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A Generic Interior-point Algorithm for Monotone Symmetric Cone Linear Complementarity Problems Based on a New Kernel Function

Behrouz Kheirfam

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Abstract Kernel functions play an important role in defining new search directions for interior-point algorithms for solving monotone linear complementarity problems. In this paper we present a new kernel function which yields the complexity bounds $\mathcal{O}(\sqrt{r}\log r \log \frac{r}{\epsilon})$ and $\mathcal{O}(\sqrt{r}\log \frac{r}{\epsilon})$ for large-and small-update methods, respectively, which are currently the best known bounds for such methods.

Keywords Monotone linear complementarity problem · Interior-point algorithms · Kernel functions · Euclidean Jordan algebra

Mathematics Subject Classification (2010) 90C51

1 Introduction

Consider the monotone symmetric cone linear complementarity problem (SCLCP) in the standard form: Given an n-dimensional Euclidean Jordan algebra $(\mathcal{J}, \circ, \langle\cdot,\cdot\rangle)$ with rank r and its associated symmetric cone of squares $\mathcal{K}$, find $(x, s) \in \mathcal{K} \times \mathcal{K}$ such that

$$s = Mx + q, \quad x \circ s = 0, \tag{SCLCP}$$

where, $M \in R^{n\times n}$ and $q \in R^n$ are given data.

Primal-dual interior-point methods (IPMs) have been well known as the most effective methods for solving wide classes of optimization problems, for example,

B. Kheirfam (✉)
Department of Mathematics, Azarbaijan Shahid Madani University, Tabriz, Iran
e-mail: b.kheirfam@azaruniv.edu

the linear optimization (LO), the quadratic optimization (QOP), the semidefinite optimization (SDO), the second-order cone optimization (SOCO), and the linear complementarity problem (LCP). Nesterov and Todd [15] provided a theoretical foundation for efficient primal-dual IPMs on a special class of convex optimization problems, where the associated cone was self-scaled. Later on, it was observed that the self-scaled cones were precisely symmetric cones [4]. The application of the Euclidean Jordan algebra as a basic tool for analyzing complexity proofs of the IPMs for symmetric cone optimization (SCO) and SCLCP was started by Faybusovich [6], who extended earlier works of Nesterov and Todd, and Kojima et al. [9, 15]. Schmieta and Alizadeh [21, 22] studied primal-dual IPMs for SCO extensively under the framework of Euclidean Jordan algebra. In addition to Faybusovich's results [6, 7], Rangarajan [18] proposed the first infeasible interior-point method (IIPM) for SCLCP. Yoshise [27] was the first to analyze IPMs for nonlinear complementarity problems over symmetric cones. Darvay [3] proposed a full-Newton step primal-dual path-following interior-point algorithm for LO. The search direction of his algorithm is introduced by using an algebraic equivalent transformation of the centering equation which define the central path and then applying Newton's method for the new system of equations. Later on, Wang [24] generalized Darvay's full-Newton step primal-dual path-following interior-point algorithm for LO to the monotone SCLCP by using Euclidean Jordan algebras. Wang and Lesaja [26] presented a full-Newton step feasible IPM for the Cartesian $P_*(\kappa)$-SCLCP. Although this method is small-update, the advantage is that the calculation of a step size at each iteration is avoided while the global convergence and the quadratic local convergence is still achieved.

The so-called barrier update parameter θ in algorithms for IPMs plays an important role in both theory and practice of IPMs. Usually, if θ is a constant independent of the dimension of the problem, then the algorithm is called a large-update method. If it depends on the dimension, then the algorithm is said to be a small-update method. Large-update methods are much more efficient than small-update methods in practice, but have a worst-case iteration bound. Such a gap between theory and practice has been referred to as irony of IPMs [19]. Recently, many authors have tried to reduce the gap of the worst-case iteration bound between the large-update IPM and the small-update IPM. Using self-regular proximity functions instead of a classical logarithmic barrier function, Peng et al. [16, 17] improved the complexity of large-update IPMs for the LO problem, the SDO problem, and the SOCO problem. Bai et al. [1] introduced a new class of eligible kernel functions. The class was defined by some simple conditions on the kernel function and its derivatives. The best iteration bound, which was given by Bai et al. [1] is $\mathcal{O}\left(\sqrt{n}\log n \log \frac{n}{\epsilon}\right)$. Cho et al. [2] proposed a new primal-dual interior-point algorithm based on a new kernel function for LO and obtained the best known results for large-update IPMs. Lee et al. [11] extended the complexity analysis for LO [2] to $P_*(\kappa)$-LCP. Wang and Bai [25] analyzed IPMs based on a parametric kernel function different from the logarithmic kernel function. Lesaja et al. [14] proposed a unified analysis of the IPMs based on the entire class of eligible kernel functions which was first introduced by Bai et al. [1]. Recently, Kheirfam [8] proposed a new primal-dual IPM for SDO based on a new kernel function which is not logarithmic and not necessarily self-regular function. Vieira [23] analyzed kernel-based IPMs for SCO and presented the iteration complexity results for ten eligible kernel functions introduced in [1]. Lesaja and Roos extended this results for $P_*(\kappa)$-LCPs [12] and SCLCP [13].

Motivated by their works, we define a new kernel function and propose a new generic interior-point algorithm based on this kernel function for SCLCP. We show that the iteration bounds are $\mathcal{O}(\sqrt{r}\log r \log \frac{r}{\epsilon})$ and $\mathcal{O}(\sqrt{r}\log \frac{r}{\epsilon})$ for large-and small-update methods, respectively, which are currently the best known bounds for such methods.

The paper is organized as follows: In Section 2 we briefly recall some properties of symmetric cones and their associated Euclidean Jordan algebra. In Section 3 we review the notions of central path, search directions and NT-steps for SCLCP problems, and remember how a given kernel function defines a generic interior-point algorithm. In Section 4 We present the generic form of the algorithm. In Section 5 we derive some properties of the barrier and proximity functions based on a new kernel function. In Section 6 we propose an expression for the decrease of the proximity during an inner iteration, and derive a default value for the step size. In Section 7 the analysis is completed by deriving the iteration complexity. Finally, we end the paper in Section 8.

2 Euclidean Jordan Algebra

In this section, we briefly describe some concepts, properties, and results from Euclidean Jordan algebras and symmetric cones that are needed in this paper. All these can be found in the book [4] by Faraut and Korányi.

A Euclidean Jordan algebra is a triple $(\mathcal{J}, \circ, \langle\cdot,\cdot\rangle)$ where $(\mathcal{J}, \langle\cdot,\cdot\rangle)$is a finite dimensional inner product space over R and $(x, y) \mapsto x \circ y : \mathcal{J} \times \mathcal{J} \to \mathcal{J}$ is a bilinear mapping satisfying the following conditions:

1. $x \circ y = y \circ x$ for all $x, y \in \mathcal{J}$,
2. $x \circ (x^2 \circ y) = x^2 \circ (x \circ y)$ for all $x, y \in \mathcal{J}$ where $x^2 = x \circ x$ and
3. $\langle x \circ y, z\rangle = \langle y, x \circ z\rangle$ for all $x, y, z \in \mathcal{J}$.

In addition, we assume that there is a unique element $e \in \mathcal{J}$ (called the identity element) such that $x \circ e = e \circ x = x$ for all $x \in \mathcal{J}$. The set $\mathcal{K} = \{x^2 : x \in \mathcal{J}\}$ is called the cone of squares of Euclidean Jordan algebra $(\mathcal{J}, \circ, \langle\cdot,\cdot\rangle)$. A cone is symmetric if and only if it is the cone of squares of some Euclidean Jordan algebra. For $x \in \mathcal{J}$, we define

$$m(x) := \min\{k > 0 : \{e, x, x^2, \cdots, x^k\} \text{ is linearly dependent}\}$$

and rank of $\mathcal{J}$ by $r = \max\{m(x) : x \in \mathcal{J}\}$. An element $c \in \mathcal{J}$ is an idempotent if $c^2 = c$; it is a primitive idempotent if it is nonzero and cannot be expressed as a sum of two nonzero idempotents. We say a finite set $\{c_1, c_2, \ldots, c_k\}$ of primitive idempotents in $\mathcal{J}$ is a Jordan frame if $c_i \circ c_j = 0$, for any $i \neq j$ and $\sum_{i=1}^{k} c_i = e$.

Theorem 1 (Theorem III.1.2 in [4]) *Let $(\mathcal{J}, \circ, \langle\cdot,\cdot\rangle)$ be a Euclidean Jordan algebra with rank$(\mathcal{J}) = r$. Then for any $x \in \mathcal{J}$, there exists a Jordan frame $\{c_1, c_2, \ldots, c_r\}$ and real numbers $\lambda_1(x), \lambda_2(x), \ldots, \lambda_r(x)$ such that $x = \sum_{i=1}^{r} \lambda_i(x)c_i$. The numbers $\lambda_i(x)$ (with their multiplicities) are the eigenvalues of x.*

Now, it is possible to extend the definition of any real-valued function $\psi(\cdot)$ to elements of the Euclidean Jordan algebra via their eigenvalues:

$$\psi(x) := \sum_{i=1}^{r} \psi(\lambda_i(x))c_i.$$

Particulary, we have some examples as follows:

- the square root: $x^{\frac{1}{2}} = \sum_{i=1}^{r} \sqrt{\lambda_i(x)}c_i$, wherever $x \in \mathcal{K}$, and undefined otherwise,
- the inverse: $x^{-1} = \sum_{i=1}^{r} \lambda_i(x)^{-1}c_i$, wherever $\lambda_i \neq 0$, for all $i = 1, 2, \ldots, r$, and undefined otherwise.
- the square: $x^2 = \sum_{i=1}^{r} \lambda_i^2(x)c_i$.

Let us denote by $\psi'(x)$ the vector-valued function induced by the derivative $\psi'(t)$ of the function $\psi(t)$:

$$\psi'(x) := \sum_{i=1}^{r} \psi'(\lambda_i(x))c_i.$$

In the Jordan algebra, we define the determinant of x and the trace of x as follows:

$$\det(x) = \Pi_{i=1}^{r}\lambda_i(x), \quad \mathrm{tr}(x) = \sum_{i=1}^{r} \lambda_i(x).$$

Since $\mathcal{J}$ is a Euclidean Jordan algebra, $\langle x, s\rangle = \mathrm{tr}(x \circ s)$ is a scalar product on $\mathcal{J}$ (Proposition III.1.5 in [4]). For $x \in \mathcal{J}$, with eigenvalues $\lambda_i(x)$, $1 \leq i \leq r$, the Frobenius norm can be defined as $\|x\|_F := \sqrt{\langle x, x\rangle} = \sqrt{\sum_{i=1}^{r} \lambda_i(x)^2}$. In a Euclidean Jordan algebra $\mathcal{J}$, we define the corresponding Lyapunov transformation $L(x) : \mathcal{J} \to \mathcal{J}$ by $L(x)y = x \circ y$. We say that elements x and y operator commute if $L(x)$ and $L(y)$ commute, i.e., $L(x)L(y) = L(y)L(x)$. It is known that x and y operator commute if and only if x and y have their spectral decompositions with respect to a common Jordan frame (Lemma X.2.2 in [4]). For each $x \in \mathcal{J}$, define

$$P(x) := 2L(x)^2 - L(x^2),$$

where, $L(x)^2 = L(x)L(x)$. The map $P(x)$ is called the quadratic representation of $\mathcal{J}$.

Another important decomposition of an element $x \in \mathcal{J}$ used in the paper is the Peirce decomposition. It is based on the fact that, for the idempotent $c \in \mathcal{J}$, the only eigenvalues of $L(c)$ are 0, $\frac{1}{2}$ and 1 (Proposition III.1.3 in [4]). Furthermore, the eigenspaces corresponding to these eigenvalues are

$$\mathcal{J}(c, \lambda) := \{x \in \mathcal{J} : \; L(x)c = c \circ x = \lambda x\}, \quad \lambda \in \left\{0, \frac{1}{2}, 1\right\}.$$

Hence, $\mathcal{J}$ is the direct sum of the corresponding subspaces $\mathcal{J}(c, 1)$, $\mathcal{J}(c, \frac{1}{2})$ and $\mathcal{J}(c, 0)$. The decomposition

$$\mathcal{J} = \mathcal{J}(c, 1) \oplus \mathcal{J}\left(c, \frac{1}{2}\right) \oplus \mathcal{J}(c, 0),$$

is called the Peirce decomposition of $\mathcal{J}$ with respect to the idempotent c.

Theorem 2 (Theorem IV.2.1 in [4]) *Let $x \in \mathcal{J}$ with the spectral decomposition as defined in Theorem 1. Then, we have*

$$\mathcal{J} = \oplus_{i \leq m} \mathcal{J}_{im},$$

where,

$$\mathcal{J}_{ii} = \{x : \; x \circ c_i = x\} \quad \text{and } \mathcal{J}_{im} = \left\{x : \; x \circ c_i = \frac{1}{2}x = x \circ c_m\right\}, \quad 1 \leq i \leq m \leq r,$$

are Peirce spaces of $\mathcal{J}$, and there exist $x_i \in R$, $c_i \in \mathcal{J}_{ii}$ and $x_{im} \in \mathcal{J}_{im} (i < m)$ such that

$$x = \sum_{i=1}^{r} x_i c_i + \sum_{i<m} x_{im}.$$

Theorem 3 (Lemma 1 in [10]) *Let $G(x) = \sum_{i=1}^{r} f(\lambda_i(x))c_i$. If f is continuously differentiable in D, then $G(x)$ is continuously differentiable at x and*

$$D_x G(x) = \sum_{i=1}^{r} f'(\lambda_i(x)) x_i c_i + \sum_{i<m} \frac{f(\lambda_i(x)) - f(\lambda_m(x))}{\lambda_i(x) - \lambda_m(x)} x_{im}, \; 1 \leq i \leq m \leq r.$$

The following lemma shows the existence and uniqueness of a scaling point w corresponding to any points $x, s \in \text{int}\mathcal{K}$ such that $P(w)$ takes s into x.

Lemma 4 (Lemma 3.2 in [7]) *Let $x, s \in \text{int}\mathcal{K}$. Then, there exists a unique $w \in \text{int}\mathcal{K}$ such that $x = P(w)s$. Moreover,*

$$w = P(x^{\frac{1}{2}})\left(P(x^{\frac{1}{2}})s\right)^{-\frac{1}{2}}\left[= P(s^{-\frac{1}{2}})\left(P(s^{\frac{1}{2}})x\right)^{\frac{1}{2}}\right].$$

The point w is called the NT-scaling point of x and s. Note that $P(w)^{\frac{1}{2}}$ and $P(w)^{-\frac{1}{2}}$ are automorphisms of $\text{int}\mathcal{K}$. Let $x, y \in \mathcal{J}$. We say that two elements x and y are similar, as denoted by $x \sim y$, if and only if x and y share the same set of eigenvalues. In what follows, we list some results regarding similarity.

Lemma 5 (Proposition 21 in [22]) *Let $x, s, u \in \text{int}\mathcal{K}$. Then*

(i) $P(x^{\frac{1}{2}})s \sim P(s^{\frac{1}{2}})x$.
(ii) $P\left((P(u)x)^{\frac{1}{2}}\right)P(u^{-1})s \sim P(x^{\frac{1}{2}})s$.

Lemma 6 (Proposition 3.2.4 in [23]) *Let $x, s \in \text{int}\mathcal{K}$, and w be the scaling point of x and s. Then*

$$\left(P(x^{\frac{1}{2}})s\right)^{\frac{1}{2}} \sim P(w^{\frac{1}{2}})s.$$

3 The Central Path

The basic idea of IPMs is to replace the second equation in (SCLCP), the so-called complementarity condition for SCLCP, by the parameterized equation

$x \circ s = \mu e$, where e is the identity element and $\mu > 0$. Thus, we have the following system:

$$\begin{aligned} s &= Mx + q, \quad x, s \in \mathrm{int}\mathcal{K}, \\ x \circ s &= \mu e. \end{aligned} \tag{1}$$

We assume that there exists strictly positive x and s that satisfy (SCLCP). For each $\mu > 0$, the system (1) has a unique solution $(x(\mu), s(\mu))$ (under given assumptions), the so-called μ-center of SCLCP. The set of μ-centers (with μ running through all positive real numbers) gives a homotopy path, which is called the central path of SCLCP [5]. If $\mu \to 0$, then the limit of the central path exists, and since the limit points satisfy the complementarity condition, the limit yields a solution of SCLCP.

At a given feasible iterate (x, s) with $x, s \in \mathrm{int}\mathcal{K}$, we are to find displacements Δx and Δs such that

$$\begin{aligned} -M\Delta x + \Delta s &= 0, \\ x \circ \Delta s + s \circ \Delta x &= \mu e - x \circ s. \end{aligned} \tag{2}$$

Due to the fact that x and s are not operator commutable in general, i.e., $L(x)L(s) \neq L(s)L(x)$, this system does not always have a unique solution. The scaling scheme is based on the following fact (Lemma 28 in [22]): Let $u \in \mathrm{int}\mathcal{K}$. Then

$$x \circ s = \mu e \Leftrightarrow P(u)x \circ P(u^{-1})s = \mu e.$$

Now, replacing the second equation of system (1) by $P(u)x \circ P(u^{-1})s = \mu e$ and applying the Newton's method we obtain the system

$$\begin{aligned} -M\Delta x + \Delta s &= 0, \\ P(u^{-1})s \circ P(u)\Delta x + P(u)x \circ P(u^{-1})\Delta s &= \mu e - P(u)x \circ P(u^{-1})s. \end{aligned} \tag{3}$$

By choosing u appropriately, this system can be used to define the search directions. Here, we focus on the scaling point $u = w^{-\frac{1}{2}}$, which w is the NT-scaling point of x and s as defined in Lemma 4. For the formulation and analysis of the generic IPM for SCLCP, the introduction of the following variance vector is critical:

$$v := \frac{P(w)^{-\frac{1}{2}}x}{\sqrt{\mu}} \left[= \frac{P(w)^{\frac{1}{2}}s}{\sqrt{\mu}} \right]. \tag{4}$$

Using the variance vector v the following scaled search directions are introduced:

$$d_x := \frac{P(w)^{-\frac{1}{2}}\Delta x}{\sqrt{\mu}}, \quad d_s := \frac{P(w)^{\frac{1}{2}}\Delta s}{\sqrt{\mu}}. \tag{5}$$

Using Eq. 5 the system (3) can be rewritten as

$$\begin{aligned} -\overline{M}d_x + d_s &= 0, \\ d_x + d_s &= v^{-1} - v, \end{aligned} \tag{6}$$

where

$$\overline{M} := P(w)^{\frac{1}{2}}MP(w)^{\frac{1}{2}}.$$

Let $v = \sum_{i=1}^{r} \lambda_i(v)c_i$ be the spectral decomposition of v with respect to the Jordan frame $\{c_1, \ldots, c_r\}$. A crucial observation is that the right-hand side $v^{-1} - v$ in the second equation of system (6) is the negative gradient of the classical logarithmic barrier function $\Psi_c(v)$, that is,

$$d_x + d_s = -\nabla\Psi_c(v),$$

where

$$\Psi_c(v) := \sum_{i=1}^{r} \psi_c(\lambda_i(v)), \qquad \psi_c(t) = \frac{t^2-1}{2} - \log(t).$$

Since $\Psi_c(v)$ is strictly convex and $\nabla\Psi_c(e) = 0$, it follows that $\Psi_c(v)$ attains its minimal value at $v = e$, with $\Psi_c(e) = 0$. Thus,

$$\Psi_c(e) = 0 \Leftrightarrow \nabla\Psi_c(e) = 0 \Leftrightarrow v = e \Leftrightarrow x \circ s = \mu e.$$

Hence, we see that the μ-center $(x(\mu), s(\mu))$ can be characterized as the minimizer of the function $\Psi_c(v)$. Thus, we can replace $\Psi_c(v)$ by any strictly convex barrier function $\Psi(v)$, $v \in \mathrm{int}\mathcal{K}$ such that

$$\Psi(v) = 0 \;\Leftrightarrow\; \nabla\Psi(v) = 0 \;\Leftrightarrow\; v = e.$$

Hence, the value of $\Psi(v)$ can be considered as a proximity measure for closeness with respect to the μ-center $(x(\mu), s(\mu))$. In what follows, we define the norm-based proximity measure $\delta(v)$ as

$$\delta(v) := \frac{1}{2}\|\nabla\Psi(v)\|_F. \tag{7}$$

Since $\Psi(v)$ is strictly convex and minimal at $v = e$, we have

$$\Psi(v) = 0 \Leftrightarrow \delta(v) = 0 \quad\Leftrightarrow\quad v = e. \tag{8}$$

The new barrier function determines the calculation of the search directions as

$$\begin{aligned} -\overline{M}d_x + d_s &= 0,\\ d_x + d_s &= -\nabla\Psi(v). \end{aligned} \tag{9}$$

By taking a step along the search direction, with the step size α defined by some line search rules, one constructs a new pair (x_+, s_+) according to

$$x_+ := x + \alpha\Delta x, \quad s_+ := s + \alpha\Delta s. \tag{10}$$

4 Algorithm

Suppose that the current iterate (x, s) is known and is in the τ-neighborhood of the corresponding μ-center, that is, $\Psi(v) \le \tau$. Next, the value of μ is reduced by the factor $1-\theta$ with $0 < \theta < 1$, which changes the value of v according to Eq. 4 and defines a new μ-center $(x(\mu), s(\mu))$. This cause that $\Psi(v) \ge \tau$. Now, we start the inner iteration by calculating new iterate (10) where Δx and Δs are calculated from system (9) and Eq. 5 and the step size α is chosen appropriately with the goal of reducing the value of barrier function $\Psi(v)$. If necessary, the procedure is repeated until we find the iterate that again belongs to the τ-neighborhood of the current

μ-center, that is, until $\Psi(v) \leq \tau$. At this point, we start a new outer iteration by reducing the value of μ again. This process is repeated until μ is small enough, say until $r\mu \leq \epsilon$ for a certain accuracy parameter ϵ, at this stage we have found an ϵ-approximate solution of SCLCP. The generic IPM outlined above is summarized in the following.

Algorithm Generic IPM for SCLCP

Input : Accuracy parameter $\epsilon > 0$;
barrier update parameter θ, $0 < \theta < 1$;
threshold parameter $\tau \geq 1$;
a starting point $x^0, s^0 \in \text{int}\mathcal{K}$, such that $\Psi(x^0, s^0, \mu^0) = \Psi(v^0) \leq \tau$
begin :
$x := x^0, s := s^0, \mu := \mu^0$;
while $r\mu \geq \epsilon$ **do**
begin
$\mu := (1-\theta)\mu$;
while $\Psi(v) > \tau$ **do**
begin
Solve the system (9) and use $Eq.$ 5 for Δx, Δs;
Determine a step size α;
$x := x + \alpha \Delta x$;
$s := s + \alpha \Delta s$;
$v = \dfrac{P(w)^{-\frac{1}{2}} x}{\sqrt{\mu}} \Big(= \dfrac{P(w)^{\frac{1}{2}} s}{\sqrt{\mu}} \Big)$
end
end
end

The aim of this paper is to investigate a new kernel function, namely

$$\psi(t) = \frac{t^2-1}{2} + \frac{e^{p(f(t)-1)}-1}{pq},$$

$$\text{where} \quad f(t) = e^{\frac{4q}{\pi}\cot(h(t))}, \quad h(t) = \frac{\pi t}{t+1}, \quad p, q \geq 1, \ t > 0 \tag{11}$$

and to show that the IPMs based on this function have the best known complexity results.

5 Properties of the New Kernel Function

Here, we present some useful properties of the kernel function $\psi(t)$ as defined by (11), that are used in the analysis of the algorithm.

5.1 Some Technical Results

For ψ we have the first three derivatives as follows:

$$\psi'(t) = t - \frac{4}{\pi}h'(t)\Big(1+\cot^2(h(t))\Big)f(t)e^{p(f(t)-1)}, \tag{12}$$

$$\psi''(t) = 1 - \frac{4}{\pi}\left(h''(t) - \left(2\cot(h(t)) + \frac{4q}{\pi}(1+\cot^2(h(t)))(1+pf(t))\right)h'(t)^2\right) \times \left(1+\cot^2(h(t))\right)f(t)e^{p(f(t)-1)}, \tag{13}$$

$$\psi'''(t) = -\frac{4}{\pi}\Big[h'''(t) - g'(t)h'(t)^2 - 2g(t)h'(t)h''(t) - \left(h''(t) - g(t)h'(t)^2\right) \times \left(2\cot(h(t)) + \frac{4q}{\pi}h'(t)\left(1+\cot^2(h(t))\right)(1+pf(t))\right)\Big] \times \left(1+\cot^2(h(t))\right)f(t)e^{p(f(t)-1)}, \tag{14}$$

where

$$g(t) = 2\cot(h(t)) + \frac{4q}{\pi}\left(1+\cot^2(h(t))\right)(1+pf(t)),$$

and

$$g'(t) = -\left(2\left(1+\frac{4q}{\pi}\cot(h(t))\right) + \frac{4pq}{\pi}\left(2\cot(h(t)) + \frac{4q}{\pi}(1+\cot^2(h(t)))\right)f(t)\right) \times h'(t)\left(1+\cot^2(h(t))\right).$$

Lemma 7 (Lemma 4 in [8]) *For the function $h(t)$ defined in (11), we have*

$$h''(t) - 2h'(t)^2\cot(h(t)) < 0, \ \ t > 0.$$

The next lemma shows that the new kernel function (11) is eligible.

Lemma 8 *Let $\psi(t)$ be as defined in (11) and $t > 0$. Then*

$$\psi''(t) > 1, \tag{15}$$

$$t\psi''(t) + \psi'(t) > 0, \quad t < 1 \tag{16}$$

$$t\psi''(t) - \psi'(t) > 0, \quad t > 1 \tag{17}$$

$$\psi'''(t) < 0. \tag{18}$$

Proof From Eq. 13, Lemma 7 and $f(t) \geq 0$, we get $\psi''(t) > 1$. This proves (15). By using Eqs. 12 and 13, we have

$$t\psi''(t) + \psi'(t) = 2t - \frac{4}{\pi}\left(th''(t) - \left(2\cot(h(t)) + \frac{4q}{\pi}\left(1+\cot^2(h(t))\right)\left(1+pf(t)\right)\right)th'(t)^2 + h'(t)\right) \times \left(1+\cot^2(h(t))\right)f(t)e^{p(f(t)-1)}. \tag{19}$$

Defining $w(t) = th''(t) - \left(2\cot(h(t)) + \frac{4q}{\pi}\big(1+\cot^2(h(t))\big)\big(1+pf(t)\big)\right)th'(t)^2 + h'(t)$, since $p, q \geq 1$ and $f(t) \geq 0$, we obtain

$$\begin{aligned} w(t) &\leq th''(t) - 2th'(t)^2\cot(h(t)) + h'(t) \\ &= \frac{-2\pi t}{(1+t)^3} - \frac{2\pi^2 t}{(1+t)^4}\cot(h(t)) + \frac{\pi}{(1+t)^2} \\ &= \frac{-2\pi^2 t}{(1+t)^4}\left(\frac{t}{2\pi} - \frac{1}{2\pi t} + \cot(h(t))\right). \end{aligned}$$

Define

$$c(t) = \cot(h(t)) + \frac{t}{2\pi} - \frac{1}{2\pi t}.$$

Using $\sin(x) \leq x$ for $x \geq 0$, we have

$$\begin{aligned} c'(t) &= -h'(t)\big(1+\cot^2(h(t))\big) + \frac{1}{2\pi} + \frac{1}{2\pi t^2} \\ &= \frac{-h'(t)}{\sin^2(h(t))} + \frac{1}{2\pi} + \frac{1}{2\pi t^2} \\ &= \frac{-\pi}{(1+t)^2\sin^2(h(t))} + \frac{1}{2\pi} + \frac{1}{2\pi t^2} \\ &\leq \frac{-\pi}{(1+t)^2 h(t)^2} + \frac{1}{2\pi} + \frac{1}{2\pi t^2} \\ &= \frac{-1}{\pi t^2} + \frac{1}{2\pi} + \frac{1}{2\pi t^2} = \frac{t^2-1}{2\pi t^2} < 0. \end{aligned}$$

This implies that $c(t)$ is strictly decreasing and hence $c(t) > c(1) = 0$ and $w(t) < 0$. Therefore

$$t\psi''(t) + \psi'(t) = 2t - \frac{4}{\pi}w(t)\big(1+\cot^2(h(t))\big)f(t)e^{p(f(t)-1)} \geq 2t > 0.$$

This implies (16). To prove (17), considering the first two derivatives of $\psi(t)$ we have

$$\begin{aligned} t\psi''(t) - \psi'(t) = \frac{4}{\pi}\Big(&h'(t) - th''(t) + 2th'(t)^2\cot(h(t)) \\ &+ \frac{4q}{\pi}t\big(1+\cot^2(h(t))\big)\big(1+pf(t)\big)h'(t)^2\Big) \\ &\times \big(1+\cot^2(h(t))\big)f(t)e^{p(f(t)-1)}. \end{aligned} \tag{20}$$

Define

$$k(t) = h'(t) - th''(t) + 2th'(t)^2\cot(h(t)) + \frac{4q}{\pi}t\big(1+\cot^2(h(t))\big)\big(1+pf(t)\big)h'(t)^2.$$

Since $p, q \geq 1$ and $f(t) \geq 0$, we get

$$\begin{aligned} k(t) &\geq h'(t) - th''(t) + 2th'(t)^2\cot(h(t)) \\ &= h'(t) - t\Big(h''(t) - 2h'(t)^2\cot(h(t))\Big), \end{aligned}$$

the last expression is positive, by $h'(t) > 0$ and Lemma 7. Therefore

$$t\psi''(t) - \psi'(t) = \frac{4}{\pi}k(t)\big(1 + \cot^2(h(t))\big) f(t)e^{p(f(t)-1)} > 0.$$

This completes the proof of (17).

The (18) holds due to Eq. 14, Lemma 7, $g(t) > 0$ and $g'(t) < 0$. This completes the proof. □

Note that $\psi'(1) = \psi(1) = 0$, and $\psi''(t) > 0$ imply that $\psi(t)$ is a nonnegative strictly convex such that $\psi(t)$ achieves its minimum at $t = 1$, i.e., $\psi(1) = 0$. This implies that, since $\psi(t)$ is twice differentiable, it is completely determined by its second derivative:

$$\psi(t) = \int_1^t \int_1^\xi \psi''(\zeta)d\zeta d\xi. \tag{21}$$

The next lemma is very useful in the analysis of interior-point algorithms based on the kernel functions (see for example [1, 16]).

Lemma 9 (Lemma 2.1.2 in [16]) *Let $\psi(t)$ be a twice differentiable function for $t > 0$. Then the following three properties are equivalent:*

(i) $\psi(\sqrt{t_1t_2}) \le \frac{1}{2}(\psi(t_1) + \psi(t_2))$ *for* $t_1, t_2 > 0$.
(ii) $\psi'(t) + t\psi''(t) \ge 0,\ t > 0$.
(iii) $\psi(e^\xi)$ *is convex.*

The third property is called exponentially convexity, or shortly e-convexity. Therefore, Lemma 9 and (16) show that the our new kernel function (11) is e-convex for $t > 0$.

As a consequence of Lemma 9, we have the following lemma, which is crucial for the analysis of the algorithm.

Lemma 10 (Theorem 4.3.2 in [23]) *If $x, s \in \operatorname{int}\mathcal{K}$, one has*

$$\Psi\Big((P(x)^{\frac{1}{2}}s)^{\frac{1}{2}}\Big) \le \frac{1}{2}(\Psi(x) + \Psi(s)).$$

The proof of the following lemma is similar to the proof of Lemma 2.6 in [1], and therefore is omitted.

Lemma 11 *If $t \ge 1$, then*

$$\psi(t) \le \frac{2 + (1 + p)q}{2}(t - 1)^2.$$

Lemma 12 (Lemma 8 in [8]) *For $t \ge 1$, one has*

$$\psi'(t) \ge \frac{\psi(t)}{t}.$$

Lemma 13 (Lemma 9 in [8]) *For $\psi(t)$, as defined in (11), we have*

$$\frac{1}{2}(t - 1)^2 \le \psi(t) \le \frac{1}{2}\psi'(t)^2.$$

Lemma 14 *Let $\varrho : [0, \infty) \to [1, \infty)$ be the inverse function of $\psi(t)$ for $t \geq 1$. Then*

$$\sqrt{1+2s} \leq \varrho(s) \leq 1 + \sqrt{2s}.$$

Proof The inverse function of $\psi(t)$ for $t \geq 1$ is obtained by solving t from

$$\psi(t) = \frac{t^2-1}{2} + \frac{e^{p(f(t)-1)}-1}{pq} = s, \quad t \geq 1.$$

Defining $\psi_b(t) = \frac{e^{p(f(t)-1)}-1}{pq}$, one has

$$\psi_b'(t) = -\left(\frac{2}{(1+t)\sin(h(t))}\right)^2 f(t)e^{p(f(t)-1)} \leq 0.$$

Therefore, $\psi_b(t)$ is monotonically decreasing for $t \geq 1$, i.e., $\psi_b(t) \leq \psi_b(1)$, and since $\psi_b(1) = 0$ we get

$$\frac{t^2-1}{2} \geq s,$$

this implies that $t = \varrho(s) \geq \sqrt{1+2s}$. This proves the first inequality. For the proof of second inequality, by Lemma 13, we have

$$s = \psi(t) \geq \frac{1}{2}(t-1)^2,$$

whence

$$t = \varrho(s) \leq 1 + \sqrt{2s}.$$

This completes the proof. □

At the start of each outer iteration, just before the update of μ with the factor $1-\theta$, we have $\Psi(v) \leq \tau$. Due to the update of μ the vector v is divided by the factor $\sqrt{1-\theta}$, with $0 < \theta < 1$, which in general leads to an increase in the value of $\Psi(v)$. It is important to estimate that increase, which the result is based on the following theorem. This is due to the fact that $\psi(t)$ satisfies (17) and (18).

Theorem 15 (Theorem 5.9.1 in [23]) *If $v \in \mathrm{int}\mathcal{K}$ and $\beta \geq 1$, then*

$$\Psi(\beta v) \leq r\psi\left(\beta\varrho\left(\frac{\Psi(v)}{r}\right)\right).$$

Lemma 16 *Let $0 \leq \theta < 1$ and $v_+ = \frac{v}{\sqrt{1-\theta}}$. If $\Psi(v) \leq \tau$, then*

$$\Psi(v_+) \leq \frac{2+(1+p)q}{2(1-\theta)}\left(\sqrt{r}\theta + \sqrt{2\tau}\right)^2.$$

Proof Since $\frac{1}{\sqrt{1-\theta}} \geq 1$ and $\varrho(\frac{\Psi(v)}{r}) \geq 1$, we have $\frac{\varrho(\frac{\Psi(v)}{r})}{\sqrt{1-\theta}} \geq 1$. Using Theorem 15 with $\beta = \frac{1}{\sqrt{1-\theta}}$, Lemmas 11 and 14, we have

$$\begin{aligned}
\Psi(v_+) \leq r\psi\left(\frac{\varrho\left(\frac{\Psi(v)}{r}\right)}{\sqrt{1-\theta}}\right) &\leq \frac{2+(1+p)q}{2} r\left(\frac{\varrho\left(\frac{\Psi(v)}{r}\right)}{\sqrt{1-\theta}} - 1\right)^2 \\
&= \frac{r(2+(1+p)q)}{2(1-\theta)}\left(\varrho\left(\frac{\Psi(v)}{r}\right) - \sqrt{1-\theta}\right)^2 \\
&\leq \frac{r(2+(1+p)q)}{2(1-\theta)}\left(1+\sqrt{\frac{2\Psi(v)}{r}} - \sqrt{1-\theta}\right)^2 \\
&\leq \frac{r(2+(1+p)q)}{2(1-\theta)}\left(1+\sqrt{\frac{2\tau}{r}} - \sqrt{1-\theta}\right)^2 \\
&\leq \frac{r(2+(1+p)q)}{2(1-\theta)}\left(\theta+\sqrt{\frac{2\tau}{r}}\right)^2 \\
&= \frac{2+(1+p)q}{2(1-\theta)}\left(\sqrt{r}\theta + \sqrt{2\tau}\right)^2, \qquad (22)
\end{aligned}$$

the last inequality follows by $1-\sqrt{1-\theta} \leq \theta,\ 0 \leq \theta < 1$. □

Define

$$\tilde{\Psi}_0 = \frac{2+(1+p)q}{2(1-\theta)}\left(\sqrt{r}\theta + \sqrt{2\tau}\right)^2, \qquad (23)$$

then $\tilde{\Psi}_0$ is an upper bound for $\Psi(v)$ during the process of the algorithm.

Remark 17 For the large-update methods with $\tau = O(r)$ and $\theta = \Theta(1)$, we have $\tilde{\Psi}_0 = O(r)$, and for small-update methods with $\tau = O(1)$ and $\theta = \Theta(\frac{1}{\sqrt{r}})$, we have $\tilde{\Psi}_0 = O((1+p)q)$.

The following theorem gives a lower bound for $\delta(v)$ in terms of $\Psi(v)$. This is due to the fact that $\psi(t)$ satisfies (18).

Theorem 18 (Theorem 5.9.12 in [23]) *If* $v \in \mathrm{int}\mathcal{K}$, *then*

$$\delta(v) \geq \frac{1}{2}\psi'(\varrho(\Psi(v))).$$

Lemma 19 *If* $\Psi(v) \geq \tau \geq 1$, *then*

$$\delta(v) \geq \frac{1}{6}\sqrt{\Psi(v)}.$$

Proof By using Theorem 18 and Lemma 12, we obtain

$$\delta(v) \geq \frac{1}{2}\psi'\Big(\varrho(\Psi(v))\Big) \geq \frac{1}{2}\frac{\psi\Big(\varrho(\Psi(v))\Big)}{\varrho(\Psi(v))} = \frac{\Psi(v)}{2\varrho(\Psi(v))}.$$

Now, by the second inequality of the Lemma 14, we have

$$\varrho(\Psi(v)) \leq 1 + \sqrt{2\Psi(v)} \leq \sqrt{\Psi(v)} + \sqrt{2\Psi(v)} < 3\sqrt{\Psi(v)}.$$

Therefore, we obtain

$$\delta(v) \geq \frac{\Psi(v)}{6\sqrt{\Psi(v)}} = \frac{1}{6}\sqrt{\Psi(v)}.$$

□

6 Analysis of the Algorithm

In this section, we show how to compute a feasible step size α of a NT-step with the decrease of the barrier function. In each inner iteration, we first compute the search directions d_x and d_s from the system (9) and then the original directions Δx and Δs calculated using Eq. 5. After a step size α is determined, the new iterate (x_+, s_+) is calculated from (10). Using Eqs. 4, 5 and (10), we obtain

$$\begin{aligned} x_+ &= x + \alpha\Delta x = \sqrt{\mu}P(w^{\frac{1}{2}})(v + \alpha d_x), \\ s_+ &= s + \alpha\Delta s = \sqrt{\mu}P(w^{-\frac{1}{2}})(v + \alpha d_s). \end{aligned} \tag{24}$$

Since $P(w^{\frac{1}{2}})$ and its inverse $P(w^{-\frac{1}{2}})$ are automorphisms of $\operatorname{int}\mathcal{K}$ (Theorem III.2.1 in [4]), x_+ and s_+ will belong to $\operatorname{int}\mathcal{K}$ if and only if $v + \alpha d_x$ and $v + \alpha d_s$ belong to $\operatorname{int}\mathcal{K}$. Note that during an inner iteration the parameter μ is fixed. Hence, after the default step the new scaled vector v_+, according to Eqs. 4 and using (24), is given by

$$v_+ = \frac{P(w_+^{-\frac{1}{2}})x_+}{\sqrt{\mu}} = P(w_+^{-\frac{1}{2}})P(w^{\frac{1}{2}})(v + \alpha d_x) = P(w_+^{\frac{1}{2}})P(w^{-\frac{1}{2}})(v + \alpha d_s),$$

where

$$w_+ = P(x_+^{\frac{1}{2}})(P(x_+^{\frac{1}{2}})s_+)^{-\frac{1}{2}}.$$

Lemma 20 (Proposition 5.9.3 in [23]) *One has*

$$v_+ \sim \left(P(v + \alpha d_x)^{\frac{1}{2}}(v + \alpha d_s)\right)^{\frac{1}{2}}.$$

This Lemma shows that the eigenvalues of v_+ are precisely the same as those of

$$\bar{v}_+ = \left(P(v + \alpha d_x)^{\frac{1}{2}}(v + \alpha d_s)\right)^{\frac{1}{2}}.$$

By Lemma 10, this implies that

$$\Psi(v_+) = \Psi(\bar{v}_+) \leq \frac{1}{2}\big(\Psi(v+\alpha d_x) + \Psi(v+\alpha d_s)\big).$$

Now, we consider the decrease in $\Psi(v)$ as a function of α and define

$$f(\alpha) := \Psi(v_+) - \Psi(v).$$

Furthermore, we define

$$f_1(\alpha) := \frac{1}{2}\big(\Psi(v+\alpha d_x) + \Psi(v+\alpha d_s)\big) - \Psi(v).$$

It follows that $f(\alpha) \leq f_1(\alpha)$ and $f(0) = f_1(0) = 0$. Taking the derivative of $f_1(\alpha)$ to α, we get

$$f_1'(\alpha) = \frac{1}{2}\big(Tr(\Psi'(v+\alpha d_x)\circ d_x) + Tr(\Psi'(v+\alpha d_s)\circ d_s)\big). \tag{25}$$

This gives, using (7) and the second equation in system (9),

$$f_1'(0) = -\frac{1}{2}Tr(\nabla\Psi(v)\circ\nabla\Psi(v)) = -\frac{1}{2}\|\nabla\Psi(v)\|^2 = -2\delta(v)^2 < 0. \tag{26}$$

Furthermore, we have the following upper bound on $f_1''(\alpha)$ [23]:

$$\begin{aligned} f_1''(\alpha) \leq \frac{1}{2}&\left(\sum_{i=1}^{r}\psi''(\eta_i)d_{xi}^2 + \sum_{i<m}\psi''(\eta_m)Tr(d_{xim}^2)\right)\\ &+\frac{1}{2}\left(\sum_{i=1}^{r}\psi''(\gamma_i)d_{si}^2 + \sum_{i<m}\psi''(\gamma_m)Tr(d_{sim}^2)\right), \end{aligned} \tag{27}$$

where $\eta_i = \lambda_i(v+\alpha d_x)$ and $\gamma_i = \lambda_i(v+\alpha d_s),\ i = 1, 2, \ldots, r.$

In what follows, we use the short notation $\delta := \delta(v)$ and state four important lemmas without proofs. These are due to the fact that $\psi''(t)$ is monotonically decreasing.

Lemma 21 (Lemma 5.2 in [13]) *One has*

$$f_1''(\alpha) \leq 2\delta^2\psi''(\lambda_{\min}(v) - 2\alpha\delta).$$

Lemma 22 (Lemma 5.3 in [13]) *If the step size α satisfies*

$$-\psi'(\lambda_{\min}(v) - 2\alpha\delta) + \psi'(\lambda_{\min}(v)) \leq 2\delta, \tag{28}$$

then

$$f_1'(\alpha) \leq 0.$$

Lemma 23 (Lemma 5.4 in [13]) *Let $\rho : [0,\infty) \to (0,1]$ denote the inverse function of the restriction of $-\frac{1}{2}\psi'(t)$ on the interval $(0,1]$, the largest possible value of the step size of α satisfying (28) is given by*

$$\bar{\alpha} := \frac{\rho(\delta) - \rho(2\delta)}{2\delta}.$$

Lemma 24 (Lemma 5.5 in [13]) *Let $\bar{\alpha}$ be as defined in Lemma 23. Then*

$$\bar{\alpha} \geq \frac{1}{\psi''(\rho(2\delta))}.$$

For the purpose of finding an upper bound of $f(\alpha)$, we need a default step size $\tilde{\alpha}$ that is the lower bound of the $\bar{\alpha}$ and consists of δ.

Lemma 25 *One has*

$$\frac{1}{\psi''(\rho(2\delta))} \geq \frac{1}{1+\frac{16}{\pi}\left(2+\pi\left(\frac{\pi}{2q}\log\gamma+\frac{4q}{\pi}\left(1+\frac{\pi^2}{16q^2}\log^2\gamma\right)(1+p\gamma)\right)\right)(4\delta+1)},$$

where $\gamma = 1 + p^{-1}\log\pi(4\delta+1)$.

Proof To obtain the inverse function $t=\rho(s)$ of $-\frac{1}{2}\psi'(t)$, we need to solve the equation

$$-\psi'(t) = -t + \frac{4}{\pi}h'(t)\left(1+\cot^2(h(t))\right)f(t)e^{p(f(t)-1)} = 2s.$$

By setting $t=\rho(2\delta)$, we have

$$-\psi'(t) = 4\delta.$$

Hence, we have

$$h'(t)\left(1+\cot^2(h(t))\right)f(t)e^{p(f(t)-1)} = \frac{\pi}{4}(4\delta+t) \leq 4\delta+1,\ 0<t\leq 1.$$

This implies

$$\begin{aligned} f(t)e^{p(f(t)-1)} &\leq \frac{4\delta+1}{h'(t)\left(1+\cot^2(h(t))\right)} \\ &= \frac{(4\delta+1)(1+t)^2}{\pi}\sin^2(h(t)) \\ &\leq \frac{(4\delta+1)(1+t)^2}{\pi}\left(\frac{\pi t}{1+t}\right)^2 \\ &= \pi(4\delta+1)t^2 \leq \pi(4\delta+1). \end{aligned} \tag{29}$$

By taking the logarithm of both sides of (29), we obtain

$$p(f(t)-1) + \frac{4q}{\pi}\cot(h(t)) \leq \log\pi(4\delta+1). \tag{30}$$

For $0<t\leq 1, 0<h(t)\leq\frac{\pi}{2}$ and we have $\cot(h(t))\geq 0$. Thus, we obtain

$$f(t) \leq 1 + p^{-1}\log\pi(4\delta+1). \tag{31}$$

Taking the logarithm of both sides of (31), we get

$$\frac{4q}{\pi}\cot(h(t)) \leq \log\left(1+p^{-1}\log(\pi(4\delta+1))\right),$$

and hence

$$\cot(h(t)) \leq \frac{\pi}{4q}\log\Big(1+p^{-1}\log(\pi(4\delta+1))\Big). \tag{32}$$

Using $h'(t)=\frac{\pi}{(1+t)^2}\leq\pi$, $h''(t)=\frac{-2\pi}{(1+t)^3}\geq-2\pi$, Eq. 13, (29), (31) and (32) we have

$$\frac{1}{\psi''\big(\rho(2\delta)\big)} \geq \frac{1}{1+\frac{16}{\pi}\Big(2+\pi\Big(\frac{\pi}{2q}\log\gamma+\frac{4q}{\pi}\big(1+\frac{\pi^2}{16q^2}\log^2\gamma\big)(1+p\gamma)\Big)\Big)(4\delta+1)}. \tag{33}$$

This completes the proof. □

In the sequel, we use the notation

$$\tilde{\alpha}=\frac{1}{1+\frac{16}{\pi}\Big(2+\pi\Big(\frac{\pi}{2q}\log\gamma+\frac{4q}{\pi}\big(1+\frac{\pi^2}{16q^2}\log^2\gamma\big)(1+p\gamma)\Big)\Big)(4\delta+1)}, \tag{34}$$

as the default step size. By Lemma 24, $\bar{\alpha}\geq\tilde{\alpha}$.

Lemma 26 (Lemma 5.6 in [13]) *If the step size α is such that $\alpha\leq\bar{\alpha}$, where $\bar{\alpha}$ is defined in Lemma 23, then*

$$f(\alpha)\leq-\alpha\delta^2.$$

Theorem 27 *If $\tilde{\alpha}$ is the default step size as given by Eq. 34 and $\Psi\geq\tau\geq1$, then*

$$f(\tilde{\alpha})\leq\frac{-\sqrt{\Psi}}{\frac{1536}{\pi}\Big(2+\pi\Big(\frac{\pi}{2q}\log\gamma_0+\frac{4q}{\pi}\big(1+\frac{\pi^2}{16q^2}\log^2\gamma_0\big)(1+p\gamma_0)\Big)\Big)},$$

where, $\gamma_0=1+p^{-1}\log\pi(\frac{2}{3}\sqrt{\Psi_0}+1)$.

Proof Using Lemma 26 with $\alpha=\tilde{\alpha}$ and Eq. 34, we have

$$f(\tilde{\alpha})\leq-\tilde{\alpha}\delta^2 \leq\frac{-\delta^2}{1+\frac{16}{\pi}\Big(2+\pi\Big(\frac{\pi}{2q}\log\gamma+\frac{4q}{\pi}\big(1+\frac{\pi^2}{16q^2}\log^2\gamma\big)(1+p\gamma)\Big)\Big)(4\delta+1)}.$$

This expresses the decrease in one inner iteration in terms of δ. Since the right-hand side of this expression is monotonically decreasing in δ, we can express the decrease in terms of $\Psi := \Psi(v)$ by Lemma 19 and after some elementary reductions as follows:

$$f(\tilde{\alpha}) \leq \frac{-\delta}{\frac{256}{\pi}\left(2+\pi\left(\frac{\pi}{2q}\log\gamma+\frac{4q}{\pi}\left(1+\frac{\pi^2}{16q^2}\log^2\gamma\right)(1+p\gamma)\right)\right)}$$

$$\leq \frac{-\sqrt{\Psi}}{\frac{1536}{\pi}\left(2+\pi\left(\frac{\pi}{2q}\log\gamma_0+\frac{4q}{\pi}\left(1+\frac{\pi^2}{16q^2}\log^2\gamma_0\right)(1+p\gamma_0)\right)\right)}, \tag{35}$$

where, the last inequality follows from $\Psi_0 \geq \Psi$. This proves the theorem. □

7 Iteration Complexity

We denote the value of $\Psi(v)$ after the μ-update as Ψ_0. The subsequent values in the same outer iteration are denoted as Ψ_k, $k = 1, 2, \ldots, K$, where K is the total number of inner iterations in the outer iteration.

According to decrease of $f(\tilde{\alpha})$, for $k = 1, 2, \ldots, K-1$, we obtain

$$\Psi_{k+1} \leq \Psi_k - \frac{\sqrt{\Psi_k}}{\frac{1536}{\pi}\left(2+\pi\left(\frac{\pi}{2q}\log\gamma_0+\frac{4q}{\pi}\left(1+\frac{\pi^2}{16q^2}\log^2\gamma_0\right)(1+p\gamma_0)\right)\right)}. \tag{36}$$

Lemma 28 (Lemma 14 in [17]) *Suppose $t_0, t_1, \ldots, t_K$ be a sequence of positive numbers such that*

$$t_{k+1} \leq t_k - \beta t_k^{1-\xi},\ k = 0, 1, \ldots, K-1,$$

where $\beta > 0$ and $0 < \xi \leq 1$. Then $K \leq \lceil \frac{t_0^{\xi}}{\beta\xi} \rceil$.

Letting $t_k = \Psi_k$, $\xi = \frac{1}{2}$ and

$$\beta = \frac{1}{\frac{1536}{\pi}\left(2+\pi\left(\frac{\pi}{2q}\log\gamma_0+\frac{4q}{\pi}\left(1+\frac{\pi^2}{16q^2}\log^2\gamma_0\right)(1+p\gamma_0)\right)\right)},$$

we can get the following theorem from Lemma 28.

Theorem 29 *Let K be the total number of inner iterations in the outer iteration. Then we have*

$$K \leq \frac{3072}{\pi}\left(2+\pi\left(\frac{\pi}{2q}\log\gamma_0+\frac{4q}{\pi}\left(1+\frac{\pi^2}{16q^2}\log^2\gamma_0\right)(1+p\gamma_0)\right)\right)\Psi_0^{\frac{1}{2}},$$

where, Ψ_0 is the value of $\Psi(v)$ after the μ-update in outer iteration.

Table 1 Number of iterations with parameters

p	θ			
	0.1	0.3	0.5	0.7
1	24	22	21	19
2	30	24	28	21
5	24728	19164	13651	3261
$q = 1$				

According to the proof of Lemma 14, it is clear that $\psi(t) \leq \frac{t^2-1}{2}$ for $t \geq 1$. Applying Theorem 15 and Lemma 14, we obtain

$$\Psi_0 \leq r\psi\left(\frac{\varrho(\frac{\tau}{r})}{\sqrt{1-\theta}}\right) \leq r\psi\left(\frac{1+\sqrt{\frac{2\tau}{r}}}{\sqrt{1-\theta}}\right) \leq \frac{2\tau+\theta r+\sqrt{8r\tau}}{2(1-\theta)}.$$

The number of outer iterations is bounded above by $\frac{1}{\theta}\log(\frac{r}{\epsilon})$ (Lemma $II.17$ in [20]). By multiplying the number of outer iterations and the number of inner iterations we get an upper bound for the total number of iterations, namely,

$$\frac{\frac{3072}{\pi}\left(2+\pi\left(\frac{\pi}{2q}\log\gamma_0+\frac{4q}{\pi}\left(1+\frac{\pi^2}{16q^2}\log^2\gamma_0\right)(1+p\gamma_0)\right)\right)}{\theta}\sqrt{\frac{2\tau+\theta r+\sqrt{8r\tau}}{2(1-\theta)}}\log\frac{r}{\epsilon}.$$

Large-update methods use $\theta = \Theta(1)$ and $\tau = O(r)$. As a consequence we then have

$$\Psi_0 \leq \frac{2\tau+\theta r+\sqrt{8r\tau}}{2(1-\theta)} = O(r).$$

Choosing

$$p = \log\pi\left(1+\frac{2}{3}\sqrt{\frac{2\tau+\theta r+\sqrt{8r\tau}}{2(1-\theta)}}\right) = O(\log r), \tag{37}$$

and

$$q = \log\gamma_0, \tag{38}$$

the iteration bound becomes

$$O\left(\sqrt{r}\log r\log\frac{r}{\epsilon}\right).$$

For small-update methods one has $\theta = \Theta(\frac{1}{\sqrt{r}})$ and $\tau = O(1)$. For get the improved iteration bound, using Eq. 23 we have

$$\Psi_0 \leq \frac{2+q(p+1)}{2(1-\theta)}\left(\sqrt{r}\theta+\sqrt{2\tau}\right)^2.$$

Using this upper bound for Ψ_0, we get the following iteration bound:

$$\frac{\frac{3072}{\pi}\left(2+\pi\left(\frac{\pi}{2q}\log\gamma_0+\frac{4q}{\pi}\left(1+\frac{\pi^2}{16q^2}\log^2\gamma_0\right)(1+p\gamma_0)\right)\right)}{\theta}\sqrt{\frac{2+q(p+1)}{2(1-\theta)}}\left(\sqrt{r}\theta+\sqrt{2\tau}\right)\log\frac{r}{\epsilon}.$$

Note that now $\Psi_0 = O(pq)$ and the iteration bound becomes

$$O\left(\sqrt{pqr}\log\frac{r}{\epsilon}\right).$$

If q and p are bounded above by constants which are independent of r, this is the best known bound for small-update methods.

Example 1 We consider the following LCP problem:

$$M = \begin{bmatrix} 0.0368 & 0.0188 & 0.0920 & 0.0211 & 0.0332 & 0.0162 \\ 0.0188 & 0.0393 & 0.0634 & 0.0176 & 0.0300 & 0.0248 \\ 0.0920 & 0.0634 & 0.4293 & 0.0617 & 0.1355 & 0.1124 \\ 0.0211 & 0.0176 & 0.0617 & 0.0203 & 0.0239 & 0.0107 \\ 0.0332 & 0.0300 & 0.1355 & 0.0239 & 0.0513 & 0.0480 \\ 0.0162 & 0.1248 & 0.0124 & 0.0107 & 0.0480 & 0.0824 \end{bmatrix}, \quad q = \begin{bmatrix} 0.1630 \\ -0.2820 \\ 0.4500 \\ -0.3560 \\ 0.2420 \\ -0.2489 \end{bmatrix},$$

with $x^0 = (2, 2, 3, 2, 2, 2)^T$ and $s^0 = (0.6912, 0.1692, 2.6679, 0.0163, 1.0213, 0.3525)^T$, as the initial point. We used $\tau = 2$ and $\epsilon = 10^{-8}$ in all experiments.

Table 1 gives the total number of iterations of the algorithm based on kernel function.

8 Conclusion

In this paper we have analyzed large and small-update methods of primal-dual interior-point algorithm based on a new kernel function for SCLCP. We obtained the best available iteration bounds for such methods.

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References

1. Bai, Y.Q., El Ghami, M., Roos, C.: A comparative study of kernel functions for primal-dual interior-point algorithms in linear optimization. SIAM J. Optim. **15**, 101–128 (2004)
2. Cho, G.M., Cho, Y.Y., Lee, Y.H.: A primal-dual interior-point algorithm based on a new kernel function. ANZIAM J. **51**, 476–491 (2010)
3. Darvay, Z.: New interior point algorithms in linear programming. Adv. Model. Optim. **5**(1), 51–92 (2003)
4. Faraut, J., Kornyi, A.: Analysis on Symmetric Cones. Oxford Mathematical Monographs. The Clarendon Press Oxford University Press, Oxford Science Publications, New York (1994)
5. Faybusovich, L.: Euclidean Jordan algebras and interior-point algorithms. Positivity **1**(4), 331–357 (1997)
6. Faybusovich, L.: Linear systems in Jordan algebras and primal-dual interior-point algorithms. J. Comput. Appl. Math. **86**, 149–175 (1997)
7. Faybusovich, L.: A Jordan-algebraic approach to potential-reduction algorithms. Math. Z. **239**(1), 117–129 (2002)
8. Kheirfam, B.: Primal-dual interior-point algorithm for semidefinite optimization based on a new kernel function with trigonometric barrier term. Numer. Algorithms **61**(4), 659–680 (2012)
9. Kojima, M., Shindoh, S., Hara, S.: Interior-point methods for the monotone semidefinite linear complementarity problem in symmetric matrices. SIAM J. Optim. **7**, 86–125 (1997)
10. Korányi A.: Monotone functions on formally real Jordan algebras. Math. Ann. **269**(1), 73–76 (1984)

11. Lee, Y.H., Cho, Y.Y., Cho, G.M.: Interior-point algorithms for $P_*(\kappa)$-LCP based on a new class of kernel functions. J. Glob. Optim. (2013). doi:10.1007/s10898-013-0072-z
12. Lesaja, G., Roos, C.: Unified analysis of kernel-based interior-point methods for $P_*(\kappa)$-linear complementarity problems. SIAM J. Optim. **20**(6), 3014–3039 (2010)
13. Lesaja, G., Roos, C.: Kernel-based interior-point methods for monotone linear complementarity problems over symmetric cones. J. Optim Theory Appl. (2013). doi:10.1007/s10957-011-9848-9
14. Lesaja, G., Wang, G.Q., Zhu, D.T.: Interior-point methods for Cartesian $P_*(\kappa)$-linear complementarity problems over symmetric cones based on the eligible kernel functions. Optim. Methods Softw. **27**(4–5), 827–843 (2012)
15. Nesterov, Y.E., Todd, M.J.: Primal-dual interior-point methods for self-scaled cones. SIAM J. Optim. **8**(2), 324–364 (1998)
16. Peng, J., Roos, C., Terlaky, T.: Self-regularity: a New Paradigm for Primal-Dual Interior-Point Algorithms. Princeton University Press, Princeton, NJ (2002)
17. Peng, J., Roos, C., Terlaky, T.: Self-regular functions and new search directions for linear and semidefinite optimization. Math. Program. **93**(1), 129–171 (2002)
18. Rangarajan, B.K.: Polynomial convergence of infeasible interior-point methods over symmetric cones. SIAM J. Optim. **16**(4), 1211–1229 (2006)
19. Renegar, J: A Mathematical View of Interior-Point Methods in Convex Optimization. MPS/SIAM Ser. Optim. SIAM, Philadelphia (2001)
20. Roos, C., Terlaky, T., Vial, J-Ph.: Theory and Algorithms for Linear Optimization. An Interior-Point Approach, Wiley, Chichester (1997)
21. Schmieta, S.H., Alizadeh, F.: Associative and Jordan algebras and polynomial time interior-point algorithms for symmetric cones. Math. Oper. Res. **26**, 543–564 (2001)
22. Schmieta, S.H., Alizadeh, F.: Extension of primal-dual interior-point algorithms to symmetric cones. Math. Program. **96**(3), 409–438 (2003)
23. Vieira, M.V.C.: Jordan algebraic approach to symmetric optimization. PhD thesis, Electrical Engineering, Mathematics and Computer Science, Delft University of Technology, The Netherlands (2007)
24. Wang, G.Q.: A new polynomial interior-point algorithm for the monotone linear complementarity problem over symmetric cones with full NT-steps. Asia-Pac. J. Oper. Res. **29**(2), 1250015 (2012)
25. Wang, G.Q., Bai, Y.Q.: A class of polynomial interior-point algorithms for the Cartesian P-matrix linear complementarity problem over symmetric cones. J. Optim. Theory Appl. **152**(3), 739–772 (2012)
26. Wang, G.Q., Lesaja, G.: Full Nesterov-Todd step feasible interior-point method for the Cartesian $P_*(\kappa)$-SCLCP. Optim. Methods Softw. **28**(3), 600–618 (2013)
27. Yoshise, A.: Homogeneous algorithms for monotone complementarity problems over symmetric cones. Pac. J. Optim. **5**, 313–337 (2009)