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A Full-Newton Step Infeasible Interior-Point Algorithm Based on Darvay Directions for Linear Optimization

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Abstract We present a full-Newton step primal-dual infeasible interior-point algorithm based on Darvay's search directions. These directions are obtained by an equivalent algebraic transformation of the centering equation. The algorithm decreases the duality gap and the feasibility residuals at the same rate. During this algorithm we construct strictly feasible iterates for a sequence of perturbations of the given problem and its dual problem. Each main iteration of the algorithm consists of a feasibility step and some centering steps. The starting point in the first iteration of the algorithm depends on a positive number ξ and it is strictly feasible for a perturbed pair, and feasibility steps find strictly feasible iterate for the next perturbed pair. By using centering steps for the new perturbed pair, we obtain strictly feasible iterate close to the central path of the new perturbed pair. The algorithm finds an ϵ-optimal solution or detects infeasibility of the given problem. The iteration bound coincides with the best known iteration bound for linear optimization problems.

Keywords Infeasible interior-point methods · Full-Newton step · Darvay search directions · Polynomial complexity

Mathematics Subject Classifications (2010) 90C05 · 90C51

1 Introduction

After the presentation of the landmark work by Karmarkar in his seminal paper [6], the interior-point methods (IPMs) have been extensively studied, due to their polynomial complexity and practical efficiency [15, 20]. One may distinguish between feasible IPMs and infeasible IPMs (IIPMs). Feasible IPMs start with a strictly feasible interior point and maintain feasibility during the solution process. Several researches have been done in this field, some of them are [15, 20]. Darvay [4] proposed a

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new technique for finding a class of search directions. Based on this technique, the author designed a full-Newton step primal-dual path-following interior-point algorithm for linear optimization (LO) with iteration bound $O(\sqrt{n}\log\frac{n}{\epsilon})$. Later on, Achache [1], Wang and Bai [16–18], Mansouri and Pirhaji [9] and Bai et al. [3] respectively extended Darvay's algorithm for LO to convex quadratic optimization (CQO), semidefinite optimization (SDO), second-order cone optimization (SOCO), symmetric optimization (SO), monotone linear complementarity problem (MLCP) and convex quadratic semidefinite optimization (CQSDO). IIPMs start with an arbitrary positive point (not necessarily feasible) and feasibility is reached as we approach to the optimal solution. The first IIPMs were proposed by Lustig [8]. Global convergence was shown by Kojima et al. [7], whereas Zhang [21] proved an $O(n^2L)$ iteration bound for IIPMs under certain conditions. For studying more details about IIPMs one can refer to [19]. In 2006, Roos [14] proposed a new IIPM to solve LO problem. Although this method is a short-step method, the advantage is that the calculation of a step size at each iteration is avoided, unlike the classical IIPMs [7, 10, 21], while global convergence and quadratic local convergence is still achieved. Furthermore, the iteration bound of the algorithm matches the best known iteration bound, namely, $O(n\log\frac{n}{\epsilon})$, for these types of algorithms.

In this paper, inspired by Darvay's and Roos' works, we combine the infeasible interior-point method for LO in [14] with a search direction proposed by Darvay, in [4]. The difference is in the search direction. Although the idea underlying the algorithm is the same as in [14], the new search direction makes the analysis far from trivial. But we prove that the new search direction works well, and yields the same iteration bound as in [14], which is currently the best known bound of IIPMs for LO. Our algorithm, at each iteration, uses only full-Newton steps no line searches are required. With the appropriate choice of parameters, our algorithm generates a sequence of iterates in the small neighborhood of the central path which implies global convergence.

The paper is organized as follows. In Section 2 we recall briefly the class of search directions given by Darvay and its properties. Section 3 consists of a full-Newton step infeasible IPM based on the algorithm proposed by Roos [14]. In Section 4 we investigate the analysis of our algorithm in more details and derive the complexity bound. In Section 5 we perform the algorithm on some numerical examples. Finally we have some concluding remarks in Section 6.

2 Full-Newton Step for Feasible IPM Based on Darvay Direction

Before presenting our full-Newton step IIPM based on Darvay direction, we need to recall the class of search directions given by Darvay [4], which gives a polynomial-time path-following feasible IPM for solving (P) and (D).

2.1 Problem Background

Let A be an $m \times n$ matrix, $b \in R^m$ and $c \in R^n$. Consider the standard form linear optimization problem

$$\begin{aligned} \min\ & c^T x \\ s.t.\ Ax &= b, \\ x &\geq 0, \end{aligned} \tag{P}$$

and its dual

$$\begin{aligned} \max\ & b^T y \\ s.t.\ & A^T y + s = c, \\ & s \geq 0. \end{aligned} \tag{D}$$

We assume that the matrix A has full row rank, i.e., $\operatorname{rank}(A) = m$. We call (x, y, s) an (infeasible) interior point if $x > 0$ and $s > 0$, and a feasible interior point if in addition $Ax = b$ and $A^T y + s = c$. We denote the sets of primal and dual interior feasible points by

$$\mathcal{F}^0(P) = \left\{x \in R^n :\ Ax = b,\ x > 0\right\},$$

and

$$\mathcal{F}^0(D) = \left\{(y, s) \in R^m \times R^n :\ A^T y + s = c,\ s > 0\right\},$$

respectively. Without loss of generality, we assume that (P) and (D) satisfy the interior point condition (IPC), i.e., there exist x^0, y^0 and s^0 such that $x^0 \in \mathcal{F}^0(P)$ and $(y^0, s^0) \in \mathcal{F}^0(D)$. It is well-known that finding an optimal solution of (P) and (D) is equivalent to solve the following system

$$\begin{aligned} Ax &= b, \quad x \geq 0, \\ A^T y + s &= c, \quad s \geq 0, \\ xs &= 0, \end{aligned} \tag{1}$$

where, xs denotes the Hadamard product of the vectors x and s. The basic idea of primal-dual IPMs is to replace the third equation in system (1), the so-called complementary condition for (P) and (D), by the parameterized equation $xs = \mu e$, with $\mu > 0$. Thus, one may consider

$$\begin{aligned} Ax &= b, \quad x \geq 0, \\ A^T y + s &= c, \quad s \geq 0, \\ xs &= \mu e. \end{aligned} \tag{2}$$

For each $\mu > 0$, the system (2) has a unique solution $(x(\mu), y(\mu), s(\mu))$, which is called the μ-center of (P) and (D) [19], and these solutions form a curve parameterized by μ. This curve is called the central path and most IPMs approximately follow the central path to reach the optimal set. If $\mu \longrightarrow 0$, then the limit of the path exists and yields optimal solutions for (P) and (D) [15].

2.2 Definition and Properties of the Darvay Directions

For solving system (2), we want to use search directions given by Darvay [4]. To this end, we consider the differentiable continuous function $\varphi : R^+ \longrightarrow R^+$, and suppose that the inverse function φ^{-1} exists, then system (2), which defines the central path, can be written in the following equivalent form

$$\begin{aligned} Ax &= b, \quad x \geq 0, \\ A^T y + s &= c, \quad s \geq 0, \\ \varphi\left(\frac{xs}{\mu}\right) &= \varphi(e). \end{aligned} \tag{3}$$

The natural way to define search direction is to follow Newton's approach and to linearize the third equation in Eq. 3. This leads to the system

$$\begin{aligned} A\Delta x &= b - Ax, \\ A^T \Delta y + \Delta s &= c - A^T y - s, \\ \frac{s}{\mu}\varphi^{'}\left(\frac{xs}{\mu}\right)\Delta x + \frac{x}{\mu}\varphi^{'}\left(\frac{xs}{\mu}\right)\Delta s &= \varphi(e) - \varphi\left(\frac{xs}{\mu}\right). \end{aligned} \tag{4}$$

If (x, y, s) is a primal-dual feasible point, then $b - Ax = 0$ and $c - A^T y - s = 0$, hence the system (4) reduces to

$$\begin{aligned} A\Delta x &= 0, \\ A^T \Delta y + \Delta s &= 0, \\ \frac{s}{\mu}\varphi^{'}\left(\frac{xs}{\mu}\right)\Delta x + \frac{x}{\mu}\varphi^{'}\left(\frac{xs}{\mu}\right)\Delta s &= \varphi(e) - \varphi\left(\frac{xs}{\mu}\right), \end{aligned} \tag{5}$$

which gives the Darvay's direction for feasible primal-dual IPMs. The new iterate is obtained by taking a full-Newton step as follows:

$$\begin{aligned} x^+ &= x + \Delta x, \\ y^+ &= y + \Delta y, \\ s^+ &= s + \Delta s. \end{aligned} \tag{6}$$

Through the first two equations of the system (5), we have

$$(\Delta x)^T \Delta s = 0.$$

Defining

$$d_x := \frac{\upsilon \Delta x}{x}, \quad d_s := \frac{\upsilon \Delta s}{s}, \quad \upsilon = \sqrt{\frac{xs}{\mu}}, \tag{7}$$

the system (5) turns to

$$\begin{aligned} \bar{A} d_x &= 0, \\ \bar{A}^T \frac{\Delta y}{\mu} + d_s &= 0, \\ d_x + d_s &= \frac{\varphi(e) - \varphi(\upsilon^2)}{\upsilon \varphi^{'}(\upsilon^2)} := p_\upsilon, \end{aligned} \tag{8}$$

where, $\bar{A} = AV^{-1}X$, $V = \text{diag}(v)$, $X = \text{diag}(x)$. Since the matrix $\bar{A}^T\bar{A}$ is positive definite, we can conclude that the system (8) has a unique solution. Recently, Peng et al. [11–13] introduced a class of search directions based on self-regular kernel functions and Bai et al. [2] also defined a class of new search directions by using the so-called eligible kernel functions. By choosing the function $\varphi(t)$ appropriately, the above system can be used to define the new search directions. We observe that $\varphi(t) = t$ yields $p_v = v^{-1} - v$, and obtain the standard primal-dual interior-point algorithm [15]. For $\varphi(t) = \sqrt{t}$, we get $p_v = 2(e - v)$. The search directions d_x and d_s obtained from system (8) with $p_v = 2(e - v)$, due to Darvay [4].

We define the proximity measure to the central path as follows

$$\delta(x, s; \mu) := \delta(v) = \frac{1}{2}\|p_v\| = \|e - v\|, \tag{9}$$

where, v defined as in Eq. 7. Using the third equation of system (8) and $d_x^T d_s = 0$, we have

$$\|p_v\|^2 = \|d_x + d_s\|^2 = \|d_x\|^2 + \|d_s\|^2.$$

Furthermore, we obtain

$$\delta(v) = 0 \Leftrightarrow v = e \Leftrightarrow d_x = d_s = 0 \Leftrightarrow xs = \mu e.$$

Defining

$$q_v = d_x - d_s,$$

we have

$$\|q_v\|^2 = \|d_x - d_s\|^2 = \|d_x\|^2 + \|d_s\|^2 = \|d_x + d_s\|^2 = \|p_v\|^2.$$

As a consequence we have the following important property that after a full Newton step, strict feasibility is still maintained.

Lemma 1 (Lemma 5.1 in [4]) *If* $\delta(x, s; \mu) < 1$, *then* $x^+ > 0$ *and* $s^+ > 0$.

The following property provides the effect of the full-Newton step on the duality gap.

Lemma 2 (Lemma 5.3 in [4]) *After a primal-dual Newton step one has*

$$\left(x^+\right)^T s^+ = \mu\left(n - \delta^2\right),$$

hence

$$(x^+)^T s^+ \leq n\mu.$$

Assume that a primal feasible $x^0 > 0$ and a dual feasible pair (y^0, s^0) with $s^0 > 0$ are given that are 'close to' μ-centers $x(\mu)$ and $(y(\mu), s(\mu))$, respectively, for some $\mu = \mu^0$. Then, one can find an ϵ-solution in $O(\sqrt{n}\log(\frac{n\mu^0}{\epsilon}))$ iterations of the Algorithm 1.

Algorithm 1 Primal-dual feasible IPM

Input:
Accuracy parameter $\epsilon > 0$;
barraier update parameter θ, $0 < \theta < 1$;
feasible (x^0, y^0, s^0) *with* $(x^0)^T s^0 = n\mu^0$, $\delta(x^0, s^0; \mu^0) < \frac{1}{2}$.
begin
$x := x^0$; $y := y^0$; $s := s^0$; $\mu := \mu^0$;
while $x^T s > \epsilon$ **do**
begin
$\mu := (1 - \theta)\mu$;
substitute $\varphi(t) = \sqrt{t}$ in (5) and compute;
$(x, y, s) := (x, y, s) + (\Delta x, \Delta y, \Delta s)$;
end
end.

The following lemma is crucial in the analysis of the algorithm, which state that the Newton process is quadratically convergent. We recall it without proof.

Lemma 3 (Lemma 5.2 in [4]) *If* $\delta = \delta(x, s, \mu) < 1$, *then*

$$\delta\left(x^+, s^+, \mu\right) \leq \frac{\delta^2}{1 + \sqrt{1 - \delta^2}}.$$

Thus $\delta(x^+, s^+, \mu) \leq \delta^2$, *which means the quadratic convergence of the full-Newton step.*

The following lemma provides an iteration bound $O(\sqrt{n}\log\frac{n\mu^0}{\epsilon})$ for the algorithm.

Lemma 4 (Lemma 5.7 in [4]) *If* $\theta = \frac{1}{2\sqrt{n}}$, $n \geq 4$, *then the algorithm requires at most*

$$2\sqrt{n}\log\frac{n\mu^0}{\epsilon}$$

iterations. The output is a primal-dual pair (x, s) *such that* $x^T s \leq \epsilon$.

3 A Full-Newton Step IIPM

In the case of an infeasible method, we call the triple (x, y, s) an ϵ-solution of (P) and (D) if the norms of the residual vectors $r_p = b - Ax$ and $r_d = c - A^T y - s$ do not exceed ϵ, and also $x^T s \leq \epsilon$. In what follows, we present an infeasible-start algorithm that generates an ϵ-solution of (P) and (D), if it exists, or establishes that no such solution exists.

3.1 The Perturbed Problems

We assume (P) and (D) have an optimal solution (x^*, y^*, s^*) with $(x^*)^T s^* = 0$. As usual for IIPMs, we start the algorithm with $(x^0, y^0, s^0) = \xi(e, 0, e)$ and $\mu^0 = \xi^2$, where ξ is a positive number such that

$$\|x^* + s^*\|_\infty \leq \xi. \tag{10}$$

The initial values of the primal and dual residual vectors are $r_p^0 = b - Ax^0$, $r_d^0 = c - A^T y^0 - s^0$. In general, $r_p^0 \neq 0$ and $r_d^0 \neq 0$. The iterates generated by the algorithm will be infeasible for (P) and (D), but they will be feasible for perturbed versions of (P) and (D) that we introduce below. For any ν with $0 < \nu \leq 1$, we consider the perturbed problem (P_ν) defined by

$$\begin{aligned} \min \ & \left(c - \nu r_d^0\right)^T x \\ s.t.\ & b - Ax = \nu r_p^0, \\ & x \geq 0, \end{aligned} \tag{P_ν}$$

and its dual problem (D_ν) which is given by

$$\begin{aligned} \max \ & \left(b - \nu r_p^0\right)^T y \\ s.t.\ & c - A^T y - s = \nu r_d^0, \\ & s \geq 0. \end{aligned} \tag{D_ν}$$

Note that if $\nu = 1$, then $x = x^0$ and $(y, s) = (y^0, s^0)$ yield strictly feasible solutions of (P_ν) and (D_ν), respectively. We conclude that if $\nu = 1$, both (P_ν) and (D_ν) are strictly feasible, which means that both perturbed problems (P_ν) and (D_ν) satisfy the IPC. More generally, one has the following lemma (Lemma 3.1 in [14]).

Lemma 5 *The original problems,* (P) *and* (D), *are feasible if and only if for each* ν *satisfying* $0 < \nu \leq 1$, *the perturbed problems* (P_ν) *and* (D_ν) *satisfy the IPC.*

Assuming that (P) and (D) are both feasible, it follows from Lemma 5 that the problems (P_ν) and (D_ν) satisfy the IPC, for each $\nu \in (0, 1]$. Then, their central paths exist, meaning that the system

$$b - Ax = \nu r_p^0, \qquad x \geq 0, \tag{11}$$

$$c - A^T y - s = \nu r_d^0, \qquad s \geq 0, \tag{12}$$

$$\varphi\left(\frac{xs}{\mu}\right) = \varphi(e), \tag{13}$$

has a unique solution, for any $\mu > 0$. For $\nu \in (0, 1]$ and $\mu = \nu\mu^0$, we denote this unique solution as $(x(\mu, \nu), y(\mu, \nu), s(\mu, \nu))$. These are the μ-centers of the perturbed problems (P_ν) and (D_ν). By taking $\nu = 1$, one has $(x(\mu^0, 1), y(\mu^0, 1), s(\mu^0, 1)) =$

$(x^0, y^0, s^0) = (\xi e, 0, \xi e)$ and $x^0 s^0 = \mu^0 e$. Hence, x^0 is the μ^0-center of the perturbed problem (P_1) and (y^0, s^0) is the μ^0-center of the perturbed problem (D_1).

3.2 An Iteration of our Algorithm

We just established that if $\nu = 1$ and $\mu = \mu^0$, then (x^0, y^0, s^0) is the μ-center of the problems (P_ν) and (D_ν). We measure proximity to the μ-center of the perturbed problems by the quantity $\delta(x, s; \mu)$ as defined in Eq. 9.

Initially, we have $\delta(x, s; \mu) = 0$. In the sequel, we assume that at the start of each iteration, just before the μ-and ν-update, $\delta(x, s; \mu) \leq \tau$, where τ is a positive threshold value. This certainly holds at the start of the first iteration. Since we then have $\delta(x, s; \mu) = 0$.

Now, we describe one main iteration of our algorithm. The algorithm begins with an infeasible interior-point (x, y, s) such that (x, y, s) is feasible for the perturbed problems (P_ν) and (D_ν), with $\mu = \nu\mu^0$ and such that $x^T s \leq n\mu$ and $\delta(x, s; \mu) \leq \tau$. We reduce ν to $\nu^+ = (1-\theta)\nu$, with $\theta \in (0, 1)$, and find new iterate (x^+, y^+, s^+) that is feasible for the perturbed problems (P_{ν^+}) and (D_{ν^+}), and such that $\delta(x^+, s^+; \mu^+) \leq \tau$. Every iteration consists of a feasibility step, a μ-update and a few centering steps, respectively. First, we find a new point (x^f, y^f, s^f) which is feasible for the perturbed problems with $\nu^+ := (1-\theta)\nu$. Then, μ is decreased to $\mu^+ := (1-\theta)\mu$. Generally, there is no guarantee that $\delta(x^f, s^f; \mu^+) \leq \tau$. So, a limited number of centering steps is applied to produce a new point (x^+, y^+, s^+) such that $\delta(x^+, s^+; \mu^+) \leq \tau$. This process is repeated until the algorithm terminates. We now summarize the steps of the algorithm as Algorithm 2 below.

Algorithm 2 A full-Newton step IIPM based on Darvay directions

Input:
  accuracy parameter $\epsilon > 0$;
  barraier update parameter θ, $0 < \theta < 1$;
  and threshold parameter $0 < \tau \leq h < 1$ ($default\ h = \frac{1}{\sqrt{2}}$).
begin
  $x := \xi e;\ y := 0;\ s := \xi e;\ \mu := \mu^0 = \xi^2;\ \nu = 1;$
  while $\max\left(x^T s, \|b - Ax\|, \|c - A^T y - s\|\right) > \epsilon$ **do**
    feasibility step:
      $(x, s, y) := (x, s, y) + (\Delta^f x, \Delta^f s, \Delta^f y);$
    μ and $\nu-$update:
      $\mu := (1-\theta)\mu,\ \nu = (1-\theta)\nu;$
    centering step:
    while $\delta(x, s; \mu) \geq \tau$ **do**
      $(x, y, s) := (x, y, s) + (\Delta x, \Delta y, \Delta s)$
    end while
  end while
end.

4 Analysis of the Feasibility Step

First, we describe the feasibility step in detail. The analysis will follow in the sequel. Suppose that we have strictly feasible iterate (x, y, s) for (P_ν) and (D_ν). This means that (x, y, s) satisfies Eqs. 11 and 12 with $\mu = \nu\xi^2$. We need displacements $\Delta^f x$, $\Delta^f y$ and $\Delta^f s$ such that

$$x^f := x + \Delta^f x, \quad y^f := y + \Delta^f y, \quad s^f := s + \Delta^f s, \tag{14}$$

are feasible for (P_{ν^+}) and (D_{ν^+}). One may easily verify that (x^f, y^f, s^f) satisfies Eqs. 11 and 12, with ν replaced by ν^+, only if the first two equations in the following system are satisfied.

$$\begin{aligned} A\Delta^f x &= \theta\nu r_p^0, \\ A^T\Delta^f y + \Delta^f s &= \theta\nu r_d^0, \\ \frac{s}{\mu}\varphi'\left(\frac{xs}{\mu}\right)\Delta^f x + \frac{x}{\mu}\varphi'\left(\frac{xs}{\mu}\right)\Delta^f s &= \varphi(e) - \varphi\left(\frac{xs}{\mu}\right). \end{aligned} \tag{15}$$

The third equation is inspired by the third equation in the system (5) that we used to define search directions for the feasible case.

According to system (15), after the feasibility step the iterates satisfy the affine equations in Eqs. 11 and 12, with ν replaced by ν^+. The hard part in the analysis will be to guarantee that x^f, s^f are positive and to guarantee that the new iterate satisfies $\delta(x^f, s^f; \mu^+) \leq h < 1$, where h is an arbitrary constant.

Let (x, y, s) denote the iterate at the start of an iteration with $x^T s \leq n\mu$ and $\delta(x, s; \mu) \leq \tau$. At the start of the first iteration this is certainly true, because $(x^0)^T s^0 = n\mu^0$ and $\delta(x^0, s^0; \mu^0) = 0$. Defining

$$d_x^f := \frac{v\Delta^f x}{x}, \qquad d_s^f := \frac{v\Delta^f s}{s}, \tag{16}$$

one can easily check that the system (15), which defines the search directions $\Delta^f x$, $\Delta^f y$ and $\Delta^f s$, can be expressed in terms of the scaled search directions d_x^f and d_s^f as follows

$$\begin{aligned} \bar{A} d_x^f &= \theta\nu r_p^0, \\ \bar{A}^T \frac{\Delta^f y}{\mu} + d_s^f &= \theta\nu v s^{-1} r_d^0, \\ d_x^f + d_s^f &= p_v. \end{aligned} \tag{17}$$

Defining $\varphi(t) = t$ yields $p_v = v^{-1} - v$ which gives the classical search directions has been studied by Roos in [14]. Here, we let $\varphi(t) = \sqrt{t}$ which gives $p_v = 2(e - v)$, and propose an infeasible interior-point algorithm based on these search directions which is exactly an extension of feasible interior-point algorithm for LO by Darvay [4]. To get the search directions $\Delta^f x$ and $\Delta^f s$ in the original x and s-space we use Eq. 16, which gives

$$\Delta^f x = x v^{-1} d_x^f, \qquad \Delta^f s = s v^{-1} d_s^f.$$

The new iterates are obtained by taking a full step, as given by Eq. 14. Hence, we have

$$x^f = x + \Delta^f x = \frac{x}{v}\left(v + d_x^f\right),$$
$$s^f = s + \Delta^f s = \frac{s}{v}\left(v + d_s^f\right). \tag{18}$$

Moreover, from $p_v = 2(e - v)$, we have

$$v + \frac{1}{2}p_v = e \Longrightarrow v^2 + p_v v = e - \frac{1}{4}p_v^2. \tag{19}$$

Let us introduce the notation

$$\tilde{p}_v := d_x^f - d_s^f. \tag{20}$$

In that case, we have

$$d_x^f d_s^f = \frac{p_v^2 - \tilde{p}_v^2}{4}.$$

Using Eqs. 14, 16, 19 and 20, we get

$$\begin{aligned} x^f s^f &= xs + \left(x\triangle^f s + s\triangle^f x\right) + \triangle^f x \triangle^f s \\ &= \mu v^2 + \mu v\left(d_x^f + d_s^f\right) + \mu d_x^f d_s^f \\ &= \mu\left(v^2 + p_v v + d_x^f d_s^f\right) \\ &= \mu\left(e - \frac{1}{4}p_v^2 + \frac{1}{4}\left(p_v^2 - \tilde{p}_v^2\right)\right) \\ &= \mu\left(e - \frac{1}{4}\tilde{p}_v^2\right). \end{aligned} \tag{21}$$

Lemma 6 *The iterates* (x^f, y^f, s^f) *are strictly feasible if and only if* $\|\frac{\tilde{p}_v}{2}\|_\infty < 1$.

Proof Note that if x^f and s^f are positive then by Eq. 21 we have $e - \frac{1}{4}\tilde{p}_v^2 > 0$, this implies $\|\frac{\tilde{p}_v}{2}\|_\infty < 1$. For proving the converse of implication, we introduce a step length $\alpha \in [0, 1]$ and define

$$x^\alpha = x + \alpha\Delta^f x = \frac{x}{v}\left(v + \alpha d_x^f\right), \quad s^\alpha = s + \alpha\Delta^f s = \frac{s}{v}\left(v + \alpha d_s^f\right).$$

We thus have $x^0 = x, x^1 = x^f, s^0 = s, s^1 = s^f$ and $x^0 s^0 = xs > 0$. From Eqs. 7, 19 and the third equation in Eq. 17, we have

$$\begin{aligned} x^\alpha s^\alpha &= \frac{xs}{v^2}\left(v + \alpha d_x^f\right)\left(v + \alpha d_s^f\right) = \mu\left(v^2 + \alpha v\left(d_x^f + d_s^f\right) + \alpha^2 d_x^f d_s^f\right) \\ &= \mu\left(v^2 + \alpha v p_v + \frac{\alpha^2}{4}\left(p_v^2 - \tilde{p}_v^2\right)\right) \\ &= \mu\left((1-\alpha)v^2 + \alpha\left(e - \frac{p_v^2}{4}\right) + \frac{\alpha^2}{4}\left(p_v^2 - \tilde{p}_v^2\right)\right). \end{aligned}$$

If $\|\frac{\tilde{p}_v}{2}\|_\infty < 1$, then we have

$$0 < e - \frac{1}{4}\tilde{p}_v^2 = e - \frac{1}{4}p_v^2 + \frac{1}{4}\left(p_v^2 - \tilde{p}_v^2\right),$$

and this implies that

$$\frac{1}{4}\left(p_v^2 - \tilde{p}_v^2\right) > -e + \frac{1}{4}p_v^2. \tag{22}$$

Therefore, we get

$$\begin{aligned} x^\alpha s^\alpha &> \mu\left((1-\alpha)v^2 + \alpha\left(e - \frac{p_v^2}{4}\right) + \alpha^2\left(-e + \frac{1}{4}p_v^2\right)\right) \\ &= \mu\left((1-\alpha)v^2 + \alpha(1-\alpha)\left(e - \frac{p_v^2}{4}\right)\right) \\ &= \mu\left((1-\alpha)v^2 + \alpha(1-\alpha)\left(2v - v^2\right)\right) \\ &= \mu(1-\alpha)\left((1-\alpha)v^2 + 2\alpha v\right). \end{aligned} \tag{23}$$

Since $(1-\alpha)\mu\left[(1-\alpha)v^2 + 2\alpha v\right] \geq 0$, it follows that $x^\alpha s^\alpha > 0$, for $\alpha \in [0, 1]$. Hence, none of the entries of x^α and s^α vanishes, for $\alpha \in [0, 1]$. Since x^0 and s^0 are positive, and x^α and s^α depend linearly on α, this implies that $x^\alpha > 0$ and $s^\alpha > 0$ for $\alpha \in [0, 1]$. Hence, x^1 and s^1 must be positive, proving the 'if' part of the statement in the lemma. □

In the sequel, we denote

$$w_i(v) := \frac{1}{2}\sqrt{|d_{x_i}^f|^2 + |d_{s_i}^f|^2},$$

and

$$w(v) := ||(w_1(v), \ldots, w_n(v))||.$$

This implies $||d_x^f|| \leq 2w(v)$ and $||d_s^f|| \leq 2w(v)$. Moreover,

$$|\left(d_x^f\right)^T d_s^f| \leq ||d_x^f||||d_s^f|| \leq \frac{1}{2}\left(||d_x^f||^2 + ||d_s^f||^2\right) = 2w(v)^2, \tag{24}$$

and for $i = 1, \ldots, n$,

$$|d_{x_i}^f d_{s_i}^f| = |d_{x_i}^f||d_{s_i}^f| \leq \frac{1}{2}\left(|d_{x_i}^f|^2 + |d_{s_i}^f|^2\right) = 2\omega_i(\upsilon)^2 \leq 2\omega(\upsilon)^2. \tag{25}$$

We proceed by deriving an upper bound for $\delta(x^f, s^f, \mu^+)$. Recall from definition (9) that

$$\delta(x^f, s^f; \mu^+) := \delta(\upsilon^f) = \|e - \upsilon^f\|, \quad \text{where } \upsilon^f = \sqrt{\frac{x^f s^f}{\mu^+}}. \tag{26}$$

Lemma 7 *If* $\|\frac{\tilde{p}_\upsilon}{2}\|_\infty < 1$, *then one has*

$$\delta(\upsilon^f) \leq \frac{\delta^2 + 2\omega(\upsilon)^2 + \theta\sqrt{n}}{1 - \theta + \sqrt{(1-\theta)(1 - \delta^2 - 2\omega(\upsilon)^2)}}.$$

Proof Using Eq. 21, after division both sides by $\mu^+ = (1-\theta)\mu$, we get

$$\left(\upsilon^f\right)^2 = \frac{e - \frac{1}{4}\tilde{p}_\upsilon^2}{1-\theta}. \tag{27}$$

On the other hand side, we have

$$\begin{aligned}\delta\left(\upsilon^f\right)^2 = \|e - \upsilon^f\|^2 &= \sum_{i=1}^n \left(1 - \upsilon_i^f\right)^2 = \sum_{i=1}^n \left(1 - \upsilon_i^f\right)^2 \frac{\left(1 + \upsilon_i^f\right)^2}{\left(1 + \upsilon_i^f\right)^2} \\ &\leq \frac{1}{\left(1 + \min_i \upsilon_i^f\right)^2} \sum_{i=1}^n \left(1 - \left(\upsilon_i^f\right)^2\right)^2. \end{aligned} \tag{28}$$

For each i, by Eq. 27, we have

$$\begin{aligned}\min_i \left(\upsilon_i^f\right)^2 = \min_i \frac{1 - \frac{1}{4}\left(\tilde{p}_\upsilon\right)_i^2}{1-\theta} &\geq \frac{1}{1-\theta}\left(1 - \frac{1}{4}\|\tilde{p}_\upsilon\|_\infty^2\right) \\ &\geq \frac{1}{1-\theta}\left(1 - \frac{1}{4}\|\tilde{p}_\upsilon\|^2\right) \\ &= \frac{1}{1-\theta}\left(1 - \frac{1}{4}\|p_\upsilon\|^2 + \left(d_x^f\right)^T d_s^f\right) \\ &\geq \frac{1}{1-\theta}\left(1 - \frac{1}{4}\|p_\upsilon\|^2 - 2\omega(\upsilon)^2\right) \\ &= \frac{1}{1-\theta}\left(1 - \delta^2 - 2\omega(\upsilon)^2\right). \end{aligned} \tag{29}$$

The last inequality follows from Eq. 24 and $\|p_v\|^2 = \|\tilde{p}_v\|^2 + 4(d_x^f)^T d_s^f$. By substitution Eq. 29 into Eq. 28 and using Eqs. 24 and 27, we get

$$\begin{aligned}
\delta\left(v^f\right)^2 &\le \frac{1-\theta}{\left(\sqrt{1-\delta^2-2w(v)^2}+\sqrt{1-\theta}\right)^2}\sum_{i=1}^{n}\left(\frac{\frac{1}{4}\left(\tilde{p}_v\right)_i^2-\theta}{1-\theta}\right)^2 \\
&= \frac{1}{(1-\theta)\left(\sqrt{1-\delta^2-2w(v)^2}+\sqrt{1-\theta}\right)^2}\sum_{i=1}^{n}\left(\frac{1}{4}\left(\tilde{p}_v\right)_i^2-\theta\right)^2 \\
&\le \frac{1}{\left(\sqrt{(1-\theta)\left(1-\delta^2-2w(v)^2\right)}+1-\theta\right)^2}\left(\left(\frac{\|\tilde{p}_v\|}{2}\right)^2+\theta\sqrt{n}\right)^2 \\
&= \frac{1}{\left(\sqrt{(1-\theta)\left(1-\delta^2-2w(v)^2\right)}+1-\theta\right)^2}\left(\left(\frac{\|p_v\|}{2}\right)^2-\left(d_x^f\right)^T d_s^f+\theta\sqrt{n}\right)^2 \\
&\le \frac{1}{\left(\sqrt{(1-\theta)\left(1-\delta^2-2w(v)^2\right)}+1-\theta\right)^2}\left(\delta^2+2w(v)^2+\theta\sqrt{n}\right)^2. \qquad (30)
\end{aligned}$$

This completes the proof. □

Because we need to have $\delta(v^f) < 1$, it follows from Lemma 7 that it suffices to have

$$\frac{\delta^2+2w(v)^2+\theta\sqrt{n}}{1-\theta+\sqrt{(1-\theta)\left(1-\delta^2-2w(v)^2\right)}} < 1. \qquad (31)$$

At this stage, we decide to choose

$$\tau = \frac{1}{8}, \quad \theta = \frac{\alpha}{2\sqrt{n}}, \quad \alpha \le 1. \qquad (32)$$

The left-hand side of Eq. 31 is monotonically increasing with respect to $w(v)^2$, then for $n \ge 1$ and $\delta(v) \le \tau$, one can verify that

$$w(v) \le \frac{1}{2\sqrt{2}} \Rightarrow \delta(v^f) \le \frac{1}{\sqrt{2}} < 1. \qquad (33)$$

4.1 Upper Bound for $w(v)$

In this section, we want to find an upper bound for $w(v)$, then we can find a default value for θ. For this purpose, consider the system (17). Let us define

$$D := X^{\frac{1}{2}} S^{-\frac{1}{2}},$$

where $X^{\frac{1}{2}}$ denotes the symmetric square root of X. From definition of $\bar{A}$, we deduce that $\bar{A} = AV^{-1}X = \sqrt{\mu}AD$. Therefore, by eliminating d_s^f from the system (17), we obtain

$$\begin{aligned}\sqrt{\mu}ADd_x^f &= \theta\nu r_p^0,\\ -(\sqrt{\mu}AD)^T\frac{\Delta^f y}{\mu} + d_x^f &= p_v - \theta\nu v s^{-1} r_d^0.\end{aligned} \tag{34}$$

Let $(\bar{x}, \bar{y}, \bar{s})$ be such that $A\bar{x} = b$ and $A^T\bar{y} + \bar{s} = c$. Then, we have

$$r_p^0 = A(\bar{x} - x^0), \;\; r_d^0 = A^T(\bar{y} - y^0) + \bar{s} - s^0. \tag{35}$$

Substituting system (35) into Eq. 34 after some computations, we obtain (for more details see [14])

$$\|d_x^f\|^2 + \|d_s^f\|^2 \le 2\|p_v\|^2 + \frac{3\theta^2\nu^2}{\mu}\left(\|D^{-1}(\bar{x} - x^0)\|^2 + \|D(\bar{s} - s^0)\|^2\right).$$

On the other hand, we have

$$\|p_v\| = \|2(e - v)\| = 2\delta(v).$$

Therefore, we have

$$\|d_x^f\|^2 + \|d_s^f\|^2 \le 8\delta(v)^2 + \frac{3\theta^2\nu^2}{\mu}\left(\|D^{-1}(\bar{x} - x^0)\|^2 + \|D(\bar{s} - s^0)\|^2\right). \tag{36}$$

Taking $\bar{x} = x^*$ and $\bar{s} = s^*$, by Eq. 10 and considering the initial iterate (x^0, y^0, s^0), we have

$$0 \le x^0 - \bar{x} \le \xi e, \quad 0 \le s^0 - \bar{s} \le \xi e.$$

Thus, it follows that

$$\begin{aligned}\|D^{-1}\left(\bar{x} - x^0\right)\|^2 + \|D\left(\bar{s} - s^0\right)\|^2 &\le \xi^2\left(\|De\|^2 + \|D^{-1}e\|^2\right)\\ &= \xi^2 e^T\left(\frac{x}{s} + \frac{s}{x}\right) = \xi^2 e^T\left(\frac{x^2 + s^2}{xs}\right)\\ &\le \xi^2\frac{e^T\left(x^2 + s^2\right)}{\mu\min_i\left(v_i^2\right)} \le \frac{\left(e^T(x + s)\right)^2}{\nu\left(1 - \delta(v)\right)^2},\end{aligned} \tag{37}$$

where, the last inequality follows from Eq. 9, $\mu = \nu\xi^2$ and $a^2 + b^2 \le (a + b)^2$ for $a, b \ge 0$.

The proof of the following lemma is the same as the proof of Lemma 4.3 in [5].

Lemma 8 *Let x and (y, s) be feasible for (P_ν) and (D_ν) respectively, with ξ as defined in Eq. 10 and $(x^0, y^0, s^0) = (\xi e, 0, \xi e)$. We then have*

$$e^T(x + s) \leq 2n\xi. \tag{38}$$

Substituting Eqs. 38 and 37 into Eq. 36, we obtain

$$\|d_x^f\|^2 + \|d_s^f\|^2 \leq 8\delta(v)^2 + \frac{12\theta^2 n^2}{(1-\delta(v))^2}. \tag{39}$$

4.2 Value for θ

At this stage, we choose $\tau = \frac{1}{8}$. Since $\delta(v) \leq \tau = \frac{1}{8}$ and the right-hand-side of Eq. 39 is monotonically increasing in $\delta(v)$, we have

$$\|d_x^f\|^2 + \|d_s^f\|^2 \leq \frac{1}{8} + \frac{768\theta^2 n^2}{49}.$$

Using $\theta = \frac{\alpha}{2\sqrt{n}}$, the above relation becomes

$$\|d_x^f\|^2 + \|d_s^f\|^2 \leq \frac{1}{8} + \frac{192n\alpha^2}{49}. \tag{40}$$

From Eq. 33 we know that $w(v) \leq \frac{1}{2\sqrt{2}}$ is needed in order to have $\delta(v^f) \leq \frac{1}{\sqrt{2}}$. Due to Eq. 40, this will hold if

$$\frac{1}{8} + \frac{192n\alpha^2}{49} \leq \frac{1}{2}.$$

If we take

$$\alpha = \frac{1}{\sqrt{11n}}, \tag{41}$$

the above inequality is satisfied. Moreover,

$$\begin{aligned}\|\tilde{p}_v\|_\infty^2 &\leq \|d_x^f - d_s^f\|^2 = \|d_x^f\|^2 + \|d_s^f\|^2 - 2(d_x^f)^T d_s^f \\ &< \frac{1}{2} - 2(d_x^f)^T d_s^f \leq \frac{1}{2} + 4w(v)^2 \leq \frac{1}{2} + \frac{1}{2} = 1,\end{aligned}$$

which, by Lemma 6, means that (x^f, y^f, s^f) is strictly feasible. Thus, we have found a desired update parameter θ.

4.3 Complexity Analysis

We have seen that if at the start of an iteration the iterate satisfies $\delta(x, s; \mu) \leq \tau$, with $\tau = \frac{1}{8}$, then after the feasibility step, with θ as defined in Eq. 32 and α as in Eq. 41, the iterate is strictly feasible and satisfies $\delta(x^f, s^f; \mu^+) \leq h < 1$.

After the feasibility step, we perform a few centering steps in order to get the iterate (x^+, y^+, s^+) which satisfies $\delta(x^+, s^+; \mu^+) \leq \tau$. By Lemma 3, after k centering steps we will have the iterate (x^+, y^+, s^+) that is still feasible for (P_{ν^+}) and (D_{ν^+}) and satisfies

$$\delta\left(x^+, s^+; \mu^+\right) \leq h^{2^k}.$$

From this, one easily deduces that $\delta(x^+, s^+; \mu^+) \leq \tau$ will hold after at most

$$\left\lceil \log_2 \frac{\log_2 \tau}{\log_2 h} \right\rceil, \tag{42}$$

centering steps. So, each main iteration consists of at most $1 + \left\lceil \log_2 \frac{\log_2 \tau}{\log_2 h} \right\rceil$ so-called inner iterations. Recall the value of τ from Eq. 32. According to Eq. 33 we may take $h = \frac{1}{\sqrt{2}}$, so the number of inner iterations in each main iteration is at most 4.

In each main iteration both the duality gap and the norms of the residual vectors are reduced by the factor $1 - \theta$. Hence, the total number of main iterations is bounded above by

$$\frac{1}{\theta} \log \frac{\max\left\{n\xi^2, \|r_p^0\|, \|r_d^0\|\right\}}{\epsilon}.$$

Due to Eqs. 32, 41 and the fact that we need at most 4 inner iterations per main iteration, we may state the main result of the paper.

Theorem 1 *If* (P) *and* (D) *are feasible and* $\xi > 0$ *is such that* $\|x^* + s^*\|_\infty \leq \xi$ *for some optimal solution* x^* *of* (P) *and* (y^*, s^*) *of* (D), *then after at most*

$$8\sqrt{11}n \log \frac{\max\left\{n\xi^2, \|r_p^0\|, \|r_d^0\|\right\}}{\epsilon},$$

inner iterations, the algorithm finds an ϵ*-optimal solution of* (P) *and* (D).

5 Numerical Results

In this section, we perform the proposed primal-dual infeasible interior-point algorithm (with an accuracy $\epsilon = 10^{-5}$) on a number of problems in the NETLIB test set for linear optimization. Numerical results were obtained by using MATLAB R2009a.

Table 1 Computational performance of the proposed algorithm

LP	Primal optimal value	Dual optimal value	Duality gap
afiro	−4.647531427504551e+002	−4.647531437146635e+002	9.642084251026972e−007
sc50a	−69.999997121037651	−70.000003851179571	6.730141919319976e−006
blend	−30.812151077217347	−30.812151304491149	2.272738015562936e−007
sc105	−52.202057234217044	−52.202063280200463	6.045983418800915e−006
adlittle	2.337683536553557e+005	2.337683549762290e+005	−0.001320873328950
kb2	3.941595241384492e−007	0	3.941595241384492e−007
stocfor1	−4.113197622387252e+004	−4.113197621904278e+004	−4.829744284506887e−006
scagr7	−2.331389826401264e+006	−2.331389823224168e+006	−0.003177095204592
share1b	−7.658931857924126e+004	−7.658931857353629e+004	−5.704961949959397e−006
share2b	−4.157322418941577e+002	−4.157322408644903e+002	−1.029667373586563e−006
sc50b	−69.999997121037651	−70.000003851179571	6.730141919319976e−006

Table 1 lists the names of the test problems, the primal optimal value $c^T x$, the dual optimal value $b^T y$ and the relative duality gap.

Based on the numerical results we have listed in the Table 1, our algorithm is reliable in terms of optimal values.

6 Concluding Remarks

We presented a full-Newton step IIPM based on Darvay search directions for linear optimization. Some good properties of our algorithm are: i) the step length need not be calculated because we have full steps, ii) both feasibility and optimality are improved at the same rate, iii) the iterates lie in the quadratic convergence neighborhood with respect to perturbed problems, iv) each main iteration of our algorithm consists of a feasibility step and at most 3 centering steps. The iteration bound obtained for this algorithm coincides with the best known bound for IIPMs.

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