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A Large-Update Interior-Point Method for Cartesian $P_*(\kappa)$-LCP Over Symmetric Cones

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Abstract In this paper, we propose a new large-update interior point algorithm for the Cartesian $P_*(\kappa)$ linear complementarity problem over symmetric cones (SCLCP) based on a parametric kernel function, which determines both search directions and the proximity measure between the iterate and the μ-center. Using Euclidean Jordan algebras, we derive the iteration bound that match the currently best known iteration bound for large update methods.

Keywords Cartesian $P_*(\kappa)$ property · Linear complementarity problem · Symmetric cone · Euclidean Jordan algebra · Interior-point method · Kernel function

1 Introduction

Recently, more and more attention has been focused on the optimization problems over symmetric cones due to the Euclidean Jordan algebraic tool. In 1997, Faybusovich [1] made the first attempt to extend an interior-point method (IPM) from semidefinite programming (SDP) to the symmetric conic linear programming (SCLP). Consequently, several IPMs for linear optimization (LO), second order cone optimization (SOCO) and SDP optimization as special cases have been successfully extended to symmetric optimization using the framework of Euclidean Jordan algebras in [2–5].

This paper is concerned with the Cartesian $P_*(\kappa)$-SCLCP, which seeks vectors $x, s \in \mathcal{J}$ such that

$$x \in \mathcal{K},\ s = \mathcal{A}(x) + q \in \mathcal{K},\ \langle x, s\rangle = 0, \tag{1}$$

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where $\langle x, s\rangle = \mathbf{Tr}(x \circ s)$ denotes the Euclidean inner product, $\mathcal{A} : \mathcal{J} \to \mathcal{J}$ is a linear transformation, $q \in \mathcal{J}$, and $\mathcal{J} = \mathcal{J}_1 \times \mathcal{J}_2 \times \ldots \times \mathcal{J}_N$ is the Cartesian product space with its cone of squares $\mathcal{K} = \mathcal{K}_1 \times \mathcal{K}_2 \times \ldots \times \mathcal{K}_N$, where each space $\mathcal{J}_\nu$ is a simple Euclidean Jordan algebra with dimensions n_ν and ranks r_ν and each $\mathcal{K}_\nu$ is the corresponding cone of squares of $\mathcal{J}_\nu$. The dimension of $\mathcal{J}$ is $n = \sum_{\nu=1}^{N} n_\nu$ and the rank is $r = \sum_{\nu=1}^{N} r_\nu$.

We call SCLCP the Cartesian $P_*(\kappa)$-SCLCP if linear transformation $\mathcal{A}$ has the Cartesian $P_*(\kappa)$ property for some nonnegative constant κ, i.e.,

$$\mathcal{A}(u) - v = 0,$$

implies

$$(1 + 4\kappa) \sum_{\nu \in I_+} \langle u^{(\nu)},\ v^{(\nu)}\rangle + \sum_{\nu \in I_-} \langle u^{(\nu)},\ v^{(\nu)}\rangle \geq 0,\ \forall\, u,\ v \in \mathcal{J},$$

where $I_+ = \{\nu\ :\ \langle u^{(\nu)},\ v^{(\nu)}\rangle \geq 0\}$ and $I_- = \{\nu\ :\ \langle u^{(\nu)},\ v^{(\nu)}\rangle < 0\}$ are two index sets. If $\mathcal{A}$ belongs to the class

$$P_* = \cup_{\kappa \geq 0} P_*(\kappa),$$

then SCLCP is said to be a Cartesian P_*-SCLCP.

The concept of the Cartesian $P_*(\kappa)$-property was first introduced by Luo and Xiu [6] in the general Euclidean Jordan algebra. Actually, it is a straightforward extension of the $P_*(\kappa)$-matrix introduced by Kojima et al. [9], where they first proved the existence and uniqueness of the central path for the $P_*(\kappa)$-LCPs and generalized the primal-dual interior point algorithm for LO to the $P_*(\kappa)$-LCPs. The Cartesian $P_*(\kappa)$-SCLCPs is a wide class of problems that contains LCPs [7, 8], Cartesian $P_*(\kappa)$-LCPs [9, 10], Cartesian $P_*(\kappa)$-SOCLCPs [11] and Cartesian $P_*(\kappa)$-SDLCPs [12] as special cases.

Recently, Choi et al. [20] presented a new complexity analysis for primal-dual IPM for SOP, using a proximity function defined by a new eligible kernel function as follows:

$$\psi(t) = \frac{t^2 - 1}{2} + \frac{e^{p(t^{-q}-1)} - 1}{pq},\ p \geq 1 \text{ and } q \geq 1 \text{ for } t > 0, \tag{2}$$

which was modified from the one in [13, 14]. A new class of eligible kernel functions was defined by some simple conditions on the kernel function and its derivatives in [15]. From [15] we recall that the kernel functions $\psi : (0, \infty) \to [0, \infty)$ that satisfy in the following conditions are called eligible kernel functions:

(a) $\psi(1) = \psi'(1) = 0$;
(b) $\psi''(t) > 0$;
(c) $\lim_{t \to 0^+} \psi(t) = \lim_{t \to \infty} \psi(t) = \infty$;
(d) $t\psi''(t) + \psi'(t) > 0,\ t < 1$;
(e) $\psi'''(t) < 0,\ t > 0$;
(f) $2\psi''(t)^2 - \psi'(t)\psi'''(t) > 0,\ t < 1$;
(g) $\psi''(t)\psi'(\beta t) - \beta\psi'(t)\psi''(\beta t) > 0,\ t > 1,\ \beta > 1$.

Several IPMs have been provided for SCLCPs and Cartesian $P_*(\kappa)$-SCLCPs based on the kernel functions, which determine both search directions and the proximity measure between the iterates and the center path (see e.g., [16–19]). The purpose of the paper is to extend the primal-dual IPM for SOP in [20] based on the kernel function (2) to the Cartesian $P_*(\kappa)$-SCLCPs. We adopt the basic analysis used in [20] and revise them to be suited for the Cartesian $P_*(\kappa)$-SCLCPs case. Finally, we derive the iteration bounds that

match the currently best known iteration bounds for large-update methods by using the analysis emphasized on the kernel function and the Euclidean Jordan algebraic techniques, namely, $O\left((1+2\kappa)\sqrt{r}\log r\log\frac{r}{\varepsilon}\right)$.

2 Preliminaries

2.1 Euclidean Jordan Algebras and Symmetric Cones

Now, we recall some basic concepts and useful results of Euclidean Jordan algebras which are found in [21]. To ease discussion, we assume that cone $\mathcal{K}$ is defined with $N = 1$.

Definition 1 Let $(\mathcal{J},\ \langle\cdot,\ \cdot\rangle)$ be an n-dimensional inner product space over $\mathbb{R}$ endowed with a bilinear mapping $\circ : (x,\ y) \mapsto x \circ y$ from $\mathcal{J} \times \mathcal{J}$ to $\mathcal{J}$. Then the triple $(\mathcal{J},\ \circ,\ \langle\cdot,\ \cdot\rangle)$ is a Euclidean Jordan algebra if the following conditions hold:

(1) $x \circ y = y \circ x$ for all $x,\ y \in \mathcal{J}$;
(2) $x \circ (x^2 \circ y) = x^2 \circ (x \circ y)$ for all $x,\ y \in \mathcal{J}$, where $x^2 := x \circ x$;
(3) $\langle x \circ y,\ z\rangle = \langle y,\ x \circ z\rangle$ for all $x,\ y,\ z \in \mathcal{J}$.

For any $x,\ y \in \mathcal{J}$ we define the canonical inner product of $x,\ y \in \mathcal{J}$ as follows:

$$\langle x,\ y\rangle := \mathbf{Tr}(x \circ y),$$

and the Frobenius norm of x as follows

$$\|x\| := \sqrt{\langle x,\ x\rangle}.$$

An Euclidean Jordan algebra is called simple iff it cannot be represented as the orthogonal direct sum of two Euclidean Jordan algebras. We assume that there exists an element $e \in \mathcal{J}$, which called the identity element, such that $x \circ e = e \circ x = x$ for all $x \in \mathcal{J}$. Note that the identity element e is unique. Denote the corresponding cone of squares by $\mathcal{K} := \left\{x^2 :\ x \in \mathcal{J}\right\}$. $\mathcal{K}$ is indeed a symmetric cone (i.e., self-dual and homogeneous). Since $\mathcal{J}$ is finite-dimensional, given $x \in \mathcal{J}$, there exists a minimal positive integer k such that the set $\left\{e,\ x,\ \ldots,\ x^k\right\}$ is linearly dependent. The rank of $\mathcal{J}$, denoted rank $(\mathcal{J})$, is the largest $\deg(x)$ of any element $x \in \mathcal{J}$.

Two idempotents c_1 and c_2 are orthogonal if $c_1 \circ c_2 = 0$. The set $\{c_1,\ c_2,\ \ldots,\ c_k\}$ is called a complete system of orthogonal idempotents if all $c_i,\ i = 1,\ 2,\ \ldots,\ k$ are idempotent, each two $c_j,\ c_l$ are orthogonal and $\sum_{j=1}^{k} c_j = e$. An idempotent element is said to be primitive if it cannot be written as the sum of two other idempotents. A complete system of orthogonal primitive idempotents forms a Jordan frame.

Theorem 1 (Theorem III.1.2 in [21]) *Let $\mathcal{J}$ be a Euclidean Jordan algebra of rank r. For any $x \in \mathcal{J}$, there exists a Jordan frame $\{c_1(x),\ c_2(x),\ \ldots,\ c_r(x)\}$ and real numbers $\lambda_1(x),\ \ldots,\ \lambda_r(x)$ such that*

$$x = \sum_{j=1}^{r} \lambda_j(x) c_j(x). \tag{3}$$

The numbers $\lambda_1(x),\ \lambda_2(x),\ \ldots,\ \lambda_r(x)$ (with their multiplicities) are called the eigenvalues of x and Eq. 3 is the spectral decomposition of x. Now, it is possible to extend the

definition of any real-valued function $\psi_{\mathcal{K}}(\cdot)$ to elements of the Euclidean Jordan algebra via their eigenvalues:

$$\psi_{\mathcal{K}}(x) := \psi_{\mathcal{K}}(\lambda_1(x))c_1(x) + \ldots + \psi_{\mathcal{K}}(\lambda_r(x))c_r(x).$$

Furthermore, some functions in eigenvalues can be generated as follows: for any $x \in \mathcal{J}$,

$$\mathbf{Tr}(x) = \sum_{i=1}^{r} \lambda_i(x), \quad \text{and} \quad \det(x) = \prod_{i=1}^{r} \lambda_i(x).$$

Let the interior of $\mathcal{K}$, denoted int$\mathcal{K}$. Moreover, we can easily verify that

$$x \in \mathcal{K} \Leftrightarrow \lambda_i(x) \geq 0, \quad x \in \text{int}\mathcal{K} \Leftrightarrow \lambda_i(x) > 0,\ i = 1, 2, \ldots, r.$$

Since " $\circ$ " is bilinear for every $x \in \mathcal{J}$, the Lyapunov transformation $L(x) : \mathcal{J} \to \mathcal{J}$ is defined as $L(x)y := x \circ y$. For each $x \in \mathcal{J}$, we define

$$P(x) := 2L(x)^2 - L(x^2),$$

where $L(x)^2 = L(x)L(x)$. The map $P(x)$ is called the quadratic representation of x in $\mathcal{J}$, which is an essential concept in the theory of Jordan algebras. In the following we first recall some results from [5].

Proposition 1 *Let $x, s \in int\mathcal{K}$. Then $P(x)s \in int\mathcal{K}$.*

Lemma 1 *Let $x, s \in int\mathcal{K}$ and w be invertible. Then $x \circ s = \mu e$ if and only if*

$$P(w)x \circ P(w^{-1})s = \mu e.$$

In the following lemma, we give the so-called Nesterov-Todd scaling of $\mathcal{J}$, which plays an important role in the design of an interior-point algorithm for Cartesian $P_*(\kappa)$-SCLCPs.

Lemma 2 (Lemma 3.2 in [22]) *Let $x, s \in int\mathcal{K}$. Then there exists a unique $w \in int\ \mathcal{K}$ such that*

$$x = P(w)s.$$

Moreover,

$$w = P(x)^{\frac{1}{2}} \left(P\left(x^{\frac{1}{2}}\right) s \right)^{-\frac{1}{2}} \left[= P\left(s^{-\frac{1}{2}}\right) \left(P\left(s^{\frac{1}{2}}\right) x \right)^{\frac{1}{2}} \right].$$

The point w is called the scaling point of x and s (in this order).

Proposition 2 (Proposition 18 in [5]) *Let $x, s \in int\mathcal{K}$. Then*

$$P(x)s \in int\mathcal{K}.$$

2.2 Back to the General Case

We proceed by adapting the definitions and properties stated so far in this section to the general case, when the cone underlying the given Cartesian $P_*(\kappa)$-SCLCPs is the Cartesian product of N symmetric cones $\mathcal{K}_\nu$, where $N > 1$. As we mentioned in Section 1, we assume that $\mathcal{J} = \mathcal{J}_1 \times \mathcal{J}_2 \times \ldots \times \mathcal{J}_N$ is the Cartesian product space with its cone of

squares $\mathcal{K} = \mathcal{K}_1 \times \mathcal{K}_2 \times \ldots \times \mathcal{K}_N$, where each space $\mathcal{J}_\nu$ is a simple Euclidean Jordan algebra with ranks r_ν. It has the following properties:

- For any $z = \left(z^{(1)T},\ z^{(2)T},\ \ldots,\ z^{(N)T}\right)^T \in \mathcal{J}$, where $z^{(\nu)} \in \mathcal{J}_\nu$, we have

$$\|z\| = \sqrt{\sum_{\nu=1}^{N} \|z^{(\nu)}\|^2},\ \ \lambda_{\min}(z) = \min_{1\le\nu\le N}\left\{\lambda_{\min}(z^{(\nu)})\right\},$$

$$\mathbf{Tr}(z) = \sum_{\nu=1}^{N} \mathbf{Tr}\left(z^{(\nu)}\right),\ \ det(z) = \prod_{\nu=1}^{N} det\left(z^{(\nu)}\right);$$

- For any $x = \left(x^{(1)T},\ x^{(2)T},\ \ldots,\ x^{(N)T}\right)^T,\ s = \left(s^{(1)T},\ s^{(2)T},\ \ldots,\ s^{(N)T}\right)^T \in \mathcal{J}$ we have

$$x \circ s = \left(\left(x^{(1)} \circ s^{(1)}\right)^T,\ \left(x^{(2)} \circ s^{(2)}\right)^T,\ \ldots,\ \left(x^{(N)} \circ s^{(N)}\right)^T\right)^T,$$

$$\langle x,\ s\rangle = \sum_{\nu=1}^{N} \left\langle x^{(\nu)},\ s^{(\nu)}\right\rangle;$$

- If $e^{(\nu)} \in \mathcal{J}_\nu$ is the identity element in the Jordan algebra for the ν-th cone, then vector

$$e = \left(e^{(1)T},\ e^{(2)T},\ \ldots,\ e^{(N)T}\right)^T$$

is the identity element in $(\mathcal{J},\ \circ)$;

- The Lyapunov transformation and the quadratic representation of $\mathcal{J}$ can be adjusted to

$$L(x) = \text{diag}\left(L\left(x^{(1)}\right),\ \ldots,\ L\left(x^{(N)}\right)\right),$$

and

$$P(x) = \text{diag}\left(P\left(x^{(1)}\right),\ \ldots,\ P\left(x^{(N)}\right)\right);$$

- For any $v = \left(v^{(1)T},\ v^{(2)T},\ \ldots,\ v^{(N)T}\right)^T \in \mathcal{J}$ we define the eligible kernel function $\psi(v)$ and the barrier function $\Psi(v)$ as follows:

$$\psi(v) = \left(\psi\left(v^{(1)}\right),\ \ldots,\ \psi\left(v^{(N)}\right)\right)^T,$$

and

$$\Psi(v) := \mathbf{Tr}(\psi(v)) = \sum_{\nu=1}^{N} \mathbf{Tr}\left(\psi\left(v^{(\nu)}\right)\right) = \sum_{\nu=1}^{N}\sum_{i=1}^{r_\nu} \psi\left(\lambda_i\left(v^{(\nu)}\right)\right), \tag{4}$$

where $\psi(v^{(\nu)})(\nu = 1,\ 2,\ \ldots,\ N)$ are defined by Eq. 2.

3 Interior-Point Algorithm for Cartesian $P_*(\kappa)$-SCLCPs

In this section we first introduce the concept of the central path for Cartesian $P_*(\kappa)$-SCLCPs. Then we mainly derive new search directions for Cartesian $P_*(\kappa)$-SCLCPs based on the eligible kernel function. A generic polynomial interior-point algorithm for Cartesian $P_*(\kappa)$-SCLCPs is also presented.

3.1 Central Path for Cartesian $P_*(\kappa)$-SCLCPs

Similarly, to the $P_*(\kappa)$-LCPs case, the concept of the central path can also be extended to the Cartesian $P_*(\kappa)$-SCLCPs. The existence and uniqueness of central path for Cartesian $P_*(\kappa)$-SCLCPs was first established by Luo and Xiu [23]. Throughout the paper we assume that the Cartesian $P_*(\kappa)$-SCLCPs satisfies the interior-point condition (IPC), i.e., there exists x^0, $s^0 \in \text{int}\mathcal{K}$ such that $s^0 = \mathcal{A}(x^0) + q$. Under the IPC condition, by relaxing the complementarity slackness $x \circ s = 0$, we obtain

$$\begin{pmatrix} \mathcal{A}(x) - s \\ x \circ s \end{pmatrix} = \begin{pmatrix} -q \\ \mu e \end{pmatrix}, \; x, \; s \in \text{int}\mathcal{K}, \tag{5}$$

where $\mu > 0$ is a parameter. Since linear transformation $\mathcal{A}$ has the Cartesian $P_*(\kappa)$-property and the IPC holds, the parameterized system (5) has a unique solution [23]. This solution is denoted as $(x(\mu), s(\mu))$ and we call $(x(\mu), s(\mu))$ the μ-center of the Cartesian $P_*(\kappa)$-SCLCPs. The set of μ-centers, that $\{(x(\mu), s(\mu)) \mid \mu > 0\}$, is called the central path of cartesian $P_*(\kappa)$-SCLCPs. Note that at μ-center we have

$$\langle x(\mu), s(\mu)\rangle = \mathbf{Tr}(x(\mu) \circ s(\mu)) = \mathbf{Tr}(\mu e) = \mu \mathbf{Tr}(e) = r\mu.$$

Therefore, if as μ tends to zero, $(x(\mu), s(\mu))$ converges to an optimal solution for cartesian $P_*(\kappa)$-SCLCPs [23].

3.2 New Search Directions for Cartesian $P_*(\kappa)$-SCLCPs

The basic idea of IPMs is to follow the central path and approach the optimal set of Cartesian $P_*(\kappa)$-SCLCPs by letting μ go to zero. To obtain the search directions for cartesian $P_*(\kappa)$-SCLCPs a typical approach is to apply Newton's method to system (5). For any strictly feasible points x, $s \in \text{int}\mathcal{K}$, we find displacements Δx and Δs such that

$$\begin{pmatrix} \mathcal{A}(x + \Delta x) - (s + \Delta s) \\ (x + \Delta x) \circ (s + \Delta s) \end{pmatrix} = \begin{pmatrix} -q \\ \mu e \end{pmatrix}. \tag{6}$$

Neglecting the term $\Delta x \circ \Delta s$ in the left-hand side expression of the second equation, we obtain the following system:

$$\begin{pmatrix} \mathcal{A}\Delta x - \Delta s \\ s \circ \Delta x + x \circ \Delta s \end{pmatrix} = \begin{pmatrix} 0 \\ \mu e - x \circ s \end{pmatrix}. \tag{7}$$

By Lemma 1 and Lemma 2, we can replace the second equation of the system (7) by

$$P(u)(x + \Delta x) \circ P(u)^{-1}(s + \Delta s) = \mu e,$$

where $P(u^{-1}) = P(u)^{-1}$ and $u = w^{-1/2}$ such that w is the Nesterov-Todd scaling point of x and s. Applying Newton's method again, and neglecting the term $P(u)\Delta x \circ P(u)^{-1}\Delta s$, we get

$$\begin{aligned} &\begin{pmatrix} \mathcal{A}\Delta x - \Delta s \\ P(u)^{-1}(s) \circ P(u)\Delta x + P(u)(x) \circ P(u)^{-1}\Delta s \end{pmatrix} \\ &\qquad = \begin{pmatrix} 0 \\ \mu e - P(u)(x) \circ P(u)^{-1}(s) \end{pmatrix}. \end{aligned} \tag{8}$$

To describe our new search directions, we define following notations:

$$\begin{aligned} \overline{\mathcal{A}} &:= \frac{\mathcal{A}P(u^{-1})}{\sqrt{\mu}}, \quad v := \frac{P(u)x}{\sqrt{\mu}} = \frac{P(u^{-1})s}{\sqrt{\mu}}, \\ dx &:= \frac{P(u)\Delta x}{\sqrt{\mu}}, \quad ds := \frac{P(u^{-1})\Delta s}{\sqrt{\mu}}. \end{aligned} \tag{9}$$

After some elementary reductions, we obtain the system (8) which is equivalent to the scaled Newton system as follows:

$$\begin{pmatrix} \overline{\mathcal{A}}(dx) - ds \\ dx + ds \end{pmatrix} = \begin{pmatrix} 0 \\ v^{-1} - v \end{pmatrix}. \tag{10}$$

Since linear transformation $\mathcal{A}$ has the Cartesian $P_*(\kappa)$-property, the system (10) has a unique solution [23]. Replacing the right-hand side of the last equation in Eq. 10 by $-\psi'(v)$, we obtain:

$$\begin{pmatrix} \overline{\mathcal{A}}(dx) - ds \\ dx + ds \end{pmatrix} = \begin{pmatrix} 0 \\ -\psi'(v) \end{pmatrix}. \tag{11}$$

The scaled search directions dx and ds are obtained by solving (11) so that Δx and Δs are computed via (9). By taking a default step size α along the search directions, we get the new iterates $(x_+,\ s_+)$ according to

$$x_+ := x + \alpha\Delta x, \quad s_+ := s + \alpha\Delta s. \tag{12}$$

3.3 Properties of the Kernel Function and the Proximity Function

In this section we present some useful properties of the kernel function (2) and the proximity function $\Psi(v)$ as defined by Eq. 4 that are used in the analysis of interior-point algorithm for Cartesian $P_*(\kappa)$-SCLCPs.

The eligible kernel function (2) satisfies

$$\psi''(t) > 1,\ \psi'''(t) < 0 \text{ and } \lim_{t\to 0^+} \psi(t) = \lim_{t\to\infty} \psi(t) = \infty.$$

Note that $\psi(1) = \psi'(1) = 0$. Then $\psi(t)$ is determined:

$$\psi(t) = \int_1^t \int_1^\xi \psi''(\zeta) d\zeta\ d\xi.$$

By using the condition (a) of the eligible kernel functions, we can easily verify that

$$x \circ s = \mu e \Leftrightarrow v = e \Leftrightarrow \psi'(v) = 0 \Leftrightarrow \Psi(v) = 0.$$

Therefore, the value of $\Psi(v)$ can be considered as a measure for the distance between the given iterate $(x,\ s)$ and the μ-center $(x(\mu),\ s(\mu))$. Hence, we call $\Psi(v)$ the proximity function for Cartesian $P_*(\kappa)$-SCLCP . Furthermore, we introduce the norm-based proximity measure as follows:

$$\sigma := \|dx + ds\| = \|\psi'(v)\| = \sqrt{\|dx\|^2 + \|ds\|^2 + 2\sum_{v=1}^{N} \langle dx^{(v)},\ ds^{(v)}\rangle}.$$

One can easily verify that $\sigma \geq 0$, and $\sigma = 0$ if and only if $\Psi(v) = 0$. The following lemma gives a lower bound of σ in terms of $\Psi(v)$.

Lemma 3 *For any $v \in int\mathcal{K}$,*

$$\sigma \geq \sqrt{2\Psi(v)}.$$

Proof From definition of the kernel function (2), we obtain $2\psi(t) \leq (\psi'(t))^2$ and

$$\sigma^2 = \sum_{\nu=1}^{N} \sum_{i=1}^{r_\nu} (\psi'(\lambda_i(v^{(\nu)})))^2,$$

therefore, we have

$$2\Psi(v) \leq \sigma^2.$$

This completes the proof. □

Now, we present Theorem 4.9 in [24] where was assumed that cone $\mathcal{K}$ is defined with $N = 1$.

Proposition 3 (Theorem 4.9 in [24]) *Let Ψ be the proximity function defined in* Eq. 4*, then for any $x,\ y \in int\mathcal{K}$,*

$$\Psi\left(\left(P\left(x^{\frac{1}{2}}\right)s\right)^{\frac{1}{2}}\right) \leq \frac{1}{2}\left(\Psi(x) + \Psi(s)\right).$$

3.4 The Generic Large-Update Interior-Point Algorithm for Cartesian $P_*(\kappa)$-SCLCPs

The algorithm works as follows. We assume that there exists a strictly feasible point $(x,\ s)$ in a τ-neighborhood of the given μ-center ($\Psi(v) \leq \tau$). Then, we will go to the outer "while loop". If μ satisfies $r\mu \geq \varepsilon$, then we decrease μ to $\mu_+ = (1-\theta)\mu$, for some fixed $\theta \in (0,\ 1)$. Then, we make use of the inner "while loop", and the procedure is repeated until we find a new iterate (x_+, s_+) that is in a τ-neighborhood of the μ_+-center, i.e., $\Psi(v_+) \leq \tau$. We enter an inner iteration by computing the search directions using Eqs. 9 and 11 at the current iterates with respect to the current value of μ and apply (12) to get new iterates. This process is repeated until μ is small enough, say until $r\mu < \varepsilon$.

The large-update interior-point algorithm for the Cartesian $P_*(\kappa)$-SCLCP is presented in Fig. 1.

4 Analysis and Complexity of the Algorithm

In this section, we first discuss the bound of the proximity function during an outer iteration. Then we compute the default step size for the algorithm presented in Fig. 1. After that we show that the default step size yields sufficient decrease of the proximity function value during each inner iteration. Finally, we present the total number of iterations and complexity of the algorithm.

4.1 Bound of the Proximity Function During an Outer Iteration

Note that at the start of each outer iteration of the algorithm, just before the update of μ with the factor $1-\theta$, we have $\Psi(v) \leq \tau$. Due to the update of μ vector v is multiplied by factor $\frac{1}{\sqrt{1-\theta}}$, with $0 < \theta < 1$, which in general leads to an increase in value of $\Psi(v)$. Then during the subsequent inner iterations, $\Psi(v)$ decreases until it passes the threshold value

Large-update interior-point algorithm for the Cartesian $P_*(\kappa)$-SCLCP

Inputs:
A proximity parameter $\tau > 1$;
an accuracy parameter $\varepsilon > 0$;
a variable damping factor α;
a fixed barrier update parameter $\theta \in (0, 1)$;
(x^0, s^0) and $\mu^0 = 1$ such that $\Psi(v^0) \leq \tau$.

begin
$x := x^0$; $s := s^0$; $\mu := \mu^0$;
while $r\mu \geq \varepsilon$ **do**
begin
$\mu := (1 - \theta)\mu$;
while $\Psi(v) > \tau$ **do**
begin
solve the system (11) and use (9) to obtain Δx and Δs;
Determine a step size α;
$x := x + \alpha\Delta x$;
$s := s + \alpha\Delta s$;
end
end
end

Fig. 1 Algorithm

τ again, hence during the course of the algorithm presented in Fig. 1 the largest values of $\Psi(v)$ occurs just after the update of μ. So next we derive an estimate for the effect of a μ-update on value of $\Psi(v)$ by using the following two lemmas, which are the same as Lemma 3.1 and 3.2 in [20].

Lemma 4 (Lemma 3.1 in [20]) *Let $\beta \geq 1$. Then*

$$\psi(\beta t) \leq \psi(t) + \frac{(\beta^2 - 1)}{2} t^2.$$

Lemma 5 *For any $v \in int\mathcal{K}$, then*

$$\|v\|^2 \leq 2(\Psi(v) + 2r).$$

Proof Since $\frac{e^{p(t^{-q}-1)}}{pq}$ is positive and $pq \geq 1$, the kernel function (2) has a lower bound as follows:

$$\psi(t) \geq \frac{t^2 - 1}{2} - \frac{1}{pq} \geq \frac{t^2}{2} - 2.$$

This implies

$$\frac{1}{2}\sum_{\nu=1}^{N}\sum_{i=1}^{r_\nu}\lambda_i^2\left(v^{(\nu)}\right) \le \sum_{\nu=1}^{N}\sum_{i=1}^{r_\nu}\psi\left(\lambda_i\left(v^{(\nu)}\right)\right) + 2\sum_{\nu=1}^{N} r_\nu.$$

This completes the proof. □

The following theorem gives an upper bound for $\Psi\left(\frac{1}{\sqrt{1-\theta}}v\right)$, which plays an important role in the analysis of the algorithm presented in Fig. 1.

Theorem 2 *Let θ be such that $0 < \theta < 1$ and $v \in int\mathcal{K}$. If $\Psi(v) \le \tau$, then*

$$\Psi\left(\frac{1}{\sqrt{1-\theta}}v\right) \le \frac{2}{1-\theta}(\tau + r).$$

Proof From Lemma 4 with $\beta = \frac{1}{\sqrt{1-\theta}} \ge 1$ and Lemma 5,

$$\begin{aligned}\Psi\left(\frac{1}{\sqrt{1-\theta}}v\right) &= \sum_{\nu=1}^{N}\sum_{i=1}^{r_\nu}\psi\left(\frac{1}{\sqrt{1-\theta}}\lambda_i\left(v^{(\nu)}\right)\right) \le \Psi(v) + \frac{1}{2}\left(\frac{1}{1-\theta} - 1\right)\|v\|^2 \\ &\le \Psi(v) + \frac{\theta}{1-\theta}(\Psi(v) + 2r) \le \frac{2}{1-\theta}(\Psi(v) + r) \le \frac{2}{1-\theta}(\tau + r).\end{aligned}$$

The proof is completed. □

4.2 Computation of the Default Step Size

In this section, we compute feasible step size α such that the proximity function is decreasing and is bounded for the decrease during inner iterations.

In each inner iteration after the default step, we have got new iterates (x_+, s_+) according to Eq. 12. We define v_+ as follows:

$$v_+ = \frac{P(u_+)x_+}{\sqrt{\mu}} = \frac{P\left(u_+^{-1}\right)s_+}{\sqrt{\mu}},$$

where $u_+ = w_+^{-1/2}$ such that w_+ is Nesterov-Todd scaling point of x_+ and s_+. Now, we consider the decrease in $\Psi(v)$ during an inner iteration as a function of α and define

$$f(\alpha) := \Psi(v_+) - \Psi(v).$$

At this stage, we invoke Proposition 3.1 in [20] where $N = 1$.

Proposition 4 (Proposition 5.6 in [24]) *Let the proximity function be as defined in* Eq. 4. *Then we have*

$$\Psi(v_+) = \Psi\left(\left(P(v + \alpha dx)^{\frac{1}{2}}(v + \alpha ds)\right)^{\frac{1}{2}}\right).$$

Proposition 3 and Proposition 4 imply that, for each $\nu,\ 1 \le \nu \le N$,

$$\Psi\left(v_+^{(\nu)}\right) = \Psi\left(\left(P\left(v^{(\nu)}+\alpha dx^{(\nu)}\right)^{\frac{1}{2}}\left(v^{(\nu)}+\alpha ds^{(\nu)}\right)\right)^{\frac{1}{2}}\right)$$
$$\le \frac{1}{2}\left(\Psi\left(v^{(\nu)}+\alpha dx^{(\nu)}\right)+\Psi\left(v^{(\nu)}+\alpha ds^{(\nu)}\right)\right).$$

Taking summation over all $\nu,\ 1 \le \nu \le N$, we get

$$\Psi(v_+) \le \frac{1}{2}\left(\Psi(v+\alpha dx)+\Psi(v+\alpha ds)\right).$$

So, we have

$$f(\alpha) \le f_1(\alpha) := \frac{1}{2}\left(\Psi(v+\alpha dx)+\Psi(v+\alpha ds)\right)-\Psi(v).$$

Since the linear transformation $\mathcal{A}$ has the Cartesian $P_*(\kappa)$-property and

$$\mathcal{A}\Delta x - \Delta s = 0,$$

from Eq. 7, we obtain

$$(1+4\kappa)\sum_{\nu\in I_+}\langle \Delta x^{(\nu)},\ \Delta s^{(\nu)}\rangle + \sum_{\nu\in I_-}\langle \Delta x^{(\nu)},\ \Delta s^{(\nu)}\rangle \ge 0, \tag{13}$$

where $I_+ = \{\nu\ :\ \langle \Delta x^{(\nu)},\ \Delta s^{(\nu)}\rangle \ge 0\},\ I_- = \{\nu\ :\ \langle \Delta x^{(\nu)},\ \Delta s^{(\nu)}\rangle < 0\}$.

Since $\langle dx,\ ds\rangle = \frac{\langle \Delta x,\ \Delta s\rangle}{\mu}$, using Eqs. 9, 13 can be rewritten as

$$(1+4\kappa)\sum_{\nu\in I_+}\langle dx^{(\nu)},\ ds^{(\nu)}\rangle + \sum_{\nu\in I_-}\langle dx^{(\nu)},\ ds^{(\nu)}\rangle \ge 0. \tag{14}$$

In the sequel, we use the following notations:

$$\sigma_+ := \sum_{\nu\in I_+}\langle dx^{(\nu)},\ ds^{(\nu)}\rangle,\ \text{and}\ \sigma_- := -\sum_{\nu\in I_-}\langle dx^{(\nu)},\ ds^{(\nu)}\rangle. \tag{15}$$

Lemma 6 *One has*

$$\sigma_+ \le \frac{1}{4}\sigma^2,\ \text{and}\ \sigma_- \le \frac{1}{4}(1+4\kappa)\sigma^2.$$

Proof We have

$$0 \le \frac{1}{4}\sum_{\nu\in I_+}\langle dx^{(\nu)}-ds^{(\nu)},\ dx^{(\nu)}-ds^{(\nu)}\rangle$$
$$= \frac{1}{4}\sum_{\nu\in I_+}\langle dx^{(\nu)}+ds^{(\nu)},\ dx^{(\nu)}+ds^{(\nu)}\rangle - \sum_{\nu\in I_+}\langle dx^{(\nu)},\ ds^{(\nu)}\rangle.$$

Hence, by Eq. 15, we have

$$\sigma_+ = \sum_{\nu\in I_+}\langle dx^{(\nu)},\ ds^{(\nu)}\rangle \le \frac{1}{4}\sum_{\nu\in I_+}\langle dx^{(\nu)}+ds^{(\nu)},\ dx^{(\nu)}+ds^{(\nu)}\rangle$$
$$\le \frac{1}{4}\sum_{\nu=1}^{N}\langle dx^{(\nu)}+ds^{(\nu)},\ dsx^{(\nu)}+ds^{(\nu)}\rangle = \frac{1}{4}\|dx+ds\|^2 = \frac{1}{4}\sigma^2.$$

It follows immediately from Eq. 14 that

$$(1+4\kappa)\sigma_+ - \sigma_- \geq 0.$$

Then

$$\sigma_- \leq (1+4\kappa)\sigma_+ \leq \frac{1}{4}(1+4\kappa)\sigma^2.$$

This proves the lemma. □

Lemma 7 *One has*

$$\|dx\|^2 + \|ds\|^2 \leq (1+2\kappa)\sigma^2.$$

Proof From Eq. 14, we have

$$\sigma^2 = \|dx+ds\|^2 = \|dx\|^2 + \|ds\|^2 + 2(\sigma_+ - \sigma_-) \geq \|dx\|^2 + \|ds\|^2 - \frac{8\kappa}{1+4\kappa}\sigma_-.$$

Thus by Lemma 6, we have

$$\|dx\|^2 + \|ds\|^2 \leq \sigma^2 + \frac{8\kappa}{1+4\kappa}\sigma_- \leq (1+2\kappa)\sigma^2.$$

The proof is completed. □

The following lemma is derived from Lemma 14 in [5] and Lemma 7.

Lemma 8 *For any* $\alpha \in \left(0, \frac{\lambda_{\min}(v)}{\sqrt{1+2\kappa}\sigma}\right)$,

$$\lambda_{\min}(v+\alpha dx) \geq \lambda_{\min}(v) - \alpha\sqrt{1+2\kappa}\sigma,$$

$$\lambda_{\min}(v+\alpha ds) \geq \lambda_{\min}(v) - \alpha\sqrt{1+2\kappa}\sigma.$$

Proof Choose $\alpha \in \left(0, \frac{\lambda_{\min}(v)}{\sqrt{1+2\kappa}\sigma}\right)$ arbitrarily. From Lemma 14 in [5],

$$\lambda_{\min}(v+\alpha dx) \geq \lambda_{\min}(v) - \alpha\|dx\|,$$

and

$$\lambda_{\min}(v+\alpha ds) \geq \lambda_{\min}(v) - \alpha\|ds\|.$$

By Lemma 7, we have

$$\|dx\| \leq \sqrt{1+2\kappa}\sigma, \text{ and } \|ds\| \leq \sqrt{1+2\kappa}\sigma.$$

Thus

$$\lambda_{\min}(v+\alpha dx) \geq \lambda_{\min}(v) - \alpha\sqrt{1+2\kappa}\sigma,$$

and

$$\lambda_{\min}(v+\alpha ds) \geq \lambda_{\min}(v) - \alpha\sqrt{1+2\kappa}\sigma.$$

The proof is completed. □

In the same way as Proposition 3.2 in [20], we have the following result.

Proposition 5 *Suppose that the functions $\psi(x)$ and $\Psi(x)$ are defined by* Eqs. 2 *and* 4, *respectively. Then, for any $\alpha \in \left(0, \frac{\lambda_{\min}(v)}{\sqrt{1+2\kappa\sigma}}\right)$,*

$$f_1'(\alpha) = \frac{1}{2}\sum_{v=1}^{N}\left(\mathbf{Tr}\left(\psi'\left(v^{(v)}+\alpha dx^{(v)}\right)\circ dx^{(v)}\right)+\mathbf{Tr}\left(\psi'\left(v^{(v)}+\alpha ds^{(v)}\right)\circ ds^{(v)}\right)\right),$$

$$f_1''(\alpha) \le \frac{3}{2}\max\left\{\Delta\psi'\left(\lambda_i\left(\hat{v}_x^{(v)}(\alpha)\right),\ \lambda_j\left(\hat{v}_x^{(v)}(\alpha)\right)\right) | 1\le v\le N,\ 1\le i,\ j\le r_v\right\}\|dx\|^2$$
$$+\frac{3}{2}\max\left\{\Delta\psi'\left(\lambda_i\left(\hat{v}_s^{(v)}(\alpha)\right),\lambda_j\left(\hat{v}_s^{(v)}(\alpha)\right)\right) | 1\le v\le N,\ 1\le i,\ j\le r_v\right\}\|ds\|^2,$$

where $\hat{v}_x^{(v)}(\alpha) = v^{(v)}+\alpha dx^{(v)}$ and $\hat{v}_s^{(v)}(\alpha) = v^{(v)}+\alpha ds^{(v)}$

$$\Delta\psi'(\lambda_i(\cdot),\lambda_j(\cdot)) = \begin{cases}\psi''(\lambda_i(\cdot)) & \lambda_i(\cdot)=\lambda_j(\cdot),\\ \frac{\psi'(\lambda_i(\cdot))-\psi'(\lambda_j(\cdot))}{\lambda_i(\cdot)-\lambda_j(\cdot)} & \lambda_i(\cdot)\ne\lambda_j(\cdot).\end{cases}$$

The following lemma gives an upper bound for the second derivative of $f_1(\alpha)$ in terms of σ and $\psi''(t)$. This plays an important role in the analysis of the algorithm presented in Fig. 1.

Proposition 6 *For any $\alpha \in \left(0, \frac{\lambda_{\min}(v)}{\sqrt{1+2\kappa\sigma}}\right)$,*

$$f_1''(\alpha) \le \frac{3}{2}(1+2\kappa)\sigma^2\psi''(\lambda_{\min}(v)-\alpha\sqrt{1+2\kappa}\sigma).$$

Proof Since $\psi''(t)$ is a decreasing function on $t\in(0,\infty)$, using Lemma 8 and the mean value theorem, we have

$$\max\left\{\Delta\psi'\left(\lambda_i\left(v^{(v)}+\alpha dx^{(v)}\right),\lambda_j\left(v^{(v)}+\alpha dx^{(v)}\right)\right) \mid v=1,\ldots,N, i, j=1,\ldots,r_v\right\}$$
$$\le \psi''\left(\lambda_{\min}\left(v+\alpha dx\right)\right) \le \psi''\left(\lambda_{\min}(v)-\alpha\sqrt{1+2\kappa}\sigma\right),$$

and

$$\max\left\{\Delta\psi'\left(\lambda_i\left(v^{(v)}+\alpha ds^{(v)}\right),\lambda_j\left(v^{(v)}+\alpha ds^{(v)}\right)\right) \mid v=1,\ldots,N, i, j=1,\ldots,r_v\right\}$$
$$\le \psi''\left(\lambda_{\min}\left(v+\alpha ds\right)\right) \le \psi''\left(\lambda_{\min}(v)-\alpha\sqrt{1+2\kappa}\sigma\right).$$

Thus, by Proposition 5 and Lemma 7,

$$f_1''(\alpha) \le \frac{3}{2}\psi''(\lambda_{\min}(v)-\alpha\sqrt{1+2\kappa}\sigma)\left(\|dx\|^2+|ds\|^2\right)$$
$$\le \frac{3}{2}(1+2\kappa)\sigma^2\psi''(\lambda_{\min}(v)-\alpha\sqrt{1+2\kappa}\sigma),$$

which completes the proof of the lemma. □

Lemma 9 *If the step size $\alpha \in \left(0, \frac{\lambda_{\min}(v)}{\sqrt{1+2\kappa\sigma}}\right)$ satisfies*

$$-\psi'(\lambda_{\min}(v)-\alpha\sqrt{1+2\kappa}\sigma)+\psi'(\lambda_{\min}(v)) \le \frac{\sigma}{3\sqrt{1+2\kappa}}, \tag{16}$$

then, $f_1'(\alpha)\le 0$.

Proof Proposition 5 gives, by Eq. 11,

$$\begin{aligned} f_1'(0) &= \frac{1}{2}\mathbf{Tr}(\psi'(v)\circ(dx+ds)) = -\frac{1}{2}\mathbf{Tr}(\psi'(v)\circ\psi'(v)) \\ &= -\frac{1}{2}\|\psi'(v)\|^2 = -\frac{\sigma^2}{2}. \end{aligned}$$

Thus, by Proposition 6, we have

$$\begin{aligned} f_1'(\alpha) &= f_1'(0) + \int_0^\alpha f_1''(\xi)d\xi \\ &\le -\frac{\sigma^2}{2} + \frac{3}{2}(1+2\kappa)\sigma^2\int_0^\alpha \psi''(\lambda_{\min}(v) - \xi\sqrt{1+2\kappa}\sigma)d\xi \\ &= -\frac{\sigma^2}{2} - \frac{3}{2}\sqrt{1+2\kappa}\sigma\left(\psi'(\lambda_{\min}(v) - \alpha\sqrt{1+2\kappa}\sigma) - \psi'(\lambda_{\min}(v))\right) \\ &\le -\frac{\sigma^2}{2} + \frac{3}{2}\sqrt{1+2\kappa}\sigma\frac{\sigma}{3\sqrt{1+2\kappa}} = 0. \end{aligned}$$

This proves the lemma. □

In the following lemma, we compute the feasible step size α such that the proximity function is decreasing when we take a new iterate for fixed μ.

Lemma 10 *Let $\rho : [0, \infty) \to (0, 1]$ denote the inverse function of the restriction of $-\psi'(t)$ in interval $(0, 1]$. Then the largest step size α that satisfies (16) is given by*

$$\alpha^* = \frac{1}{\sqrt{1+2\kappa}\sigma}\left(\rho(\sigma) - \rho\left(\left(1 + \frac{1}{3\sqrt{1+2\kappa}}\right)\sigma\right)\right). \tag{17}$$

Proof We want to compute the step size α such that (16) holds with α as large as possible. Since $\psi''(t)$ is monotonically decreasing, the derivative of the left hand side in Eq. 16 with respect to α is

$$\sqrt{1+2\kappa}\sigma\psi''(\lambda_{\min}(v) - \alpha\sqrt{1+2\kappa}\sigma) > 0.$$

So the largest possible value of α satisfying (16) occurs when

$$-\psi'(\lambda_{\min}(v) - \alpha\sqrt{1+2\kappa}\sigma) + \psi'(\lambda_{\min}(v)) = \frac{\sigma}{3\sqrt{1+2\kappa}}. \tag{18}$$

The derivative of the left hand side in Eq. 16 with respect to $\lambda_{\min}(v)$ is

$$-\psi''(\lambda_{\min}(v) - \alpha\sqrt{1+2\kappa}\sigma) + \psi''(\lambda_{\min}(v)) < 0.$$

This implies that with σ fixed if $\lambda_{\min}(v)$ gets smaller, then α gets smaller. Note that

$$\sigma = \|\psi'(v)\| \ge |\psi'(\lambda_{\min}(v))| \ge -\psi'(\lambda_{\min}(v)). \tag{19}$$

The equality holds if and only if $\lambda_{\min}(v)$ is the only coordinate which is different from 1 and $\lambda_{\min}(v) < 1$, i.e., $\psi'(\lambda_{\min}(v)) < 0$. Hence, the worst situation for the largest step size occurs when $\lambda_{\min}(v)$ satisfies

$$-\psi'(\lambda_{\min}(v)) = \sigma. \tag{20}$$

In the worst case by Eq. 20 and the definition of ρ,

$$\lambda_{\min}(v) = \rho(\sigma). \tag{21}$$

From Eqs. 18 and 20,

$$-\psi'(\lambda_{\min}(v) - \alpha\sqrt{1+2\kappa}\sigma) = \left(1 + \frac{1}{3\sqrt{1+2\kappa}}\right)\sigma. \tag{22}$$

Thus by the definition of ρ and Eq. 20, the largest step size α of the worse case is given as follows:

$$\alpha^* = \frac{1}{\sqrt{1+2\kappa}\sigma}\left(\rho(\sigma) - \rho\left(\left(1 + \frac{1}{3\sqrt{1+2\kappa}}\right)\sigma\right)\right). \tag{23}$$

This completes the proof. □

Let $x, s \in \text{int}\mathcal{K}$ and $\hat{\alpha} \in (0, \alpha^*]$, we want to prove that $x_+ \in \text{int}\mathcal{K}$ and $s_+ \in \text{int}\mathcal{K}$ and Eq. 16 is satisfied for all $\hat{\alpha} \in (0, \alpha^*]$. We know that $x_+ = \sqrt{\mu}P(u^{-1})(v + \hat{\alpha}dx)$ and $s_+ = \sqrt{\mu}P(u)(v + \hat{\alpha}ds)$. By the definition of ρ and Eq. 19, we have

$$\lambda_{\min}(v) \geq \rho(\sigma). \tag{24}$$

So we obtain

$$\hat{\alpha} \leq \alpha^* = \frac{1}{\sqrt{1+2\kappa}\sigma}\left(\rho(\sigma) - \rho\left(\left(1 + \frac{1}{3\sqrt{1+2\kappa}}\right)\sigma\right)\right) \leq \frac{\lambda_{\min}(v)}{\sqrt{1+2\kappa}\sigma}. \tag{25}$$

By Lemma 8 and Eq. 25, we have $v + \hat{\alpha}dx \in \text{int}\mathcal{K}$ and $v + \hat{\alpha}ds \in \text{int}\mathcal{K}$. Thus by Proposition 2, we obtain $x_+ \in \text{int}\mathcal{K}$ and $s_+ \in \text{int}\mathcal{K}$. Since the left-hand side of inequality (16) is increasing in α and decreasing in $\lambda_{\min}(v)$. Thus we obtain

$$\begin{aligned}
&-\psi'(\lambda_{\min}(v) - \hat{\alpha}\sqrt{1+2\kappa}\sigma) + \psi'(\lambda_{\min}(v)) \\
&\quad \leq -\psi'(\lambda_{\min}(v) - \alpha^*\sqrt{1+2\kappa}\sigma) + \psi'(\lambda_{\min}(v)) \\
&\quad \leq -\psi'(\rho(\sigma) - \alpha^*\sqrt{1+2\kappa}\sigma) + \psi'(\rho(\sigma)) \\
&\quad = \left(1 + \frac{1}{3\sqrt{1+2\kappa}}\right)\sigma - \sigma = \frac{\sigma}{3\sqrt{1+2\kappa}}.
\end{aligned}$$

Now, we compute the lower bound for α^* in Lemma 10. It can be easily obtained by using the following result.

Lemma 11 (Lemma 3.6 in [20]) *Let $\sigma \geq 1$. Then, for $0 < t \leq \rho\left(\frac{4}{3}\sigma\right)$,*

$$\psi''(t) \leq 1 + 3\sigma(1 + pq + q)\left(1 + \frac{1}{p}\log 3\sigma\right)^{\frac{q+1}{q}}.$$

Theorem 3 *Let α^* be as defined in* Eq. 23, *Then*

$$\alpha^* \geq \frac{1}{3(1+2\kappa)\psi''\left(\rho\left(\left(1 + \frac{1}{3\sqrt{1+2\kappa}}\right)\sigma\right)\right)}.$$

Proof Due to the definition of ρ, we have

$$-\psi'(\rho(\sigma)) = \sigma.$$

Taking the derivative of σ at both sides, we get

$$\rho'(\sigma) = -\frac{1}{\psi''(\rho(\sigma))} < 0.$$

Hence ρ is monotonically decreasing. Moreover, we have

$$\begin{aligned}\alpha^* &= \frac{1}{\sqrt{1+2\kappa}\sigma}\int_{\left(1+\frac{1}{3\sqrt{1+2\kappa}}\right)\sigma}^{\sigma}\rho'(\xi)d\xi = \frac{1}{\sqrt{1+2\kappa}\sigma}\int_{\sigma}^{\left(1+\frac{1}{3\sqrt{1+2\kappa}}\right)\sigma}\frac{1}{\psi''(\rho(\xi))}d\xi \\ &\geq \frac{1}{\sqrt{1+2\kappa}\sigma}\left[\frac{\xi}{\psi''\left(\rho\left(\left(1+\frac{1}{3\sqrt{1+2\kappa}}\right)\sigma\right)\right)}\right]_{\sigma}^{\left(1+\frac{1}{3\sqrt{1+2\kappa}}\right)\sigma} \\ &= \frac{1}{3(1+2\kappa)\psi''\left(\rho\left(\left(1+\frac{1}{3\sqrt{1+2\kappa}}\right)\sigma\right)\right)},\end{aligned}$$

where the inequality follows from ρ and ψ'' are monotonically decreasing. □

Since $\psi''(t)$ is monotonically decreasing, Theorem 3 and Lemma 11 imply that

$$\alpha^* \geq \frac{1}{3(1+2\kappa)\left(1+3\sigma(1+pq+q)\left(1+\frac{1}{p}\log 3\sigma\right)^{\frac{q+1}{q}}\right)}.$$

In the sequel we define the $\bar{\alpha}$ as follows:

$$\bar{\alpha} = \frac{1}{3(1+2\kappa)\left(1+3\sigma(1+pq+q)\left(1+\frac{1}{p}\log 3\sigma\right)^{\frac{q+1}{q}}\right)}, \tag{26}$$

and we will use $\bar{\alpha}$ as the default step size in the algorithm for the Cartesian $P_*(\kappa)$-SCLCPs.

4.3 Decrease of the Proximity Function During an Inner Iteration

In what follows, we will show that proximity function Ψ in each inner iteration with the default step size $\bar{\alpha}$, as defined by Eq. 26, is decreasing. It can be easily established by using the following technical result.

Lemma 12 (Lemma 3.12 in [25]) *Let $h(t)$ be a twice differentiable convex function with $h(0) = 0$, $h'(0) < 0$ and let $h(t)$ attain its (global) minimum at $t^* > 0$. If $h''(t)$ is increasing for $t \in [0,\ t^*]$, then*

$$h(t) \leq \frac{th'(0)}{2},\quad 0 \leq t \leq t^*.$$

By Lemma 12, we have the following lemma, which gives an upper bound for the decreasing value of the proximity function Ψ in each inner iteration.

Lemma 13 *If the step size α is such that $\alpha \leq \alpha^*$, then*

$$f(\alpha) \leq -\frac{\alpha}{4}\sigma^2.$$

Proof By Proposition 6, we have

$$\begin{aligned} f_1(\alpha) &= f_1(0) + f_1'(0)\alpha + \int_0^\alpha \int_0^\xi f_1''(\zeta)d\zeta d\xi \\ &\leq h(\alpha) := f_1(0) + f_1'(0)\alpha \\ &\quad + \frac{3}{2}(1+2\kappa)\sigma^2 \int_0^\alpha \int_0^\xi \psi''(\lambda_{\min}(v) - \zeta\sqrt{1+2\kappa}\sigma)d\zeta d\xi. \end{aligned}$$

We can easily verify that

$$h(0) = f_1(0) = 0, \quad h'(0) = f_1'(0) = -\frac{\sigma^2}{2}.$$

Taking $\alpha \leq \alpha^*$, we derive

$$\begin{aligned} h'(\alpha) &= -\frac{\sigma^2}{2} - \frac{3}{2}\sqrt{1+2\kappa}\sigma\left(\psi'(\lambda_{\min}(v) - \alpha\sqrt{1+2\kappa}\sigma) - \psi'(\lambda_{\min}(v))\right) \\ &\leq -\frac{\sigma^2}{2} + \frac{3}{2}\sqrt{1+2\kappa}\sigma\frac{\sigma}{3\sqrt{1+2\kappa}} = 0. \end{aligned}$$

The last inequality holds due to the definition of α^*, which guarantees that if $\alpha \leq \alpha^*$ then inequality (16) holds. Since $\psi'''(t) < 0$ for $t > 0$, $h''(\alpha)$ is increasing in α. Therefore $h(\alpha)$ is satisfying assumptions of Lemma 12,

$$f(\alpha) \leq f_1(\alpha) \leq h(\alpha) \leq \frac{\alpha h'(0)}{2} = -\frac{\alpha}{4}\sigma^2.$$

The result follows. □

Theorem 4 *With $\bar{\alpha}$ being the default step size as given by* Eq. 26*, one has*

$$f(\bar{\alpha}) \leq \left(-\frac{1}{6(1+2\kappa)}\right)\frac{\sqrt{\Psi}}{1+3\sqrt{2}(1+pq+q)\left(1+\frac{1}{p}\log 3\sqrt{2\Psi_0}\right)^{\frac{q+1}{q}}},$$

where Ψ_0 is the value of $\Psi(\alpha)$ after μ-update in outer iteration.

Proof Using Lemma 13 with $\alpha = \bar{\alpha}$ and Eq. 26, we have

$$f(\bar{\alpha}) \leq -\frac{\bar{\alpha}}{4}\sigma^2 = \left(-\frac{1}{12(1+2\kappa)}\right)\frac{\sigma^2}{1+3\sigma(1+pq+q)\left(1+\frac{1}{p}\log 3\sigma\right)^{\frac{q+1}{q}}}.$$

Since the last expression is monotonically decreasing in σ, we can express the decrease in terms of $\Psi = \Psi(v)$ by Lemma 3 as follows:

$$\begin{aligned} f(\bar{\alpha}) &\le \left(-\frac{1}{6(1+2\kappa)}\right) \frac{\Psi}{1+3\sqrt{2\Psi}(1+pq+q)\left(1+\frac{1}{p}\log 3\sqrt{2\Psi}\right)^{\frac{q+1}{q}}} \\ &\le \left(-\frac{1}{6(1+2\kappa)}\right) \frac{\sqrt{\Psi}\sqrt{\Psi}}{\sqrt{\Psi}+3\sqrt{2\Psi}(1+pq+q)\left(1+\frac{1}{p}\log 3\sqrt{2\Psi_0}\right)^{\frac{q+1}{q}}} \\ &= \left(-\frac{1}{6(1+2\kappa)}\right) \frac{\sqrt{\Psi}}{1+3\sqrt{2}(1+pq+q)\left(1+\frac{1}{p}\log 3\sqrt{2\Psi_0}\right)^{\frac{q+1}{q}}}, \end{aligned}$$

where the inequality follows from $\Psi_0 \ge \Psi \ge \tau \ge 1$. The proof of the theorem is completed. □

4.4 Iteration Bound

For the analysis of the algorithm presented in Fig. 1, we need to count how many inner iterations are required to return to the situation where $\Psi(v) \le \tau$ after a μ-update. We denote the value of $\Psi(v) \le \tau$ after a μ-update as Ψ_0, the subsequent values in the same outer iteration are denoted as $\Psi_k,\ k = 1, 2, \ldots, K$, where K denotes the total number of inner iterations in the outer iteration. Then we have

$$\Psi_0 \le \frac{2}{1-\theta}(\tau + r) = O(r), \quad \Psi_{k-1} > \tau, \quad 0 \le \Psi_k \le \tau,$$

where the inequality and equality follow from Theorem 2 with $\tau = O(r)$ and $\theta = \Theta(1)$. According to decrease of $f(\bar{\alpha})$ obtained in Theorem 4,

$$\Psi_{k+1} \le \Psi_k - \frac{1}{(1+2\kappa)\left(6+18\sqrt{2}(1+pq+q)\left(1+\frac{1}{p}\log 3\sqrt{2\Psi_0}\right)^{\frac{q+1}{q}}\right)}\sqrt{\Psi}.$$

At this stage, we invoke Lemma 14 in [25].

Lemma 14 (Lemma 14 in [25]) *Let $t_0, t_1, \ldots, t_K$ be a sequence of positive numbers such that*

$$t_{k+1} \le t_k - \beta t_k^{1-\gamma}, \ k = 0, 1, \ldots, K-1,$$

where $\beta > 0$ and $0 < \gamma \le 1$. Then

$$K \le \frac{t_0^{\gamma}}{\beta\gamma}.$$

Letting $t_k = \Psi_k,\ \beta = \frac{1}{(1+2\kappa)\left(6+18\sqrt{2}(1+pq+q)\left(1+\frac{1}{p}\log 3\sqrt{2\Psi_0}\right)^{\frac{q+1}{q}}\right)}$ and $\gamma = \frac{1}{2}$, we can get the following lemma from Lemma 4. This gives an upper bound for the number of iterations of the algorithm presented in Fig. 1.

Lemma 15 *Let K be the total number of inner iterations in outer iteration. Then we have*

$$K \leq 2(1+2\kappa)\left(6+18\sqrt{2}(1+pq+q)\left(1+\frac{1}{p}\log 3\sqrt{2\Psi_0}\right)^{\frac{q+1}{q}}\right)\Psi_0^{\frac{1}{2}},$$

where Ψ_0 is the value of $\Psi(\alpha)$ after μ-update in outer iteration.

After some elementary reductions, we have the following theorem, which yields the iterations bound of the algorithm for the Cartesian $P_*(\kappa)$-SCLCPs.

Theorem 5 *If $\tau \geq 1$ and $0 < \theta < 1$, the total number of iterations is not more than*

$$\left\lceil 2(1+2\kappa)\left(6+18\sqrt{2}(1+pq+q)\left(1+\frac{1}{p}\log 3\sqrt{2\Psi_0}\right)^{\frac{q+1}{q}}\right)\Psi_0^{\frac{1}{2}}\right\rceil\left\lceil\frac{1}{\theta}\log\frac{r}{\varepsilon}\right\rceil.$$

Remark 1 If one takes $p = \log r$ and $q = 1$ (when $\tau = O(r)$ and $\theta = \Theta(1)$), it shows clearly that gives the currently best known iteration bound for the large-update methods, namely

$$O\left((1+2\kappa)\sqrt{r}\log r\log\left(\frac{r}{\varepsilon}\right)\right).$$

5 Concluding Remarks

In this paper we have extended a new complexity analysis of primal-dual interior-point algorithms for SOP based on the parametric kernel function $\psi(t)$ defined by Eq. 2 proposed in [20] to Cartesian $P_*(\kappa)$-SCLCPs. By using Euclidean Jordan algebraic techniques, we derived the iteration bound that match the currently best known iteration bound for large-update methods. Some interesting topics remain for further research. First, the search directions used, are all based on NT-symmetrization scheme. It may be possible to design similar algorithms using other symmetrization schemes to obtain polynomial-time iteration bounds. Second, the numerical test is an interesting topic for investigating the behavior of the algorithm so as to be compared with other approaches.

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