i) $h(x, x) \geq 0$ for all $x \in C$,
(ii) For each $y \in C$ fixed, the function $x \to h(x, y)$ is upper semicontinuous,
(iii) For each $x \in C$ fixed, the function $y \to h(x, y)$ is convex and lower semicontinuous,

and assume that for fixed $r > 0$ and $z \in C$, there exists a nonempty compact convex subset K of H_1 and $x \in C \cap K$ such that

$$F(y, x) + h(y, x) + \frac{1}{r}\langle y - x, x - z\rangle < 0, \quad \forall y \in C \setminus K. \tag{28}$$

The proof of the following lemma is similar to the proof of Lemma 2.13 in [1] and hence omitted.

Lemma 2 *Assume that $F_1, h_1 : C \times C \to \mathbf{R}$ satisfying Lemma 1. Let $r > 0$ and $x \in H_1$. Then, there exists $z \in C$ such that*

$$F_1(z, y) + h_1(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \quad \forall y \in C. \tag{29}$$

Lemma 3 *Assume that the bifunctions $F_1, h_1 : C \times C \to \mathbf{R}$ satisfying Lemma 1 and h_1 is monotone. For $r > 0$ and for all $x \in H_1$, define a mapping $T_r^{(F_1,h_1)} : H_1 \to C$ as follows:*

$$T_r^{(F_1,h_1)}(x) = \left\{z \in C : F_1(z, y) + h_1(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \quad \forall y \in C\right\}. \tag{30}$$

Then, the following hold:

(1) $T_r^{(F_1,h_1)}$ *is single-valued.*
(2) $T_r^{(F_1,h_1)}$ *is firmly nonexpansive, i.e.,*

$$\|T_r^{(F_1,h_1)}x - T_r^{(F_1,h_1)}y\|^2 \leq \langle T_r^{(F_1,h_1)}x - T_r^{(F_1,h_1)}y, x - y\rangle, \quad \forall x, y \in H_1. \tag{31}$$

(3) $Fix(T_r^{(F_1,h_1)}) = GEP(F_1, h_1)$.
(4) $GEP(F_1, h_1)$ *is compact and convex.*

Further, assume that $F_2, h_2 : Q \times Q \to \mathbf{R}$ satisfying Lemma 1. For $s > 0$ and for all $w \in H_2$, define a mapping $T_s^{(F_2,h_2)} : H_2 \to Q$ as follows:

$$T_s^{(F_2,h_2)}(w) = \left\{d \in Q : F_2(d, e) + h_2(d, e) + \frac{1}{s}\langle e - d, d - w\rangle \geq 0, \quad \forall e \in Q\right\}. \tag{32}$$

Then, we easily observe that $T_s^{(F_2,h_2)}$ is single-valued and firmly nonexpansive, $GEP(F_2, h_2, Q)$ is compact and convex, and $Fix(T_s^{(F_2,h_2)}) = GEP(F_2, h_2, Q)$, where $GEP(F_2, h_2, Q)$ is the solution set of the following generalized equilibrium problem:

Find $y^* \in Q$ such that $F_2(y^*, y) + h_2(y^*, y) \geq 0, \forall y \in Q$.

We observe that $GEP(F_2, h_2) \subset GEP(F_2, h_2, Q)$. Further, it is easy to prove that Γ is a closed and convex set.

Remark 1 Lemmas 2 and 3 are slight generalizations of Lemma 3.5 in [3] where the equilibrium condition $F_1(\hat{x}, x) = h_1(\hat{x}, x) = 0$ has been relaxed to $F_1(\hat{x}, x) \geq 0$ and $h_1(\hat{x}, x) \geq 0$ for all $x \in C$. Further, the monotonicity of h_1 in Lemma 2 is not required.

Lemma 4 [7] *(Demiclosedness principle) Let C be a closed convex subset of a real Hilbert space H_1 and let $T : C \to C$ be a nonexpansive mapping. Then $I - T$ is demiclosed at zero, that is, $x_n \rightharpoonup x$, $x_n - Tx_n \to 0$ implies $x = Tx$.*

Lemma 5 [9] *Let C be a nonempty bounded closed and convex subset of Hilbert space H_1 and $\{T_n\}$ a sequence of mappings of C into itself. Suppose that*

$$\lim_{k,l\to\infty} \omega_l^k = 0,$$

where $\omega_l^k = \sup\{\|T_k z - T_l z\| : z \in C\} < \infty$, for all $k, l \in \mathbf{N}$. Then for each $x \in C$, $\{T_n x\}$ converges strongly to some point of C. Moreover, let T be a mapping from C into itself defined by

$$Tx = \lim_{n\to\infty} T_n x, \ \forall x \in C.$$

Then, $\lim_{n\to\infty} \sup\{\|Tz - T_n z\| : z \in C\} = 0$.

Lemma 6 [10] *For given $x^*, y^* \in C$, (x^*, y^*) is a solution of a general system of variational inequality problems if and only if x^* is a fixed point of the mapping $\bar{G} : C \to C$ defined by*

$$\bar{G}(x) = P_C[P_C(x - \lambda_2 B_2 x) - \lambda_1 B_1 P_C(x - \lambda_2 B_2 x)], \ \ \forall x \in C,$$

where $y^* = P_C(x - \lambda_2 B_2 x)$, λ_1, λ_2 are positive constants and $B_1, B_2 : C \to H_1$ are two mappings.

Lemma 7 *Let $\bar{G} : C \to C$ be defined in Lemma 6. If $B_1, B_2 : C \to H_1$ be β_1, β_2-inverse-strongly monotones, and $\lambda_1 \in (0, 2\beta_1), \lambda_2 \in (0, 2\beta_2)$, respectively, then $\bar{G}$ is a nonexpansive mapping.*

Proof For any $x, y \in C$, we have

$$\begin{aligned}
\|\bar{G}(x) - \bar{G}(y)\|^2 &= \|P_C[P_C(x - \lambda_2 B_2 x) - \lambda_1 B_1 P_C(x - \lambda_2 B_2 x)] \\
&\quad - P_C[P_C(y - \lambda_2 B_2 y) - \lambda_1 B_1 P_C(y - \lambda_2 B_2 y)]\|^2 \\
&\le \|[P_C(x - \lambda_2 B_2 x) - \lambda_1 B_1 P_C(x - \lambda_2 B_2 x)] \\
&\quad - [P_C(y - \lambda_2 B_2 y) - \lambda_1 B_1 P_C(y - \lambda_2 B_2 y)]\|^2 \\
&= \|[P_C(x - \lambda_2 B_2 x) - P_C(y - \lambda_2 B_2 y)] \\
&\quad - \lambda_1[B_1 P_C(x - \lambda_2 B_2 x) - B_1 P_C(y - \lambda_2 B_2 y)]\|^2 \\
&= \|P_C(x - \lambda_2 B_2 x) - P_C(y - \lambda_2 B_2 y)\|^2 \\
&\quad - 2\lambda_1 \langle P_C(x - \lambda_2 B_2 x) - P_C(y - \lambda_2 B_2 y), \\
&\quad B_1 P_C(x - \lambda_2 B_2 x) - B_1 P_C(y - \lambda_2 B_2 y)\rangle \\
&\quad + \lambda_1^2 \|B_1 P_C(x - \lambda_2 B_2 x) - B_1 P_C(y - \lambda_2 B_2 y)\|^2 \\
&\le \|P_C(x - \lambda_2 B_2 x) - P_C(y - \lambda_2 B_2 y)\|^2 \\
&\quad - 2\lambda_1 \beta_1 \|B_1 P_C(x - \lambda_2 B_2 x) - B_1 P_C(y - \lambda_2 B_2 y)\|^2 \\
&\quad + \lambda_1^2 \|B_1 P_C(x - \lambda_2 B_2 x) - B_1 P_C(y - \lambda_2 B_2 y)\|^2 \\
&= \|P_C(x - \lambda_2 B_2 x) - P_C(y - \lambda_2 B_2 y)\|^2 \\
&\quad + \lambda_1(\lambda_1 - 2\beta_1) \|B_1 P_C(x - \lambda_2 B_2 x) - B_1 P_C(y - \lambda_2 B_2 y)\|^2 \\
&\le \|P_C(x - \lambda_2 B_2 x) - P_C(y - \lambda_2 B_2 y)\|^2 \\
&\le \|(x - \lambda_2 B_2 x) - (y - \lambda_2 B_2 y)\|^2 \\
&= \|(x - y) - \lambda_2(B_2 x - B_2 y)\|^2 \\
&= \|x - y\|^2 - 2\lambda_2 \langle x - y, B_2 x - B_2 y\rangle + \lambda_2^2 \|B_2 x - B_2 y\|^2 \\
&\le \|x - y\|^2 - 2\beta_2 \lambda_2 \|B_2 x - B_2 y\|^2 + \lambda_2^2 \|B_2 x - B_2 y\|^2 \\
&= \|x - y\|^2 + \lambda_2(\lambda_2 - 2\beta_2) \|B_2 x - B_2 y\|^2 \\
&\le \|x - y\|^2.
\end{aligned}$$

This show that $\bar{G}$ is nonexpansive on C. □

3 Strong Convergence Theorem

Theorem 1 *Let H_1 and H_2 be two real Hilbert spaces and $C \subset H_1$ and $Q \subset H_2$ be nonempty closed convex subsets of H_1 and H_2, respectively. Let B_1, B_2 be β_1, β_2-inverse* *strongly monotone mappings from C into H_1, respectively. Let $A : H_1 \to H_2$ be a bounded linear operator. Let $F_1, h_1 : C \times C \to \mathbf{R}$ and $F_2, h_2 : Q \times Q \to \mathbf{R}$ satisfying Lemma 1; h_1, h_2 are monotone and F_2 is upper semicontinuous and S_n be a sequence of non-expansive mappings from C into itself and let S be a mapping of C into itself defined by $Sx = \lim_{n\to\infty} S_n x, \forall x \in C$ such that*

$$\Omega_1 := \cap_{n=1}^{\infty} Fix(S_n) \cap \Gamma \cap GVI(C, B_1, B_2) \neq \emptyset. \tag{33}$$

For a given $x_0 \in H_1, C_1 = C, x_1 = P_C x_0, u_n \in C$, let the iterative sequences $\{u_n\}, \{z_n\}, \{y_n\}$ and $\{x_n\}$ be generated by

$$\begin{cases} u_n = T_{r_n}^{(F_1,h_1)}(x_n + \xi A^*(T_{r_n}^{(F_2,h_2)} - I)Ax_n), \\ z_n = P_C(u_n - \lambda_2 B_2 u_n), \\ y_n = \alpha_n x_n + (1-\alpha_n) S_n P_C(z_n - \lambda_1 B_1 z_n), \\ C_{n+1} = \{z \in C_n : \|y_n - z\| \leq \|x_n - z\|\}, \\ x_{n+1} = P_{C_{n+1}} x_0, \quad \forall n \in \mathbf{N}, \end{cases} \tag{34}$$

where $\{\alpha_n\} \in (0,1), \lambda_1 \in [a_1, b_1] \subset (0, 2\beta_1), \lambda_2 \in [a_2, b_2] \subset (0, 2\beta_2), \{r_n\} \subset (0, \infty)$ and $\xi \in (0, \frac{1}{L})$, L is the spectral radius of the operator A^*A and A^* is the adjoint of A satisfying the following conditions:

(C1) $0 < a_1 \leq \lambda_1 \leq b_1 < 2\beta_1$;
(C2) $0 < a_2 \leq \lambda_2 \leq b_2 < 2\beta_2$;
(C3) $\liminf_{n\to\infty} r_n > 0$.

Then, the sequence $\{x_n\}$ converges strongly to $P_{\Omega_1} x_0$ and (x^*, y^*) is a solution of a general system of variational inequalities problems, where $y^* = P_C(x^* - \lambda_2 B_2 x^*)$.

Proof We will divide the proof into six steps.

Step 1. We will show that $\{x_n\}$ is well-defined and C_n is closed and convex for any $n \in \mathbf{N}$. From the assumption, we see that $C_1 = C$ is closed and convex. Suppose that C_k is closed and convex for some $k \geq 1$. Now, we will show that C_{k+1} is closed and convex for some k. For any $x^* \in C_k$, we obtain

$$\begin{aligned} & \|y_k - x^*\| \leq \|x_k - x^*\| \\ \Leftrightarrow\ & \|y_k - x^*\|^2 \leq \|x_k - x^*\|^2 \\ \Leftrightarrow\ & \|y_k - x_k + x_k - x^*\|^2 \leq \|x_k - x^*\|^2 \\ \Leftrightarrow\ & \|y_k - x^*\|^2 + 2\langle y_k - x_k, x_k - x^*\rangle + \|x_k - x^*\|^2 \leq \|x_k - x^*\|^2. \end{aligned}$$

This implies that $\|y_k - x^*\| \leq \|x_k - x^*\|$ is equivalent to $\|y_k - x_k\|^2 + 2\langle y_k - x_k, x_k - x^2\rangle \leq 0$. Thus, C_{k+1} is closed and convex. Then, C_n is closed and convex for any $n \in \mathbf{N}$. This implies that $\{x_n\}$ is well-defined.

Step 2. We will show that $\Omega_1 \subset C_n$, for all $n \in \mathbf{N}$ by mathematical induction. Since $x^* \in \Omega_1$, we get $x^* = S_n x^* = P_C[P_C(x^* - \lambda_2 B_2 x^*) - \lambda_1 B_1 P_C(x^* - \lambda_2 B_2 x^*)] = T_{r_n}^{(F_1,h_1)} x^*$ and $Ax^* = T_{r_n}^{(F_2,h_2)} Ax^*$. From the assumption, we see that $\Omega_1 \subset C = C_1$. Suppose that $\Omega_1 \subset C_k$, for some $k \geq 1$.

Put $y^* = P_C(x^* - \lambda_2 B_2 x^*)$ and $v_k = P_C(z_k - \lambda_1 B_1 z_k)$. Then $x^* = P_C(y^* - \lambda_1 B_1 y^*)$ and

$$\begin{aligned} \|v_k - x^*\| &= \|P_C(z_k - \lambda_1 B_1 z_k) - P_C(y^* - \lambda_1 B_1 y^*)\| \\ &\leq \|(z_k - \lambda_1 B_1 z_k) - (y^* - \lambda_1 B_1 y^*)\| \\ &= \|(I - \lambda_1 B_1)z_k - (I - \lambda_1 B_1)y^*\| \\ &\leq \|z_k - y^*\| \\ &= \|P_C(u_k - \lambda_2 B_2 u_k) - P_C(x^* - \lambda_2 B_2 x^*)\| \\ &\leq \|(u_k - \lambda_2 B_2 u_k) - (x^* - \lambda_2 B_2 x^*)\| \\ &= \|(I - \lambda_2 B_2)u_k - (I - \lambda_2 B_2)x^*\| \\ &\leq \|u_k - x^*\|. \end{aligned} \tag{35}$$

Since $x^* \in \Omega_1$, i.e., $x^* \in \Gamma$, and we have $x^* = T_{r_k}^{(F_1,h_1)} x^*$ and $Ax^* = T_{r_k}^{(F_2,h_2)} Ax^*$. We estimate

$$\begin{aligned} \|u_k - x^*\|^2 &= \|T_{r_k}^{(F_1,h_1)}(x_k + \xi A^*(T_{r_k}^{(F_2,h_2)} - I)Ax_k) - x^*\|^2 \\ &= \|T_{r_k}^{(F_1,h_1)}(x_k + \xi A^*(T_{r_k}^{(F_2,h_2)} - I)Ax_k) - T_{r_k}^{(F_1,h_1)} x^*\|^2 \\ &\leq \|x_k + \xi A^*(T_{r_k}^{(F_2,h_2)} - I)Ax_k - x^*\|^2 \\ &\leq \|x_k - x^*\|^2 + \xi^2 \|A^*(T_{r_k}^{(F_2,h_2)} - I)Ax_k\|^2 \\ &\quad +2\xi \langle x_k - x^*, A^*(T_{r_k}^{(F_2,h_2)} - I)Ax_k \rangle. \end{aligned} \tag{36}$$

Thus, we have

$$\begin{aligned} \|u_k - x^*\|^2 \leq\ & \|x_k - x^*\|^2 + \xi^2 \langle (T_{r_k}^{(F_2,h_2)} - I)Ax_k, AA^*(T_{r_k}^{(F_2,h_2)} - I)Ax_k \rangle \\ & +2\xi \langle x_k - x^*, A^*(T_{r_k}^{(F_2,h_2)} - I)Ax_n \rangle. \end{aligned} \tag{37}$$

Now, we have

$$\begin{aligned} & \xi^2 \langle (T_{r_k}^{(F_2,h_2)} - I)Ax_k, AA^*(T_{r_k}^{(F_2,h_2)} - I)Ax_k \rangle \\ \leq\ & L\xi^2 \langle (T_{r_k}^{(F_2,h_2)} - I)Ax_k, (T_{r_k}^{(F_2,h_2)} - I)Ax_k \rangle \\ =\ & L\xi^2 \|(T_{r_k}^{(F_2,h_2)} - I)Ax_k\|^2. \end{aligned} \tag{38}$$

Denoting $\Lambda := 2\xi \langle x_k - x^*, A^*(T_{r_k}^{(F_2,h_2)} - I)Ax_n \rangle$ and using (25), we have

$$\begin{aligned} \Lambda &= 2\xi \langle x_k - x^*, A^*(T_{r_k}^{(F_2,h_2)} - I)Ax_k \rangle \\ &= 2\xi \langle A(x_k - x^*), (T_{r_k}^{(F_2,h_2)} - I)Ax_k \rangle \\ &= 2\xi \langle A(x_k - x^*) + (T_{r_k}^{(F_2,h_2)} - I)Ax_k - (T_{r_k}^{(F_2,h_2)} - I)Ax_k, (T_{r_k}^{(F_2,h_2)} - I)Ax_k \rangle \\ &= 2\xi \left\{ \langle T_{r_k}^{(F_2,h_2)} Ax_k - Ax^*, (T_{r_k}^{(F_2,h_2)} - I)Ax_k \rangle - \|(T_{r_k}^{(F_2,h_2)} - I)Ax_k\|^2 \right\} \\ &\leq 2\xi \left\{ \frac{1}{2} \|(T_{r_k}^{(F_2,h_2)} - I)Ax_k\|^2 - \|(T_{r_k}^{(F_2,h_2)} - I)Ax_k\|^2 \right\} \\ &\leq -\xi \|(T_{r_k}^{(F_2,h_2)} - I)Ax_k\|^2. \end{aligned} \tag{39}$$

Using Equations (37), (38) and (39), we obtain

$$\|u_k - x^*\|^2 \leq \|x_k - x^*\|^2 + \xi(L\xi - 1)\|(T_{r_k}^{(F_2,h_2)} - I)Ax_k\|^2. \tag{40}$$

Since $\xi \in (0, \frac{1}{L})$, we obtain

$$\|u_k - x^*\|^2 \leq \|x_k - x^*\|^2. \tag{41}$$

So from Equations (35) and (41), we have

$$\|v_k - x^*\| \leq \|z_k - y^*\| \leq \|u_k - x^*\| \leq \|x_k - x^*\|. \tag{42}$$

Hence, we have

$$\begin{aligned}
\|y_k - x^*\| &= \|\alpha_k x_k + (1-\alpha_k)S_k v_k - x^*\| \\
&\leq \alpha_k\|x_k - x^*\| + (1-\alpha_k)\|S_k v_k - x^*\| \\
&\leq \alpha_k\|x_k - x^*\| + (1-\alpha_k)\|v_k - x^*\| \\
&\leq \alpha_k\|x_k - x^*\| + (1-\alpha_k)\|x_k - x^*\| \\
&= \|x_k - x^*\|.
\end{aligned}$$

Thus, we get $x^* \in C_{k+1}$. This implies that $\Omega_1 \subset C_n$ for all $n \in \mathbf{N}$.

Step 3. We will show that $\lim_{n\to\infty}\|x_n - x_0\|$ exists. From $x_n = P_{C_n}x_0$, we have

$$\langle x_0 - x_n, y - x_n\rangle \leq 0, \tag{43}$$

then

$$\langle x_0 - x_n, x_n - y\rangle \geq 0, \tag{44}$$

for each $y \in C_n$. Using $\Omega_1 \subset C_n$, we also have

$$\langle x_0 - x_n, x_n - x^*\rangle \geq 0 \text{ for each } x^* \in \Omega_1 \text{ and } n \in \mathbf{N}. \tag{45}$$

Then, for $x^* \in \Omega_1$, we obtain

$$\begin{aligned}
0 &\leq \langle x_0 - x_n, x_n - x^*\rangle \\
&= \langle x_0 - x_n, x_n - x_0 + x_0 - x^*\rangle \\
&= \langle x_0 - x_n, x_n - x_0\rangle + \langle x_0 - x_n, x_0 - x^*\rangle \\
&\leq -\langle x_0 - x_n, x_0 - x_n\rangle + \langle x_0 - x_n, x_0 - x^*\rangle \\
&\leq -\|x_0 - x_n\|^2 + \langle x_0 - x_n, x_0 - x^*\rangle \\
&\leq -\|x_0 - x_n\|^2 + \|x_0 - x_n\|\|x_0 - x^*\|.
\end{aligned}$$

This implies that

$$\|x_0 - x_n\| \leq \|x_0 - x^*\| \text{ for all } x^* \in \Omega_1 \text{ and } n \in \mathbf{N}. \tag{46}$$

From $x_n = P_{C_n}x_0$ and $x_{n+1} = P_{C_n+1}x_0 \in C_{n+1} \subset C_n$, we have

$$\langle x_0 - x_n, x_n - x_{n+1}\rangle \geq 0. \tag{47}$$

Hence

$$\begin{aligned}
0 &\leq \langle x_0 - x_n, x_n - x_{n+1}\rangle \\
&= \langle x_0 - x_n, x_n - x_0 + x_0 - x_{n+1}\rangle \\
&= \langle x_0 - x_n, x_n - x_0\rangle + \langle x_0 - x_n, x_0 - x_{n+1}\rangle \\
&= -\langle x_0 - x_n, x_0 - x_n\rangle + \langle x_0 - x_n, x_0 - x_{n+1}\rangle \\
&\leq -\|x_0 - x_n\|^2 + \langle x_0 - x_n, x_0 - x_{n+1}\rangle \\
&\leq -\|x_0 - x_n\|^2 + \|x_0 - x_n\|\|x_0 - x_{n+1}\|.
\end{aligned}$$

It follows that

$$\|x_0 - x_n\| \leq \|x_0 - x_{n+1}\|, \text{ for all } n \in \mathbf{N}. \tag{48}$$

Thus, the sequence $\{\|x_n - x_0\|\}$ is a bounded and nondecreasing sequence, so $\lim_{n\to\infty} \|x_n - x_0\|$ exists, and then there exists m such that

$$\lim_{n\to\infty} \|x_n - x_0\| = m. \tag{49}$$

Step 4. We will show the following :

(i) $\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0,$
(ii) $\lim_{n\to\infty} \|x_n - y_n\| = 0,$
(iii) $\lim_{n\to\infty} \|x_n - u_n\| = 0,$
(iv) $\lim_{n\to\infty} \|x_n - v_n\| = 0.$

From (47), we get

$$\begin{aligned}
&\|x_n - x_{n+1}\|^2 \\
&= \|x_n - x_0 + x_0 - x_{n+1}\|^2 \\
&= \|x_n - x_0\|^2 + 2\langle x_n - x_0, x_0 - x_{n+1}\rangle + \|x_0 - x_{n+1}\|^2 \\
&= \|x_n - x_0\|^2 + 2\langle x_n - x_0, x_0 - x_n + x_n - x_{n+1}\rangle + \|x_0 - x_{n+1}\|^2 \\
&= \|x_n - x_0\|^2 + 2\langle x_n - x_0, x_0 - x_n\rangle + 2\langle x_n - x_0, x_n - x_{n+1}\rangle + \|x_0 - x_{n+1}\|^2 \\
&\le \|x_n - x_0\|^2 + 2\langle x_n - x_0, x_0 - x_n\rangle + \|x_0 - x_{n+1}\|^2 \\
&= \|x_n - x_0\|^2 - 2\langle x_n - x_0, x_n - x_0\rangle + \|x_0 - x_{n+1}\|^2 \\
&\le \|x_n - x_0\|^2 - 2\|x_n - x_0\|^2 + \|x_0 - x_{n+1}\|^2 \\
&= -\|x_n - x_0\|^2 + \|x_0 - x_{n+1}\|^2.
\end{aligned}$$

From (49), we obtain $\|x_{n+1} - x_0\|^2 - \|x_n - x_0\|^2 \to 0$, therefore

$$\lim_{n\to\infty} \|x_n - x_{n+1}\| = 0. \tag{50}$$

By $x_{n+1} = P_{C_{n+1}} x_0 \in C_{n+1} \subset C_n$, we have

$$\|x_{n+1} - y_n\| \le \|x_{n+1} - x_n\|. \tag{51}$$

Furthermore, we also obtain

$$\|x_n - y_n\| \le \|x_n - x_{n+1}\| + \|x_{n+1} - y_n\| \le 2\|x_n - x_{n+1}\|. \tag{52}$$

From (50), we get

$$\lim_{n\to\infty} \|x_n - y_n\| = 0, \tag{53}$$

and we also have

$$\begin{aligned}
\|y_n - x_n\| &= \|\alpha_n x_n + (1-\alpha_n) S_n v_n - x_n\| = \| - (1-\alpha_n) x_n + (1-\alpha_n) S_n v_n\| \\
&= \|(1-\alpha_n)(S_n v_n - x_n)\| = (1-\alpha_n)\|S_n v_n - x_n\|.
\end{aligned}$$

Since $\alpha_n \in (0, 1)$ and (53), we get

$$\lim_{n\to\infty} \|S_n v_n - x_n\| = 0. \tag{54}$$

Consider from Equations (34), (40) and (42), we obtain

$$\begin{aligned}
&\|y_n - x^*\|^2 \\
\leq\ & \alpha_n\|x_n - x^*\|^2 + (1-\alpha_n)\|S_n v_n - x^*\|^2 \\
\leq\ & \alpha_n\|x_n - x^*\|^2 + (1-\alpha_n)\|v_n - x^*\|^2 \\
\leq\ & \alpha_n\|x_n - x^*\|^2 + (1-\alpha_n)\|u_n - x^*\|^2 \\
\leq\ & \alpha_n\|x_n - x^*\|^2 + (1-\alpha_n)(\|x_n - x^*\|^2 + \xi(L\xi - 1)\|(T_{r_n}^{(F_2,h_2)} - I)Ax_n\|^2) \\
=\ & \alpha_n\|x_n - x^*\|^2 + \|x_n - x^*\|^2 - \alpha_n\|x_n - x^*\|^2 \\
& +(1-\alpha_n)\xi(L\xi - 1)\|(T_{r_n}^{(F_2,h_2)} - I)Ax_n\| \\
=\ & \|x_n - x^*\|^2 - (1-\alpha_n)\xi(1 - L\xi)\|(T_{r_n}^{(F_2,h_2)} - I)Ax_n\|^2.
\end{aligned}$$

This implies that

$$\begin{aligned}
& (1-\alpha_n)\xi(1 - L\xi)\|(T_{r_n}^{(F_2,h_2)} - I)Ax_n\|^2 \\
\leq\ & \|x_n - x^*\|^2 - \|y_n - x^*\|^2 \\
=\ & (\|x_n - x^*\| - \|y_n - x^*\|)(\|x_n - x^*\| + \|y_n - x^*\|) \\
=\ & (\|x_n - x^* - y_n + x^*\|)(\|x_n - x^*\| + \|y_n - x^*\|) \\
=\ & \|x_n - y_n\|(\|x_n - x^*\| + \|y_n - x^*\|).
\end{aligned}$$

From (53), $\alpha_n \in (0, 1)$, $(1-\alpha_n)\xi(1 - L\xi) > 0$, we obtain

$$\lim_{n\to\infty}\|(T_{r_n}^{(F_2,h_2)} - I)Ax_n\| = 0. \tag{55}$$

For $x^* \in \Omega_1$ and $T_{r_n}^{(F_1,h_1)}$ is firmly nonexpansive, we get

$$\begin{aligned}
&\|u_n - x^*\|^2 \\
=\ & \|T_{r_n}^{(F_1,h_1)}(x_n + \xi A^*(T_{r_n}^{(F_2,h_2)} - I)Ax_n) - x^*\|^2 \\
=\ & \|T_{r_n}^{(F_1,h_1)}(x_n + \xi A^*(T_{r_n}^{(F_2,h_2)} - I)Ax_n) - T_{r_n}^{(F_1,h_1)}x^*\|^2 \\
\leq\ & \langle u_n - x^*, x_n + \xi A^*(T_{r_n}^{(F_2,h_2)} - I)Ax_n - x^*\rangle \\
=\ & \frac{1}{2}\Big\{\|u_n - x^*\| + \|x_n + \xi A^*(T_{r_n}^{(F_2,h_2)} - I)Ax_n - x^*\|^2 \\
& -\|(u_n - x^*) - [x_n + \xi A^*(T_{r_n}^{(F_2,h_2)} - I)Ax_n - x^*]\|^2\Big\} \\
=\ & \frac{1}{2}\Big\{\|u_n - x^*\|^2 + \|x_n - x^*\|^2 - \|u_n - x_n - \xi A^*(T_{r_n}^{(F_2,h_2)} - I)Ax_n\|^2\Big\} \\
=\ & \frac{1}{2}\Big\{\|u_n - x^*\|^2 + \|x_n - x^*\|^2 - \|u_n - x_n\|^2 + \xi^2\|A^*(T_{r_n}^{(F_2,h_2)} - I)Ax_n\|^2 \\
& -2\xi\langle u_n - x_n, A^*(T_{r_n}^{(F_2,h_2)} - I)Ax_n\rangle\Big\}.
\end{aligned}$$

Hence, we obtain

$$\|u_n - x^*\|^2 \leq \|x_n - x^*\|^2 - \|u_n - x_n\|^2 + 2\xi\|A(u_n - x_n)\|\|(T_{r_n}^{(F_2,h_2)} - I)Ax_n\|. \tag{56}$$

By Equations (34), (42) and (56), it follows that

$$\begin{aligned}
&\|y_n - x^*\|^2 \\
\leq\ & \alpha_n\|x_n - x^*\|^2 + (1-\alpha_n)\|S_n v_n - x^*\|^2 \\
\leq\ & \alpha_n\|x_n - x^*\|^2 + (1-\alpha_n)\|v_n - x^*\|^2 \\
\leq\ & \alpha_n\|x_n - x^*\|^2 + (1-\alpha_n)\|u_n - x^*\|^2 \\
\leq\ & \alpha_n\|x_n - x^*\|^2 + (1-\alpha_n)(\|x_n - x^*\|^2 - \|u_n - x_n\|^2 \\
& +2\xi\|A(u_n - x_n)\|\|(T_{r_n}^{(F_2,h_2)} - I)Ax_n\|) \\
=\ & \|x_n - x^*\|^2 - (1-\alpha_n)\|x_n - u_n\|^2 + 2(1-\alpha_n)\xi\|A(u_n - x_n)\|\|(T_{r_n}^{(F_2,h_2)} - I)Ax_n\|.
\end{aligned}$$

So,

$$\begin{aligned}
&(1-\alpha_n)\|x_n - u_n\|^2 \\
\leq\ & \|x_n - x^*\|^2 - \|y_n - x^*\|^2 + 2(1-\alpha_n)\xi\|A(u_n - x_n)\|\|(T_{r_n}^{(F_2,h_2)} - I)Ax_n\| \\
=\ & \|x_n - y_n\|(\|x_n - x^*\| + \|y_n - x^*\|) + 2(1-\alpha_n)\xi\|A(u_n - x_n)\|\|(T_{r_n}^{(F_2,h_2)} - I)Ax_n\|.
\end{aligned}$$

From Equations (53), (55) and $\alpha_n \in (0, 1)$, we get

$$\lim_{n\to\infty} \|x_n - u_n\| = 0. \tag{57}$$

Since $x^* \in \Omega_1$ from Equations (34) and (42), we obtain

$$\begin{aligned}
\|y_n - x^*\|^2 &\leq \alpha_n\|x_n - x^*\|^2 + (1-\alpha_n)\|S_n v_n - x^*\|^2 \\
&\leq \alpha_n\|x_n - x^*\| + (1-\alpha_n)\|v_n - x^*\|^2 \\
&= \alpha_n\|x_n - x^*\|^2 + \|v_n - x^*\|^2 - \alpha_n\|x_n - x^*\|^2 \\
&= \|P_C(z_n - \lambda_1 B_1 z_n) - P_C(y^* - \lambda_1 B_1 y^*)\|^2 \\
&\leq \|(I - \lambda_1 B_1)z_n - (I - \lambda_1 B_1)y^*\|^2 \\
&\leq \|z_n - y^*\|^2 + \lambda_1(\lambda_1 - 2\beta_1)\|B_1 z_n - B_1 y^*\|^2 \\
&\leq \|x_n - x^*\|^2 + a_1(b_1 - 2\beta_1)\|B_1 z_n - B_1 y^*\|^2.
\end{aligned}$$

and also

$$\begin{aligned}
\|y_n - x^*\|^2 &\leq \alpha_n\|x_n - x^*\|^2 + (1-\alpha_n)\|S_n v_n - x^*\|^2 \\
&\leq \alpha_n\|x_n - x^*\| + (1-\alpha_n)\|v_n - x^*\|^2 \\
&= \alpha_n\|x_n - x^*\|^2 + \|z_n - y^*\|^2 - \alpha_n\|x_n - x^*\|^2 \\
&= \|P_C(u_n - \lambda_2 B_2 u_n) - P_C(x^* - \lambda_2 B_2 x^*)\|^2 \\
&\leq \|(I - \lambda_2 B_2)u_n - (I - \lambda_2 B_2)x^*\|^2 \\
&\leq \|u_n - x^*\|^2 + \lambda_2(\lambda_2 - 2\beta_2)\|B_2 u_n - B_2 x^*\|^2 \\
&\leq \|x_n - x^*\|^2 + a_2(b_2 - 2\beta_2)\|B_2 u_n - B_2 x^*\|^2.
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
a_1(2\beta_1 - b_1)\|B_1 z_n - B_1 y^*\|^2 &\leq \|x_n - x^*\|^2 - \|y_n - x^*\|^2 \\
&\leq \|x_n - y_n\|(\|x_n - x^*\| + \|y_n - x^*\|)
\end{aligned}$$

and

$$\begin{aligned}
a_2(2\beta_2 - b_2)\|B_2 u_n - B_2 x^*\|^2 &\leq \|x_n - x^*\|^2 - \|y_n - x^*\|^2 \\
&\leq \|x_n - y_n\|(\|x_n - x^*\| + \|y_n - x^*\|).
\end{aligned}$$

From conditions (C1), (C2) and (53), we get

$$\lim_{n\to\infty}\|B_1 z_n - B_1 y^*\| = \lim_{n\to\infty}\|B_2 u_n - B_2 x^*\| = 0. \tag{58}$$

On the other hand, by (20), we have

$$\begin{aligned}
&\|z_n - y^*\|^2\\
&= \|P_C(u_n - \lambda_2 B_2 u_n) - P_C(x^* - \lambda_2 B_2 x^*)\|^2\\
&\leq \langle (u_n - \lambda_2 B_2 u_n) - (x^* - \lambda_2 B_2 x^*), z_n - y^*\rangle\\
&= \frac{1}{2}\Big\{\|(u_n - \lambda_2 B_2 u_n) - (x^* - \lambda_2 B_2 x^*)\|^2 + \|z_n - y^*\|^2\\
&\quad -\|[(u_n - \lambda_2 B_2 u_n) - (x^* - \lambda_2 B_2 x^*)]^2 - (z_n - y^*)\|^2\Big\}\\
&\leq \frac{1}{2}\Big\{\|u_n - x^*\|^2 + \|z_n - y^*\|^2 - \|(u_n - z_n) - (x^* - y^*) - \lambda_2(B_2 u_n - B_2 x^*)\|^2\Big\}\\
&\leq \frac{1}{2}\Big\{\|x_n - x^*\|^2 + \|z_n - y^*\|^2 - \|(u_n - z_n) - (x^* - y^*)\|^2\\
&\quad +2\lambda_2\langle (u_n - z_n) - (x^* - y^*), B_2 u_n - B_2 x^*\rangle - \lambda_2^2\|B_2 u_n - B_2 x^*\|^2\Big\}.
\end{aligned}$$

So, we obtain

$$\begin{aligned}
\|z_n - y^*\|^2 \leq\; & \|x_n - x^*\|^2 - \|(u_n - z_n) - (x^* - y^*)\|^2\\
& +2\lambda_2\|(u_n - z_n) - (x^* - y^*)\|\|B_2 u_n - B_2 x^*\|\\
& -\lambda_2^2\|B_2 u_n - B_2 x^*\|^2.
\end{aligned} \tag{59}$$

From (42) and (59), it follows that

$$\begin{aligned}
&\|y_n - x^*\|^2\\
&= \|\alpha_n x_n + (1-\alpha_n) S_n v_n - x^*\|^2\\
&\leq \alpha_n\|x_n - x^*\|^2 + (1-\alpha_n)\|v_n - x^*\|^2\\
&\leq \alpha_n\|x_n - x^*\|^2 + (1-\alpha_n)\|z_n - y^*\|^2\\
&\leq \alpha_n\|x_n - x^*\|^2 + (1-\alpha_n)\{\|x_n - x^*\|^2 - \|(u_n - z_n) - (x^* - y^*)\|^2\\
&\quad +2\lambda_2\|(u_n - z_n) - (x^* - y^*)\|\|B_2 u_n - B_2 x^*\| - \lambda_2^2\|B_2 u_n - B_2 x^*\|^2\}\\
&= \|x_n - x^*\|^2 - (1-\alpha_n)\|(u_n - z_n) - (x^* - y^*)\|^2\\
&\quad +2\lambda_2(1-\alpha_n)\|(u_n - z_n) - (x^* - y^*)\|\|B_2 u_n - B_2 x^*\|\\
&\quad -\lambda_2^2(1-\alpha_n)\|B_2 u_n - B_2 x^*\|^2.
\end{aligned}$$

Thus, we get

$$\begin{aligned}
&(1-\alpha_n)\|(u_n - z_n) - (x^* - y^*)\|^2\\
&\leq \|x_n - x^*\|^2 - \|y_n - x^*\|^2 - \lambda_2^2(1-\alpha_n)\|B_2 u_n - B_2 x^*\|^2\\
&\quad +2\lambda_2(1-\alpha_n)\|(u_n - z_n) - (x^* - y^*)\|\|B_2 u_n - B_2 x^*\|\\
&\leq \|x_n - y_n\|(\|x_n - x^*\| + \|y_n - x^*\|) - \lambda_2^2(1-\alpha_n)\|B_2 u_n - B_2 x^*\|^2\\
&\quad +2\lambda_2(1-\alpha_n)\|(u_n - z_n) - (x^* - y^*)\|\|B_2 u_n - B_2 x^*\|.
\end{aligned}$$

Since $\alpha_n \in (0, 1)$, Equations (53) and (58), we obtain

$$\lim_{n\to\infty}\|(u_n - z_n) - (x^* - y^*)\| = 0. \tag{60}$$

Now, from Equations (16) and (23), we observe that

$$\begin{aligned}
&\|(z_n - v_n) + (x^* - y^*)\|^2 \\
&= \|(z_n - y^*) - (v_n - x^*)\|^2 \\
&= \|(z_n - \lambda_1 B_1 z_n) - (y^* - \lambda_1 B_1 y^*) - [P_C(z_n - \lambda_1 B_1 z_n) - P_C(y^* - \lambda_1 B_1 y^*)] \\
&\quad + \lambda_1 (B_1 z_n - B_1 y^*)\|^2 \\
&\le \|(z_n - \lambda_1 B_1 z_n) - (y^* - \lambda_1 B_1 y^*) - [P_C(z_n - \lambda_1 B_1 z_n) - P_C(y^* - \lambda_1 B_1 y^*)]\|^2 \\
&\quad + 2\lambda_1 \langle B_1 z_n - B_1 y^*, (z_n - v_n) + (x^* - y^*)\rangle \\
&\le \|(z_n - \lambda_1 B_1 z_n) - (y^* - \lambda_1 B_1 y^*)\|^2 - \|P_C(z_n - \lambda_1 B_1 z_n) - P_C(y^* - \lambda_1 B_1 y^*)\|^2 \\
&\quad + 2\lambda_1 \|B_1 z_n - B_1 y^*\| \|(z_n - v_n) + (x^* - y^*)\| \\
&\le \|(z_n - \lambda_1 B_1 z_n) - (y^* - \lambda_1 B_1 y^*)\|^2 - \|S_n P_C(z_n - \lambda_1 B_1 z_n) \\
&\quad - S_n P_C(y^* - \lambda_1 B_1 y^*)\|^2 + 2\lambda_1 \|B_1 z_n - B_1 y^*\| \|(z_n - v_n) + (x^* - y^*)\| \\
&\le \|(z_n - \lambda_1 B_1 z_n) - (y^* - \lambda_1 B_1 y^*)\|^2 - \|S_n v_n - S_n x^*\|^2 \\
&\quad + 2\lambda_1 \|B_1 z_n - B_1 y^*\| \|(z_n - v_n) + (x^* - y^*)\| \\
&\le \{\|(z_n - \lambda_1 B_1 z_n) - (y^* - \lambda_1 B_1 y^*)\| - \|S_n v_n - x^*\|\} \\
&\quad \times \{\|(z_n - \lambda_1 B_1 z_n) - (y^* - \lambda_1 B_1 y^*)\| + \|S_n v_n - x^*\|\} \\
&\quad + 2\lambda_1 \|B_1 z_n - B_1 y^*\| \|(z_n - v_n) + (x^* - y^*)\| \\
&\le \|(z_n - \lambda_1 B_1 z_n) - (y^* - \lambda_1 B_1 y^*) - (S_n v_n - x^*)\| \\
&\quad \times \{\|(z_n - \lambda_1 B_1 z_n) - (y^* - \lambda_1 B_1 y^*)\| + \|S_n v_n - x^*\|\} \\
&\quad + 2\lambda_1 \|B_1 z_n - B_1 y^*\| \|(z_n - v_n) + (x^* - y^*)\| \\
&= \|(z_n - u_n + u_n - S_n v_n) + (x^* - y^*) - \lambda_1 (B_1 z_n - B_1 y^*)\| \\
&\quad \times \{\|(z_n - \lambda_1 B_1 z_n) - (y^* - \lambda_1 B_1 y^*)\| + \|S_n v_n - x^*\|\} \\
&\quad + 2\lambda_1 \|B_1 z_n - B_1 y^*\| \|(z_n - v_n) + (x^* - y^*)\| \\
&= \|(u_n - S_n v_n) + (x^* - y^*) - (u_n - z_n) - \lambda_1 (B_1 z_n - B_1 y^*)\| \\
&\quad \times \{\|(z_n - \lambda_1 B_1 z_n) - (y^* - \lambda_1 B_1 y^*)\| + \|S_n v_n - x^*\|\} \\
&\quad + 2\lambda_1 \|B_1 z_n - B_1 y^*\| \|(z_n - v_n) + (x^* - y^*)\| \\
&\le \{\|u_n - S_n v_n\| + \|(x^* - y^*) - (u_n - z_n)\| + \lambda_1 \|B_1 z_n - B_1 y^*\|\} \\
&\quad \times \{\|(z_n - \lambda_1 B_1 z_n) - (y^* - \lambda_1 B_1 y^*)\| + \|S_n v_n - x^*\|\} \\
&\quad + 2\lambda_1 \|B_1 z_n - B_1 y^*\| \|(z_n - v_n) + (x^* - y^*)\|.
\end{aligned}$$

Since

$$\|u_n - S_n v_n\| \le \|u_n - x_n\| + \|x_n - S_n v_n\|. \tag{61}$$

From Equations (54) and (57), implies that

$$\lim_{n\to\infty} \|u_n - S_n v_n\| = 0. \tag{62}$$

We have from Equations (58), (60) and (62), it follows that

$$\lim_{n\to\infty} \|(z_n - v_n) + (x^* - y^*)\| = 0. \tag{63}$$

Also, observe that

$$\begin{aligned}
&\|S_n v_n - v_n\| \\
&= \|S_n v_n - u_n + u_n - z_n + z_n - x^* + x^* - y^* + y^* - v_n\| \\
&\le \|S_n v_n - u_n\| + \|(u_n - z_n) - (x^* - y^*)\| + \|(z_n - v_n) + (x^* - y^*)\|.
\end{aligned}$$

From Equations (60), (62) and (63), we get

$$\lim_{n\to\infty} \|S_n v_n - v_n\| = 0. \tag{64}$$

Since

$$\|x_n - v_n\| \le \|x_n - S_n v_n\| + \|S_n v_n - v_n\|. \tag{65}$$

By Equations (54) and (64), we get

$$\lim_{n\to\infty} \|x_n - v_n\| = 0. \tag{66}$$

Step 5. We will show that $z \in \Omega_1 := \cap_{n=1}^{\infty} Fix(S_n) \cap \Gamma \cap GVI(C, B_1, B_2)$.

First, we show that $z \in \cap_{n=1}^{\infty} Fix(S_n)$. From $\|S_n v_n - v_n\| \to 0$, we obtain that $S_n v_{n_i} \rightharpoonup z$ as $i \to \infty$. By Lemma 4, we conclude that $z \in \cap_{n=1}^{\infty} Fix(S_n)$.

Next, we will show that $z \in \Gamma$.

First, we will show $z \in GEP(F_1, h_1)$.

Since $u_n = T_{r_n}^{(F_1,h_1)} x_n$, we have

$$F_1(u_n, w) + h_1(u_n, w) + \frac{1}{r_n}\langle w - u_n, u_n - x_n\rangle \ge 0, \quad \forall w \in C. \tag{67}$$

It follows from the monotonicity of F_1 that

$$h_1(u_n, w) + \frac{1}{r_n}\langle w - u_n, u_n - x_n\rangle \ge F_1(w, u_n), \tag{68}$$

and hence replacing n by n_i, we get

$$h_1(u_{n_i}, w) + \left\langle w - u_{n_i}, \frac{u_{n_i} - x_{n_i}}{r_{n_i}} \right\rangle \ge F_1(w, u_{n_i}). \tag{69}$$

Since $\|u_n - x_n\| \to 0$ we get $u_{n_i} \rightharpoonup z$ and $\frac{u_{n_i} - x_{n_i}}{r_{n_i}} \to 0$. It follows by Lemma 1 (iv) that $0 \ge F_1(w, z)$, $\forall z \in C$. For any t with $0 < t \le 1$ and $w \in C$, let $w_t = tw + (1-t)z$. Since $w \in C$, $z \in C$, we have $w_t \in C$, and hence, $F_1(w_t, z) \le 0$. So, from Lemma 1 (i) and (iv), we have

$$\begin{aligned}
0 &= F_1(w_t, w_t) + h_1(w_t, w_t) \\
&\le t[F_1(w_t, w) + h_1(w_t, w)] + (1-t)[F_1(w_t, z) + h_1(w_t, z)] \\
&\le t[F_1(w_t, w) + h_1(w_t, w)] + (1-t)[F_1(z, w_t) + h_1(z, w_t)] \\
&\le [F_1(w_t, w) + h_1(w_t, w)].
\end{aligned}$$

Therefore, $0 \le F_1(w_t, w) + h_1(w_t, w)$. From Lemma 1 (iii), we have $0 \le F_1(z, w) + h_1(z, w)$. This implies that $z \in GEP(F_1, h_1)$.

Next, we show that $Az \in GEP(F_2, h_2)$. Since $\|u_n - x_n\| \to 0$, $u_n \rightharpoonup z$ as $n \to \infty$ and $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \rightharpoonup z$, and since A is bounded linear operator, so $Ax_{n_i} \rightharpoonup Az$.

Now, setting $k_{n_i} = Ax_{n_i} - T_{r_{n_i}}^{(F_2,h_2)} Ax_{n_i}$. It follows from (55) that $\lim_{i\to\infty} k_{n_i} = 0$ and $Ax_{n_i} - k_{n_i} = T_{r_{n_i}}^{(F_2,h_2)} Ax_{n_i}$.

Therefore, from Lemma 3, we have

$$F_2(Ax_{n_i} - k_{n_i}, \tilde{z}) + h_2(Ax_{n_i} - k_{n_i}, \tilde{z}) + \frac{1}{r_{n_i}}\langle \tilde{z} - (Ax_{n_i} - k_{n_i}), (Ax_{n_i} - k_{n_i}) - Ax_{n_i}\rangle \ge 0, \tag{70}$$

$\forall \tilde{z} \in Q$. Since F_2 and h_2 are upper semicontinuous, taking lim sup to above inequality as $i \to \infty$ and using condition (C3), we obtain

$$F_2(Az, \tilde{z}) + h_2(Az, \tilde{z}) \ge 0, \quad \forall \tilde{z} \in Q, \tag{71}$$

which means that $Az \in GEP(F_2, h_2)$ and hence $z \in \Gamma$.

Last, we show that $z \in GVI(C, B_1, B_2)$. Since $\lim_{n\to\infty} \|S_n v_n - v_n\| = 0, \lim_{n\to\infty} \|S_n v_n - x_n\| = 0, \lim_{n\to\infty} \|x_n - u_n\| = 0$, then we get

$$\|v_n - u_n\| \leq \|v_n - S_n v_n\| + \|S_n v_n - x_n\| + \|x_n - u_n\|, \tag{72}$$

we conclude that $\lim_{n\to\infty} \|v_n - u_n\| = 0$. Furthermore, by the nonexpansivity of $\bar{G}$ in Lemma 6, we have

$$\begin{aligned}\|v_n - \bar{G}(v_n)\| &= \|P_C[P_C(u_n - \lambda_2 B_2 u_n) - \lambda_1 B_1 P_C(u_n - \lambda_2 B_2 u_n)] - \bar{G}(v_n)\| \\ &= \|\bar{G}(u_n) - \bar{G}(v_n)\| \\ &\leq \|u_n - v_n\|.\end{aligned}$$

Since $\|u_n - v_n\| \leq \|u_n - x_n\| + \|x_n - v_n\|$, and $\|u_n - x_n\| \to 0, \|x_n - v_n\| \to 0$, we get $\|u_n - v_n\| \to 0$ as $n \to \infty$. Thus, $\lim_{n\to\infty} \|v_n - \bar{G}(v_n)\| = 0$. According to Lemma 4, we obtain that $z \in GVI(C, B_1, B_2)$. Hence, $z \in \Omega_1$.

Step 6. We will show that $x_n \to z$, where $z = P_{\Omega_1} x_0$. Since $x_n = P_{C_n} x_0$ and $z \in \Omega_1 \subset C_n$, we have

$$\|x_n - x_0\| \leq \|z - x_0\|. \tag{73}$$

It follows from $\acute{z} = P_{\Omega_1} x_0$ and the lower semicontinuity of the norm that

$$\|\acute{z} - x_0\| \leq \|z - x_0\| \leq \liminf_{i\to\infty} \|x_{n_i} - x_0\| \leq \limsup_{i\to\infty} \|x_{n_i} - x_0\| \leq \|\acute{z} - x_0\|. \tag{74}$$

Thus, we obtain that $\lim_{i\to\infty} \|x_{n_i} - x_0\| = \|z - x_0\| = \|\acute{z} - x_0\|$. Using the Kadec-Klee property of H_1, we obtain that

$$\lim_{i\to\infty} x_{n_i} = z = \acute{z}. \tag{75}$$

Since $\{x_{n_i}\}$ is an arbitrary weakly convergent subsequence of $\{x_n\}$, we can conclude that $\{x_n\}$ converges strongly to z, where $z = P_{\Omega_1} x_0$. This complete the proof. □

Corollary 1 *Let H_1 and H_2 be two real Hilbert spaces and $C \subset H_1$ and $Q \subset H_2$ be nonempty closed convex subsets of H_1 and H_2, respectively. Let B be β-inverse strongly monotone mappings from C into H_1, respectively. Let $A : H_1 \to H_2$ be a bounded linear operator. Let $F_1, h_1 : C \times C \to \mathbf{R}$ and $F_2, h_2 : Q \times Q \to \mathbf{R}$ satisfying Lemma 1; h_1, h_2 are monotone and F_2 is upper semicontinuous and S be a nonexpansive mapping from C into itself such that*

$$\Omega_1 := \cap_{n=1}^{\infty} Fix(S) \cap \Gamma \cap VI(C, B) \neq \emptyset. \tag{76}$$

For a given $x_0 \in H, C_1 = C, x_1 = P_C x_0, u_n \in C$, let the iterative sequences $\{u_n\}, \{z_n\}, \{y_n\}$ and $\{x_n\}$ be generated by

$$\begin{cases} u_n = T_{r_n}^{(F_1,h_1)}(x_n + \xi A^*(T_{r_n}^{(F_2,h_2)} - I)Ax_n), \\ z_n = P_C(u_n - \lambda B u_n), \\ y_n = \alpha_n x_n + (1 - \alpha_n) S P_C(z_n - \lambda B z_n), \\ C_{n+1} = \{z \in C_n : \|y_n - z\| \leq \|x_n - z\|\}, \\ x_{n+1} = P_{C_{n+1}} x_0, \quad \forall n \in \mathbf{N}, \end{cases} \tag{77}$$

where $\{\alpha_n\} \in (0, l]$ for some $l \in (0, 1), \lambda_1 \in [a, b] \subset (0, 2\beta), \xi \in (0, \frac{1}{L}), L$ is the spectral radius of the operator A^*A and A^* is the adjoint of A and $\{r_n\} \subset (0, \infty)$ satisfying $\liminf_{n\to\infty} r_n > 0$. Then the sequence $\{x_n\}$ converges strongly to $P_{\Omega_1} x_0$ and (x^*, y^*) is a solution of a general system of variational inequalities problems, where $y^* = P_C(x^* - \lambda B x^*)$.

Proof Setting $S_n \equiv S$ for all $n \in \mathbf{N}$, $B \equiv B_1 \equiv B_2$ and $\lambda = \lambda_1 = \lambda_2$ in Theorem 1. □

4 Conclusion

The results presented in Theorem 1 and Corollary 1 of this paper extend and improve the results of Kazmi and Rivi [18], Kumam [21] and Kumam and Kumam [22].

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