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# Efficiency Improvement Strategy Under Constant Sum of Inputs

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**Abstract** In this paper, we have formulated Data Envelopment Analysis (DEA) models to reduce the inputs in an inefficient Decision Making Unit (DMU) when the specific inputs are under the constant sum constraint. We have also extended the models to reallocate the excess input without any reduction in efficiency of other DMUs. These DEA models and methods developed in this work will help decision makers in developing an optimal strategy to transfer excess input to other DMUs. Theoretical results have been illustrated with the help of a case study.

**Keywords** DEA · Constant sum of inputs · Efficiency · Linear programming

**Mathematics Subject Classifications (2010)** 90C32 · 90C90

## 1 Introduction

Data Envelopment Analysis (DEA) is a widely applied non-parametric mathematical programming technique to calculate the relative efficiency of firms/organizations or Decision Making Units (DMUs) operating in a similar environment and utilizing multiple inputs to produce multiple outputs. Based on Farrell's [9] work on productive efficiency, DEA was first introduced by Charnes et al. [3]. Efficiency obtained using DEA, in its simplest form, is the comparison of the weighted output to weighted input ratio of the observing DMU with that of the best practice in the group. Measurement of efficiency is important to shareholders, managers, and investors for any future course of action. DEA has been extensively applied to a wide spectrum of practical problems. Examples include financial institutions [16], bank failure prediction [1], electric utilities evaluation [12], textile industry performance [2], and portfolio evaluation [15].

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The basic DEA models start with the Charnes Cooper Rhodes (CCR) models [3]. The CCR models assume that the production function exhibits Constant Returns to Scale (CRS). These models use linear programming to determine the optimum virtual weights for the parameters in order to calculate the best possible efficiency score for each DMU in comparison to the other DMUs. These models can have the orientation of minimizing the inputs while keeping outputs constant (input oriented model) or vice versa (output-oriented). The Banker-Charnes-Cooper (BCC) models [7], an extension of the basic CCR models, are used when the production function exhibits Variable Returns to Scale (VRS). Other models, like Additive models [18] focus directly on the slacks in the system as a means of determining efficiency. A further difference is that CCR and BCC models focus on radial projection whereas additive models look at non-radial projection. The two approaches are combined in the Hybrid model [7]. Further developments in DEA involve generalizing these models for wider applications under different sets of assumptions.

One of the assumptions underlying basic DEA models is the assumption of free disposability of inputs and outputs, i.e., a DMU is free to change its inputs and outputs without any conflict with the other DMUs. However, this assumption cannot always hold. One situation where this assumption fails is the case when the DMUs are under the Constant Sum of Input (CSOI) constraint. In this situation, the total amount of input in one or more input parameters is a fixed quantity. CSOI problems in the real world can be seen whenever there is a limit to the amount of resources available. This could be a fixed cost [5] or office space [11] to be allocated to different departments of an organization. It could even be an undesirable output, like carbon-dioxide emissions by different countries allocated using carbon credits [10]. Considerable prior research has already been done based on this scenario. Cook and Kress [5] developed a fixed-cost allocation model in 1999, and Yan et al. [20] utilized inverse DEA models for resource allocation. More recent work includes Guedes de Avellar et al. [13] Spherical Frontier Model of fixed cost allocation. Yang et al. [21] work on competition strategy under fixed-sum outputs is used as the basis of the models developed in this paper.

The methods described in existing works [5, 11, 13] on CSOI generally assume the CSOI scenario is a fixed cost allocation problem where the fixed input in question is to be freely allocated among the various DMUs. We have discussed this problem from a different perspective - we assume that the fixed inputs have already been distributed among various DMUs, and now an inefficient DMU will improve its efficiency by transferring excess input to other DMUs. In this paper, we have formulated DEA models to find a strategy to reduce the inputs in an inefficient DMU from a given set of DMUs, when the multiple inputs in question are under the CSOI constraint. This strategy has been further developed to the situations when the efficiency improvement takes place without reducing the efficiency of other DMUs. It differs from the existing research on CSOI in the sense that it assumes the resources have already been allocated, and now an inefficient DMU needs to reallocate excess input while under the CSOI constraint. The models and algorithm developed in the paper can carry out the reallocation without reducing the efficiency of the receiving DMUs. This is an improvement over the original method in Yang et al. [21] as their model does not take into account the effect of reallocation on the efficiency of other DMUs.

This paper is organized as follows. In Section 2, we have formulated various DEA models and theoretical results have been developed. Section 3 contains a case study to illustrate the theoretical results developed in the paper. In Section 4, we provide the concluding remarks.

## 2 Model Formulation and Theoretical Results

The following notations have been used throughout this paper. Other notations, used in certain sections, will be defined at appropriate places when used.

$n$: The number of DMUs.
$m$: The number of inputs.
$s$: The number of outputs.
$x_{ij}$: $j^{th}$ input of the $i^{th}$ DMU.
$y_{ir}$: $r^{th}$ output of the $i^{th}$ DMU.
$u_0$: Value representing the variable part of variable returns to scale DEA models like BCC.
$u_r$: The weight assigned to the $r^{th}$ output.
$v_j$: The weight assigned to the $j^{th}$ input.
$\theta_k$: Efficiency of the $k^{th}$ DMU.
$f_{kj}$: The amount being reduced from the $j^{th}$ input under CSOI from the $k^{th}$ DMU.
$s_{ij}$: The amount being added to the $j^{th}$ input of the $i^{th}$ DMU, $i \neq k$.
$X_{kl}$: The maximum value to which the $l^{th}$ input of the $k^{th}$ DMU must be reduced, for the $k^{th}$ DMU to become efficient.
$\epsilon$: An infinitesimally small positive value.

### 2.1 Model Formulation Under Variable Returns to Scale (VRS)

Consider the following input-oriented BCC model to evaluate the efficiency of the $k^{th}$ DMU:

$$\theta_k = Max \frac{\sum_{r=1}^{s} u_r y_{kr} + u_0}{\sum_{j=1}^{m} v_j x_{kj}}$$

subject to

$$\frac{\sum_{r=1}^{s} u_r y_{ir} + u_0}{\sum_{j=1}^{m} v_j x_{ij}} \leq 1, \; i = 1, \ldots, n,$$

$$v_j, \; u_r \geq 0, \; j = 1, \ldots, m, \; r = 1, \ldots, s, \; u_0 \text{ free in sign.}$$

Without any loss of generality, it can be assumed that the first $d$ input parameters are under CSOI constraint. Since they are under CSOI constraint, therefore the sum of all changes must be 0. Thus, if the $j^{th}$ input of the $k^{th}$ DMU i.e., $x_{kj}$, is reduced by a certain amount $f_{kj}$ then the value of the $j^{th}$ input of the other DMUs will have to be increased. Let $s_{ij} (i \neq k, i = 1, \ldots, n)$ be the amount by which the $j^{th}$ input of the $i^{th} (i \neq k)$ DMU is increased, then $f_{kj} = \sum_{i=1, i \neq k}^{n} s_{ij}$, $f_{kj} < x_{kj}$. After incorporating these changes into the BCC model, the new efficiency of the $k^{th}$ DMU can be obtained by solving the following model $(M1)$:

$$(M1)\ \ \theta_k^{new} = Max \frac{\sum_{r=1}^{s} u_r y_{kr} + u_0}{\sum_{j=1}^{d} v_j (x_{kj} - f_{kj}) + \sum_{j=d+1}^{m} v_j x_{kj}}$$

subject to

$$\frac{\sum_{r=1}^{s} u_r y_{kr} + u_0}{\sum_{j=1}^{d} v_j (x_{kj} - f_{kj}) + \sum_{j=d+1}^{m} v_j x_{kj}} \leq 1,$$

$$\frac{\sum_{r=1}^{s} u_r y_{ir} + u_0}{\sum_{j=1}^{d} v_j (x_{ij} + s_{ij}) + \sum_{j=d+1}^{m} v_j x_{ij}} \leq 1,\ i \neq k,\ i = 1, \ldots, n,$$

$$f_{kj} = \sum_{i=1, i\neq k}^{n} s_{ij},\ j = 1, \ldots, d,$$

$$f_{kj} \leq x_{kj} - \epsilon,\ j = 1, \ldots, d,$$

$$f_{kj},\ s_{ij} \geq 0,\ j = 1, \ldots, d,\ i = 1, \ldots, n,$$

$$v_j,\ u_r \geq 0,\ j = 1, \ldots, m,\ r = 1, \ldots, s,\ u_0 \text{ free in sign.}$$

**Theorem 1** *In the model* $(M1)$*, for any* $k^{th}$ *observed DMU from a set of DMUs, there exists at least one feasible solution at which* $\theta_k^{new} = 1$.

*Proof* To prove this, we need to show that there exists at least one feasible solution such that $\theta_k^{new} = 1$. Let us set the values $\hat{u}_0 = 0$ and $\hat{v}_j = 0$ (representing the values of $u_0$ and $v_j$ respectively) for $j = 2, \ldots, m$. Now, let $\hat{u}_r$ (representing the value of $u_r;\ r = 1, \ldots, s$), and $\hat{v}_1$ (representing the value of $v_1$) be any positive values such that

$$\frac{\sum_{r=1}^{s} \hat{u}_r y_{ir}}{\hat{v}_1 x_{i1}} \leq 1,\ i = 1, \ldots, n. \tag{2.1}$$

Using the values set in Eq. 2.1, let there be a positive value $\hat{f}_{k1} < x_{k1}$ (representing the value of $f_{k1}$), such that

$$\frac{\sum_{r=1}^{s} \hat{u}_r y_{kr}}{\hat{v}_1 (x_{k1} - \hat{f}_{k1})} = 1$$

$$\Rightarrow \hat{f}_{k1} = x_{k1} - \frac{\sum_{r=1}^{s} \hat{u}_r y_{kr}}{\hat{v}_1} \leq x_{k1} - \epsilon \tag{2.2}$$

Let us choose the values of $\hat{s}_{i1}$ such that $\hat{f}_{k1} = \sum_{i=1, i \neq k}^{n} \hat{s}_{i1}$ and let

$$\hat{s}_{ij},\ \hat{f}_{kj} = 0;\ j = 2, \ldots, d. \tag{2.3}$$

Substituting these assumed values $\hat{s}_{ij}$, $\hat{f}_{kj}(i \neq k, i = 1, \ldots, n, j = 1, \ldots, d)$, $\hat{u}_0$, $\hat{u}_r(r = 1, \ldots, s)$, $\hat{v}_j(j = 1, \ldots, m)$ in model $(M1)$, we get

$$\theta_k^{new} = \frac{\sum_{r=1}^{s} \hat{u}_r y_{kr} + \hat{u}_0}{\sum_{j=1}^{d} \hat{v}_j (x_{kj} - \hat{f}_{kj}) + \sum_{j=d+1}^{m} \hat{v}_j x_{kj}} = 1, \qquad \text{(Using (2.2))}$$

$$\frac{\sum_{r=1}^{s} \hat{u}_r y_{ir} + \hat{u}_0}{\sum_{j=1}^{d} \hat{v}_j (x_{ij} + \hat{s}_{ij}) + \sum_{j=d+1}^{m} \hat{v}_j x_{ij}} \leq \frac{\sum_{r=1}^{s} \hat{u}_r y_{ir}}{\hat{v}_1 x_{i1}} \leq 1,\ i \neq k,\ i = 1, \ldots, n, \qquad \text{(Using (2.1))}$$

$$\hat{f}_{kj} = \sum_{i=1, i \neq k}^{n} \hat{s}_{ij}, \qquad \text{(Using (2.3))}$$

$$\begin{gathered} \hat{f}_{kj} \leq x_{kj} - \epsilon, \\ \hat{f}_{kj},\ \hat{s}_{ij} \geq 0,\ j = 1, \ldots, d,\ i = 1, \ldots, n, \\ \hat{v}_j,\ \hat{u}_r \geq 0,\ j = 1, \ldots, m,\ r = 1, \ldots, s,\ \hat{u_0} \text{ free in sign.} \end{gathered}$$

(Using (2.2 and 2.3))

Thus, we have proved that the assumed values $\hat{s}_{ij},\ \hat{f}_{kj}, \hat{u}_0,\ \hat{u}_r, \hat{v}_j$ represent a feasible solution according to the constraints in model $(M1)$ and for which $\theta_k^{new} = 1$. This concludes the proof. $\square$

By Theorem 1, we have shown that there exists at least one feasible solution for which $\theta_k^{new} = 1$. Since $\theta_k^{new}$ in model $(M1)$ is the objective function of a maximizing optimization with a maximum possible value of 1, it can be concluded that for any set of DMUs, $\theta_k^{new} = 1$. This implies that the observed DMU will always achieve efficiency when model $(M1)$ is solved. Next, we analyze the problem of minimizing the amount of reduction. This would mean minimizing the weighted sum of reduction amount, $\sum_{j=1}^{d} v_j f_{kj}$. However, it is problematic to use this as an objective function because the weights $v_j$'s tend to be lowered to 0. Instead, the objective function is changed to maximize the new weighted sum of inputs

$\sum_{j=1}^{d} v_j(x_{kj} - f_{kj})$ of the observed DMU. We have already shown that there always exist values of $f_{kj}$ for which $\theta_k^{new} = 1$, and therefore, this is taken as a constraint. After applying these modifications, the new model $(M2)$ is as follows:

$$
\begin{aligned}
(M2)\quad & Max \sum_{j=1}^{d} v_j(x_{kj} - f_{kj}) \\
& \text{subject to} \\
& \frac{\sum_{r=1}^{s} u_r y_{kr} + u_0}{\sum_{j=1}^{d} v_j(x_{kj} - f_{kj}) + \sum_{j=d+1}^{m} v_j x_{kj}} = 1, \\
& \frac{\sum_{r=1}^{s} u_r y_{ir} + u_0}{\sum_{j=1}^{d} v_j(x_{ij} + s_{ij}) + \sum_{j=d+1}^{m} v_j x_{ij}} \leq 1,\ i \neq k,\ i = 1, \ldots, n, \\
& f_{kj} = \sum_{i=1, i \neq k}^{n} s_{ij}, \\
& f_{kj} \leq x_{kj} - \epsilon, \\
& f_{kj},\ s_{ij} \geq 0,\ j = 1, \ldots, d,\ i = 1, \ldots, n, \\
& v_j,\ u_r \geq 0,\ j = 1, \ldots, m,\ r = 1, \ldots, s,\ u_0 \text{ free in sign.}
\end{aligned}
$$

The first constraint of model $(M2)$ sets the weighted output to input ratio of the observed DMU equal to 1 after decrease in input, thus ensuring the observed DMU will achieve efficiency after the decrease. Also, it can be easily shown that if a DMU is already efficient, applying model $(M2)$ will result in an objective value of $\sum_{j=1}^{d} v_j x_{kj}$, as no input reduction is necessary.

However, model $(M2)$ is a non-linear programming problem. To transform it to a linear programming problem, we set $v_j f_{kj} = \tau_{kj}$ and $v_j s_{ij} = \phi_{ij}$, then carry out cross-multiplication on the constraints. We add the constraint $\sum_{j=1}^{m} v_j x_{kj} = 1$ to prevent the weights from reaching extreme values. It should also be noted that $\sum_{j=1}^{d}(v_j x_{kj} - \tau_{kj}) + \sum_{j=d+1}^{m} v_j x_{kj} = \sum_{j=1}^{m} v_j x_{kj} - \sum_{j=1}^{d} \tau_{kj}$, which allows us to simplify some of the constraints. The linear programming model $(M3)$ is shown below.

$$
\begin{aligned}
(M3)\quad & Max \sum_{j=1}^{d}(v_j x_{kj} - \tau_{kj}) \\
& \text{subject to} \\
& \sum_{r=1}^{s} u_r y_{kr} + u_0 - \sum_{j=1}^{m} v_j x_{kj} + \sum_{j=1}^{d} \tau_{kj} = 0, \\
& \sum_{r=1}^{s} u_r y_{ir} + u_0 - \sum_{j=1}^{m} v_j x_{ij} - \sum_{j=1}^{d} \phi_{ij} \le 0,\ i \neq k,\ i = 1, \ldots, n, \\
& \sum_{j=1}^{m} v_j x_{kj} = 1, \\
& \tau_{kj} = \sum_{i=1, i\neq k}^{n} \phi_{ij}, \\
& \tau_{kj} \le v_j x_{kj} - \epsilon, \\
& \tau_{kj},\ \phi_{ij} \ge 0,\ j = 1, \ldots, d,\ i = 1, \ldots, n, \\
& v_j,\ u_r \ge 0,\ j = 1, \ldots, m,\ r = 1, \ldots, s,\ u_0 \text{ free in sign.}
\end{aligned}
$$

## 2.2 Improving the Efficiency of the Observed DMU Without Reducing Efficiency of the Other DMUs

So far, we have discussed the models where the excess input is redistributed to the other DMUs, without considering the effect on the efficiency of the receiving DMUs. We will now analyze how to redistribute the excess input of the observed DMU so that the efficiency scores of all the other DMUs are not adversely affected.

In order to achieve this, our aim is to identify those conditions under which we may increase the value of an input without reducing efficiency of the observed DMU. In the next theorem, we have shown that if the number of input parameters is reduced by one, then the new efficiency will not be better than the original efficiency.

Let $\theta_k^{PART}$ denote the efficiency of the $k^{th}$ DMU if the values of any one $j^{th}$ input parameter $x_{ij}$ $(i = 1, \ldots, n, j = 1, \ldots, m)$ is removed from the data set of all DMUs.

**Theorem 2** *If the original efficiency of the observed DMU is $\theta_k$ and the efficiency after one input parameter is removed is $\theta_k^{PART}$ then $\theta_k^{PART} \leq \theta_k$.*

*Proof* Without any loss of generality, it can be assumed that the first input parameter is missing from the data. Thus, the formulation to calculate $\theta_k^{PART}$ can be written as:

$$\theta_k^{PART} = Max \frac{\sum_{r=1}^{s} u_r y_{kr} + u_0}{\sum_{j=2}^{m} v_j x_{kj}}$$

subject to

$$\frac{\sum_{r=1}^{s} u_r y_{ir} + u_0}{\sum_{j=2}^{m} v_j x_{ij}} \leq 1;\ i = 1, \ldots, n,$$

$$v_j,\ u_r \geq 0,\ \ j = 2, \ldots, m,\ r = 1, \ldots, s,\ u_0 \text{ free in sign.}$$

If $v_1 = 0$, and all other values of $v_j$ $(j = 2, \ldots, m)$, $u_r$ $(r = 1, \ldots, s)$ and $u_0$ remain the same then $\theta_k^{PART} = \theta_k$. The actual solutions are obtained by a maximization problem, so we have proved that $\theta_k$ can never be less than $\theta_k^{PART}$ as there always exist a solution whereby it can at least be of equal value. □

*Remark 1* According to Theorem 2, $\theta_k^{PART} \leq \theta_k$. It can be shown that if we assume

$$\theta_k^{PART} = \theta_k \tag{2.4}$$

then in the optimal solution for the standard DEA model, $v_1 = 0$.

*Remark 2* The managerial implications of Theorem 2 are twofold. Firstly, Theorem 2 proves that the efficiency of any DMU will either reduce or stay the same if the number of input parameters is decreased. It also proves that increasing the number of input parameters will either increase efficiency of a DMU, or allow it to stay the same. Secondly, if Eq. 2.4 is satisfied by any $k^{th}$ DMU, then the input parameter that was removed for calculating $\theta_k^{PART}$ can be increased without reducing $\theta_k$, since its weight is 0. This means that Theorem 2 allows us to identify which input parameters in a DMU can be increased without reducing the DMU's efficiency.

Now, using these arguments, we describe an algorithm for distributing the excess input under CSOI in Section 2.3.

### 2.3 Algorithm for Observed DMU Achieving Efficiency Without Reducing Efficiency of the Other DMUs (All DMUs Having Two or More Inputs)

**Algorithm 1**

Step 1. Determine the maximum amount $X_{kl}$, to which the $l^{th}$ input of the $k^{th}$ DMU can be reduced so that the $k^{th}$ DMU becomes efficient. It can be assumed that the first $d$ inputs ($d \leq m$) are under CSOI. Choose any single value of $l(l = 1, \ldots, d)$, and solve the following model:

$$\begin{aligned}
(M4)\quad & Max\ X_{kl} \\
& \text{subject to} \\
& \sum_{r=1}^{s} u_r y_{kr} + u_0 - \sum_{j=1, j \neq l}^{m} v_j x_{kj} - v_l X_{kl} = 0, \\
& \sum_{r=1}^{s} u_r y_{ir} + u_0 - \sum_{j=1}^{m} v_j x_{ij} \leq 0,\ i \neq k,\ i = 1, \ldots, n, \\
& 0 \leq X_{kl} \leq x_{kl}, \\
& v_l \geq \epsilon, \\
& v_j,\ u_r \geq 0,\ j = 1, \ldots, m,\ r = 1, \ldots, s,\ u_0\ \text{free in sign.}
\end{aligned}$$

Model ($M4$) is repeated for all values of $l(l = 1, \ldots, d)$, to get the corresponding values of $X_{kl}(l = 1, \ldots, d)$. Since it is necessary for $X_{kl}$ to have a non-zero weight for a meaningful result, the weight $v_l$ is constrained to be greater than 0.

Step 2. Determine the DMUs which satisfy Eq. 2.4 for Input 1. Let set $A_1$ be the set of these DMUs. Repeat this step for the $2^{nd}, 3^{rd}, \ldots, d^{th}$ inputs, and store the results to sets $A_2, A_3, \ldots, A_d$.

Step 3. Model ($M3$) is modified so that the excess input is redistributed only to those DMUs that satisfy (2.4) for that particular input, as determined in Step 2, and the inputs are not reduced to less than the values determined in Step 1. This modified model ($M5$) is as follows:

$$\begin{aligned}
(M5)\quad & Max \sum_{j=1}^{d} (v_j x_{kj} - \tau_{kj}) \\
& \text{subject to} \\
& \sum_{r=1}^{s} u_r y_{kr} + u_0 - \sum_{j=1}^{m} v_j x_{kj} + \sum_{j=1}^{d} \tau_{kj} = 0, \\
& \sum_{r=1}^{s} u_r y_{ir} + u_0 - \sum_{j=1}^{m} v_j x_{ij} - \sum_{j=1, i \in A_j}^{d} \phi_{ij} \leq 0,\ i \neq k,\ i = 1, \ldots, n, \\
& \sum_{j=1}^{m} v_j x_{kj} = 1,
\end{aligned}$$

$$\tau_{kj} = \sum_{i=1, i\neq k, i\in A_j}^{n} \phi_{ij}, \ j = 1, \ldots, d,$$
$$\tau_{kj} \leq v_j(x_{kj} - X_{kj}), \ j = 1, \ldots, d,$$
$$\phi_{ij} \geq 0, i \in A_j, \ j = 1, \ldots, d,$$
$$\tau_{kj} \geq 0, \ j = 1, \ldots, d,$$
$$v_j, \ u_r \geq 0, \ j = 1, \ldots, m, \ r = 1, \ldots, s, \ u_0 \text{ free in sign.}$$

**Theorem 3** *Model* $(M4)$ *always has a feasible solution.*

*Proof* Let $u_r(r = 1, \ldots, s)$ and $u_0$ have any positive values. Set $v_j = 0(j = 1, \ldots, l, j \neq l)$. Now, select a value of $v_l$ such that $\sum_{r=1}^{s} u_r y_{ir} + u_0 - v_l x_{il} \leq 0$ for all $i = 1, \ldots, n, i \neq k$. This means a feasible value of $v_l$ is

$$v_l = Max \left( \frac{\sum_{r=1}^{s} u_r y_{ir} + u_0}{x_{il}} \right), i = 1 \ldots, n, i \neq k.$$

Now, the value of $X_{kl}$ can be calculated as

$$X_{kl} = \frac{\sum_{r=1}^{s} u_r y_{kr} + u_0}{v_l}.$$

Thus, we have proved that by setting $v_j = 0(j = 1, \ldots, l, j \neq l)$ and $u_r, u_0 \geq 0(r = 1, \ldots, s)$ we can get feasible solutions to $v_l$ and $X_{kl}$ for any set of input and output values. □

*Remark 3* It is possible for the above model $(M5)$ to have alternative optimal solutions. Although results developed in this paper are mainly concerned with the efficiency scores of a set of DMUs, analysts may be interested in eliminating the alternate optimal solutions to have more insights into the performance of various DMUs. In case of model $(M5)$ having alternate optima, efficient DMUs may have alternative parameter weights that give the same efficiency score.

Here, we eliminate the alternate optimal solutions using the approach proposed by Cooper et al. [6]. In this, we choose to eliminate optima where the weights of one or more input parameter $(v_j)$ or output parameter $(u_r)$ has a value of zero. A zero parameter weight implies that particular parameter is not being considered at all while calculating the observed DMU's efficiency, and this is undesirable as efficiency of a DMU is meant to be calculated using all chosen parameters. The objective of model $(M6)$ is to identify an optimum solution where the maximum number of parameter weights are strictly positive.

The model for eliminating alternate optima is a mixed integer linear programming problem. Let $s_{ij}^*(i = 1, \ldots, n,\ i \neq k,\ j = 1, \ldots, d)$ and $f_{kj}^* = \sum_{i=1, i\neq k, i\in A_j}^{n} s_{ij}^*(j = 1, \ldots, d)$ be the results for input transfer calculated from the solution of model $(M5)$.

Let $z_0$ be the variable that checks how many parameter weight are strictly positive.

Let $I_p$ be the integer variable checking if a parameter is strictly positive.

Let M be an arbitrarily large positive value.

The model $(M6)$ is as follows:

$$(M6)\quad Max\, z_0 = \sum_{p=1}^{m+s} I_p$$

subject to

$$\sum_{r=1}^{s} u_r y_{kr} + u_0 - \sum_{j=1}^{m} v_j x_{kj} + \sum_{j=1}^{d} \tau_{kj} = 0,$$

$$\sum_{r=1}^{s} u_r y_{ir} + u_0 - \sum_{j=1}^{m} v_j x_{ij} - \sum_{j=1, i\in A_j}^{d} \phi_{ij} \leq 0,\ i \neq k,\ i = 1, \ldots, n,$$

$$\sum_{j=1}^{m} v_j x_{kj} = 1,$$

$$\tau_{kj} = \sum_{i=1, i\neq k, i\in A_j}^{n} \phi_{ij},\ j = 1, \ldots, d,$$

$$\tau_{kj} \leq v_j(x_{kj} - X_{kj}),\ j = 1, \ldots, d,$$

$$\sum_{j=1}^{d} \tau_{kj} \leq \sum_{j=1}^{d} v_j f_{kj}^*,$$

$$I_p \in \{0, 1\},\ p = 1, \ldots, m+s,$$

$$I_r \leq M u_r,\ r = 1, \ldots, s,$$

$$I_{s+j} \leq M v_j,\ j = 1, \ldots, m,$$

$$\tau_{kj} \geq 0,\ j = 1, \ldots, d,$$

$$\phi_{ij} \geq 0,\ i \in A_j,\ j = 1, \ldots, d,$$

$$v_j,\ u_r,\ z_0 \geq 0,\ j = 1, \ldots, m,\ r = 1, \ldots, s,\ u_0 \text{ free in sign.}$$

Model $(M6)$ ensures that the largest number of parameter weights have strictly positive values. However, it is possible for a parameter weight to have such a small value that even though it is positive, it does not have any effect on the model. To try and ensure that as many parameter weights as possible have meaningful values, we need to maximize the minimum value of all non-zero parameter weights. Let $I_p^*$ $(p = 1, \ldots, m+s)$ be the solution to model $(M6)$. Let $z_0$ be the minimum value of all parameter weights. The model to maximize the minimum value of the non-zero parameter weights is as shown in model $(M7)$.

$$
\begin{aligned}
(M7)\quad & Max\, z_0 \\
& \text{subject to} \\
& \sum_{r=1}^{s} u_r y_{kr} + u_0 - \sum_{j=1}^{m} v_j x_{kj} + \sum_{j=1}^{d} \tau_{kj} = 0, \\
& \sum_{r=1}^{s} u_r y_{ir} + u_0 - \sum_{j=1}^{m} v_j x_{ij} - \sum_{j=1, i \in A_j}^{d} \phi_{ij} \le 0,\ i \ne k,\ i = 1, \ldots, n, \\
& \sum_{j=1}^{m} v_j x_{kj} = 1, \\
& \tau_{kj} = \sum_{i=1, i \ne k, i \in A_j}^{n} \phi_{ij},\ j = 1, \ldots, d, \\
& \tau_{kj} \le v_j (x_{kj} - X_{kj}),\ j = 1, \ldots, d, \\
& \sum_{j=1}^{d} \tau_{kj} \le \sum_{j=1}^{d} v_j f_{kj}^*, \\
& I_r^* z_0 \le u_r,\ r = 1, \ldots, s, \\
& I_{s+j}^* z_0 \le v_j,\ j = 1, \ldots, m, \\
& \tau_{kj} \ge 0,\ j = 1, \ldots, d, \\
& \phi_{ij} \ge 0,\ i \in A_j,\ j = 1, \ldots, d, \\
& v_j,\ u_r,\ z_0 \ge 0,\ j = 1, \ldots, m,\ r = 1, \ldots, s,\ u_0 \text{ free in sign.}
\end{aligned}
$$

Through above we have described the process to eliminate the alternate optima for efficient DMUs as only the efficient DMUs in DEA are used to calculate the efficiency scores of all other DMUs. In addition, it may also be noted that the inefficient DMUs rarely have alternative optimal solution in practice [6].

*Remark 4* Algorithm 1 can only be utilized when there are at least two input parameters in the original data for the DMUs. When there is only one input, Step 1 of the algorithm cannot be carried out. This is because step 1 involves determining the efficiency of the observed DMU after the input under CSOI is removed from consideration. If there is only one input and it is removed, there is no input data and the efficiency cannot be calculated using the standard CCR or BCC DEA models.

*Remark 5* The DEA models described in this paper can be easily adapted to DMUs which operate under Constant Return to Scale (CRS). In order to adapt the model to CRS, we set $u_0 = 0$.

### 2.4 Algorithm for Observed DMU Achieving Efficiency Without Reducing Efficiency of the Other DMUs (All DMUs Having Only One Input)

The algorithm described in Section 2.3 can only be applied when $m \geq 2$. To obtain the desired result when there is only one input, we use the following algorithm.

**Algorithm 2**

Step 1. First, the amount of excess input $f_s$ in the observed DMU is determined by applying model $(M4)$ with $l = 1$. The excess input in the observed DMU $f_s = x_{k1} - X_{k1}$.

Step 2. The excess input $f_s$ as determined from Step 1, is now redistributed across all DMUs in proportion to their input, in order to ensure none of the DMUs drop in efficiency. Let the input values $X_{k1},\ x_{i1}(i \neq k)$ all be multiplied by value $\alpha(\alpha \geq 1)$, such that the sum of the new input values is equal to the original sum of input $\sum_{i=1}^{n} x_{i1}$.

$$\text{Thus,}\quad \alpha\left(\sum_{i=1}^{n} x_{i1} - f_s\right) = \sum_{i=1}^{n} x_{i1}$$

$$\Rightarrow \quad \alpha = \frac{\sum_{i=1}^{n} x_{i1}}{\sum_{i=1}^{n} x_{i1} - f_s} = 1 + \frac{f_s}{\sum_{i=1}^{n} x_{i1} - f_s} \tag{2.5}$$

Multiplying the input values of step 1 with $\alpha$, the new input values are

$$\Rightarrow x_{k1}^{*} = (x_{k1} - f_s)\left(1 + \frac{f_s}{\sum_{i=1}^{n} x_{i1} - f_s}\right) \tag{2.6}$$

$$x_{i1}^{*} = x_{i1}\left(1 + \frac{f_s}{\sum_{i=1}^{n} x_{i1} - f_s}\right),\ i = 1, \ldots, n,\ i \neq \tag{2.7}$$

## 3 A Case Study of $CO_2$ Emissions in 2012

We use a case study on carbon dioxide emissions by various countries to demonstrate how a DMU will use Algorithm 1 to improve efficiency under CSOI without reducing the efficiency of other DMUs. $CO_2$ emissions can be transferred from one country to another through the use of carbon credits. The data for this case study has been collected from a variety of public sources. For this study, we have collected data on 32 countries with the highest estimated $CO_2$ emissions in 2012 [8]. The countries use three inputs - $CO_2$ emissions [8], population [14], and energy consumption [17], in the year 2012, and the output is the 2012 national GDP, adjusted for Purchasing Power Parity (PPP), as estimated by the International Monetary Fund [19]. While $CO_2$ emissions are a product, of a country's industrial process,

**Table 1** Output/Input and data for 32 countries

| Country | Input 1 $CO_2$ emission (thou. ton.) | Input 2 Population (millions) | Input 3 Energy used (mil. ton. oil) | Output 1 GDP(PPP) ($ billion) | VRS Eff. |
|---|---|---|---|---|---|
| China | 9700000 | 1361.24 | 2735.2 | 12261 | 0.592 |
| USA | 5420000 | 317.13 | 2208.8 | 16244 | 1 |
| India | 1970000 | 1236.84 | 563.5 | 4716 | 0.886 |
| Russian Federation | 1830000 | 143.6 | 694.2 | 2486 | 0.406 |
| Japan | 1240000 | 127.29 | 478.2 | 4575 | 1 |
| Germany | 810000 | 80.55 | 311.7 | 3167 | 1 |
| South Korea | 610000 | 50.22 | 271.1 | 1622 | 0.73 |
| Canada | 560000 | 35.16 | 328.8 | 1446 | 0.882 |
| Indonesia | 490000 | 237.64 | 159.4 | 1212 | 0.735 |
| United Kingdom | 470000 | 63.71 | 203.6 | 2312 | 1 |
| Saudi Arabia | 460000 | 30 | 222.2 | 741 | 0.579 |
| Brazil | 450000 | 201.03 | 274.7 | 2330 | 0.895 |
| Mexico | 450000 | 118.4 | 187.7 | 1758 | 0.851 |
| Australia | 430000 | 23.26 | 125.7 | 961 | 0.953 |
| Iran | 410000 | 77.08 | 234.2 | 988 | 0.491 |
| Italy | 410000 | 59.83 | 162.5 | 1813 | 1 |
| South Africa | 360000 | 52.98 | 123.8 | 579 | 0.602 |
| France | 360000 | 65.81 | 245.4 | 2252 | 1 |
| Poland | 350000 | 38.5 | 97.6 | 802 | 0.911 |
| Ukraine | 320000 | 45.46 | 125.3 | 335 | 0.591 |
| Malaysia | 310429.33 | 29.79 | 76.3 | 492 | 0.926 |
| Spain | 300000 | 46.7 | 144.8 | 1407 | 0.971 |
| Turkey | 278866.33 | 75.63 | 119.2 | 1125 | 0.936 |
| Taiwan | 270000 | 23.36 | 109.4 | 902 | 0.951 |
| Thailand | 230000 | 65.93 | 117.6 | 646 | 0.717 |
| Kazakhstan | 222990.58 | 17.1 | 58.1 | 232 | 1 |
| Egypt | 208864.56 | 83.66 | 87.1 | 538 | 0.864 |
| Argentina | 195212.22 | 40.12 | 82.1 | 747 | 1 |
| Venezuela | 178217.22 | 29.28 | 86.8 | 402 | 0.957 |
| Pakistan | 174912.11 | 184.88 | 69.3 | 515 | 1 |
| United Arab Emirates | 170376.43 | 8.26 | 89.3 | 271 | 1 |
| Netherlands | 160000 | 16.81 | 89.1 | 710 | 1 |

since it is an undesirable product, it may be reclassified as an input [10] for the purposes of efficiency calculation. Since the countries have different sizes and populations and must operate at different scales, we use Variable Returns to Scale (VRS) models for our calculations. The efficiency scores of the DMUs in the numerical illustration are calculated using Coelli's [4] DEAP software. The data for the 32 countries is shown in Table 1.

Of the inefficient countries, we choose to improve the efficiency of Russia since it has the lowest efficiency.

**Table 2** New $CO_2$ emissions and VRS efficiency

| Country | Old $CO_2$ emissions (thou. ton.) | Old VRS Eff. | New $CO_2$ emissions (thou. ton.) | New VRS Eff. |
|---|---|---|---|---|
| China | 9700000 | 0.592 | 9700000 | 0.592 |
| USA | 5420000 | 1 | 6646846.174 | 1 |
| India | 1970000 | 0.886 | 1970000 | 0.886 |
| Russian Federation | 1830000 | 0.406 | 444839.792 | 1 |
| Japan | 1240000 | 1 | 1398314.034 | 1 |
| Germany | 810000 | 1 | 810000 | 1 |
| South Korea | 610000 | 0.73 | 610000 | 0.755 |
| Canada | 560000 | 0.882 | 560000 | 0.883 |
| Indonesia | 490000 | 0.735 | 490000 | 0.735 |
| United Kingdom | 470000 | 1 | 470000 | 1 |
| Saudi Arabia | 460000 | 0.579 | 460000 | 0.579 |
| Brazil | 450000 | 0.895 | 450000 | 0.9 |
| Mexico | 450000 | 0.851 | 450000 | 0.851 |
| Australia | 430000 | 0.953 | 430000 | 0.953 |
| Iran | 410000 | 0.491 | 410000 | 0.491 |
| Italy | 410000 | 1 | 410000 | 1 |
| South Africa | 360000 | 0.602 | 360000 | 0.602 |
| France | 360000 | 1 | 360000 | 1 |
| Poland | 350000 | 0.911 | 350000 | 0.911 |
| Ukraine | 320000 | 0.591 | 320000 | 0.591 |
| Malaysia | 310429.33 | 0.926 | 310429.33 | 0.926 |
| Spain | 300000 | 0.971 | 300000 | 0.971 |
| Turkey | 278866.33 | 0.936 | 278866.33 | 0.936 |
| Taiwan | 270000 | 0.951 | 270000 | 0.951 |
| Thailand | 230000 | 0.717 | 230000 | 0.717 |
| Kazakhstan | 222990.58 | 1 | 222990.58 | 1 |
| Egypt | 208864.56 | 0.864 | 208864.56 | 0.864 |
| Argentina | 195212.22 | 1 | 195212.22 | 1 |
| Venezuela | 178217.22 | 0.957 | 178217.22 | 0.957 |
| Pakistan | 174912.11 | 1 | 174912.11 | 1 |
| United Arab Emirates | 170376.43 | 1 | 170376.43 | 1 |
| Netherlands | 160000 | 1 | 160000 | 1 |

In step 1 of Algorithm 1, we calculate Russia's highest $CO_2$ emission quantity that guarantees efficiency, $X_{41} = 444839.792$.

Following Step 2 of Algorithm 1, we determine the DMUs which satisfy Eq. 2.4 for the input under CSOI, $CO_2$ emissions. The DMUs that fulfill the condition are listed in set $A_1$ as follows:

$$A_1 = \{1, 2, 3, 4, 5, 6, 7, 9, 10, 11, 14, 16, 17, 19, 21, 24, 26, 28, 30, 31, 32\}.$$

**Table 3** $CO_2$ Emissions and efficiency after eliminating alternate optima

| Country | Original $CO_2$ emissions (thou. ton.) | Original VRS Eff. | Final $CO_2$ emissions (thou. ton.) | Final VRS Eff. |
|---|---|---|---|---|
| China | 9700000 | 0.592 | 9700000 | 0.592 |
| USA | 5420000 | 1 | 6639577.088 | 1 |
| India | 1970000 | 0.886 | 1970000 | 0.886 |
| Russian Federation | 1830000 | 0.406 | 444839.7923 | 1 |
| Japan | 1240000 | 1 | 1405583.12 | 1 |
| Germany | 810000 | 1 | 810000 | 1 |
| South Korea | 610000 | 0.73 | 610000 | 0.755 |
| Canada | 560000 | 0.882 | 560000 | 0.883 |
| Indonesia | 490000 | 0.735 | 490000 | 0.735 |
| United Kingdom | 470000 | 1 | 470000 | 1 |
| Saudi Arabia | 460000 | 0.579 | 460000 | 0.579 |
| Brazil | 450000 | 0.895 | 450000 | 0.9 |
| Mexico | 450000 | 0.851 | 450000 | 0.851 |
| Australia | 430000 | 0.953 | 430000 | 0.953 |
| Iran | 410000 | 0.491 | 410000 | 0.491 |
| Italy | 410000 | 1 | 410000 | 1 |
| South Africa | 360000 | 0.602 | 360000 | 0.602 |
| France | 360000 | 1 | 360000 | 1 |
| Poland | 350000 | 0.911 | 350000 | 0.911 |
| Ukraine | 320000 | 0.591 | 320000 | 0.591 |
| Malaysia | 310429.33 | 0.926 | 310429.33 | 0.926 |
| Spain | 300000 | 0.971 | 300000 | 0.971 |
| Turkey | 278866.33 | 0.936 | 278866.33 | 0.936 |
| Taiwan | 270000 | 0.951 | 270000 | 0.951 |
| Thailand | 230000 | 0.717 | 230000 | 0.717 |
| Kazakhstan | 222990.58 | 1 | 222990.58 | 1 |
| Egypt | 208864.56 | 0.864 | 208864.56 | 0.864 |
| Argentina | 195212.22 | 1 | 195212.22 | 1 |
| Venezuela | 178217.22 | 0.957 | 178217.22 | 0.957 |
| Pakistan | 174912.11 | 1 | 174912.11 | 1 |
| United Arab Emirates | 170376.43 | 1 | 170376.43 | 1 |
| Netherlands | 160000 | 1 | 160000 | 1 |

In step 3, we use the results of the first two steps to apply model ($M5$). The results are shown in Table 2 on the following page.

Table 2 shows that Russia has now become efficient, and all the other countries still have the same efficiency score or better. Thus, Algorithm 1 has fulfilled its objective of making the chosen DMU efficient without reducing the efficiency of the other DMUs. Now, we need to eliminate alternate optima and select an optimum solution with as few extreme parameter weight values as possible. To do this we first apply model ($M6$) to the results of Algorithm 1. Model ($M6$) gives the results $z_0 = 4$ and $I_p = 1, 1, 1, 1$, meaning that there exists a solution where all parameter weights have strictly positive values. Using these results of model ($M6$) and the earlier results from model ($M5$), we apply model ($M7$) to maximize the minimum non-zero weight. The results of model ($M7$) are shown in Table 3.

Model ($M7$) causes the same reduction in input as model ($M5$), but the redistribution of the input is different. After model ($M5$), the parameter weights were $v_1 = 5.3487E-07$, $v_2 = 0.000147554$, $v_3 = 0$, $u_1 = 0.000242976$, meaning the LP placed no weight on the third input parameter. After applying ($M7$), the weights were $v_1 = 5.36327E{-}07$, $v_2 = 2.21069E-05$, $v_3 = 2.21069E-05$, $u_1 = 0.000244202$, and at least some weight has been placed on all parameters. Thus, the results in Table 3 represent an improved optimal solution over that shown in Table 2.

## 4 Conclusion

We have formulated DEA models to reduce the inputs in an inefficient DMU from a given set of DMUs, when the multiple inputs are under CSOI constraint. It is different from earlier works, which treated CSOI inputs as fixed costs to be allocated across all DMUs with maximum efficiency. The models developed here focus only on improving the efficiency of one DMU at a time. First, a model is developed to achieve the objective with a minimum amount of reduction in the original inputs of the observed DMU. Subsequently, algorithms are developed to increase the efficiency of the observed DMU without any reduction in efficiency of the other DMUs. Theoretical results developed in this paper have been illustrated with the help of a case study example on $CO_2$ emmission in 2012. One of the main advantages of this work is that it allows a single DMU to develop a strategy to improve its efficiency and the strategy will require the cooperation of only a few chosen DMUs. Existing work often treats the CSOI problem as a fixed cost allocation, assuming cooperation among all the DMUs, something that might not be feasible. Furthermore, the models and algorithms in this paper allow the improvement of efficiency of one DMU without reducing the efficiency of others. The limitation of this work is that the models in this paper are designed to work under the assumption that there is no restriction on parameter weights, and improve only the technical efficiency. Future developments may include addressing situations where the inputs are not freely disposable. Considerations of allocative efficiency or weight restrictions are beyond the scope of this work, but may be an area of future research.

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