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A Recursive Algorithm for State Dependent $GI/M/1/N$ Queue with Bernoulli-Schedule Vacation

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Abstract In this paper, we study a renewal input working vacations queue with state dependent services and Bernoulli-schedule vacations. The model is analyzed with single and multiple working vacations. The server goes for exponential working vacation whenever the queue is empty and the vacation rate is state dependent. At the instant of a service completion, the vacation is interrupted and the server resumes a regular busy period with probability $1-q$ (if there are customers in the queue), or continues the vacation with probability q $(0 \leq q \leq 1)$. We provide a recursive algorithm using the supplementary variable technique to numerically compute the stationary queue length distribution of the system. Finally, using some numerical results, we present the parameter effect on the various performance measures.

Keywords State dependent · Bernoulli-schedule · Vacation interruption · Supplementary variable · Working vacations

1 Introduction

In the study of vacation queues generally it is assumed that the server stops service during vacation period. More details on vacation queues can be found in Doshi [7], Tian and Zhang [20] and the references therein. Recently, Servi and Finn [18] introduced a class of semi vacation policy. The server will not completely remain inactive during the vacation period; rather he will render service to the queue with a different rate. When a vacation ends and if there are customers in the queue, a service period begins with the original service rate; otherwise on return from a vacation if there are no customers in the queue, the server goes for another vacation and

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continues to do so till on return from a vacation he finds at least one customer. Such type of vacation is called multiple working vacation (MWV). On the other hand, under the single working vacation (SWV) policy, the server takes only one working vacation whenever the system becomes empty. Therefore, if the system is empty on return from a SWV, the server stays in the system waiting for customers to arrive instead of taking another vacation; otherwise, he changes the service rate back to the regular rate as under the MWV policy. Later Servi and Finn's [18] work was extended to $GI/M/1$ queue by Baba [1]. A finite buffer $GI/M/1$ queue with SWV and MWVs have been discussed by Banik [2] and Banik et al. [3], respectively. Tian et al. [21] have investigated an $M/M/1$ queue with SWV using matrix-geometric method. The performance analysis of a $GI/M/1$ queue with SWV has been presented by Li and Tian [12].

The single server queueing system with Bernoulli-schedule vacation was first introduced by Keilson and Servi [10], wherein after each service completion during vacation the server may take another vacation with probability q or starts regular service with probability $1-q,\ 0 \leq q \leq 1$. The phenomena wherein the server resumes the regular service without taking the remaining vacation is called vacation interruption. The analysis of a $GI/M/1$ queue with working vacations and vacation interruption has been presented by Li et al. [13]. An $M/M/1$ queue with Bernoulli-schedule-controlled vacation and vacation interruption was studied by Zhang and Shi [22]. Ramaswami and Servi [17] studied the busy period of the $M/G/1$ vacation model with a Bernoulli-schedule vacation. A two-phase batch arrival queueing system with a vacation time under Bernoulli-schedule has been described by Choudhury and Madan [6]. Gao and Liu [8] have generalized an $M/G/1$ queue with SWV and vacation interruption under Bernoulli-schedule. Madan and Anabosi [15] discussed a single server queue with two types of services, Bernoulli-schedule server vacations and a single vacation policy.

Queueing models with finite capacity and state dependent services are more appropriate in queueing networks and often increase the complexity of solutions of these systems. Chao and Rahman [4, 5] have analyzed state dependent vacation queues by computational algorithm and presented various types of station vacations as special cases. An extensive work on state dependent queueing models in emergency evacuation networks has been done by Macgregor Smith [14]. Kijima and Makimoto [11] have studied the stationary queue length distributions of $M(n)/G/1/K$ and $GI/M(n)/1/K$ queues via the Neut's method [16]. An efficient algorithm for the state dependent services and state dependent MWV for $GI/M(n)/1/N$ has been presented by Goswami et al. [9].

The above literature survey clearly indicates that state dependent services are useful to model numerous real life applications. It may be observed that except Goswami et al. [9] no much study has been focussed on state dependent queues with working vacations. Motivated by the above observations, this paper aims to contribute to the theory of working vacation models with Bernoulli-schedule vacation interruption. The aim of the paper is two fold. First to analyze the state dependent $GI/M(n)/1/N$ model with Bernoulli-schedule vacation interruption, and the second one is to present computational algorithm by which one can get the queue length distributions in a very efficient way. By introducing a parameter β, the model is analyzed both with SWV ($\beta = 1$) and MWV ($\beta = 0$). By taking $\beta = 0,\ q = 1$, [9] becomes the special case of our present model. Further, the introduction of

Bernoulli-schedule vacation interruption brings more utility to the present model. Using the supplementary variable technique, we have developed the steady state system length distributions at pre-arrival and arbitrary epochs. Some performance measures such as the blocking probability, the expected queue length, the expected waiting time, etc., have been evaluated. Numerical results have been illustrated in the form of tables and graphs.

This paper is organized as follows. Section 2 presents the description and analysis of the model. Computational algorithm to compute the stationary system length distribution is presented in Section 3. Various performance measures and some special cases of our model are presented in Sections 4 and 5, respectively. Section 6 contains numerical results to show the effectiveness of the model parameters followed by conclusions in Section 7.

2 Description and Analysis of the Model

Let us consider a renewal input $GI/M(n)/1/N$ queue where N is the finite capacity of the system. We assume that the inter-arrival times of successive arrivals are independent and identically distributed random variables with cumulative distribution function $A(x)$, probability density function $a(x)$, $x \geq 0$, Laplace-Stiletjes (L.-S.) transform $A^*(\theta)$ and mean inter-arrival time $1/\lambda = -A^{*(1)}(0)$, where $h^{(1)}(0)$ denotes the first derivative of $h(\theta)$ evaluated at $\theta = 0$. The model is analyzed with SWV and MWV and for that we have introduced a parameter β which is assumed 0 for MWV and 1 for SWV. The server takes vacation whenever the system becomes empty. At the instants of service completion, the vacation is interrupted and the server resumes a regular busy period with probability $1-q$ (if there are customers in the queue), or continues the vacation with probability q, $0 \leq q \leq 1$. The vacation times follow exponential distribution and are state dependent with rate γ_i, $1-\beta \leq i \leq N$. The service times during regular busy period (vacation) are also exponentially distributed with rates μ_i (η_i), $1 \leq i \leq N$, when there are i customers present in the system before beginning a service. After returning from SWV he remains idle in the system if no customer is present in the queue otherwise, a regular service period resumes. The customers are served by a single server in the order of first come first served (FCFS) discipline. The model is denoted by $GI/M(n)/1/N/BS-VI$. Let $\mu,\ \eta,\ \gamma$ be the mean service rates during regular busy period, working vacation period and mean vacation rate, respectively, and are given by $\mu = \sum_{i=1}^{N} \mu_i/N,\ \eta = \sum_{i=1}^{N} \eta_i/N,\ \gamma = \sum_{i=1-\beta}^{N} \gamma_i/N$. The traffic intensity is given by $\rho = \lambda/\mu$.

Let us define the state of the system and the joint probabilities as

- $N_s(t) =$ Number of customers present in the system,
- $U(t) =$ Remaining inter-arrival time for the next arrival (supplementary variable),
- $\zeta(t) =\ 0\ (1)$, if the server is in working vacation (regular busy) period.

$$\pi_{i,j}(u,t)du = Pr\{N_s(t) \leq i,\ u < U(t) \leq u + du,\ \zeta(t) = j\},\ u \geq 0,\ j = 0, 1,$$
$$1-\beta \leq i \leq N,$$

where $\pi_{i,0}(u,t),\ 0 \le i \le N$ and $\pi_{i,1}(u,t),\ 1-\beta \le i \le N$ denote the probability of i customers in the system when the server is in working vacation and regular service, respectively. In SWV, $\pi_{0,1}(u,t)$ denotes the dormant state of the server.

2.1 Analysis of the Model

Let us first develop the differential-difference equations that relate the distribution of number of customers in the system at the end of vacation period and regular service period. For this, we use the supplementary variable technique and relate the state of the system at time epochs t and $t+dt$. Using probabilistic arguments, we have the following equation for the state (0, 0).

$$\pi_{0,0}(x-dx,t+dt) = (1-\beta\gamma_0 dt)\pi_{0,0}(x,t) + \mu_1\pi_{1,1}(x,t)dt + \eta_1\pi_{1,0}(x,t)dt.$$

Applying Taylor's series expansion on the left hand side of the equation, assuming $dx = dt$ and neglecting higher order derivatives, we obtain

$$\left(\frac{\partial}{\partial t} - \frac{\partial}{\partial x}\right)\pi_{0,0}(x,t) = \mu_1\pi_{1,1}(x,t) + \eta_1\pi_{1,0}(x,t) - \beta\gamma_0\pi_{0,0}(x,t).$$

Taking limit as $t \to \infty$ in the above equation, we get the following steady state differential-difference equation

$$-\pi^{(1)}_{0,0}(x) = \mu_1\pi_{1,1}(x) + \eta_1\pi_{1,0}(x) - \beta\gamma_0\pi_{0,0}(x). \tag{1}$$

Similarly, we obtain the remaining equations as

$$-\pi^{(1)}_{i,0}(x) = -(\gamma_i+\eta_i)\pi_{i,0}(x) + q\eta_{i+1}\pi_{i+1,0}(x) + a(x)\pi_{i-1,0}(0),$$
$$1 \le i \le N-1, \tag{2}$$

$$-\pi^{(1)}_{N,0}(x) = -(\gamma_N+\eta_N)\pi_{N,0}(x) + a(x)\left(\pi_{N-1,0}(0) + \pi_{N,0}(0)\right), \tag{3}$$

$$-\beta\pi^{(1)}_{0,1}(x) = \beta\gamma_0\pi_{0,0}(x), \tag{4}$$

$$-\pi^{(1)}_{1,1}(x) = -\mu_1\pi_{1,1}(x) + \mu_2\pi_{2,1}(x) + \gamma_1\pi_{1,0}(x) \tag{5}$$

$$-\pi^{(1)}_{i,1}(x) = -\mu_i\pi_{i,1}(x) + \mu_{i+1}\pi_{i+1,1}(x) + \gamma_i\pi_{i,0}(x) + (1-q)\eta_{i+1}\pi_{i+1,0}(x)$$
$$+ a(x)\pi_{i-1,1}(0), \qquad 2 \le i \le N-1, \tag{6}$$

$$-\pi^{(1)}_{N,1}(x) = -\mu_N\pi_{N,1}(x) + \gamma_N\pi_{N,0}(x) + a(x)\left(\pi_{N-1,1}(0) + \pi_{N,1}(0)\right), \tag{7}$$

where $\pi_{i,0}(0)$ and $\pi_{i,1}(0)$ are the respective rates of arrivals i.e., an arrival is about to occur. Let us define the Laplace transforms of $\pi_{i,0}(x)$ and $\pi_{i,1}(x)$ as $\pi^*_{i,0}(\theta) = \int_0^\infty e^{-\theta x}\pi_{i,0}(x)dx,\ \ \pi^*_{i,1}(\theta) = \int_0^\infty e^{-\theta x}\pi_{i,1}(x)dx,\ \ Re\,\theta \ge 0$. Therefore, we have

$$\pi_{i,0} \equiv \pi^*_{i,0}(0) = \int_0^\infty \pi_{i,0}(x)dx,\ \pi_{i,1} \equiv \pi^*_{i,1}(0) = \int_0^\infty \pi_{i,1}(x)dx,$$

where $\pi_{i,0}$ ($\pi_{i,1}$) is the joint probability that there are i customers in the system and the server is in working vacation (dormant or in busy period) at an arbitrary epoch.

Multiplying Eqs.1–7 by $e^{-\theta x}$ and integrating with respect to x from 0 to ∞ yields

$$(\beta\gamma_0-\theta)\pi^*_{0,0}(\theta)=\mu_1\pi^*_{1,1}(\theta)+\eta_1\pi^*_{1,0}(\theta)-\pi_{0,0}(0), \tag{8}$$

$$\begin{aligned}(\gamma_i+\eta_i-\theta)\pi^*_{i,0}(\theta)=\;&A^*(\theta)\pi_{i-1,0}(0)+q\eta_{i+1}\pi^*_{i+1,0}(\theta)-\pi_{i,0}(0),\\ &1\le i\le N-1,\end{aligned} \tag{9}$$

$$(\gamma_N+\eta_N-\theta)\pi^*_{N,0}(\theta)=A^*(\theta)\left(\pi_{N-1,0}(0)+\pi_{N,0}(0)\right)-\pi_{N,0}(0), \tag{10}$$

$$-\beta\theta\pi^*_{0,1}(\theta)=\beta(\gamma_0\pi^*_{0,0}(\theta)-\pi_{0,1}(0)), \tag{11}$$

$$(\mu_1-\theta)\pi^*_{1,1}(\theta)=\mu_2\pi^*_{2,1}(\theta)+\gamma_1\pi^*_{1,0}(\theta)-\pi_{1,1}(0), \tag{12}$$

$$\begin{aligned}(\mu_i-\theta)\pi^*_{i,1}(\theta)=\;&\mu_{i+1}\pi^*_{i+1,1}(\theta)+\gamma_i\pi^*_{i,0}(\theta)+(1-q)\eta_{i+1}\pi^*_{i+1,0}(\theta)\\ &+A^*(\theta)\pi_{i-1,1}(0)-\pi_{i,1}(0),\ 2\le i\le N-1,\end{aligned} \tag{13}$$

$$\begin{aligned}(\mu_N-\theta)\pi^*_{N,1}(\theta)=\;&\gamma_N\pi^*_{N,0}(\theta)+A^*(\theta)\Big(\pi_{N,1}(0)+\pi_{N-1,1}(0)\Big)\\ &-\pi_{N,1}(0).\end{aligned} \tag{14}$$

Now using Eqs. 8–14 we obtain the following important lemma.

Lemma 1

$$\sum_{i=0}^{N}\pi_{i,0}(0)+\sum_{i=1-\beta}^{N}\pi_{i,1}(0)=\lambda. \tag{15}$$

The left hand side denotes the mean number of entrances into the system per unit time and is equal to the mean arrival rate λ.

Proof Adding Eqs. 8–14 and taking limit as $\theta\to 0$, we obtain the desired result using the normalization condition $\sum_{i=0}^{N}\pi_{i,0}+\sum_{i=1-\beta}^{N}\pi_{i,1}=1$. □

2.2 Derivations of $\pi_{i,j}(0)$ and $\pi^*_{i,j}(\theta)$

Substituting $\theta=(\gamma_N+\eta_N)$ in Eq. 10, we get

$$\pi_{N-1,0}(0)=\left(\frac{1-A^*(\gamma_N+\eta_N)}{A^*(\gamma_N+\eta_N)}\right)\pi_{N,0}(0). \tag{16}$$

From Eq. 10, we have

$$\pi^*_{N,0}(\theta)=\frac{(A^*(\theta)-A^*(\gamma_N+\eta_N))}{(\gamma_N+\eta_N-\theta)A^*(\gamma_N+\eta_N)}\pi_{N,0}(0). \tag{17}$$

Substituting $\theta = (\gamma_i + \eta_i)$ in Eq. 9, we get

$$\pi_{i-1,0}(0) = -\left[\frac{q\eta_{i+1}\pi^*_{i+1,0}(\gamma_i + \eta_i) - \pi_{i,0}(0)}{A^*(\gamma_i + \eta_i)}\right], \quad i = N-1, \ldots, 1. \tag{18}$$

From Eq. 9, we obtain

$$\pi^*_{i,0}(\theta) = \frac{q\eta_{i+1}\pi^*_{i+1,0}(\theta) + A^*(\theta)\pi_{i-1,0}(0) - \pi_{i,0}(0)}{(\gamma_i + \eta_i - \theta)}, \quad i = N-1, \ldots, 1. \tag{19}$$

Substituting $\theta = \mu_N$ in Eq. 14 and $\theta = \mu_i$ in Eq. 13, we get

$$\pi_{N-1,1}(0) = \frac{1 - A^*(\mu_N)}{A^*(\mu_N)}\pi_{N,1}(0) - \frac{\gamma_N}{A^*(\mu_N)}\pi_{N,0}(0), \tag{20}$$

$$\begin{aligned}\pi_{i-1,1}(0) = {} & \frac{\pi_{i,1}(0)}{A^*(\mu_i)} - \frac{\mu_{i+1}\pi^*_{i+1,1}(\mu_i)}{A^*(\mu_i)} - \frac{(1-q)\eta_{i+1}\pi^*_{i+1,0}(\mu_i)}{A^*(\mu_i)} - \frac{\gamma_i\pi^*_{i,0}(\mu_i)}{A^*(\mu_i)}, \\ & i = N-1, \ldots, 1-\beta,\end{aligned} \tag{21}$$

where $\pi^*_{i,1}(\theta)$ are given by the following:

$$\pi^*_{N,1}(\theta) = \frac{\gamma_N\pi^*_{N,0}(\theta) + A^*(\theta)(\pi_{N-1,1}(0) + \pi_{N,1}(0)) - \pi_{N,1}(0)}{(\mu_N - \theta)}, \tag{22}$$

$$\begin{aligned}\pi^*_{i,1}(\theta) = {} & \frac{\gamma_i\pi^*_{i,0}(\theta) + \mu_{i+1}\pi^*_{i+1,1}(\theta) + (1-q)\eta_{i+1}\pi^*_{i+1,0}(\theta)}{(\mu_i - \theta)} \\ & + \frac{A^*(\theta)\pi_{i-1,1}(0) - \pi_{i,1}(0)}{(\mu_i - \theta)}, \; i = N-1, \ldots, 1.\end{aligned} \tag{23}$$

For $\theta = \gamma_i + \eta_i$ $(1 \le i \le N)$, $\pi^*_{i,0}(\theta)$ are given by

$$\pi^*_{N,0}(\theta) = -A^{*(1)}(\theta)\left(\pi_{N-1,0}(0) + \pi_{N,0}(0)\right), \tag{24}$$

$$\pi^*_{i,0}(\theta) = -\left(q\eta_{i+1}\pi^{*(1)}_{i+1,0}(\theta) + A^{*(1)}(\theta)\pi_{i-1,0}(0)\right), \quad 1 \le i \le N-1. \tag{25}$$

For $\theta = \mu_i$ $(1-\beta \le i \le N)$, $\pi^*_{i,1}(\theta)$ are given by

$$\pi^*_{N,1}(\theta) = -\left(\gamma_N\pi^{*(1)}_{N,0}(\theta) + A^{*(1)}(\theta)(\pi_{N-1,1}(0) + \pi_{N,1}(0))\right), \tag{26}$$

$$\begin{aligned}\pi^*_{i,1}(\theta) = -\Big(& \gamma_i\pi^{*(1)}_{i,0}(\theta) + \mu_{i+1}\pi^{*(1)}_{i+1,0}(\theta) + (1-q)\eta_{i+1}\pi^{*(1)}_{i+1,0}(\theta) \\ & + A^{*(1)}(\theta)\pi_{i-1,1}(0)\Big), \quad 1-\beta \le i \le N-1.\end{aligned} \tag{27}$$

2.3 Relation Between Steady State Distribution at Pre-arrival and Arbitrary Epochs

Let $\pi^-_{i,j}$, $1-\beta \le i \le N$, $j = 0, 1$ denote the pre-arrival epoch probabilities, that is, an arrival finds i customers in the system and the server is in state j at an arrival epoch. Applying Bayes' theorem, we have

$$\pi^-_{i,j} = \lim_{t\to\infty} \frac{P[N_s(t) = i, \zeta(t) = j, U(t) = 0]}{P[U(t) = 0]}.$$

Further, using Eq. 15 in the above expression, we obtain

$$\pi^-_{i,j} = \frac{\pi_{i,j}(0)}{\lambda},\ 1-\beta \le i \le N, \quad j = 0, 1. \tag{28}$$

From the above set of expressions one can evaluate the pre-arrival epoch probabilities. To obtain the steady state probabilities at arbitrary epochs, we develop a relation between pre-arrival and arbitrary epoch probabilities. Setting $\theta = 0$ in Eqs. 8–14 and using Eq. 28, we obtain

$$\pi_{N,0} = \frac{\lambda}{\gamma_N + \eta_N}\pi^-_{N-1,0}, \tag{29}$$

$$\begin{aligned}\pi_{i,0} = \frac{\lambda}{\gamma_i + \eta_i}\Bigg(&\pi^-_{i-1,0} + \left(\frac{(q-1)\eta_{i+1} - \gamma_{i+1}}{\gamma_{i+1} + \eta_{i+1}}\right)\pi^-_{i,0} \\ &+ \sum_{j=i+1}^{N-1}\left(\frac{(q-1)\eta_{j+1} - \gamma_{j+1}}{\gamma_{j+1} + \eta_{j+1}}\right)\prod_{k=i+1}^{j}\left(\frac{q\eta_k}{\gamma_k + \eta_k}\right)\pi^-_{j,0}\,, \\ &1-\beta \le i \le N-1,\end{aligned} \tag{30}$$

$$\pi_{N,1} = \frac{\lambda}{\mu_N}\left[\left(\frac{\gamma_N}{\gamma_N + \eta_N}\right)\pi^-_{N-1,0} + \pi^-_{N-1,1}\right], \tag{31}$$

$$\begin{aligned}\pi_{i,1} = \frac{\lambda}{\mu_i}\Bigg[&\pi^-_{i-1,1} + \frac{\gamma_i}{\gamma_i + \eta_i}\pi^-_{i-1,0} - \sum_{j=i}^{N-1}\left(\frac{(q-1)\eta_{j+1} - \gamma_{j+1}}{\gamma_{j+1} + \eta_{j+1}}\right) \\ &\times \prod_{k=n}^{j} q^{k-i}\frac{\eta_k}{\gamma_k + \eta_k}\pi^-_{j,0}\Bigg], 1 \le i \le N-1.\end{aligned} \tag{32}$$

If $\beta = 0$, that is, for MWV

$$\begin{aligned}\pi_{1,1} = \frac{\lambda}{\mu_1}\Bigg[&\frac{\gamma_1}{\gamma_1 + \eta_1}\pi^-_{0,0} - \left(\frac{(q-1)\eta_2 - \gamma_2}{\gamma_2}\frac{\gamma_1}{\gamma_1 + \eta_1} + \frac{\gamma_2}{\gamma_1 + \eta_1}\right)\pi^-_{1,0} \\ &+ \sum_{j=2}^{N-1}\left(\frac{(q-1)\eta_{j+1} - \gamma_{j+1}}{\gamma_{j+1} + \eta_{j+1}}\right)\prod_{k=2}^{j}\left(\frac{\eta_k}{\gamma_k + \eta_k}\right)\left(\frac{q\gamma_1}{\gamma_1 + \eta_1} - q^{k-2}\right)\pi^-_{j,0}\Bigg],\end{aligned} \tag{33}$$

$$\pi_{0,0} = 1 - \sum_{i=1}^{N}(\pi_{i,0} + \pi_{i,1}). \tag{34}$$

If $\beta = 1$, that is, for SWV

$$\pi_{0,1} = 1 - \sum_{i=0}^{N}\pi_{i,0} - \sum_{i=1}^{N}\pi_{i,1}. \tag{35}$$

As the state probabilities at pre-arrival epochs are known from Eq. 28, we can evaluate the arbitrary epoch probabilities using Eqs. 29–35.

3 Computational Algorithm

In this section, we give a computational algorithm for evaluation of pre-arrival and arbitrary epoch probabilities. The algorithm is based on the analysis of Section 2, that is, we compute arrival rates $\pi_{i,j}(0),\ 0 \le i \le N,\ j=0;\ 1-\beta \le i \le N,\ j=1$ in terms of $\pi_{N,0}(0)$. We determine $\pi_{N,0}(0)$ using Eq. 15. After computing the probabilities $\pi_{i,j}(0)$, we can evaluate pre-arrival epoch and the arbitrary epoch probabilities.

Step 1: Calculate $\pi_{i,0}(0)$ and $\pi^*_{i,0}(\theta)$ in terms of $\pi_{N,0}(0)$ as follows.
Using the Eqs. 16 and 18 we get

$$\pi_{i,0}(0) = \psi_i \pi_{N,0}(0),\ 0 \le i \le N.$$

– Calculate ψ_i $(1 \le i \le N)$ as

$$\psi_N = 1,\ \psi_{N-1} = \frac{1 - A^*(\delta_N)}{A^*(\delta_N)},\ \ \psi_{i-1} = \frac{\psi_i - q\eta_{i+1}\zeta_{i+1,\delta_i}}{A^*(\delta_i)},$$

From Eqs. 17 and 19 we compute

$$\pi^*_{i,0}(\theta) = \zeta_{i,\theta}\pi_{N,0}(0),\ 1 \le i \le N.$$

The lth derivatives are obtained using Eqs. 24 and 25 where $l = 1, 2, \ldots,$

$$\pi^{*(l)}_{i,0}(\theta) = \zeta^{(l)}_{i,\theta}\pi_{N,0}(0),\ 1 \le i \le N,$$

where $\zeta_{i,\theta}$ and $\zeta^{(l)}_{i,\theta}$ are given below.

– Calculate $\zeta_{i,\theta}$ $(0 \le i \le N)$ as follows
if $i = N$ **then**
if $\theta = \delta_N$ **then**

$$\zeta_{N,\theta} = -A^{*(1)}(\theta)(\psi_{N-1} + \psi_N)$$

else

$$\zeta_{N,\theta} = \frac{A^*(\theta)(\psi_{N-1} + \psi_N) - \psi_N}{\delta_N - \theta}$$

end if
else if $0 \le i \le N-1$ **then**
if $\theta = \delta_i$

$$\zeta_{i,\theta} = -\left(A^{*(1)}(\theta)\psi_{i-1} + q\eta_{i+1}\zeta^{(1)}_{i+1,\theta}\right)$$

else

$$\zeta_{i,\theta} = \frac{A^*(\theta)\psi_{i-1} + q\eta_{i+1}\zeta_{i+1,\theta} - \psi_i}{\delta_i - \theta}$$

end if
end if

– Calculate $\zeta^{(l)}_{n,\theta}$ as follows
if $i = N$ **then**

if $\theta = \delta_N$ **then**

$$\zeta^{(l)}_{N,\theta} = -\frac{A^{*(l+1)}(\theta)(\psi_{N-1}+\psi_N)}{l+1}$$

else

$$\zeta^{(l)}_{N,\theta} = \frac{A^{*(l)}(\theta)(\psi_{N-1}+\psi_N)+l\zeta^{(l-1)}_{N,\theta}}{\delta_N-\theta}$$

end if $0 \le i \le N-1$ **then**

if $\theta = \delta_i$ **then**

$$\zeta^{(l)}_{i,\theta} = -\frac{A^{*(l+1)}(\theta)\psi_{i-1}+q\eta_{i+1}\zeta^{(l+1)}_{i+1,\theta}}{l+1}$$

else

$$\zeta^{(l)}_{i,\theta} = \frac{A^{*(l)}(\theta)\psi_{i-1}+q\eta_{i+1}\zeta^{(l)}_{i+1,\theta}+l\zeta^{(l-1)}_{i,\theta}}{\delta_i-\theta}$$

end if

end if

Step 2: Using Eqs. 20 and 21, find $\pi_{i,1}(0)$ in terms of $\pi_{N,0}(0)$ and $\pi_{N,1}(0)$ as

$$\pi_{i,1}(0) = t_i\pi_{N,0}(0) + d_i\pi_{N,1}(0),\ 1-\beta \le i \le N,$$

where t_i, d_i are computed as follows.

– Calculate t_i and d_i for $1-\beta \le i \le N$

$$t_N = 0,\ \ d_N = 1,\ \ t_{N-1} = -\frac{\gamma_N\zeta_{N,\mu_N}}{A^*(\mu_N)},\ \ d_{N-1} = \frac{1-A^*(\mu_N)}{A^*(\mu_N)},$$

$$t_{i-1} = \frac{t_i-\mu_{i+1}e_{i+1,\mu_i}-(1-q)\eta_{i+1}\zeta_{i+1,\mu_i}-\gamma_i\zeta_{i,\mu_i}}{A^*(\mu_i)},$$

$$d_{i-1} = \frac{d_i-\mu_{i+1}f_{i+1,\mu_i}}{A^*(\mu_i)}.$$

Using Eqs. 22 and 23 we evaluate $\pi^*_{i,1}(\theta)$ in terms of $\pi_{N,0}(0)$ and $\pi_{N,1}(0)$ as

$$\pi^*_{i,1}(\theta) = e_{i,\theta}\pi_{N,0}(0) + f_{i,\theta}\pi_{N,1}(0),\ 1-\beta \le i \le N,$$

The lth($l \ge 1$) derivatives are obtained using Eqs. 26 and 27.

$$\pi^{*(l)}_{i,1}(\theta) = e^{(l)}_{i,\theta}\pi_{N,0}(0) + f^{(l)}_{i,\theta}\pi_{N,1}(0),\ 1-\beta \le i \le N.$$

– Calculate $e_{i,\theta}$ and $f_{i,\theta}$ as follows

if i = N **then**

if $\theta = \mu_N$, **then**

$$e_{N,\theta} = -\gamma_N\zeta^{(1)}_{N,\theta} - A^{*(1)}(\theta)t_{N-1}$$

$$f_{N,\theta} = -A^{*(1)}(\theta)\,(d_{N-1}+d_N)$$

else

$$e_{N,\theta} = \frac{\gamma_N \zeta_{N,\theta} + A^*(\theta)(t_{N-1} + t_N) - t_N}{\mu_N - \theta}$$

$$f_{N,\theta} = \frac{A^*(\theta)(d_{N-1} + d_N) - d_N}{\mu_N - \theta}$$

end if
else if $1 - \beta \leq i \leq N - 1$ **then**
if $\theta = \mu_i$ **then**

$$e_{i,\theta} = -\left(\gamma_i \zeta_{i,\theta}^{(1)} + \mu_{i+1} e_{i+1,\theta}^{(1)} + (1-q)\eta_{i+1}\zeta_{i+1,\theta}^{(1)} + A^{*(1)}(\theta)t_{i-1}\right)$$

$$f_{i,\theta} = -\left(\mu_{i+1} f_{i+1,\theta}^{(1)} + A^{*(1)}(\theta)d_{i-1}\right)$$

else

$$e_{i,\theta} = \frac{\gamma_i \zeta_{i,\theta} + \mu_{i+1} e_{i+1,\theta} + (1-q)\eta_{i+1}\zeta_{i+1,\theta} + A^*(\theta)t_{i-1} - t_i}{\mu_i - \theta}$$

$$f_{i,\theta} = \frac{\mu_{i+1} f_{i+1,\theta} + A^*(\theta)d_{i-1} - d_i}{\mu_i - \theta}$$

end if
end if

– Calculate $e_{i,\theta}^{(l)}$ and $f_{i,\theta}^{(l)}$ as follows
if i = N **then**
if $\theta = \mu_N$, **then**

$$e_{N,\theta}^{(l)} = -\frac{\gamma_N \zeta_{N,\theta}^{(l+1)} + A^{*(l+1)} t_{N-1}}{l+1}$$

$$f_{N,\theta}^{(l)} = -\frac{A^{*(l+1)}(d_{N-1} + d_N)}{l+1}$$

else

$$e_{N,\theta}^{(l)} = -\frac{\gamma_N \zeta_{N,\theta}^{(l)} + A^{*(l)}(t_{N-1} - t_N) + l e_{N,\theta}^{(l-1)}}{\mu_N - \theta}$$

$$f_{N,\theta}^{(l)} = \frac{A^{*(l)}(\theta)(d_{N-1} - d_N) + l f_{N,\theta}^{(l-1)}}{\mu_N - \theta}$$

end if
else if $1 - \beta \leq i \leq N - 1$ **then**
if $\theta = \mu_i$, **then**

$$e_{i,\theta}^{(l)} = -\frac{\gamma_i \zeta_{i,\theta}^{(l+1)} + \mu_{i+1} e_{i+1,\theta}^{(l+1)} + (1-q)\eta_{i+1}\zeta_{i+1,\theta}^{(l+1)} + A^{*(l+1)}(\theta)t_{i-1}}{l+1}$$

$$f_{i,\theta}^{(l)} = -\frac{\mu_{i+1} f_{i+1,\theta}^{(l+1)} + A^{*(l+1)}(\theta)d_{i-1}}{l+1}$$

else

$$e_{i,\theta}^{(l)} = \frac{\gamma_i \zeta_{i,\theta}^{(l)} + \mu_{i+1} e_{i+1,\theta}^{(l)} + (1-q)\eta_{i+1}\zeta_{i+1,\theta}^{(l)} + A^{*(l)}(\theta) t_{i-1} + l e_{i,\theta}^{(l-1)}}{\mu_i - \theta}$$

$$f_{i,\theta}^{(l)} = \frac{\mu_{i+1} f_{i+1,\theta}^{(l)} + A^{*(l)}(\theta) d_{i-1} + l f_{i,\theta}^{(l-1)}}{\mu_i - \theta}$$

end if
end if

Step 3: Using Eqs. 12 and 8 we compute $\pi_{N,1}(0)$ in terms of $\pi_{N,0}(0)$ as $\pi_{N,1}(0) = k\pi_{N,0}(0)$, where

$$k = \left(\frac{\mu_2 e_{2,\mu_1} + \gamma_1 \zeta_{1,\mu_1} - t_1}{d_1 - \mu_2 f_{2,\mu_1}}\right), \text{ if } \beta = 0,$$

$$k = \left(\frac{\psi_0 - \eta_1 \zeta_{1,\gamma_0} - \mu_1 e_{1,\gamma_0}}{\mu_2 f_{1,\gamma_0}}\right), \text{ if } \beta = 1.$$

Step 4: For $1-\beta \leq i \leq N$, calculate $\pi_{i,1}(0)$ in terms of $\pi_{N,0}(0)$ as follows

$$\pi_{i,1}(0) = (t_i + kd_i)\pi_{N,0}(0), \; 1-\beta \leq i \leq N.$$

Step 5: Determine $\pi_{N,0}(0)$ from Eq. 15 as

$$\pi_{N,0}(0) = \lambda \left[\sum_{i=0}^{N} \psi_n + \sum_{i=1-\beta}^{N} (t_i + kd_i)\right]^{-1}.$$

Step 6: Compute the pre-arrival epoch probabilities using the relation (28).
Step 7: For $\beta = 1$, the arbitrary epoch probabilities $\pi_{i,j}$ are determined from Eqs. 29–32 and 35. We use Eqs. 29–34 for $\beta = 0$.

4 Performance Measures

In this section, we discuss some operating characteristics such as the average number of customers in the queue (L_q), average number of customers in the system (L_s), probability that the server is in idle period (P_{idle}), probability that the server is in regular busy period (P_b), probability that the server is in working vacation period (P_{wv}) and the blocking probability of the server (P_{loss}). They are given by:

$$L_q = \sum_{i=1}^{N}(i-1)\pi_{i,0} + \sum_{i=1}^{N}(i-1)\pi_{i,1}; \quad L_s = \sum_{i=1}^{N} i\pi_{i,0} + \sum_{i=1}^{N} i\pi_{i,1};$$

$$P_{\text{idle}} = \pi_{0,1}; \quad P_b = \sum_{i=1}^{N}\pi_{i,1}; \quad P_{wv} = \sum_{i=0}^{N}\pi_{i,0}; \quad P_{\text{loss}} = \pi_{N,0}^{-} + \pi_{N,1}^{-}.$$

The average waiting time of a customer in the queue (W_q) using Little's rule is given by $W_q = L_q/\hat{\lambda}$, where $\hat{\lambda} = \lambda(1 - P_{\text{loss}})$ is the effective arrival rate.

5 Special Cases

In this section, some special cases of our model have been derived by taking some particular values of the parameters.

Case I: If $\beta = 0,\ q = 1$, our model reduces to $GI/M(n)/1/MWV$, Goswami et al. [9].

Case II: If $\beta = 0,\ q = 1,\ \eta_i = \eta,\ \gamma_i = \gamma,\ \mu_i = \mu$ for all i, which means the model with constant services. Our results match with Banik et al. [3]. If $q = 0$, our model reduces to constant services with vacation interruption i.e., $GI/M/1/MWV - VI$. If $\beta = 1,\ q = 1$, our results match with Banik [2]. If $q = 0$, i.e., $GI/M/1/SWV - VI$ queue results.

Case III: If $\beta = 0, \mu_i = \mu,\ \gamma_i = \gamma,\ \eta_i \to 0$ for all i, our results match with Tian et al. [19]. $\beta = 1$ yields constant services with SV, Tian and Zhang [20].

Case IV: If β=0, 1. $q = 0$ and $q = 1,\ \mu_i = \mu,\ \eta \to 0,\ \gamma_i \to \infty$, for $1 \le i \le N$. The model reduces to $GI/M/1/N$ queue without vacations and our results match with the results available in literature.

6 Numerical Results

The model we consider has certain implications in practice. For example, in a packet-switched network, the router is an interconnection device that attaches two or more networks. It takes charge of receiving packets and forwarding them to the next hop, according to the some routing information in its routing table. If the routes are available in the routing table, router will serve the packets depending on the number of packets which is termed as state dependent routing. This procedure routes the packets to the least loaded disks. This offers significant throughput benefits over state independent routing. To collect the complete routing information, the router may exchange its routing information with the other routers. The router is in idle state when the last packet is served and no packet arrives. To keep the router functioning well some special maintenance such as routing information backup and virus scan are performed in idle state. This maintenance duration of the router can be seen as vacation periods. Meanwhile, under the maintenance, the router can serve the packet at the slower speed which can economize the system. In queueing terminology router, routing at slower speed, state dependent routing correspond to the server, working vacation, state dependent services. Therefore, the queueing systems with working vacation and vacation interruption are used to reconfigure the communication networks.

In order to validate the computational algorithm of Section 3, some numerical computations are carried out some of which are presented in the form of tables and graphs. Let us assume that $\mu_i = ln[i + 0.6],\ \eta_i = ln[i + 0.3]$ for $1 \le i \le N,\ \gamma_i = ln[i + 0.4]$ for $1 - \beta \le i \le N,\ \rho = 0.7,\ \lambda = \rho\mu$ and $N = 10$ with mean rates $\mu = 1.66449$, $\eta = 1.59224$ and $\gamma = 1.61722$. The comparisons and effect of accuracy are derived from numerical results.

Tables 1 and 2 present the sensitivity analysis for varying values of λ and μ (mean service rate during busy period) for state dependent queue. We present the comparison among multiple vacation (MV) (single vacation (SV)), multiple (single)

Table 1 Sensitivity analysis of $M/M(n)/1/10$ queue with MWV with and without vacation interruption, multiple vacations (MV) and no vacations (NV)

μ	λ		MWV	MWV-VI	MV	NV
1.64128	0.5	L_s	1.25209	1.23316	1.46694	1.19148
		P_{loss}	0.00000	0.00000	0.00010	0.00000
		W_s	2.50419	2.46633	2.9339	2.38297
1.75023	1.0	L_s	2.33157	2.29244	2.56892	2.19155
		P_{loss}	0.00121	0.00118	0.00156	0.00112
		W_s	2.33441	2.29517	2.57294	2.19402
1.84497	1.5	L_s	3.76780	3.72893	3.96347	3.62598
		P_{loss}	0.01587	0.01566	0.01759	0.01517
		W_s	2.5524	2.52553	2.68963	2.45458

working vacation with and without vacation interruption (MWV-VI (SWV-VI), MWV(SWV)) and no vacation (NV). The inter-arrival time is exponentially distributed. It is observed that for a fixed μ, the performance measures L_s, P_{loss} and W_s are least for no vacation models, but with working vacations, the vacation interruption is always recommended. Further, the SWV-VI is better than the corresponding MWV-VI model, with the gap being widening with decreasing values of λ and μ. As μ and λ increase the performance measures also increase.

Figure 1 depicts the effect of mean service rate during vacation (η) on the expected queue length (L_q) for vacation rates $\gamma = 1.64128, 1.75023, 1.84497$ (by taking $\gamma_i = ln[i + 0.5]$, $\gamma_i = ln[i + 1.0]$, $\gamma_i = ln[i + 1.5]$) for exponential inter-arrival times. The other parameters are as considered above with $\mu = 1.66449$. We observe that as η increases the expected queue length L_q decreases. Further, we observe that as γ increases, the L_q decreases up to a certain level (near the point 1.66449 where η crosses μ) and then reverses its trend. This is due to the fact that with increase of η and γ the number of customers getting service during vacation increases resulting in the decrease of L_q. However, when η coincides with that of the regular service rate μ, it is beneficial to render regular service. Therefore, working vacation queue utilizes the idle time effectively when $\eta \leq \mu$.

Figure 2 provides a comparative effect of arrival rate (λ) on L_q among the models: (i) $E_2/M(n)/1/10/SWV-VI$, (ii) $E_2/M/1/10/SWV-VI$, (iii) $E_2/M(n)/1/10/SWV$ (iv) $E_2/M/1/10/SWV$ (v) $E_2/M(n)/1/10/SV$ and (vi)

Table 2 Sensitivity analysis of $M/M(n)/1/10$ queue with SWV with and without vacation interruption, for single vacation (SV) and no vacation (NV)

μ	λ		SWV	SWV-VI	SV	NV
1.64128	0.5	L_s	1.24290	1.18331	1.43412	1.19148
		P_{loss}	0.00000	0.00000	0.00010	0.00000
		W_s	2.48581	2.36663	2.86827	2.38297
1.75023	1.0	L_s	2.32181	2.25141	2.54816	2.19155
		P_{loss}	0.00120	0.00116	0.00153	0.00112
		W_s	2.32462	2.25405	2.55208	2.19402
1.84497	1.5	L_s	3.76160	3.70335	3.95156	3.62598
		P_{loss}	0.01584	0.01557	0.01750	0.01517
		W_s	2.54812	2.50796	2.68131	2.45458

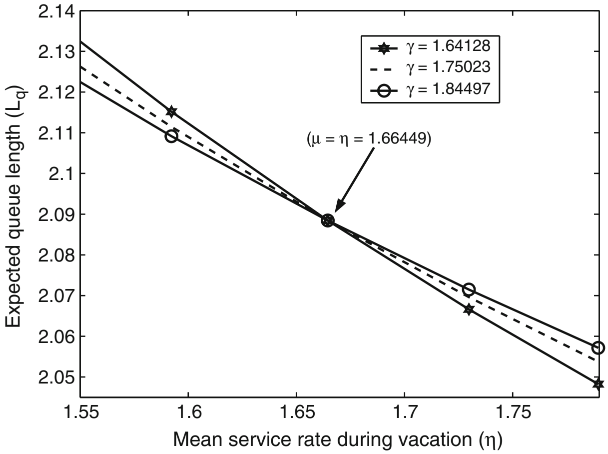

Fig. 1 Effect of η on L_q for SWV-VI

$E_2/M/1/10/SV$. Clearly, L_q increases as the arrival rate increases. Further, we observe that the model with state dependent services and vacation interruption i.e., model (i) performs best among all models. Similar effect is observed among the models with MWV and vacation interruption from Fig. 3. We observe from Figs. 2 and 3 that state dependent models are better as compared to the corresponding state independent models.

Figure 4 illustrates the effect of λ on W_q for different inter-arrival time distributions. It can be observed that W_q increases with λ. Further, for a fixed λ, HE_2 distribution yields highest and deterministic distribution the lowest average waiting times. The expected queue lengths of the following models: (i) $M/M(n)/1/10/SWV-VI$ (ii) $M/M(n)/1/10/SWV$ (iii) $M/M(n)/1/10/MWV-VI$ and (iv) $M/M(n)/1/10/MWV$ are compared in Fig. 5. Intuitively, the expected queue length increases as the arrival rate increases. Moreover, the models with vacation interruption give us better results than the models without vacation interruption.

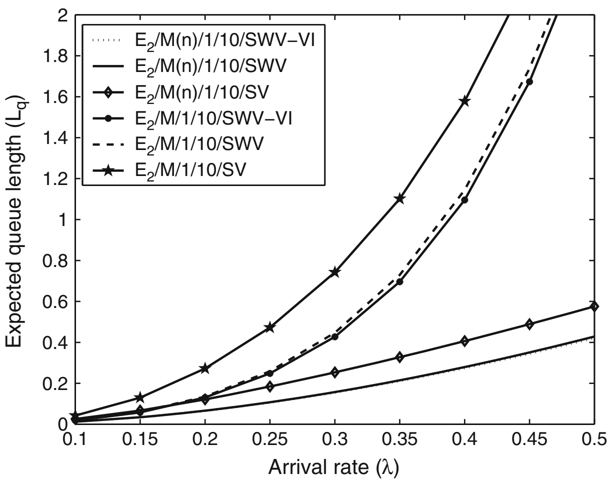

Fig. 2 Effect of λ on L_q

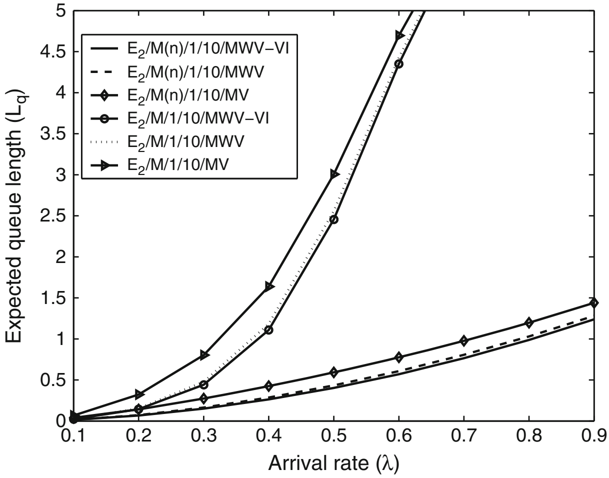

Fig. 3 Effect of λ on L_q

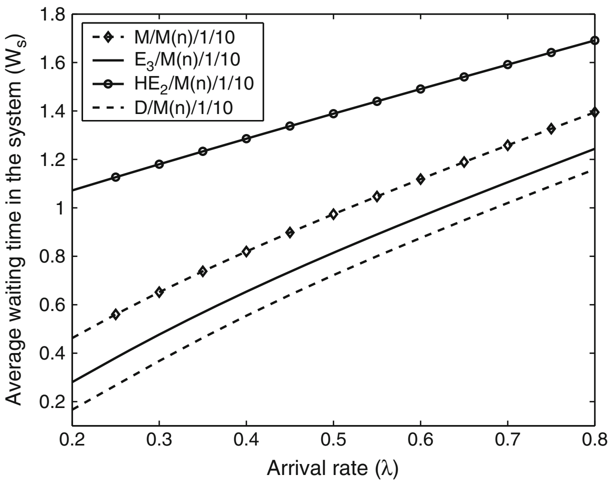

Fig. 4 Effect of λ on W_q for various distributions in SWV ($\beta = 1$)

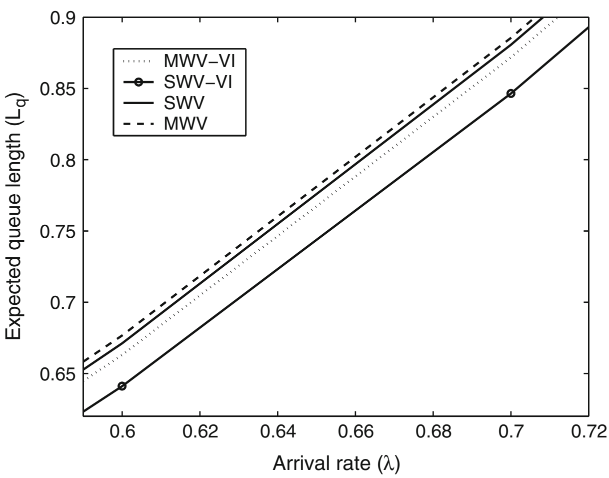

Fig. 5 Effect of λ on L_q

Finally, from the figures and tables we can observe that:

- The expected queue length decreases with increase of η and γ for state dependent queueing models.
- The queue lengths in state dependent models are better than those of state independent models.
- For better service we can consider working vacation models with vacation interruption that utilizes the server more and decreases the waiting lines effectively.
- Among the four models considered, model (iv) performs best.

7 Conclusions

This paper presents the analysis of single server state dependent queue with working vacations and Bernoulli-schedule vacation interruption. The model is analyzed both for single and multiple working vacations. The inter-arrival time is arbitrarily distributed while the service times during regular busy period, during working vacation and vacation times are exponentially distributed and are state dependent. The paper also establishes the explicit relations between pre-arrival and arbitrary queue length distributions. Further, the theoretical results have been presented in a very tractable algorithmic form so that numerical computations can be carried out directly. By taking particular parameters of the model say β and q, the present model reduces to earlier works as special cases, as pointed out in Section 5. This paper extends the earlier works by the introduction of Bernoulli-Schedule vacation interruption for both MWV and SWV models. Our study may be helpful in many areas like restaurants, ATM center, setting traffic management strategies, railway routes, modeling circulation systems in buildings, pedestrian/vehicular circulation systems, etc. The recursive algorithm may be helpful to analyze more complicated models such as $GI/M^{(a,b)}/1$, $GI/M/c$ queues with various vacation policies that are left for future investigations.

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