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Wolfe Type Higher Order Multiple Objective Nondifferentiable Symmetric Dual Programming with Generalized Invex Function

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Abstract In this paper, a new class of higher order (ϕ, ρ)-invex function is introduced with an example, in which the sublinearity and convexity assumption on ϕ with respect to third argument is relaxed. A pair of higher order Wolfe type multiobjective symmetric dual for a class of nondifferentiable multiobjective programming involving square root term is presented and the weak duality, strong duality and converse duality theorems are established with their proofs under higher order (ϕ, ρ)-invexity and (ϕ, ρ)-incavity assumption. Self duality theorem is proved for the proposed dual program. These results are used to discuss Wolfe type higher-order symmetric minimax mixed integer dual problems. A numerical example is developed where the results of weak and strong duality theorems can be applied. Discussion on some particular cases shows that our results generalize earlier results in related domain.

Keywords Multiple objective programming · Higher order (ϕ, ρ)-invexity · Pareto optimal solution · Schwartz inequality · Square root term

1 Introduction

Duality is a fruitful theory in mathematical programming and is useful both theoretically and practically. Duality as used in our daily life means the sort of harmony of two opposite or complementary parts through which they integrate into a whole. Symmetry is bound up with duality and in particular, is significant in mathematics. Duality principle relates to

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constrained minimization and maximization problems. According to this principle the existence of a solution to one primal or dual ensures a solution to other and optimum value of the two problems is equal. The problem of optimizing a numerical function of one or more variable subject to constraints on the variables is called the mathematical programming or constrained optimization problem. When either the objective function or one or more of the constraints are nonlinear, the programming problem is called nonlinear programming problem. Nonlinear programming discipline plays an increasingly imperative role in such diverse fields as operations research and management science, engineering, economics, system analysis and computer science. Again, most of the optimizations problems arising in practice have several objectives which have to be optimized simultaneously. We generally aim at minimizing all the objective functions at the same time if there is no conflict between the objective functions. However, for general multiobjective programming, the objective functions are in contradiction to each other. For that case, Pareto optimality is a measure of efficiency. For details, readers can see [12].

Duality for nonlinear programming problem has been studied by many researchers in past. Symmetric duality in nonlinear programming was first introduced by. Dorn [18] who defined a mathematical programming problem and its dual to be symmetric if the dual of the dual is the primal problem. Later Dantzig et al. [16], Mond [31], Bazarra and Goode [8] and Mond and Weir [33] established a pair of symmetric dual programs involving scalar function $f(x, y)$, $x \in R^n$, $y \in R^m$ under the condition that $f(\cdot, y)$ is convex for each y and $f(x, \cdot)$ is concave for each x. Devi [17], Weir and Mond [36], Mond and Schechter [32] studied non differentiable symmetric duality for a class of optimization problem in which the objective function consist of support function. Husain et al. [25] formulate a pair of Mond Weir type second order symmetric dual and establish the duality results under pseudo convexity –pseudo concavity assumption.

In recent years, several extension and generalization have been considered for classical convexity. A significant generalization of convex function is that of in-vex function introduced by Hanson [23] and Craven [15]. After the work of Hanson and Craven, other types of differentiable functions have been introduced with the intent of generalizing in vex function from different point of view. Hanson and Mond [24] introduced the concept of F-convexity. The concept of generalized (F, ρ) convexity is introduced by Preda [35]. The (F, ρ)-convexity was recently generalized to (ϕ, ρ)-invexity by Caristi et al [9] in which ϕ is convex in its third argument and replaces the sub linear property of F in third argument. Liang et al. [26] introduced a unified formulation of generalized convexity called (F, α, ρ, d)-convexity and Yuan et al. [38] introduced the concept of (C, α, ρ, d)-convexity which is the generalization of (F, α, ρ, d)-convexity.

Higher order duality in nonlinear programming has been studied in last few years by many researchers. Mangasarian [27] formulated a class of higher order dual problems for nonlinear programming problems. Mond and Zhang [34] obtained duality results for various higher order dual programming problems under higher order invexity assumptions. Mishra and Rueda [29, 30] established duality results under higher order generalized invexity whereas they generalized the results of Zhang [39] to higher order type 1 function in their paper [30]. Later on Yang et al [37] discussed higher order duality results under generalized convexity assumption for multiobjective programming problems involving support functions. Chen [10] presented Mond-Weir type higher order symmetric duality for scalar and multiobjective nondifferentiable programming problem under F-convexity, where as

Mishra [28] presented Mond-Weir type higher order symmetric dual program under generalized invexity. Chinchuluun and Pardalos [11] gave a survey of recent developments in multiobjective optimization. Ahmad et al. [5] formulated a general Mond-Weir type higher order dual and established duality results under (F, α, ρ, d) - $type1$ function. Higher order symmetric multiobjective duality involving generalized (F, ρ, γ, b) - *convexity* was given by Batatoresue et al. [7]. Chinchuluun et al. [13] established the optimality condition and duality for nondifferentiable multiobjective fractional programming with generalized convexity. Gulati and Gupta [21] formulated Wolfe type higher order nondifferentiable symmetric duality for scalar programming problem containing support function and established the duality results with generalized F-convexity. Agarwal et al. [1] gave a note on higher order nondifferentiable symmetric duality in multiobjective programming. Gupta et al. [22] formulated a pair of symmetric higher order Wolfe type nondifferentiable multiobjective programs over arbitrary cones and proved weak, strong and converse duality theorems under higher order K-(F, α, ρ, d)-convexity assumption. Ahmad [2] introduced a unified higher order duality in nondifferentiable multiobjective programming involving cones. Ahmad and Husain [4] presented second order symmetric duality in multiobjective programming involving cones. Recently, Ahmad et al. [3] introduced higher order duality in nondifferentiable fractional programming involving generalized convexity.

In the present paper, we have introduced the concept of higher order (ϕ, ρ)-invexity which is a generalization of higher order (F, ρ)-convexity and higher order (C, ρ)-convexity by relaxing the assumption of sublinearity and convexity of F and C respectively with respect to third argument with an example. Then we have formulated a pair Wolfe type higher nondifferentiable multiobjective symmetric dual programs using square root term and established the duality results with higher order (ϕ, ρ)-invexity assumption. Also we have established higher order Wolfe type self duality theorem with its proof. A numerical example is developed in which the above duality results can be applied. Again the duality results are used to discuss Wolfe type higher order nondifferentiable symmetric minimax mixed integer dual programs. Special cases are discussed to show that this study extends some of the known results.

2 Notations and Preliminaries

Let R^n and R^m denote the n-dimensional and m-dimensional Euclidean space respectively. Also, let R^n_+ and R^m_+ be the nonnegative orthants of R^n and R^m respectively. The following conventions for vectors $x, u \in R^n$ will be followed throughout this paper; $x < u \Leftrightarrow x_i < u_i$, $i = 1, 2, .., n$ and $x \leq u \Leftrightarrow x_i \leq u_i$, $i = 1, 2, .., n$. Further for any vector, we denote $x^T u = \sum_{i=1}^{n} x_i u_i$.

Let X and Y are open subset sets of R^n and R^m respectively. Let $f_i(x, y)$ be a real valued twice differentiable function defined on $X \times Y$. Let $\nabla_x f_i(x, y)$ and $\nabla_y f_i(x, y)$ denote the gradient vectors of $f_i(x, y)$ with respect to first variable x and second variable y respectively. Also let $\nabla_{xx} f_i(x, y)$ and $\nabla_{yy} f_i(x, y)$ denote the Hessian matrix of $f_i(x, y)$ with respect to the first variable x and second variable y respectively. Let $r \in R$, $\rho \in R$. Assume that ϕ_0 and $\phi_1$are a real valued function defined on $X \times X \times R^{n+1}$ and $Y \times Y \times R^{m+1}$ respectively such that $\phi_0(x, u, (0, 0)) = 0$, $\phi_1(v, y, (0, 0)) = 0$.

Now we define the following definition;

Definition 2.1 A real-valued twice differentiable function $f_i(\ *\ , y)$: $X \times Y \to R$ is said to be ***higher order*** (ϕ, ρ)***-invex*** at $u \in X$ with respect to the differentiable function g_i: $X \times Y \times R^n \to R$, if there exist ϕ_0: $X \times X \times R^{n+1} \to R$, $\rho_i \in R$ such that

$$f_i(x, y) - f_i(u, y) \geq \phi_0 \left(x, u; \left(\nabla_x f_i(u, y) + \nabla_{q_i} g_i(u, y, q_i), \rho_i\right)\right) + g_i(x, y, q_i) - q_i^T \nabla_{q_i} g_i(x, y, q_i).$$

Definition 2.2 The function $f_i(x, *\ )$ is said to be higher order (ϕ, ρ)***-invex*** at $y \in Y$ with respect to a differentiable function h_i: $X \times Y \times R^m \to R$, if there exist ϕ_1: $Y \times Y \times R^{m+1} \to R$, $\rho_i \in R$ such that

$$f_i(x, v) - f_i(x, y) \geq \phi_1 \left(v, y; \left(\nabla_y f_i(x, y) + \nabla_{p_i} h_i(x, y, p_i), \rho_i\right)\right) + h_i(x, y, p_i) - p_i^T \nabla_{p_i} h_i(x, y, p_i).$$

Definition 2.3 A real-valued twice differentiable function $f_i(\ *\ , y)$: $X \times Y \to R$ is said to be ***higher order*** (ϕ, ρ)***-pseudo invex*** at $u \in X$ with respect to the differentiable function g_i: $X \times Y \times R^n \to R$, if there exist ϕ_0: $X \times X \times R^{n+1} \to R$, $\rho_i \in R$ such that

$$\phi_0(x, u; (\nabla_x f_i(u, y) + \nabla_{q_i} g_i(u, y, q_i), \rho_i)) \geq 0$$
$$\Rightarrow f_i(x, y) - f_i(u, y) \geq g_i(x, y, q_i) - q_i^T \nabla_{q_i} g_i(x, y, q_i).$$

Definition 2.4 A real-valued twice differentiable function $f_i(x, *\ )$: $X \times Y \to R$ is said to be ***higher order*** (ϕ, ρ)***-pseudo invex*** at $y \in Y$ with respect to the differentiable function h_i: $X \times Y \times R^m \to R$, if there exist ϕ_1: $Y \times Y \times R^{m+1} \to R$, $\rho_i \in R$ such that

$$\phi_1(v, y; (\nabla_y f_i(x, y) + \nabla_{p_i} h_i(x, y, p_i), \rho_i)) \geq 0$$
$$\Rightarrow f_i(x, v) - f_i(x, y) \geq h_i(x, y, p_i) - p_i^T \nabla_{p_i} h_i(x, y, p_i).$$

Definition 2.5 A real valued twice differentiable function f is ***higher order*** (ϕ, ρ)***- incave*** with respect to h and ***higher order*** (ϕ, ρ)***-pseudo incave*** with respect to h if - f is higher order (ϕ, ρ)-invex with respect to –h and higher order (ϕ, ρ) pseudo invex with respect to -h respectively.

Remark 2.1

i If $\phi(x, u; (\nabla_x f(u, y) + \nabla_q g(u, y, q), \rho)) = F(x, u; (\nabla_x f(u, y) + \nabla_q g(u, y, q)) + \rho d^2(x, u)$ with $\rho = 0$ and F is sub linear in third argument, then the above definition reduce to ***higher order F-convexity*** and ***higher order F-pseudo convexity*** introduced by Chen [10].

ii If $\phi(x, u; (\nabla_x f(u, y) + \nabla_q g(u, y, q), \rho)) = F(x, u; (\nabla_x f(u, y) + \nabla_q g(u, y, q)) + \rho d^2(x, u)$ with $\rho > 0$ and F is sub linear in third argument, then the above definition reduce to ***higher order (F,*** ρ***)convexity*** and ***higher order F-pseudo convexity*** introduced Batatorescu [7] with $b = 1$ and $\gamma = 1$.

iii If $\phi(x, u; (\nabla f(u) + \nabla_p h(u, p), \rho)) = \eta(x, u)^T [\nabla f(u) + \nabla_p h(u, p)] + \rho d^2(x, u)$, where η: $X \times X \to R^n$ and $\rho = 0$ then above definition reduces to ***higher order invex*** function given by Mishra [28].

iv If $\phi(x, u; (\nabla_x f(u, y) + \nabla_q g(u, y, q), \rho)) = F(x, u; \alpha(x, u)(\nabla_x f(u, y) + \nabla_q g(u, y, q)) + \rho d^2(x, u)$ and F is sub linear in third argument, then the above definitions reduce to

higher order (F, α, ρ, d)-convexity and **higher order (F, α, ρ, d)-pseudo convexity** introduced by Liang et al. [26].

v If ϕ is convex with respect to third argument, then we obtain the definition of higher order (C, α, ρ, d)-convex function introduced by Yuan et al. [38] with $\alpha = 1$, $d = 0$.

Example 2.1 Let $X = (0, \infty) = Y$. $P \subset R_+$. Let f: $X \times Y \to R$ be defined as $f(x, y) = x^4 + x^2 + y^2 + 1$ and $h : X \times Y \times P \to R$ be defined as $h(x, y, p) = 3p(x^2 + 1) - y$, where $x \in X$, $y \in Y$, $p \in P$, Let assume $\phi : X \times X \times P \times R \to R$ defined by $\phi(x, u; (a, \rho)) = (1 - 2^\rho)(x^4 + x^2) - u^T a$ and $F : X \times X \times R \to R$ defined by $F(x, u; a) = \frac{|a|}{2} x^2 (u + 1)^2$.

Clearly F is sub linear in third argument and also F is convex in third argument, but (i) ϕ is not sub linear with respect to third argument, because

$$\phi(x, u; (a_1 + a_2, \rho_1 + \rho_2)) \leq \phi(x, u; (a_1, \rho_1)) + \phi(x, u; (a_2, \rho_2))$$

is not true for

$$\rho_1 = 1, \rho_2 = -1.$$

and (ii) ϕ is not convex with respect to third argument, because

$$\phi(x, u; (\lambda(a_1, \rho_1) + (1 - \lambda)(a_2, \rho_2)) \ \leq \lambda\phi(x, u; (a_1, \rho_1)) + (1 - \lambda)\phi(x, u; (a_2, \rho_2))$$

is not true for

$$\lambda = \frac{1}{2}, \rho_1 = 1, \rho_2 = 0.$$

Now let

$$\alpha = 4, \rho = -2, d^2(x, u) = x^2 + u^2.$$

So at

$$u = 0, f(x, y) - f(u, y) - h(u, y, p) + p^T \nabla_p h(u, y, p) = x^4 + x^2 + y,$$
$$F(x, u; \alpha(x, u)(\nabla_x f(u, y) + \nabla_p h(u, y, p))) + \rho d^2(x, u) = 6x^2 - 2x^2 = 4x^2,$$

and

$$\begin{aligned}\phi(x, u; (\nabla_x f(u, y) + \nabla_p h(u, y, p), \rho)) &= \left(1-2^\rho\right)\left(x^4 + x^2\right) - u^T \\ &\quad \times \left(\nabla_x f(u, y) + \nabla_p h(u, y, p)\right) \\ &= (1 - 2^\rho)(x^4 + x^2).\end{aligned}$$

Hence $\forall x \in X$, $\forall y \in Y$, $\rho \in R$,

$$\begin{aligned}&f(x, y) - f(u, y) - h(u, y, p) + p^T \nabla_p h(u, y, p) \\ &\geq \phi\left(x, u; \left(\nabla_x f(u, y) + \nabla_p h(u, y, p), \rho\right)\right) - u^T \left(\nabla_x f(u, y) + \nabla_p h(u, y, p)\right.\end{aligned}$$

Therefore $f(\cdot, y)$: $X \times Y \to R$ is said to be ***higher order*** (ϕ, ρ)***-invex*** at $u = 0$ with respect to $h : X \times Y \times R \to R$.

But $\alpha = 4$, $\rho = -2$, $x = 1$, $y = 1$,

$$\begin{aligned}&f(x, y) - f(u, y) - h(u, y, p) + p^T \nabla_p h(u, y, p) = 3 \\ &< F\left(x, u; \alpha(x, u)\left(\nabla_x f(u, y) + \nabla_p h(u, y, p)\right)\right) + \rho d^2(x, u) = 4.\end{aligned}$$

So $f(\cdot, y)$: $X \times Y \to R$ is not **higher order** (F, α, ρ, d)**-convex** at $u = 0$ with respect to $h: X \times Y \times R \to R$.

Again since F is convex in third argument, we can replace F by C.

So $f(\cdot, y): X \times Y \to R$ is not **higher order** (C, α, ρ, d)**-convex** at $u = 0$ with respect to $h: X \times Y \times R \to R$.

Therefore the above examples clearly illustrate that the class of higher order **higher order** (ϕ, ρ)**-invex** is more generalized than the **higher order** (F, α, ρ, d)**-convex introduced by Liang et al.** [26] **and higher order** (C, α, ρ, d)**-convex introduced by Yuan et al.** [37]

Consider the following multiobjective programming problem (MP):

MP :(**Primal**) Minimize $f(x) = (f_1(x), f_2(x), \ldots, f_k(x))$

Subject to $h(x) \le 0, x \in X \subseteq R^n$, where $f: X \to R^k, h: X \to R^m$.

Definition 2.6 [12, 13] A vector $\overline{x} \in X_0$ is said to be an efficient solution (Pareto Optimal) of problem (P) if there exists no $x \in X_0$ such that $f_i(x) \le f_i(\overline{x})$, $\forall i = 1, 2, \ldots, k$ and $f_j(x) < f_j(\bar{x})$, for at least one index $j \in \{1, 2, \ldots, k\}$.

Definition 2.7 [12, 13] A vector $\bar{x} \in X_0$ is said to be weakly efficient solution (weakly Pareto Optimal) of problem (P) if there exists no $x \in X_0$ such that $f_i(x) < f_i(\overline{x})$, $\forall i = 1, 2, \ldots, k$.

Definition 2.8 (**Schwartz Inequality**) Let $x, y \in R^n$ and $A \in R^n \times R^n$ be a positive semi definite matrix, $x^T A y \le \left(x^T A x\right)^{\frac{1}{2}} \left(y^T A y\right)^{\frac{1}{2}}$, equality holds if for some $\lambda \ge 0$, $Ax = \lambda Ay$.

3 Wolfe Type Higher Order Symmetric Dual Programs

Let $f_i: R^n \times R^m \to R$, $g_i: R^n \times R^m \times R^n \to R$ and $h_i: R^n \times R^m \times R^m \to R$. are twice differentiable functions. $\lambda_i \in R$, $p_i \in R^m$, $q_i \in R^n$, $w_i \in R^m$, $z_i \in R^n$, $i = 1, 2, \ldots k$. B_i and C_i are positive semi definite matrices of order $n \times n$ and $m \times m$ respectively. And also $w = (w_1, w_2, \ldots w_k)$, $z = (z_1, z_2, \ldots z_k)$.

Now we formulate the following pair of Wolfe type higher order nondifferentiable multiobjective symmetric dual programs and prove duality theorems.

Wolfe type higher order nondifferentiable multiobjective symmetric primal

- Primal (WHNMSP):

$$L(x, y, w, p) = Minimize \begin{pmatrix} f_i(x, y) + \left(x^T B_i x\right)^{\frac{1}{2}} + h_i(x, y, p_i) - p_i^T \left(\nabla_{p_i} h_i(x, y, p_i)\right) \\ -y^T \left[\nabla_y f_i(x, y) + \nabla_{p_i} h_i(x, y, p_i)\right], i = 1, 2, ..k. \end{pmatrix}$$

Subject to $\sum_{i=1}^{k} \lambda_i \left[\nabla_y f_i(x, y) - C_i w_i + \nabla_{p_i} h_i(x, y, p_i)\right] \le 0,$ (3.1)

$$w_i^T C_i w_i \le 1, i = 1, 2, \ldots k. \tag{3.2}$$

$$\lambda > 0, \sum_{i=1}^{k} \lambda_i = 1, \tag{3.3}$$

- Dual (WHNMSD):

$$M(u,v,z,q) = Maximize\begin{pmatrix} f_i(u,v) - \left(v^T C_i v\right)^{\frac{1}{2}} + g_i(u,v,q_i) - q_i^T\left(\nabla_{q_i} g_i(u,v,q_i)\right) \\ -u^T\left[\nabla_u f_i(u,v) + \nabla_{q_i} g_i(u,v,q_i)\right], i = 1,2,..k. \end{pmatrix}$$

Subject to $\sum_{i=1}^{k}\lambda_i\left[\nabla_u f_i(u,v) + B_i z_i + \nabla_{q_i} g_i(u,v,q_i)\right] \geq 0,$

$$\tag{3.4}$$

$$z_i^T B_i z_i \leq 1, i = 1,2,...k. \tag{3.5}$$

$$\lambda > 0, \sum_{i=1}^{k}\lambda_i = 1, \tag{3.6}$$

Remark 3.1 Since the objective function of (WHNMSP) and (WHNMSD) contains the square root terms like $\left(x^T Ax\right)^{\frac{1}{2}}$, these problems are nondifferentiable programming problems.

Theorem 3.1 (**Weak Duality**) *Let (x, y, λ, w, p) be feasible solution of (WHNMSP) and (u, v, λ, z, q) be feasible solution of (WHNMSD) and assume that*

i. $\sum_{i=1}^{k}\lambda_i\left[f_i(\cdot,v) + (\cdot)^T B_i z_i\right]$ *is higher order (φ, ρ)-invex at u with respect to*

$$g_i(u,v,q_i), i = 1,2,..,k;$$

ii. $\sum_{i=1}^{k}\lambda_i\left[f_i(x,\cdot) - (\cdot)^T C_i w_i\right]$ *is higher order (φ, ρ)-incave at y with respect to*

$$h_i(x,y,p_i), i = 1,2,..,k;$$

iii $\phi_o(x,u;(\xi,\rho)) + u^T\xi \geq 0, \forall\xi \in R_+^n,$
iv $\phi_1(v,y;(\varsigma,\rho)) + y^T\varsigma \geq 0, \forall\varsigma \in R_+^m.$

Then $L(x,y,w,\lambda,p) \nleq M(u,v,z,\lambda,q)$.

Proof Since (u, v, λ, z, q) is feasible solution for dual, by dual constraint (3.4), the vector $\xi = \sum_{i=1}^{k}\lambda_i\left[\nabla_u f_i(u,v) + B_i z_i + \nabla_{q_i} g_i(u,v,q_i)\right] \in R_+^n$ and so from the hypothesis (iii), we have

$$\phi_o(x,u;(\xi,\rho)) + u^T\xi \geq 0 \Rightarrow \phi_o\left(x,u;\left(\sum_{i=1}^{k}\lambda_i\left[\nabla_u f_i(u,v) + B_i z_i + \nabla_{q_i} g_i(u,v,q_i)\right],\rho\right)\right) \geq -u^T\left(\sum_{i=1}^{k}\lambda_i\left[\nabla_u f_i(u,v) + B_i z_i + \nabla_{q_i} g_i(u,v,q_i)\right]\right) \tag{3.7}$$

Again since (x, y, λ, w, p) is feasible solution for primal, by primal constraint (3.1)

$$\varsigma = -\left(\sum_{i=1}^{k}\lambda_i\left[\nabla_y f_i(x,y) - C_i w_i + \nabla_{p_i} h_i(x,y,p_i)\right]\right) \in R_+^m.$$

So hypothesis (iv) implies

$$\phi(x,u;(\varsigma,\rho))+y^T\varsigma \ \geq\ 0 \Rightarrow \phi_1\left(x,u;\left(-\left(\sum\nolimits_{i=1}^{k}\lambda\left[\nabla_y f_i(x,y)\right.\right.\right.\right.$$
$$\left.\left.\left.\left.-C_iw_i+\nabla_{p_i}h_i(x,y,p_i)\right]\right),p\right)\right)$$
$$\geq\ y^T\left(\sum\nolimits_{i=1}^{k}\left[\nabla_y f_i(x,y)-C_iw_i+\nabla_{p_i}h_i(x,y,p_i)\right]\right) \quad (3.8)$$

Now from hypothesis (i) we have $\sum_{i=1}^{k}\lambda_i\left[f_i\,(*,v)+(*)^T\ B_iz_i\right]$ is higher order $(\phi_0,\ \rho)$-invex at u for fixed v and $\lambda>0$. So for ϕ_0: $R^n\times R^n\times R^{n+1}\to R$ and $\rho\in R$, we have

$$\sum_{i=1}^{k}\lambda_i\left[f_i\,(x,v)+x^TB_iz_i\right]-\sum_{i=1}^{k}\lambda_i\left[f_i\,(u,v)+u^TB_iz_i\right]$$
$$\geq\phi_0\left(x,u,\left(\sum\nolimits_{i=1}^{k}\lambda_i\left[\nabla_u f_i\,(u,v)+B_iz_i+\nabla_{q_i}g_i\,(u,v,q_i)\right],\rho\right)\right) \quad (3.9)$$
$$+\sum_{i=1}^{k}\lambda_i\left[g_i\,(u,v,q_i)-q_i^T\nabla_{q_i}g_i\,(u,v,q_i)\right]$$

Again the higher order $(\phi_1,\ \rho)$ -incavity of $\sum_{i=1}^{k}\lambda_i\left[f_i\,(x,\cdot)-(\cdot)^T\ C_iw_i\right]$ at y for fixed x, $\lambda>0$, ϕ_1: $R^m\times R^m\times R^{m+1}\to R$ and $\rho\in R$, implies that

$$\sum_{i=1}^{k}\lambda_i\left[f_i\,(x,y)-y^TC_iw_i\right]-\sum_{i=1}^{k}\lambda_i\left[f_i\,(x,v)-v^TC_iw_i\right]$$
$$\geq\phi_1\left(y,v;\left(-\sum_{i=1}^{r}\left(\lambda_i\left[\nabla_y f_i\,(x,y)-C_iw_i+\nabla_{p_i}h_i\,(x,y,p_i)\right],\rho\right)\right)\right. \quad (3.10)$$
$$-\sum_{i=1}^{k}\lambda_i\left[h_i\,(x,y,p_i)-p_i^T\nabla_{p_i}h_i\,(x,y,p_i)\right].$$

Adding (3.9) and (3.10)

$$\sum_{i=1}^{k}\lambda_i\left[f_i\,(x,y)+x^TB_iz_i-y^TC_iw_i\right]-\sum_{i=1}^{k}\lambda_i\left[f_i\,(u,v)+u^TB_iz_i-v^TC_iw_i\right]$$
$$\geq\phi_0\left(x,u,\left(\sum\nolimits_{i=1}^{k}\lambda_i\left[\nabla_u f_i\,(u,v)+B_iz_i+\nabla_{q_i}g_i\,(u,v,q_i)\right],\rho\right)\right)$$
$$+\phi_1\left(y,v;\left(-\sum_{i=1}^{r}\left(\lambda_i\left[\nabla_y f_i\,(x,y)-C_iw_i+\nabla_{p_i}h_i\,(x,y,p_i)\right],\rho\right)\right)\right. \quad (3.11)$$
$$+\sum_{i=1}^{k}\lambda_i\left[g_i\,(u,v,q_i)-q_i^T\nabla_{q_i}g_i\,(u,v,q_i)\right]$$
$$-\sum_{i=1}^{k}\lambda_i\left[h_i\,(x,y,p_i)-p_i^T\nabla_{p_i}h_i\,(x,y,p_i)\right]$$

Using (3.7) and (3.8), in (3.11), we get

$$\begin{aligned}
&\sum_{i=1}^{k}\lambda_i\left[f_i(x,y)+x^TB_iz_i-y^TC_iw_i\right]-\sum_{i=1}^{k}\lambda_i\left[f_i(u,v)+u^TB_iz_i-v^TC_iw_i\right]\\
&\geq -u^T\sum_{i=1}^{k}\lambda_i\left[\nabla_u f_i(u,v)+B_iz_i+\nabla_{q_i}g_i(u,v,q_i)\right]\\
&\quad+y^T\sum_{i=1}^{k}\lambda_i\left[\nabla_y f_i(x,y)-C_iw_i+\nabla_{p_i}h_i(u,v,p_i)\right]\\
&\quad+\sum_{i=1}^{k}\lambda_i\left[g_i(u,v,q_i)-q_i^T\nabla_{q_i}g_i(u,v,q_i)\right]\\
&\quad-\sum_{i=1}^{k}\lambda_i\left[h_i(x,y,p_i)-p_i^T\nabla_{p_i}h_i(x,y,p_i)\right]\\
&\Rightarrow \sum_{i=1}^{k}\lambda_i\left[f_i(x,y)+x^TB_iz_i\right]-\sum_{i=1}^{k}\lambda_i\left[f_i(u,v)-v^TC_iw_i\right]\\
&\geq -u^T\sum_{i=1}^{k}\lambda_i\left[\nabla_u f_i(u,v)+\nabla_{q_i}g_i(u,v,q_i)\right]+y^T\sum_{i=1}^{k}\lambda_i\left[\nabla_y f_i(x,y)+\nabla_{p_i}h_i(u,v,p_i)\right]\\
&\quad+\sum_{i=1}^{k}\lambda_i\left[g_i(u,v,q_i)-q_i^T\nabla_{q_i}g_i(u,v,q_i)\right]-\sum_{i=1}^{k}\lambda_i\left[h_i(x,y,p_i)-p_i^T\nabla_{p_i}h_i(x,y,p_i)\right]\\
&\Rightarrow \sum_{i=1}^{k}\lambda_i\left[f_i(x,y)+x^TB_iz_i+h_i(x,y,p_i)-p_i^T\nabla_{p_i}h_i(x,y,p_i)\right.\\
&\qquad\qquad\left.-y^T\left(\nabla_y f_i(x,y)-\nabla_{p_i}h_i(x,y,p_i)\right)\right]\\
&\geq \sum_{i=1}^{k}\lambda_i\left[f_i(u,v)-v^TC_iw_i+g_i(u,v,q_i)-q_i^T\nabla_{q_i}g_i(u,v,q_i)\right.\\
&\qquad\qquad\left.-u^T\left(\nabla_u f_i(u,v)-\nabla_{q_i}g_i(u,v,q_i)\right)\right]
\end{aligned}\tag{3.12}$$

From Schwartz inequality, (3.2) and (3.5) we get

$$x^TB_iz_i=\left(x^TB_ix\right)^{\frac{1}{2}}\left(z_i^TB_iz_i\right)^{\frac{1}{2}}\leq\left(x^TB_ix\right)^{\frac{1}{2}}.\tag{3.13}$$

$$v^TC_iw_i=\left(v^TC_iv\right)^{\frac{1}{2}}\left(w_i^TC_iw_i\right)^{\frac{1}{2}}\leq\left(v^TC_iv\right)^{\frac{1}{2}}.\tag{3.14}$$

Using (3.13) and (3.14) in (3.12), we obtain

$$\begin{aligned}
&\sum_{i=1}^{k}\lambda_i\left[f_i(x,y)+\left(x^TB_ix\right)^{\frac{1}{2}}+h_i(x,y,p_i)-p_i^T\nabla_{p_i}h_i(x,y,p_i)\right.\\
&\qquad\qquad\left.-y^T\left(\nabla_y f_i(x,y)-\nabla_{p_i}h_i(x,y,p_i)\right)\right]\\
&\geq \sum_{i=1}^{k}\lambda_i\left[f_i(u,v)-\left(v^TC_iv\right)^{\frac{1}{2}}+g_i(u,v,q_i)-q_i^T\nabla_{q_i}g_i(u,v,q_i)\right.\\
&\qquad\qquad\left.-u^T\left(\nabla_u f_i(u,v)-\nabla_{q_i}g_i(u,v,q_i)\right)\right]\\
&\Rightarrow \textstyle\sum_{i=1}^{k}\lambda_iL_i(x,y,z,p)\geq\sum_{i=1}^{k}\lambda_iM_i(u,v,w,q)
\end{aligned}$$

That is $L(x,y,\lambda,z,p)\geq M(u,v,\lambda,w,q)$. □

Theorem 3.2 (**Strong Duality**) *Let f_i: $R^n\times R^m\to R$ be thrice differentiable function, g_i: $R^n\times R^m\times R^n\to R$ and h_i: $R^n\times R^m\times R^m\to R$ be differentiable function, B_i and C_i for i = 1,2, .,k; are $n\times n$ and $m\times m$ positive semi definite matrices respectively.*

Let $(\bar{x}, \bar{y}, \bar{w}, \bar{\lambda}, \bar{p})$ *be a weakly Pareto optimal solution of Primal (WHNMSP). Assume the following conditions are satisfied;*

(i) *for all* $i \in \{1, 2, ..k\}$, $\nabla_{p_i p_i} h_i\left(\overline{x}, \overline{y}, \overline{p}_i\right)$ *are nonsingular,*

(ii) *the vector* $\sum_{i=1}^{k} \overline{\lambda}_i\left[\nabla_y h_i\left(\overline{x}, \overline{y}, \overline{p}_i\right) - \nabla_{p_i} h_i\left(\overline{x}, \overline{y}, \overline{p}_i\right) + \nabla_{yy} f_i\left(\overline{x}, \overline{y}\right)\overline{p}_i\right] \notin$

$$span\left\{\nabla_y f_1\left(\overline{x}, \overline{y}\right) - C_1\overline{w}_1, .., \nabla_y f_k\left(\overline{x}, \overline{y}\right) - C_k\overline{w}_k\right\}.$$

(iii) $\sum_{i=1}^{k} \overline{\lambda}_i\left[\nabla_y h_i\left(x, \overline{y}, \overline{p}_i\right) - \nabla_{p_i} h_i\left(\overline{x}, \overline{y}, \overline{p}_i\right) + \nabla_{yy} f_i\left(\overline{x}, \overline{y}\right)\overline{p}_i\right] = 0 \Rightarrow \overline{p}_i = 0, \forall i.$

(iv) *for all i,* $h_i\left(\overline{x}, \overline{y}, 0\right) = 0, g_i\left(\overline{x}, \overline{y}, 0\right) = 0, \nabla_{p_i} h_i\left(\overline{x}, \overline{y}, 0\right) = 0, \nabla_y h_i\left(\overline{x}, \overline{y}, 0\right) = 0,$

$$\nabla_x h_i\left(\overline{x}, \overline{y}, 0\right) = \nabla_{q_i} g_i\left(\overline{x}, \overline{y}, 0\right)$$

(v) *The set of vectors* $\left\{\nabla_y f_1\left(\overline{x}, \overline{y}\right) - C_1\overline{w}_1, ..., \nabla_y f_k\left(\overline{x}, \overline{y}\right) - C_k\overline{w}_k\right\}$ *are linearly independent.*

Then (a) $\overline{p}_i = 0, \forall i$. *(b) there exist* $z_i \in R^n$ *such that* $\left(\overline{x}, \overline{y}, \overline{\lambda}, \overline{z}, \overline{q} = 0\right)$ *is a feasible solution for dual (WHNMSD) and two objective values are equal. Also if the hypothesis of theorem 3.1 are satisfied for all feasible solution of (WHNMSP) and (WHNMSD), then* $\left(\overline{x}, \overline{y}, \overline{w}, \overline{\lambda}, \overline{p}\right)$ *and* $\left(\overline{x}, \overline{y}, \overline{\lambda}, \overline{z}, \overline{q}\right)$ *are Pareto optimal solution for (WHNMSP) and (WHNMSD) respectively.*

Proof Since $\left(\overline{x}, \overline{y}, \overline{\lambda}, \overline{w}, \overline{p}\right)$ is weakly Pareto optimal solution (WHNMSP), there exist $\alpha \in R^k, \beta \in R^m, \nu \in R^k, \mu = \sum_{i=1}^{k} \alpha_i \in R$ such that the following Fritz-John optimality condition stated by Craven [14] are satisfied at $\left(\overline{x}, \overline{y}, \overline{\lambda}, \overline{w}, \overline{p}\right)$ for $w \in R^m$;

$$\sum_{i=1}^{k} \alpha_i\left[\nabla_x f_i\left(\overline{x}, \overline{y}\right) + B_i\overline{z}_i\right] + \mu\sum_{i=1}^{k} \overline{\lambda}_i\left[\nabla_x h_i\left(\overline{x}, \overline{y}, \overline{p}_i\right)\right] + \sum_{i=1}^{k} \overline{\lambda}_i\left[\nabla_{xy} f_i\left(\overline{x}, \overline{y}\right)\right] \times(\beta - \mu\overline{y}) + \sum_{i=1}^{k} \overline{\lambda}_i\left[\nabla_{xp_i} h_i\left(\overline{x}, \overline{y}, \overline{p}_i\right)\right]\left(\beta - \mu\left(\overline{y} + \overline{p}_i\right)\right) = 0, \tag{3.15}$$

$$\sum_{i=1}^{k}\left(\alpha_i - \mu\overline{\lambda}_i\right)\left[\nabla_y f_i\left(\overline{x}, \overline{y}\right) - C_i\overline{w}_i\right] + \sum_{i=1}^{k} \mu\overline{\lambda}_i\left[\nabla_y h_i\left(\overline{x}, \overline{y}, \overline{p}_i\right) - \nabla_{p_i} h_i\left(\overline{x}, \overline{y}, \overline{p}_i\right)\right] + \sum_{i=1}^{k} \overline{\lambda}_i\left(\nabla_{yy} f_i\left(\overline{x}, \overline{y}\right)\left(\beta - \mu\overline{y}\right) + \sum_{i=1}^{k} \overline{\lambda}_i\left[\nabla_{p_i y} h_i\left(\overline{x}, \overline{y}, \overline{p}_i\right)\right]\left(\beta - \mu\overline{y} - \mu\overline{p}_i\right)\right) = 0. \tag{3.16}$$

$$\sum_{i=1}^{k} \overline{\lambda}_i\left[\nabla_{p_i p_i} h_i\left(\overline{x}, \overline{y}, \overline{p}_i\right)\right]\left[\beta - \mu\left(\overline{p}_i + \overline{y}\right)\right] = 0, \tag{3.17}$$

$$\beta^T\sum_{i=1}^{k} \overline{\lambda}_i\left[\nabla_y f_i\left(\overline{x}, \overline{y}\right) - C_i\overline{w}_i + \nabla_{p_i} h_i\left(\overline{x}, \overline{y}, \overline{p}_i\right)\right] = 0, \tag{3.18}$$

$$\alpha_i C_i\overline{y} + \left(\beta - \mu\overline{y}\right)^T\lambda_i C_i = 2\nu_i C_i\overline{w}_i, i = 1, 2, ..k. \tag{3.19}$$

$$\overline{x}^T B_i\overline{z}_i = \left(\overline{x}^T B_i\overline{x}\right)^{\frac{1}{2}}, i = 1, 2, ..k \tag{3.20}$$

$$\overline{z}_i^T B_i\overline{z}_i \leq 1, i = 1, 2, ..k \tag{3.21}$$

$$\nu_i\left(\overline{w}_i^T C_i\overline{w}_i - 1\right) = 0, i = 1, 2, ..k \tag{3.22}$$

$$(\alpha, \beta, \mu, \nu) \geq \mathbf{0}, \tag{3.23}$$

$$(\alpha, \beta, \mu, \nu) \neq \mathbf{0}. \tag{3.24}$$

By hypothesis (i), (3.17) gives

$$\beta = \mu\left(\overline{p}_i + \overline{y}\right) \tag{3.25}$$

Suppose $\alpha_i = 0$, for $i = 1, 2, ..k$. So

$$\mu = \sum_{i=1}^{k}\alpha_i = 0. \tag{3.26}$$

So (25) gives,

$$\beta = 0. \tag{3.27}$$

Again from (3.19) we get

$$\nu_i = 0, \forall i. \tag{3.28}$$

So from (3.26), (3.27) and (3.28), we get $(\alpha, \beta, \mu, \nu) = \mathbf{0}$. This contradicts (3.24).

So $\alpha_i > 0$ for at least one i which implies $\alpha \geq 0$.

Hence

$$\mu = \sum_{i=1}^{k}\alpha_i > 0. \tag{3.29}$$

Using (3.25) in (3.16), we get

$$\begin{aligned}
&\sum_{i=1}^{k}\left(\alpha_i - \mu\overline{\lambda}_i\right)\left[\nabla_y f_i\left(\overline{x}, \overline{y}\right) - C_i\overline{w}_i\right] + \sum_{i=1}^{k}\mu\overline{\lambda}_i\left[\nabla_y h_i\left(\overline{x}, \overline{y}, \overline{p}_i\right) - \nabla_{p_i}h_i\left(\overline{x}, \overline{y}, \overline{p}_i\right)\right] \\
&\qquad + \sum_{i=1}^{k}\mu\overline{\lambda}_i\nabla_{yy}f_i\left(\overline{x}, \overline{y}\right)\overline{p}_i = 0. \\
&\Rightarrow \sum_{i=1}^{k}\left(\alpha_i - \mu\overline{\lambda}_i\right)\left[\nabla_y f_i\left(\overline{x}, \overline{y}\right) - C_i\overline{w}_i\right] \\
&\qquad + \mu\sum_{i=1}^{k}\overline{\lambda}_i\left[\nabla_y h_i\left(\overline{x}, \overline{y}, \overline{p}_i\right) - \nabla_{p_i}h_i\left(\overline{x}, \overline{y}, \overline{p}_i\right) + \nabla_{yy}f_i\left(\overline{x}, \overline{y}\right)\overline{p}_i\right] = 0 \\
&\Rightarrow \sum_{i=1}^{k}\overline{\lambda}_i\left[\nabla_y h_i\left(\overline{x}, \overline{y}, \overline{p}_i\right) - \nabla_{p_i}h_i\left(\overline{x}, \overline{y}, \overline{p}_i\right) + \nabla_{yy}f_i\left(\overline{x}, \overline{y}\right)\overline{p}_i\right] \\
&\qquad = -\frac{1}{\mu}\sum_{i=1}^{k}\left(\overline{\alpha}_i - \mu\overline{\lambda}_i\right)\left[\nabla_y f_i\left(\overline{x}, \overline{y}\right) - C_i\overline{w}_i\right]
\end{aligned} \tag{3.30}$$

So by hypothesis (ii), we get

$$\sum_{i=1}^{k}\overline{\lambda}_i\left[\nabla_y h_i(\overline{x}, \overline{y}, \overline{p}_i)_i - \nabla_{p_i}h_i(\overline{x}, \overline{y}, \overline{p}_i) + \nabla_{yy}f_i(\overline{x}, \overline{y})\overline{p}_i\right] = 0. \tag{3.31}$$

Hence by hypothesis (iii) we get

$$\overline{p}_i = 0, \forall i. \tag{3.32}$$

So (3.25) implies

$$\beta = \mu\overline{y}. \tag{3.33}$$

Now from (3.30) and (3.31), we have

$$\sum_{i=1}^{k}\left(\alpha_i-\mu\overline{\lambda}_i\right)\left[\nabla_y f_i\left(\overline{x},\overline{y}\right)-C_i\overline{w}_i\right]=0. \tag{3.34}$$

Since the vectors $\left\{\nabla_y f_1\left(\overline{x},\overline{y}\right)-C_1\overline{w}_1,\ldots,\nabla_y f_k\left(\overline{x},\overline{y}\right)-C_k\overline{w}_k\right\}$ are linearly independent, from (3.34), we obtain

$$\alpha_i=\mu\overline{\lambda}_i. \tag{3.35}$$

Using (3.32) and (3.33) in (3.15), we get

$$\sum_{i=1}^{k}\alpha_i\left[\nabla_x f_i\left(\overline{x},\overline{y}\right)+B_i\overline{z}_i\right]+\sum_{i=1}^{k}\mu\overline{\lambda}_i\left(\nabla_x h_i\left(\overline{x},\overline{y},\overline{p}_i\right)\right)=0.$$

Using (3.35) in above equation, we get

$$\begin{aligned}&\mu\sum_{i=1}^{k}\overline{\lambda}_i\left[\nabla_x f_i\left(\overline{x},\overline{y}\right)+B_i\overline{z}_i+\nabla_x h_i\left(\overline{x},\overline{y},\overline{p}_i\right)\right]=0.\\ &\Rightarrow\sum_{i=1}^{k}\overline{\lambda}_i\left[\nabla_x f_i\left(\overline{x},\overline{y}\right)+B_i\overline{z}_i+\nabla_x h_i\left(\overline{x},\overline{y},\overline{p}_i\right)\right]=0.\end{aligned} \tag{3.36}$$

From the hypothesis (iv), for $\overline{q}=0$, (3.36) yields

$$\sum\nolimits_{i=1}^{k}\lambda_i\left[\nabla_x f_i\left(\overline{x},\overline{y}\right)+B_i\overline{z}_i+\nabla_{q_i}g_i\left(\overline{x},\overline{y},\overline{q}_i=0\right)\right]=0. \tag{3.37}$$

Again whenever $\nu_i>0$, (3.22) reduces to

$$\overline{w}_i C_i\overline{w}_i=1. \tag{3.38}$$

So $\overline{y}^T C_i\overline{w}_i=\left(\overline{y}^T C_i\overline{y}\right)^{\frac{1}{2}}$.
When $\nu_i=0$, (3.19) gives $\alpha_i C_i\overline{y}+\left(\beta-\mu\overline{y}\right)^T\lambda_i C_i=0, i=1,2,..k$.
which by (3.33) reduces to $\alpha_i C_i\overline{y}=0, \forall i$.

Since $\alpha_i>0$ for at least one i, the above equation implies $\overline{y}=0$. So $\overline{y}^T C_i\overline{w}_i=\left(\overline{y}^T C_i\overline{y}\right)^{\frac{1}{2}}$. Hence in either case

$$\overline{y}^T C_i\overline{w}_i=\left(\overline{y}^T C_i\overline{y}\right)^{\frac{1}{2}}. \tag{3.39}$$

Now from (3.37) and (3.39), we obtain that $\left(\overline{x},\overline{y},\overline{\lambda},\overline{z},\overline{q}=0\right)$ satisfies the dual constraint (3.4) and (3.5).
Hence $\left(\overline{x},\overline{y},\overline{\lambda},\overline{z},\overline{q}=0\right)$ is feasible for (WHNMSD).
From (3.18), (3.33), we get

$$\begin{aligned}&\overline{y}^T\left(\sum\nolimits_{i=1}^{k}\lambda_i\left[\nabla_y f_i\left(\overline{x},\overline{y}\right)-C_i w_i+\nabla_{p_i}h_i\left(\overline{x},\overline{y},\overline{p}_i\right)\right]\right)=0.\\ &\Rightarrow\sum_{i=1}^{k}\overline{\lambda}_i\left(\overline{y}^T C_i\overline{w}_i\right)=\overline{y}^T\left(\sum_{i=1}^{k}\lambda_i\left[\nabla_y f_i\left(\overline{x},\overline{y}\right)+\nabla_{p_i}h_i\left(\overline{x},\overline{y},\overline{p}_i\right)\right]\right).\end{aligned} \tag{3.40}$$

Hence

$$L(\bar{x},\bar{y},\bar{w},0)=\sum_{i=1}^{k}\bar{\lambda}_i\left[f_i(\bar{x},\bar{y})+\left(\bar{x}^TB_i\bar{x}\right)^{\frac{1}{2}}-\bar{y}^T\nabla_y f_i(\bar{x},\bar{y})+h_i(\bar{x},\bar{y},0)-\bar{y}^T\nabla_{p_i}h_i(\bar{x},\bar{y},0)\right]$$

$$=\sum_{i=1}^{k}\bar{\lambda}_i\left[f_i(\bar{x},\bar{y})+\left(\bar{x}^TB_i\bar{z}_i\right)-\bar{y}^T\nabla_y f_i(\bar{x},\bar{y})+h_i(\bar{x},\bar{y},0)-\bar{y}^T\nabla_{p_i}h_i(\bar{x},\bar{y},0)\right]\ \text{(using (\textbf{3.21}))}$$

$$=\sum_{i=1}^{k}\bar{\lambda}_i\left[f_i(\bar{x},\bar{y})-\bar{x}^T\left[\nabla_x f_i(\bar{x},\bar{y})+\nabla_{q_i}g_i(\bar{x},\bar{y},0)\right]\right.$$
$$\left.-\bar{y}^T\left(\nabla_y f_i(\bar{x},\bar{y})+\bar{y}^T\nabla_{p_i}h_i(\bar{x},\bar{y},0)\right)+h_i(\bar{x},\bar{y},0)\right]$$

$$=\sum_{i=1}^{k}\bar{\lambda}_i\left[f_i(\bar{x},\bar{y})-\bar{x}^T\left[\nabla_x f_i(\bar{x},\bar{y})+\nabla_{q_i}g_i(\bar{x},\bar{y},0)\right]\right.$$
$$\left.-\bar{y}^TC_i\bar{w}_i+h_i(\bar{x},\bar{y},0)\right]\quad\text{(using (\textbf{3.40}))}$$

$$=\sum_{i=1}^{k}\bar{\lambda}_i\left[f_i(\bar{x},\bar{y})-\bar{x}^T\left[\nabla_x f_i(\bar{x},\bar{y})+\nabla_{q_i}g_i(\bar{x},\bar{y},0)\right]-\left(\bar{y}^TC_i\bar{y}\right)^{\frac{1}{2}}+g_i(\bar{x},\bar{y},0)\right]$$

$$=f_i(\bar{x},\bar{y})-\bar{x}^T\left(\sum_{i=1}^{k}\lambda_i\left[\nabla_x f_i(\bar{x},\bar{y})+\nabla_{q_i}g(\bar{x},\bar{y},0)\right]\right)-\left(\bar{y}^TC_i\bar{y}\right)^{\frac{1}{2}}+g_i(\bar{x},\bar{y},0)$$
$$\text{(using hypothesis (iii) and (\textbf{3.39}))}$$

$$=M(\bar{x},\bar{y},\bar{z},0). \tag{3.41}$$

Now we claim that $\left(\bar{x},\bar{y},\bar{w},\bar{p}_i=0\right)$ is Pareto optimal solution for (WHNMSP). If this would not be the case, then there would exist a feasible solution $\left(\bar{u},\bar{v},\bar{w},\bar{p}_i=0\right)$ for (WHNMSP) such that for $L(\bar{u},\bar{v},\bar{w},\bar{p}=0)<L(\bar{x},\bar{y},\bar{w},\bar{p}=0)$

$$\Rightarrow L(\bar{u},\bar{v},\bar{w},\bar{p}=0)<M(\bar{x},\bar{y},\bar{z},\bar{q}=0).\ \text{(using (\textbf{3.41}))}$$

This is a contradiction to Theorem 3.1.
Hence $\left(\bar{x},\bar{y},\bar{w},\bar{p}_i=0\right)$ is a Pareto optimal solution of (WHNMSP).
Similarly by Theorem 3.1 $\left(\bar{x},\bar{y},\bar{\lambda},\bar{z},\bar{q}\right)$ is Pareto optimal solution of (WHNMSD). □

Theorem 3.3 (**Converse Duality**) *Let f_i: $R^n\times R^m\to R$ be thrice differentiable function,g_i: $R^n\times R^m\times R^n\to R$ and h_i: $R^n\times R^m\times R^m\to R$ be differentiable function, B_i and C_i for $i=1, 2,., k$; are $n\times n$ and $m\times m$ positive semi definite matrices respectively. Let $\left(\bar{u},\bar{v},\bar{\lambda},\bar{z},\bar{q}\right)$ be a weakly Pareto optimal solution of Dual (WHNMSD). Assume the following conditions are satisfied;*

(i) *for all $i\in\{1,2,..k\}$, $\nabla_{q_iq_i}g_i\left(\bar{x},\bar{y},\bar{q}_i\right)$ are nonsingular,*

(ii) *the vector* $\sum_{i=1}^{k}\bar{\lambda}_i\left[\nabla_u g_i\left(\bar{u},\bar{v},\bar{q}_i\right)-\nabla_{q_i}g_i\left(\bar{u},\bar{v},\bar{q}_i\right)+\nabla_{uu}f_i(\bar{u},\bar{v})\bar{q}_i\right]\notin$

$$span\left\{\nabla_u f_1(\bar{u},\bar{v})+B_1\bar{z}_1,..,\nabla_u f_k(\bar{u},\bar{v})+B_k\bar{z}_k\right\}.$$

(iii) $\sum_{i=1}^{k}\bar{\lambda}_i\left[\nabla_u g_i\left(\bar{u},\bar{v},\bar{q}_i\right)-\nabla_{q_i}g_i\left(\bar{u},\bar{v},\bar{q}_i\right)+\nabla_{uu}f_i(\bar{u},\bar{v})\bar{q}_i\right]=0\Rightarrow\bar{q}_i=0,\forall i;$

(iv) *for all* $h_i(\bar{u},\bar{v},0)=0$, $g_i(\bar{u},\bar{v},0)=0$, $\nabla_{q_i}g_i(\bar{u},\bar{v},0)=0$, $\nabla_u g_i(\bar{u},\bar{v},0)=0$, $\nabla_v g_i(\bar{u},\bar{v},0)=\nabla_{p_i}h_i(\bar{u},\bar{v},0)$.

(v) *The set of vectors* $\left\{\nabla_u f_1(\bar{u},\bar{v})+B_1\bar{z}_1,...,\nabla_u f_k(\bar{u},\bar{v})+B_k\bar{z}_k\right\}$ *are linearly independent.*

Then (a) $\bar{q}_i=0,\forall i$. (b) there exist $\bar{w}_i\in R^m$ such that $\left(\bar{u},\bar{v},\bar{\lambda},\bar{w},p=0\right)$ is a feasible solution for primal (WHNMSP) and two objective values are equal. Also if the hypothesis of Theorem 3.1 are satisfied for all feasible solution of (WHNMSP) and (WHNMSD), then

$\left(\overline{u}, \overline{v}, \overline{\lambda}, \overline{z}, \overline{q}=0\right)$ *and* $\left(\overline{x}, \overline{y}, \overline{\lambda}, \overline{z}, \overline{q}=0\right)$ *are Pareto optimal solution for (WHNMSP) and (WHNMSD) respectively.*

Proof The proof follows along the lines of Theorem 3.2. □

4 Self Dual

A mathematical programming problem is said to be self dual when the dual is recast in the form of the primal and new program constructed is same as the primal problem.

We now prove the following self duality theorem for the primal and dual.

Theorem 4.4 ***(Self Duality)*** *Assume $m = n$, $B_i = C_i$, $p_i = q_i$, $z_i = w_i$. If f_i, h_i and g_i are skew symmetric with respect to x and y with $h_i(x, y, p_i) = g_i(x, y, q_i)$, then (WHNMSP) is a self dual. Furthermore, if (WHNMSP) and (WHNMSD) are dual programs and the hypothesis of Theorem 6.3.2 are satisfied, then $(\overline{x}, \overline{y}, \overline{w}, \overline{p})$ is Pareto optimal solution for (WHNMSP), implies (i) $\overline{p} = 0$ and (ii) $(\overline{y}, \overline{x}, \overline{w}, q = 0)$ is Pareto optimal solution for (WHNMSD). Also the values of two objective functions are equal to zero.*

Proof Rewriting the dual as minimization problem, we have

$$M^*(u, v, z, q) = Minimize\,(-M(u, v, z, q))$$

$$= Minimize \begin{pmatrix} -f_i(u, v) + \left(v^T C_i v\right)^{\frac{1}{2}} - g_i(u, v, q_i) + q_i^T\left(\nabla_{q_i} g_i(u, v, q_i)\right) \\ + u^T\left[\nabla_u f_i(u, v) + \nabla_{q_i} g_i(u, v, q_i)\right], i = 1, 2, .., k. \end{pmatrix}$$

Subject to

$$\sum_{i=1}^{k} \lambda_i\left[-\nabla_u f_i(u, v) + B_i z_i - \nabla_{q_i} g_i(u, v, q_i)\right] \leq 0,$$
$$z_i^T B_i z_i \leq 1, i = 1, 2, ...k.$$
$$\lambda > 0, \sum_{i=1}^{p} \lambda_i = 1,$$

Since $f_i(u, v)$ and $g_i(u, v, q_i)$ are skew symmetric with respect to u and v, we have $f_i(u, v) = -f_i(v, u)$, $g_i(u, v, q_i) = -g_i(v, u, q_i)$, $\nabla_u f_i(u, v) = -\nabla_v f_i(v, u)$ and $\nabla_{q_i} g_i(u, v, q_i) = -\nabla_{q_i} g_i(v, u, q_i)$.

Hence the above dual program becomes

$$M^*(u, v, z, q) = Minimize \begin{pmatrix} f_i(v, u) + \left(v^T C_i v\right)^{\frac{1}{2}} - u^T B_i z_i + g_i(v, u, q_i) \\ - q_i^T(\nabla_{q_i} g_i(v, u, q_i)), i = 1, 2, ..k. \end{pmatrix}$$

Subject to

$$\sum_{i=1}^{k} \lambda_i\left[\nabla_u f_i(v, u) - B_i z_i + \nabla_{q_i} g_i(v, u, q_i)\right] \leq 0,$$
$$z_i^T B_i z_i \leq 1, i = 1, 2, ...k.$$
$$\lambda > 0, \sum_{i=1}^{p} \lambda_i = 1,$$

Again since $B_i = C_i$, $w_i = z_i$ and $h_i(x, y, p_i) = g_i(x, y, q_i)$ the above problem is just primal problem. Hence the dual program is self dual.

Thus $\left(\overline{x}, \overline{y}, \overline{w}, \overline{p}_i\right)$ is Pareto optimal solution for (WHNMSP) implies $\left(\overline{y}, \overline{x}, \overline{w}, \overline{p}_i\right)$ is Pareto optimal solution for (WHNMSD). By similar argument $\left(\overline{x}, \overline{y}, \overline{z}, \overline{q}_i\right)$ is optimal for (WHNMSD) implies $\left(\overline{y}, \overline{x}, \overline{z}, \overline{q}_i\right)$ is Pareto optimal for (WHNMSP).

Since $\left(\overline{x}, \overline{y}, \overline{w}, \overline{p}_i\right)$ is Pareto optimal solution for (WHNMSP), then by Theorem 3.2, there exist $z \in R^n$ such that $\left(\overline{x}, \overline{y}, \overline{z}, \overline{q}_i\right)$ is Pareto optimal solution for (WHNMSD) and

$$L\left(\overline{x}, \overline{y}, \overline{w}, \overline{p} = 0\right) = M\left(\overline{x}, \overline{y}, \overline{z}, \overline{q} = 0\right) \tag{4.1}$$

Hence $\left(\overline{x}, \overline{y}, \overline{w}, \overline{p}_i\right)$ and $\left(\overline{y}, \overline{x}, \overline{z}, \overline{q}_i\right)$ are Pareto optimal solution for (WHNMSP)

$$\text{So } L_i\left(\overline{x}, \overline{y}, \overline{w}, \overline{p}_i = 0\right) = f_i\left(\overline{x}, \overline{y}\right) + \left(\overline{x}^T B_i x\right)^{\frac{1}{2}} + h_i\left(\overline{x}, \overline{y}, \overline{p}_i = 0\right)$$

$$-y^T\left[\nabla_y f_i\left(\overline{x}, \overline{y}\right) - \nabla_{p_i} h_i\left(\overline{x}, \overline{y}, \overline{p} = 0\right)\right]$$

$$= f_i\left(\overline{y}, \overline{x}\right) + \left(\overline{y}^T C_i \overline{y}\right)^{\frac{1}{2}} + h_i\left(\overline{y}, \overline{x}, \overline{q}_i = 0\right) - \overline{x}^T\left[\nabla_x f_i\left(\overline{y}, \overline{x}\right) - \nabla_{p_i} h_i\left(\overline{y}, \overline{x}, \overline{q} = 0\right)\right]$$

$$= -f_i\left(\overline{x}, \overline{y}\right) + \left(\overline{y}^T C_i \overline{y}\right)^{\frac{1}{2}} - h_i\left(\overline{x}, \overline{y}, \overline{q}_i = 0\right) + \overline{x}^T\left[\nabla_x f_i\left(\overline{x}, \overline{y}\right) - \nabla_{p_i} h_i\left(\overline{x}, \overline{y}, \overline{q} = 0\right)\right]$$

($\because$ $f_i(x, y)$ and $h_i(x, y, p)$ are skew symmetric with respect to x and y.)

$$= -f_i\left(\overline{x}, \overline{y}\right) + \left(\overline{y}^T C_i \overline{y}\right)^{\frac{1}{2}} - g_i\left(\overline{x}, \overline{y}, \overline{q}_i = 0\right) + \overline{x}^T\left[\nabla_x f_i\left(\overline{x}, \overline{y}\right) - \nabla_{p_i} g_i\left(\overline{x}, \overline{y}, \overline{q} = 0\right)\right]$$

$$(\because h\left(\overline{x}, \overline{y}, 0\right) = g\left(\overline{x}, \overline{y}, 0\right))$$

$$= -M\left(\overline{x}, \overline{y}, \overline{z}, \overline{q} = 0\right) \tag{4.2}$$

From (4.1) and (4.2), we obtain $H\left(\overline{x}, \overline{y}, \overline{w}, \overline{p} = 0\right) = G\left(\overline{x}, \overline{y}, \overline{z}, \overline{q} = 0\right) = 0.$ □

5 Wolfe Type Higher Order Minimax Mixed Integer Programming

Let U and V be two arbitrary sets of integers in R^{n_1} and R^{m_1} respectively. Throughout this section, we constraint some of the components of x and y are belong to arbitrary sets of integers as in Balas [6]. Suppose that the first $n_1 (0 \le n_1 \le n)$ components of x belong to U and the first $m_1 (0 \le m_1 \le m)$ components of y belongs to v, then we write $(x, y) = (x^1, x^2, y^1, y^2)$ where $x^1 = \left(x_1, x_2, .., x_{n_1}\right)$ and $y^1 = \left(y_1, y_2, .., y_{n_1}\right)$. x^2 and y^2 belong to R^{n-n_1} and R^{m-m_1}.

Definition 5. 1 Let s^1, s^2,. . .,s^k be elements of an arbitrary vector space. A vector function G(s^1, s^2 ,. . .,s^k) will be called additively separable with respect to s^1, if there exist vector function H(s^1) (independent of s^2 ,. . .,s^k) and K(s^2 ,. . .,s^k) (independent of s^1) such that G(s^1, s^2 ,. . .,s^k)= H(s^1)+ K(s^2 ,. . .,s^k).

We now consider the following Wolfe type higher order multiobjective minimax mixed integer symmetric dual program.

- Primal (WHNMSIP):

$$L(x, y, w, p) = \max_{x^1} \min_{x^2, y, w, \lambda, p}$$
$$\times \begin{pmatrix} f_i(x, y) + (x^T B_i x)^{\frac{1}{2}} + h_i(x, y, p_i) - p_i^T\left(\nabla_{p_i} h_i(x, y, p_i)\right) \\ -y^T\left[\nabla_y f_i(x, y) + \nabla_{p_i} h_i(x, y, p_i)\right], i = 1, 2, ...k \end{pmatrix}$$

Subject to

$$\sum_{i=1}^{k} \lambda_i \left[\nabla_y f_i(x, y) - C_i w_i + \nabla_{p_i} h_i(x, y, p_i)\right] \leq 0, \tag{5.1}$$

$$w_i^T C_i w_i \leq 1, i = 1, 2, ...k. \tag{5.2}$$

$$\lambda > 0, \sum_{i=1}^{k} \lambda_i = 1, \tag{5.3}$$

$$x^1 \in U, y^1 \in V, p \in R^{m-m_1}. \tag{5.4}$$

- Dual (WHNMSID):

$$M(u, v, z, q) = \min_{v^1} \max_{u,v^2,z,\lambda,q} \begin{pmatrix} f_i(u, v) - \left(v^T C_i v\right)^{\frac{1}{2}} + g_i(u, v, q_i) - q_i^T \left(\nabla_{q_i} g_i(u, v, q_i)\right) \\ -u^T \left[\nabla_u f_i(u, v) + \nabla_{q_i} g_i(u, v, q_i)\right], i = 1, 2, ...k. \end{pmatrix}$$

Subject to

$$\sum_{i=1}^{k} \lambda_i \left[\nabla_u f_i(u, v) + B_i z_i + \nabla_{q_i} g_i(u, v, q_i)\right] \geq 0, \tag{5.5}$$

$$z_i^T B_i z_i \leq 1, i = 1, 2, ...k. \tag{5.6}$$

$$\lambda > 0, \sum_{i=1}^{p} \lambda_i = 1, \tag{5.7}$$

$$u^1 \in U, v^1 \in V, q \in R^{n-n_1}. \tag{5.8}$$

Theorem 5.5 (Symmetric Duality) *Let* $\left(\overline{x}, \overline{y}, \overline{w}, \overline{\lambda}, \overline{p}\right)$ *be Weakly Pareto optimal solution of primal. Suppose that the following conditions are satisfied:*

1. *f(x, y) and h(x, y, p) are additively separable with respect to* x^1 *or* y^1.
2. *For any feasible solution* $\left(\overline{x}, \overline{y}, \overline{w}, \overline{\lambda}, \overline{p}\right)$ *in primal and* $\left(\overline{u}, \overline{v}, \overline{z}, \overline{\lambda}, \overline{q}\right)$ *in dual;* $\sum_{i=1}^{k} \lambda_i \left[f_i(u^2, v) + (u^2)^T B_i z_i\right]$ *is higher order (*ϕ*,* ρ*) - invex at* u^2 *with respect to g(u, v, q) with* $q \in R^{n-n_1}$ *for each (*u^1*, v) and* $\sum_{i=1}^{k} \lambda_i \left[f_i(x^2, y) - (y^2)^T C_i w_i\right]$ *is higher order (*ϕ*,* ρ*) - incave at* y^2 *with respect to h(x, y, p) with* $p \in R^{m-m_1}$ *for each (x,* y^1*).*
3. $\nabla_{pp} h\left(\overline{x}, \overline{y}, \overline{p}\right)$ *is nonsingular.*
4. *The vector* $h_i\left(\overline{x}, \overline{y}, 0\right) = g\left(\overline{x}, \overline{y}, 0\right) \nabla_{x^2} h_i\left(\overline{x}, \overline{y}, 0\right) = \nabla_{q_i} g\left(\overline{x}, \overline{y}, 0\right)$
5. $\phi_0(x^2, u^2; (a, \rho)) + (u^2)^T a \geq 0$, $\phi_1(v^2, y^2; (b, \rho)) + (y^2)^T b \geq 0$, *for all* $a \in R_+^{n-n_1}$ *and* $b \in R_+^{m-m_1}$

Then there exist $z_i \in R^{n-n_1}$ *such that* $\left(\overline{x}, \overline{y}, \overline{\lambda}, \overline{z}, \overline{q} = 0\right)$ *is Pareto optimal solution of dual and the values of two objective functions are equal.*

Proof The proof follows along the lines of Theorem 3.1 in Gulati and Ahmad [19] by using Theorem 3.1 and 3.2 stated in section -3. □

6 Numerical Example

Let $X = [1, \infty) = Y.P \subset R_+$. Let $f: X^2 \times Y^2 \to R$ be defined as
$f(x, y) = e^{x_1^2} + e^{x_2^2} - e^{y_1^2} - e^{y_2^2}$, $h: X^2 \times Y^2 \times P^2 \to R$ be defined as
$h(x, y, p) = -\left(p_1^2 + p_2^2\right)\left(x_1^2 + x_2^2 + y_1^2 + y_2^2\right)$, $g: X^2 \times Y^2 \times P^2 \to R$ defined as

$$g(x, y, q) = \left(q_1^2 + q_2^2\right)\left(x_1^2 + x_2^2 + y_1^2 + y_2^2\right).$$

Where $x = (x_1, x_2) \in X^2$, $y = (y_1, y_2) \in Y^2$, $p = (p_1, p_2) \in P^2$, $B = C = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$,

$$p = \begin{pmatrix} p_1 \\ p_2 \end{pmatrix}, p_1, p_2 \in P, q = \begin{pmatrix} q_1 \\ q_2 \end{pmatrix}, q_1, q_2 \in P, w = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix},$$

$$z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}, z_1, z_2, w_1, w_2 \in [1, \infty),$$

$$\nabla_x f(x, y) = \begin{pmatrix} 2x_1 e^{x_1^2} \\ 2x_2 e^{x_2^2} \end{pmatrix}, \nabla_y f(x, y) = \begin{pmatrix} -2y_1 e^{y_1^2} \\ -2y_2 e^{y_2^2} \end{pmatrix},$$

$$\nabla_p h(x, y, p) = \begin{pmatrix} -2p_1\left(x_1^2 + x_2^2 + y_1^2 + y_2^2\right) \\ -2p_2\left(x_1^2 + x_2^2 + y_1^2 + y_2^2\right) \end{pmatrix},$$

$$\text{Primal: } Maximize\begin{pmatrix} e^{x_1^2} + e^{x_2^2} - e^{y_1^2} - e^{y_2^2} + \left(p_1^2 + p_2^2 + 2p_1y_1 + 2p_2y_2\right)\left(x_1^2 + x_2^2 + y1^2 + y_1^2\right) \\ +2y_1^2 e^{y_1^2} + 2y_2^2 e^{y_2^2} + \sqrt{x_1^2 + x_2^2} \end{pmatrix}$$

$$\begin{aligned} \text{Subject to } & -2y_1 e^{y_1^2} - w_1 - 2p_1\left(x_1^2 + x_2^2 + y_1^2 + y_2^2\right) \le 0, \\ & -2y_2 e^{y_2^2} - w_2 - 2p_2\left(x_1^2 + x_2^2 + y_1^2 + y_2^2\right) \le 0, \\ & w_1^2 + w_2^2 \le 1. \end{aligned}$$

$$\text{Dual: } Maximize\begin{pmatrix} e^{u_1^2} + e^{u_2^2} - e^{v_1^2} - e^{v_2^2} + \left(q_1^2 + q_2^2 - 2q_1u_1 - 2q_2u_2\right)\left(u_1^2 + u_2^2 + v_1^2 + v_1^2\right) \\ -2u_1^2 e^{u_1^2} - 2u_2^2 e^{u_2^2} - \sqrt{v_1^2 + v_2^2} \end{pmatrix}$$

$$\begin{aligned} \text{Subject to } & 2u_1 e^{u_1^2} + z_1 + 2q_1\left(u_1^2 + u_2^2 + v_1^2 + v_2^2\right) \ge 0, \\ & 2u_2 e^{u_2^2} + z_2 + 2q_2\left(u_1^2 + u_2^2 + v_1^2 + v_2^2\right) \ge 0, \\ & z_1^2 + z_2^2 \le 1. \end{aligned}$$

Now for the above problem let assume $\phi_0 : X \times X \times P \times R_+ \to R$ defined by $\phi_0(x, u; (a, \rho)) = (1 - 2^\rho)\left(e^{u_1^2} + e^{u_2^2}\right) + 2u_1^2 e^{u_1^2} + 2u_2^2 e^{u_2^2} - u^T a$ and $\phi_1 : Y \times Y \times P \times R_+ \to R$ defined by $\phi_1(v, y; (b, \rho)) = (1 - 2^\rho)\left(e^{y_1^2} + e^{y_2^2}\right) + 2y_1^2 e^{y_1^2} + 2y_2^2 e^{y_2^2} - y^T b.$

Now for $\rho = 1$

$$
\begin{aligned}
&(f(x, v) + x^T z) - (f(u, v) + u^T z) - \phi_0(x, u; (\nabla_u f(u, v) + z + \nabla_q g(u, v, q), \rho)) \\
&\qquad -g(u, v, q) + q^T (\nabla_q g(u, v, q)) \\
&= \left(e^{x_1^2} + e^{x_2^2} - e^{y_1^2} - e^{y_2^2} + x_1 z_1 + x_2 z_2\right) - \left(e^{u_1^2} + e^{u_2^2} - e^{y_1^2} - e^{y_2^2} + u_1 z_1 + u_2 z_2\right) \\
&- \Big[(1-2^\rho)\left(e^{u_1^2}+e^{u_2^2}\right) + 2u_1^2 e^{u_1^2} + 2u_2^2 e^{u_2^2} - 2u_1^2 e^{u_1^2} - u_1 z_1 - 2q_1 u_1 \left(u_1^2 + u_2^2 + v_1^2 + v_2^2\right) \\
&\quad -2u_2^2 e^{u_2^2} - u_2 z_2 - 2q_2 u_2 \left(u_1^2 + u_2^2 + v_1^2 + v_2^2\right)\Big] + \left(q_1^2 + q_1^2\right)\left(u_1^2 + u_2^2 + v_1^2 + v_2^2\right) \\
&= e^{x_1^2} + e^{x_2^2} + x_1 z_1 + x_2 z_2 + \left(q_1^2 + q_2^2 + 2q_1 u_1 + 2q_2 u_2\right)\left(u_1^2 + u_2^2 + v_1^2 + v_2^2\right) \\
&\geq x_1 + x_2 + x_1 z_1 + x_2 z_2 + \left(q_1^2 + q_2^2 + 2q_1 u_1 + 2q_2 u_2\right)\left(u_1^2 + u_2^2 + v_1^2 + v_2^2\right) \geq 0 \text{ for all} \\
&\qquad x_1, x_2, u_1, u_2, v_1, v_2 \in [1, \infty), q_1, q_2 \in [0, \infty), z_1, z_2 \in [-1, 1]. \\
&\qquad (\because x \geq -xz \text{ for } z \in [-1, 1] \;\&\; x \in [1, \infty).)
\end{aligned}
$$

So $f(\cdot, v) + (\cdot)^T Bz$ is higher order (ϕ_0, ρ) - *invex* at $u \in X$ with respect to g.

Again for $\rho = 1$

$$
\begin{aligned}
&(f(x, y) - y^T w) - (f(x, v) - v^T w) - \phi_1(v, y; (-(\nabla_y f(x, y) - w + \nabla_p h(x, y, p)), \rho)) \\
&\qquad +h(x, y, p) - p^T (\nabla_p h(x, y, p)) \\
&= \left(e^{x_1^2} + e^{x_2^2} - e^{y_1^2} - e^{y_2^2} - y_1 w_1 - y_2 w_2\right) - \left(e^{x_1^2} + e^{x_2^2} - e^{v_1^2} - e^{v_2^2} - v_1 w_1 - v_2 w_2\right) \\
&- \Big[\left(1-2^1\right)\left(e^{y_1^2}+e^{y_2^2}\right) + 2y_1^2 e^{y_1^2} + 2y_2^2 e^{y_2^2} - 2y_1^2 e^{y_1^2} - y_1 w_1 - 2p_1 y_1 \left(x_1^2 + x_2^2 + y_1^2 + y_2^2\right) \\
&\quad -2y_2^2 e^{y_2^2} - y_2 w_2 - 2p_2 y_2 \left(x_1^2 + x_2^2 + y_1^2 + y_2^2\right)\Big] + \left(p_1^2 + p_2^2\right)\left(x_1^2 + x_2^2 + y_1^2 + y_2^2\right) \\
&= e^{v_1^2} + e^{v_2^2} + v_1 w_1 + v_2 w_2 + \left(p_1^2 + p_2^2 + 2p_1 y_1 + 2p_2 y_2\right)\left(x_1^2 + x_2^2 + y_1^2 + y_2^2\right) \\
&\geq v_1 + v_2 + v_1 w_1 + v_2 w_2 + \left(p_1^2 + p_2^2 + 2p_1 y_1 + 2p_2 y_2\right)\left(x_1^2 + x_2^2 + y_1^2 + y_2^2\right) \geq 0 \text{ for all} \\
&\qquad x_1, x_2, y_1, y_2, v_1, v_2 \in [1, \infty), p_1, p_2 \in [0, \infty), w_1, w_2 \in [-1, 1]. \\
&\qquad (\because v \geq -vw \text{ for } w \in [-1, 1] \;\&\; v \in [1, \infty).)
\end{aligned}
$$

So $f(x, \cdot) - (\cdot)^T Cw$ is higher order (ϕ_1, ρ) - *incave* at $y \in Y$ with respect to h.

Therefore all the conditions of Theorem 3.1 are satisfied. Hence the result of Weak duality theorem (Theorem 3.1) is applicable.

Again (1) $\nabla_{pp} h(\overline{x}, \overline{y}, \overline{p})$ is nonsingular,

(II) $\nabla_y h(\overline{x}, \overline{y}, \overline{p}) - \nabla_p h(\overline{x}, \overline{y}, \overline{p}) + \nabla_{yy} f(\overline{x}, \overline{y})\, \overline{p} = 0 \Rightarrow \overline{p} = 0.$

And (III) $h(\overline{x}, \overline{y}, 0) = g(\overline{x}, \overline{y}, 0)$, $\nabla_x h(\overline{x}, \overline{y}, 0) = \nabla_p h(\overline{x}, \overline{y}, 0)$, So all the condition of strong duality theorem are satisfied. Hence the result of Strong duality theorem (Theorem 3.2) is applicable. Also $f(x, y)$ and $h(x, y, p)$ are skew symmetric with respect to x and y. Thus the above dual problem becomes a self dual and by self duality theorem the optimal value of both primal and dual problem is 0 for p = 0.

7 Special Cases

(1) If we take $(x^T Ax)^{\frac{1}{2}} = s(x|C)$ and $(y^T By)^{\frac{1}{2}} = s(y|D)$ for
$C = \{Ay : y^T Ay \leq 1\}$, $D = \{Bx : x^T Bx \leq 1\}$ for k = 1, then our model reduces to the model proposed by Gulati and Gupta [21] as follows;

- Primal1: $L(x, y, w, p) = \text{minimize} \begin{pmatrix} f(x, y) + s(x|C) + h(x, y, p) - p^T(\nabla_p h(x, y, p)) \\ -y^T \left[\nabla_y f(x, y) + \nabla_p h(x, y, p)\right] \end{pmatrix}$

$$\text{Subject to } \nabla_y f(x, y) - z + \nabla_p h(x, y, p) \leq 0$$
$$z \in D$$

- Dual1: $M(u, v, z, q) = \text{maximize} \begin{pmatrix} f(u, v) - s(v|D) + g(u, v, q) - q^T(\nabla_q g(u, v, q)) \\ -u^T \left[\nabla_u f(u, v) + \nabla_q g(u, v, q)\right) \end{pmatrix}$

$$\text{Subject to } \nabla_u f(u, v) + w + \nabla_q g(u, v, q) \geq 0,$$
$$w \in C$$

(II) If we take $(x^T Ax)^{\frac{1}{2}} = s(x|C)$ and $(y^T By)^{\frac{1}{2}} = s(y|D)$ for $C = \{Ay : y^T Ay \leq 1\}$, $D = \{Bx : x^T Bx \leq 1\}$ and $h(x, y, p) = \frac{1}{2} p^T \nabla_{yy} f(x, y) p$, $g(u, v, q) = \frac{1}{2} q^T \nabla_{xx} f(u, v) q$ for k = 1, then our model reduces to the model proposed by Gulati and Gupta [20] as follows;

- Primal2: $L(x, y, z, p) = \text{minimize} \begin{pmatrix} f(x, y) + s(x|C) - \frac{1}{2} p^T(\nabla_{yy} f(x, y) p) \\ -y^T \left[\nabla_y f(x, y) + \nabla_{yy} f(x, y) p)\right] \end{pmatrix}$

$$\text{Subject to } \nabla_y f(x, y) - z + \nabla_{yy} f(x, y) p \leq 0$$
$$z \in D$$

- Dual2: $M(u, v, z, q) = \text{maximize} \begin{pmatrix} f(u, v) - s(v|D) - \frac{1}{2} q^T(\nabla_{uu} f(u, v) q) \\ -u^T \left[\nabla_u f(u, v) + \nabla_{uu} f(u, v) q\right) \end{pmatrix}$

$$\text{Subject to } \nabla_u f(u, v) + w + \nabla_{uu} f(u, v) q \geq 0,$$
$$w \in C.$$

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