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Dijkstra's Algorithm for Solving the Shortest Path Problem on Networks Under Intuitionistic Fuzzy Environment

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Abstract In this paper, a well known problem called the Shortest Path Problem (SPP) has been considered in an uncertain environment. The cost parameters for traveling each arc have been considered as Intuitionistic Fuzzy Numbers (IFNs) which are the more generalized form of fuzzy numbers involving a degree of acceptance and a degree of rejection. A heuristic methodology for solving the SPP has been developed, which aim to exploit tolerance for imprecision, uncertainty and partial truth to achieve tractability, robustness and low cost solution corresponding to the minimum-cost path or the shortest path. The Modified Intuitionistic Fuzzy Dijkstra's Algorithm (MIFDA) has been proposed in this paper for solving Intuitionistic Fuzzy Shortest Path Problem (IFSPP) using the Intuitionistic Fuzzy Hybrid Geometric (IFHG) operator. A numerical example illustrates the effectiveness of the proposed method.

Keywords Shortest path problem · Intuitionistic fuzzy sets (IFSs) · Intutionistic fuzzy value (IFV) · Intuitionistic fuzzy numbers (IFNs) · Intutionistic fuzzy hybrid geometric (IFHG) operator · Decision making problem · Dijkstra's algorithm

Mathematics Subject Classifications (2010) 05C85 · 05C22 · 05C30 · 90B10 · 68T20 · 68T30 · 68T37 · 03E72 · 47S40 · 90B50

1 Introduction

Shortest path problem where the costs have vague values is one of the most studied problems in fuzzy sets and systems area [23]. The authors in [27, 33, 50] outlined a

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model and an algorithm for computing a shortest path in a network having various types of fuzzy arc lengths by a dynamic programming method. Variation of Shortest path problem can be found in the papers [18, 43]. Shortest Path Problem with fuzzy arc lengths have been solved by different methods by many authors [13, 14, 22, 23, 25–27, 29, 31, 36, 39–41]. However, it seems that in the literature there is no investigation on SPP with data in the form of intuitionistic fuzzy numbers which is a more generalized form of fuzzy number. The major objective of this paper is to bridge this gap, by posing and proposing a methodology for solving the SPP with IF arc parameters, applying the different properties of IFSs.

Different operators for aggregating fuzzy numbers have been studied by many authors. The authors in the papers [21, 59] present a wide range of fuzzy induced generalized aggregation operators. Many authors have worked on aggregation operators of IFNs. In the paper [54], Xu and Yager have developed some geometric aggregation operators, such as the intuitionistic fuzzy weighted geometric (IFWG) operator, the intuitionistic fuzzy ordered weighted geometric (IFOWG) operator, and the intuitionistic fuzzy hybrid geometric (IFHG) operator. A new aggregation operator called induced generalized intuitionistic fuzzy ordered weighted averaging (IG-IFOWA) operator is proposed in the paper [45]. The authors in [57] develop a series of operators for aggregating IFNs. Two new aggregation operators: induced intuitionistic fuzzy ordered weighted geometric (I-IFOWG) operator and induced interval-valued intuitionistic fuzzy ordered weighted geometric (I-IIFOWG) operator are proposed in the paper [20].

The concept of IFS can be viewed as an appropriate/alternative approach to define a fuzzy set in case where available information is not sufficient for the definition of an imprecise concept by means of a conventional fuzzy set [58]. In fuzzy sets the degree of acceptance is considered only but IFS is characterized by a membership function and a non-membership function so that the sum of both values is less than or equal to one. The grade of a membership function indicates a subjective degree of preference of a decision maker within a given tolerance and grade of a non-membership function indicates a subjective degree of negative response of a decision maker within a given tolerance.

Presently intuitionistic fuzzy sets are being studied and used in different fields of science. Using this concept many authors have solved many practical problems [1–8, 24, 30, 44, 46–49, 53–57]. Burillo et al. [10] proposed definition of intuitionistic fuzzy number and studied perturbations of intuitionistic fuzzy number and the first properties of the correlation between these numbers. The intuitionistic fuzzy set has received more and more attention since its appearance. Gau and Buehrer [19] introduced the concept of vague set. Chen and Tan [12] and Hong and Choi [24] presented some techniques for handling multicriteria fuzzy decision making problems based on vague sets. But Bustince and Burillo [11] showed that vague sets are intuitionistic fuzzy sets. Mitchell [35] considered the problem of ranking a set of IFNs to define a fuzzy rank and a characteristic vagueness factor for each IFN.

Dijkstra's algorithm, conceived by Dutch computer scientist Edsger Dijkstra in 1956 and published in 1959, [15, 16] is a graph search algorithm that solves the single-source shortest path problem for a graph with nonnegative edge path costs, producing a shortest path tree. This algorithm is often used in routing and as a subroutine in other graph algorithms. Many authors have studied and worked on Dijkstra's algorithm for solving shortest path problems [9, 17, 28, 32, 34, 37, 38, 42, 51, 52].

In this paper a SPP has been considered in an uncertain (IF) environment. The costs (or time or distance etc.) required for traveling along the arcs of the network are taken as the IFNs. The problem is to find the shortest path between two nodes. This shortest path corresponds to that path which requires minimum total cost (or time or distance etc.) of traveling from the source to the sink. Minimum total IF cost corresponds to maximum aggregate degree of acceptance μ_{ij} and the minimum aggregate degree of rejection v_{ij} of the cost. IFHG operator has been used for deducing the weighted aggregated IF value of two IFNs. In this paper, a modified Dijkstra's method has been developed in the Intuitionistic Fuzzy environment for finding the weighted aggregated IFV of the minimum-cost path or the shortest path. The shortest path can be easily constructed by working backwards from the terminal vertex such that we go to that predecessor from whom the current vertex has got its permanent label. Shortest Path Problem with fuzzy arc lengths has been solved by different methods by many authors. However, it seems that in the literature there is no investigation on SPP with data in the form of IFNs, which is more generalized form of fuzzy numbers. This depicts the major contribution of this paper. The proposed new methodology for solving IFSPP cannot be found in the literature so far. The properties applied here have not been applied earlier for solving IFSPP. These are the main significances of this paper.

The rest of the paper is organized as follows. Section 2 depicts the preliminary concepts of IFSs, IF value, IFNs, ranking of IF value and IFHG operator. Section 3 describes the proposed method. In Section 4, a numerical example has been solved using the proposed method, Section 5 depicts the results and discussions and Section 6 concludes the paper.

2 Some Preliminary Concepts

2.1 Intuitionistic Fuzzy Set (IFS)

Atanassov [2] generalized the concept of fuzzy set [58], and defined the concept of intuitionistic fuzzy set as follows.

Let a set X be fixed. An intuitionistic fuzzy set A in X is an object having the form:

$$A = \{\langle x, \mu_A(x), \nu_A(x)\rangle \, | x \in X\} \tag{1}$$

where the functions $\mu_A : X \to [0, 1]$ and $\nu_A : X \to [0, 1]$ define the degree of membership and the degree of non-membership of the element $x \in X$ to $A \subset X$, respectively, and for every $x \in X$, $\mu_A + \nu_A(x) \leq 1$.

2.2 Intuitionistic Fuzzy Value (IFV)

The definition of Intuitionistic Fuzzy Value has been given by Xu [54].

If a membership function t_A and a non-membership function f_A be used to denote the lower bounds on μ_A, then, the degree of membership of x in the intuitionistic fuzzy set A is bounded to a subinterval $[t_A(x), 1-f_A(x)]$ of [0, 1]. Gau and Buehrer [19] called the interval $[t_A(x), 1-f_A(x)]$, a vague value. However, Bustince and

Burillo [11] showed that vague sets are intuitionistic fuzzy sets. For computational convenience, in this paper, the interval $[t_A(x), 1-f_A(x)]$ is called an intuitionistic fuzzy value, and replaces Eq. 1 with

$$A = \left\{ \left\langle x, \left[t_A(x), 1 - f_A(x) \right] \right\rangle | x \in X \right\} \tag{2}$$

correspondingly.

The intuitionistic fuzzy value $[t_A(x)$, 1-$f_A(x)]$ indicates that the exact degree of membership $\mu_A(x)$ of x may be unknown. But it is bounded by $t_A(x) \leq \mu_A(x) \leq 1 - f_A(x)$ where $t_A(x) + f_A(x) \leq 1$.

2.3 Definition: Intuitionistic Fuzzy Number (IFN)

IFN was introduced by Burillo et al. [10]. According to them an IFN is an intuitionistic fuzzy subset of the real line, normal, convex for the membership and concave for the non-membership. With this idea every fuzzy number is an IFN. The definition of IFN is given below

An intuitionistic fuzzy number $\widetilde{A}^i$ is defined as follows:

(i) an intuitionistic fuzzy sub set of the real line
(ii) normal i.e., there is any $x_0 \in R$ such that $\mu_{\widetilde{A}^i}(x_0)$=1 $\left(\text{so}\, \nu_{\widetilde{A}^i}(x_0) = 0\right)$
(iii) a convex set for the membership function $\mu_{\widetilde{A}^i}(x)$ i.e.,

$$\mu_{\widetilde{A}^i}(\lambda x_1 + (1-\lambda)x_2) \geq \min\left(\mu_{\widetilde{A}^i}(x_1), \mu_{\widetilde{A}^i}(x_2)\right) \forall x_1, x_2 \in R, \lambda \in [0, 1]$$

(iv) a concave set for the non-membership function $\nu_{\widetilde{A}^i}(x)$ i.e.,

$$\nu_{\widetilde{A}^i}(\lambda x_1 + (1-\lambda)x_2) \leq \max\left(\nu_{\widetilde{A}^i}(x_1), \nu_{\widetilde{A}^i}(x_2)\right) \forall x_1, x_2 \in R, \lambda \in [0, 1]$$

2.4 Ranking of Intuitionistic Fuzzy Value

Chen and Tan [12] introduced a score function S of an IFV, which is represented as follows.

Let $\widetilde{a} = \left[t_{\widetilde{a}}, 1 - f_{\widetilde{a}}\right]$, be an intuitionistic fuzzy value, where

$$t_{\widetilde{a}} \in [0, 1],\ f_{\widetilde{a}} \in [0, 1], t_{\widetilde{a}_j} + f_{\widetilde{a}_j} \leq 1$$

The score of $\widetilde{a}$ can be evaluated by the score function S shown as:

$$S(\widetilde{a}) = t_{\widetilde{a}} - f_{\widetilde{a}} \tag{3}$$

where $S(\widetilde{a}) \in [-1, 1]$. The larger the score $S(\widetilde{a})$, the greater the intuitionistic fuzzy value $\widetilde{a}$.

Hong and Choi [24] defined an accuracy function H to evaluate the degree of accuracy of the intuitionistic fuzzy value $\widetilde{a}$ where $H(\widetilde{a}) \in [0, 1]$.

$$H(\widetilde{a}) = t_{\widetilde{a}} + f_{\widetilde{a}} \tag{4}$$

is the accuracy degree of $\widetilde{a}$ where $H(\widetilde{a}) \in [0, 1]$. The larger the value of $H(\widetilde{a})$, the more is the degree of accuracy of the degree of membership of the IFV $\widetilde{a}$.

Based on the score function S and the accuracy degree H, in the following, Xu and Yager [54] give an order relation between two IFVs, which is defined as follows:

Let $\widetilde{a} = \left[t_{\widetilde{a}}, 1 - f_{\widetilde{a}}\right]$ and $\widetilde{b} = \left[t_{\widetilde{b}}, 1 - f_{\widetilde{b}}\right]$ be two IFVs, $S\left(\widetilde{a}\right) = t_{\widetilde{a}} - f_{\widetilde{a}}$ and $S\left(\widetilde{b}\right) = t_{\widetilde{b}} - f_{\widetilde{b}}$ be the scores of $\widetilde{a}$ and $\widetilde{b}$ respectively, and let $H\left(\widetilde{a}\right) = t_{\widetilde{a}} + f_{\widetilde{a}}$ and $H\left(\widetilde{b}\right) = t_{\widetilde{b}} + f_{\widetilde{b}}$ be the accuracy degrees of $\widetilde{a}$ and $\widetilde{b}$ respectively, then

- If $S\left(\widetilde{a}\right) < S\left(\widetilde{b}\right)$, then $\widetilde{a}$ is smaller than $\widetilde{b}$, denoted by $\widetilde{a} < \widetilde{b}$ (5)
- If $S\left(\widetilde{a}\right) > S\left(\widetilde{b}\right)$, then $\widetilde{a}$ is greater than $\widetilde{b}$, denoted by $\widetilde{a} > \widetilde{b}$ (6)
- If $S\left(\widetilde{a}\right) = S\left(\widetilde{b}\right)$, then
  1. if $H\left(\widetilde{a}\right) = H\left(\widetilde{b}\right)$, then $\widetilde{a}$ and $\widetilde{b}$ represent the same information, denoted by $\widetilde{a} = \widetilde{b}$
  2. if $H\left(\widetilde{a}\right) < H\left(\widetilde{b}\right)$, then $\widetilde{a}$ is smaller than $\widetilde{b}$, denoted by $\widetilde{a} < \widetilde{b}$
  3. if $H\left(\widetilde{a}\right) > H\left(\widetilde{b}\right)$, then $\widetilde{a}$ is greater than $\widetilde{b}$, denoted by $\widetilde{a} > \widetilde{b}$.

2.5 The IFHG Operator

The definition of IFHG operator was introduced by Xu in [54]. An IFHG operator is a mapping IFHG: $\Omega^n \to \Omega$, which has an associated vector $w = (w_1, w_2, w_3, \ldots, w_n)^T$ with $w_j > 0, \sum_{j=1}^{n} w_j = 1$ such that

$$IFHG_{\omega,w}\left(\widetilde{a}_1, \widetilde{a}_2, \ldots, \widetilde{a}_n\right) = \left(\dot{\widetilde{a}}_{\sigma(1)}\right)^{w_1} \otimes \left(\dot{\widetilde{a}}_{\sigma(2)}\right)^{w_2} \otimes \ldots \otimes \left(\dot{\widetilde{a}}_{\sigma(n)}\right)^{w_n}$$

where $\dot{\widetilde{a}}_{\sigma(j)}$ is the jth largest of the weighted intuitionistic fuzzy values $\dot{\widetilde{a}}_j$

$$\left(\dot{\widetilde{a}}_j = \widetilde{a}_j^{n\omega_j}, \; j = 1, 2, 3, \ldots, n\right), \tag{7}$$

$\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$ is the weight vector of $\widetilde{a}_j\,(j = 1, 2, \ldots.., n)$, with $\omega_j > 0$, $\sum_{j=1}^{n} \omega_j = 1$, and n is the balancing coefficient, which plays a role of balance.

Xu [53] developed a normal distribution based method for determining the associated weights $w = (w_1, w_2, w_3, \ldots, w_n)^T$ with $\omega_j > 0$, $\sum_{j=1}^{n} w_j = 1$ of the IFHG operator, which is defined as follows:

$$w_j = \frac{e^{-\frac{(j-\mu_n)^2}{2\sigma_n^2}}}{\sum_{i=1}^{n} e^{-\frac{(i-\mu_n)^2}{2\sigma_n^2}}}, \quad j = 1, 2, \ldots, n. \tag{8}$$

where μ_n is the mean of the collection of $1, 2, 3, \ldots, n$, and $\sigma_n (\sigma_n > 0)$ is the standard deviation of the collection of $1, 2, 3, \ldots, n$, i.e,

$$\mu_n = \frac{1}{n}\frac{n(n+1)}{2} = \frac{n+1}{2}, \quad \sigma_n = \sqrt{\frac{1}{n}\sum_{i=1}^{n}(i - \mu_n)^2} \tag{9}$$

The prominent characteristic of the method is that it can relieve the influence of unfair arguments on the final results by assigning low weights to those "false" or "biased" ones.

Let $\dot{\widetilde{a}}_{\sigma(j)} = \left[t_{\dot{\widetilde{a}}_{\sigma(j)}}, 1 - f_{\dot{\widetilde{a}}_{\sigma(j)}} \right]$ then we have

$$IFHG_{\omega,w}\left(\widetilde{a}_1, \widetilde{a}_2, \ldots, \widetilde{a}_n\right) = \left[\prod_{j=1}^{n} t_{\dot{\widetilde{a}}_{\sigma(j)}}^{w_j}, \prod_{j=1}^{n} \left(1 - f_{\dot{\widetilde{a}}_{\sigma(j)}}\right)^{w_j} \right] \tag{10}$$

and the aggregated value derived by using the IFHG operator is also an intuitionistic fuzzy value.

3 The Proposed Method

A connected network with given arcs and nodes in which s is the source node and e is the end (sink) node is considered. The problem is to find the shortest path between s and e with respect to the cost (or time or distance etc.) parameter related to each arc. This parameter is considered to be in terms of IFNs $\widetilde{c}_{ij} = \left(\mu_{ij}, \nu_{ij}\right)$, where μ_{ij} represents the degree of acceptance that the arc $i - j$ will be included in the shortest path with respect to the cost for traveling along the arc $i - j$. Similarly ν_{ij} represents the degree of rejection that the arc $i - j$ will be included in the shortest path with respect to the cost for traveling along the arc $i - j$. From the IFSs, the IF value for each arc of the network is considered in the form of $[t_A(x), 1\text{-}f_A(x)]$ described earlier in Section 2.2. The weight vector $\omega = \left(\omega_{12}, \omega_{13}, \ldots, \omega_{23}, \omega_{24}, \ldots, \omega_{ij}, \ldots, \omega_n\right)^T$, where each ω_{ij} related to arc $i - j$ is considered as the opinion of the expert regarding the IF cost $\widetilde{c}_{ij}$ required for traveling along the arc $i - j$, $0 \leq \omega_j \leq 1$.

3.1 The Modified Intuitionistic Fuzzy Dijkstra's Algorithm (MIFDA)

Input: Let G = (V, E) be a simple weighted network with intuitionistic fuzzy parameters for the arcs. 's' is the starting point and 'e' is the terminal point.

Output: (a) Weighted aggregated IFV of the minimum-cost path or the shortest path w.r.t. the total intuitionistic fuzzy cost for traveling through the shortest path.
(b) the shortest path.

Let $L(x)$ denote the label of the node 'x' which represents the weighted aggregated IFV for the path from the node 's' to the node 'x'. The weight vector $\omega = \left(\omega_{12}, \omega_{13}, \ldots, \omega_{23}, \omega_{24}, \ldots, \omega_{ij}, \ldots, \omega_n\right)^T$, where each ω_{ij} related to arc $i - j$ is considered as the opinion of the expert regarding the IF cost $\widetilde{c}_{ij}$ required for traveling along the arc $i - j$, $0 \leq \omega_j \leq 1$.

Step1: Let $P = \phi$, where P is the set of those nodes which have permanent labels and $T = \{$all nodes of the network G$\}$. At first, the permanent label to 's' has been assigned as $L(s) = (1, 0)$, (initially),'s' is the starting node, so definitely it will be present in the shortest path. This is represented by the IFN (1, 0) where 1 represents the degree of acceptance and 0 represents the degree of rejection of the fact that node 's' is in the shortest path.

Also, $L(x) = (0, 1) \forall x \in T$ and $x \neq$ ‘s’

Step2: That node ‘v’ in T is selected which has the highest score value of its label, called the permanent label of ‘v’ (i.e., $L(v)$). Then $P = P \cup \{v\}$ and $T = T - (v)$. Again the node in T with highest score value of its label is selected. The new label of a node‘x’ in T is given by

$$L\left(x\right) = \max\left\{\text{old } L\left(x\right), \ \text{IFHG}\left(L\left(v\right), \ c\left(v, x\right)\right)\right\}, \tag{11}$$

where $c(v, x)$ is the IF cost for traveling along the arc $v - x$.

The associated weights $w_j > 0$ of the IFHG operator are evaluated using Eqs. 8 and 9 where $\sum_{j=1}^{n} w_j = 1$. Then using the IFHG operator of the Eq. 10 of Section 2.5 described in this paper, the aggregated value of $L(v)$and $c(v, x)$ are derived which are also in terms of intuitionistic fuzzy value. The max function is used for evaluating the maximum of the two IFNs using the Eqs. 3, 5 and 6. If there is no difference between two scores, then the accuracy degrees are calculated by using Eq. 4.

It has been assumed that $c(v, x) = (0, 1)$, if there is no edge joining the node ‘v’ directly to the node ‘x’.

Step3: STOP. The process of finding the nodes with permanent label is repeated until ‘e’ gets a permanent label.

The above steps do not actually list the shortest path from the starting node to the terminal node; it only gives the weighted aggregated IFV of the IF cost of traveling the shortest path giving its degree of acceptance and the degree of rejection for the shortest path.

Step4: The shortest path can be easily constructed by working backwards from the terminal node such that one moves to that predecessor from whom the current node has got its permanent label.

Step5: End

In Step 2 for calculating the IFHG $(L(v), c(v, x))$, one has to proceed as follows:

ω_j = the weight to be considered as the opinion of the expert regarding the IF cost $\widetilde{c}_{ij}$ required for traveling along the arc $i - j$. The normalized weight vectors for evaluating IFHG $(L(v), c(v, x))$ are calculated whenever required.

Considering the IF values $\widetilde{a}_p = \left[t_{\widetilde{a}_p}, 1 - f_{\widetilde{a}_p}\right]$ of the $p = 1, 2, \ldots, q$ arcs, the weighted IF values $\dot{\widetilde{a}}_p$, $p = 1, 2, \ldots, q$ for the q arcs are calculated using Eq. 7 as

$$\dot{\widetilde{a}}_p = \left[t_{\widetilde{a}_p}^{qx\widetilde{\omega}_{jp}}, \ \left(1 - f_{\widetilde{a}_p}\right)^{qx\widetilde{\omega}_{jp}}\right] \tag{12}$$

Now, by using the ranking method (utilizing the scores $S\left(\dot{\widetilde{a}}_p\right)$ by using Eq. 3) of the IFVs described in Section 2.3, the pth largest of the weighted IFVs $\dot{\widetilde{a}}_p$ $\left(\dot{\widetilde{a}}_p = \widetilde{a}_p^{n\widetilde{\omega}_{jp}}, p = 1, 2, 3, \ldots\ldots, q\right)$ is identified as $\dot{\widetilde{a}}_{\sigma(p)}$. (If there is no difference between two scores, then it is required to calculate the accuracy degrees $H\left(\dot{\widetilde{a}}_p\right)$ by using Eq. 4. After that the alternatives $\dot{\widetilde{a}}_p$ are ranked in accordance with the accuracy degrees.) Then using the IFHG operator shown in Eq. 10, the weighted aggregated IFV for $(L(v), c(v, x))$ are derived in the form of $[t_j, 1 - f_j]$ and hence the corresponding weighted aggregated IFN $(r_j) = (t_j, f_j)$ for IFHG $(L(v), c(v, x))$ are obtained which can be used for evaluating $L(x)$ using Eq. 11.

4 Numerical Illustration

Example A network has been considered with nodes s, a, b, c, d, e as shown in Fig. 1. The Intuitionistic Fuzzy Costs for traveling along the respective arcs and their weights (ω_j) decided by experts are given in Table 1. The objective is to find the shortest path from the node s to the node e so that the total IF cost of traveling is minimum. Here the given costs are in the form of IFNs $\widetilde{c}_{ij} = (\mu_{ij}, \nu_{ij})$, where μ_{ij} represents the degree of acceptance that the arc $i - j$ will be in the shortest path with respect to the cost for traveling along the arc $i - j$. Similarly v_{ij} represents the degree of rejection that the arc $i - j$ will be in the shortest path with respect to the cost for traveling along the arc $i - j$.

Here $\omega_j =$ the weight to be considered as the opinion of the expert regarding the IF cost $\widetilde{c}_{ij}$ required for traveling along the arc $i - j$.

The normalized weight vectors for evaluating IFHG $(L(v), c(v, x))$ are calculated whenever required.

Solution The proposed algorithm MIFDA described in Section 3.1 of this paper has been applied for solving this example.

Iteration 1

Step1: At first $P = \phi$, $T = \{s, a, b, c, d, e\}$. Let the label of 's' i.e., $L(s) = (1, 0)$ $L(x) = (0, 1) \forall x \in T$ and $x \neq$'s'.
Then $v = s$, $P = \{s\}$, $T = \{a, b, c, d, e\}$

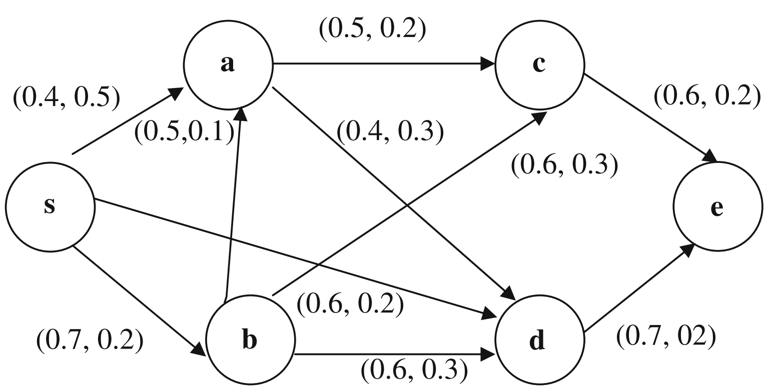

Fig. 1 Network with IF costs

Table 1 Data for IF Costs for traveling along the respective arcs of the given network

Arcs $(i-j)$	Intuitionistic Fuzzy Costs along these arcs $c(i, j)$	Weights (ω_j)
sa	(0.4, 0.5)	0.3
sb	(0.7, 0.2)	0.6
bd	(0.6, 0.3)	0.6
de	(0.7, 0.2)	0.8
ac	(0.5, 0.2)	0.5
ce	(0.6, 0.2)	0.6
sd	(0.6, 0.2)	0.6
ba	(0.5, 0.1)	0.4
ad	(0.4, 0.3)	0.4
bc	(0.6, 0.3)	0.5

Now, the permanent labels of the nodes present in T are evaluated using Eq. 11.

$$L(a) = \max\{\text{old } L(a), \text{ IFHG } (L(s), c(s,a))\}$$

$$\text{or } L(a) = \max\{(0,1), \text{ IFHG } ((1,0), (0.4, 0.5))\}$$

$$\text{or } L(a) = \max\{[0,1], \text{ IFHG } ([1,1], [0.4, 0.5])\} \tag{13}$$

Now for finding IFHG $([1, 1], [0.4, 0.5])$, the corresponding normalized weight is given by $\omega = \left(\frac{1}{1.3}, \frac{0.3}{1.3}\right) = (0.769, 0.231)$.

Therefore, $\dot{\tilde{a}}_1 = \left[1^{2x0.769}, 1^{2x0.769}\right] = [1, 1]$, $\dot{\tilde{a}}_2 = \left[0.4^{2x0..231}, 0.5^{2x0.231}\right] = [0.655, 0.726]$

$$S\left(\dot{\tilde{a}}_1\right) = 1 - (1-1) = 1, \; S\left(\dot{\tilde{a}}_2\right) = 0.655 - (1 - 0.726) = 0.381$$

Hence, $S\left(\dot{\tilde{a}}_1\right) > S\left(\dot{\tilde{a}}_2\right)$

Thus, $\dot{\tilde{a}}_{\sigma(1)} = \left[1{,}1\right], \dot{\tilde{a}}_{\sigma(2)} = [0.655, 0.726]$,
Now since there are two IFVs for $L(v)$ and $c(v, x)$, taking $n = 2$, using the normal distribution based method developed by Xu [53], for determining the associated weights $w = (w_1, w_2)$ i.e., by using Eqs. 8 and 9 the following associated weights for the IFHG operator are obtained:

$$\text{For } n = 2, \mu_n = 1.5, \sigma_n^2 = 0.250, w = (w_1, w_2) = (0.5, 0.5). \tag{14}$$

Then the weighted aggregated IFV for IFHG([1, 1], [0.4, 0.5]) in the form of $\left[t_1, 1 - f_1\right]$ and hence the corresponding weighted aggregated IFN $r_1 = (t_1, f_1)$ derived by using the IFHG operator (Eq. 10) and the weights $w = (w_1, w_2) = (0.5, 0.5)$ of Eq. 14 has been deduced as follows:

$$\begin{aligned} IFHG_{\omega,w}(\tilde{a}_1, \tilde{a}_2, \tilde{a}_3) &= \left[\prod_{j=1}^{2} t_{\dot{\tilde{a}}_{\sigma(j)}}^{w_j}, \prod_{j=1}^{2}\left(1 - f_{\dot{\tilde{a}}_{\sigma(j)}}\right)^{w_j}\right] \\ &= \left[1^{0.5} X 0.655^{0.5}, 1^{0.5} X 0.726^{0.5}\right] \\ &= [0.809, 0.852] = \left[t_1, 1 - f_1\right] \end{aligned}$$

The corresponding weighted aggregated IFN for IFHG ([1, 1], [0.4, 0.5]) is given by $r_1 = (t_1, f_1) = (0.809, 0.148)$.
Then from Eq. 13 and using Eqs. 3–6,

$$L(a) = \max\{(0, 1), (0.809, 0.148)\} = (0.809, 0.148)$$

The score of $L(a)$ is given by $S(L(a)) = 0.661$
Similarly, $L(b) = \max\{\text{old } L(b), \text{IFHG}(L(s), c(s, b))\}$

$$\begin{aligned} &\text{or } L(b) = \max\{(0, 1), \text{IFHG}((1, 0), (0.7, 0.2))\} \\ &\text{or } L(b) = \max\{[0, 0], \text{IFHG}([1, 1], [0.7, 0.8])\} \end{aligned} \tag{15}$$

Now for finding IFHG ([1, 1], [0.7, 0.8]), the corresponding normalized weight is given by $\omega = \left(\frac{1}{1.6}, \frac{0.6}{1.6}\right) = (0.625, 0.375)$.

Therefore, $\dot{\tilde{a}}_1 = \left[1^{2x0.625}, 1^{2x0.625}\right] = [1, 1]$, $\dot{\tilde{a}}_2 = \left[0.7^{2x0..375}, 0.8^{2x0.375}\right] = [0.765, 0.846]$

$$\therefore S\left(\dot{\tilde{a}}_1\right) = 1 - (1 - 1) = 1,\ S\left(\dot{\tilde{a}}_2\right) = 0.765 - (1 - 0.846) = 0.611$$

Hence, $S\left(\dot{\tilde{a}}_1\right) > S\left(\dot{\tilde{a}}_2\right)$

Thus, $\dot{\tilde{a}}_{\sigma(1)} = [1{,}1]$, $\dot{\tilde{a}}_{\sigma(2)} = [0.765, 0.846]$,
Now since there are two IFVs for $L(v)$ and $c(v, x)$, taking $n = 2$, using the normal distribution based method developed by Xu [53], for determining the associated weights $w = (w_1, w_2)$ i.e., by using Eqs. 8 and 9 the following associated weights for the IFHG operator are obtained, which is the same as in Eq. 14:

$$\text{For } n = 2, \mu_n = 1.5, \sigma_n^2 = 0.250, w = (w_1, w_2) = (0.5, 0.5).$$

Then the weighted aggregated IFV for IFHG([1, 1], [0.7, 0.8]) in the form of $[t_2, 1 - f_2]$ and hence the corresponding weighted aggregated IFN $r_2 = (t_2, f_2)$ derived by using the IFHG operator (Eq. 10) and the weights $w = (w_1, w_2) = (0.5, 0.5)$ of Eq. 14 has been deduced as follows:

$$\begin{aligned} IFHG_{\omega,w}(\tilde{a}_1, \tilde{a}_2, \tilde{a}_3) &= \left[\prod_{j=1}^{2} t_{\dot{\tilde{a}}_{\sigma(j)}}^{w_j}, \prod_{j=1}^{2}\left(1 - f_{\dot{\tilde{a}}_{\sigma(j)}}\right)^{w_j}\right] \\ &= \left[1^{0.5}X0.765^{0.5}, 1^{0.5}X0.846^{0.5}\right] \\ &= [0.875, 0.920] \\ &= \left[t_2, 1 - f_2\right] \end{aligned}$$

The corresponding weighted aggregated IFN for IFHG ([1, 1], [0.7, 0.8]) is given by$r_2 = (t_2, f_2) = (0.875, 0.080)$.
Then from Eq. 15, $L(b) = \max\{(0, 1), (0.875, 0.080)\} = (0.875, 0.080)$. The score of $L(b)$ is given by $S(L(b)) = 0.795$.
Proceeding similarly, we can get the labels of the other nodes present in T and their scores. These values are given in Table 2.

Table 2 Results of Iteration 1

Nodes (x)	ω	$w = (w_1, w_2)$	L(x)	S(L(x))
a	$\omega = \left(\frac{1}{1.3}, \frac{0.3}{1.3}\right) = (0.769, 0.231)$	(0.5, 0.5)	(0.809, 0.148)	0.661
b	$\omega = \left(\frac{1}{1.6}, \frac{0.6}{1.6}\right) = (0.625, 0.375)$	(0.5,0.5)	(0.875, 0.080)	**0.795**
c	$\omega = (1, 0)$	(0.5, 0.5)	(0, 1)	−1
d	$\omega = \left(\frac{1}{1.6}, \frac{0.6}{1.6}\right) = (0.625, 0.375)$	(0.5, 0.5)	(0.826, 0.080)	0.746
e	$\omega = (1, 0)$	(0.5, 0.5)	(0, 1)	−1

The significance of bold entry represent the highest scores and the corresponding node gets the permanent label in that iteration

Step2: The node 'b' in T is selected which has the highest score value 0.795 of its label $L(b) = (0.875, 0.080)$. Hence $L(b)$ is the permanent label of 'b'. Then $v = b$, $P = P \cup \{b\}$ and $T = T - (b)$ i.e., $P = \{s, b\}$ and $T = \{a, c, d, e\}$.

Iteration 2

Step1: Then using Eq. 11, the new labels of all the nodes in T are evaluated proceeding in the same manner. The results are shown in Table 3.

Step2: The node 'd' in T is selected which has the highest score value 0.746 of its label $L(d) = (0.826, 0.080)$. Hence $L(d)$ is the permanent label of 'd'. Then $v = d$, $P = P \cup \{d\}$ and $T = T - (d)$ i.e., $P = \{s, b, d\}$ and $T = \{a, c, e\}$.

Iteration 3

Step1: Then using Eq. 11, the new labels of all the nodes in T are evaluated proceeding in the same manner. The results are shown in Table 4.

Step2: The node 'a' in T is selected which has the highest score value 0.661 of its label $L(a) = (0.809, 0.148)$. Hence $L(a)$ is the permanent label of 'a'. Then $v = a$, $P = P \cup \{a\}$ and $T = T - (a)$ i.e., $P = \{s, b, d, a\}$ and $T = \{c, e\}$.

Iteration 4

Step1: Then using Eq. 11, the new labels of all the nodes in T are evaluated proceeding in the same manner. The results are shown in Table 5.

Step2: The node 'e' in T is selected which has the highest score value 0.601 of its label $L(e) = (0.708, 0.185)$. Hence $L(e)$ is the permanent label of 'e' and $v = e$.

Table 3 Results of Iteration 2

Nodes (x)	ω	w= (w_1,w_2)	L(x)	S(L(x))
a	$\omega = (0.6, 0.4)$	(0.5, 0.5)	(0.809, 0.148)	0.661
c	$\omega = \left(\frac{06}{1.1}, \frac{0.5}{1.1}\right) = (0.545, 0.455)$	(0.5, 0.5)	(0.737, 0.187)	0.550
d	$\omega = \left(\frac{0.6}{1.2}, \frac{0.6}{1.2}\right) = (0.5, 0.5)$	(0.5, 0.5)	(0.826, 0.080)	**0.746**
e	$\omega = (1, 0)$	(0.5, 0.5)	(0, 1)	−1

The significance of bold entry represent the highest scores and the corresponding node gets the permanent label in that iteration

Table 4 Results of Iteration 3

Nodes (x)	ω	w= (w_1,w_2)	L(x)	S(L(x))
a	$\omega = (1,0)$	(0.5,0.5)	(0.809, 0.148)	**0.661**
c	$\omega = (1,0)$	(0.5,0.5)	(0.737, 0.187)	0.550
e	$\omega = \left(\frac{0.6}{1.4}, \frac{0.8}{1.4}\right) = (0.429, 0.571)$	(0.5,0.5)	(0.751, 0.150)	0.601

The significance of bold entry represent the highest scores and the corresponding node gets the permanent label in that iteration

Table 5 Results of Iteration 4

Nodes (x)	ω	w= (w_1,w_2)	L(x)	S(L(x))
c	$\omega = \left(\frac{0.3}{0.8}, \frac{0.5}{0.8}\right) = (0.375, 0.625)$	(0.5,0.5)	(0.737, 0.187)	0.550
e	$\omega = (1,0)$	(0.5,0.5)	(0.751, 0.150)	**0.601**

The significance of bold entry represent the highest scores and the corresponding node gets the permanent label in that iteration

Step3: Since 'e' is the terminal node which has got its permanent label, so we STOP the process here. $L(e) = (0.751,\ 0.150)$ represents the weighted aggregated IFV of the minimum-cost path or the shortest path w.r.t. the total intuitionistic fuzzy cost for traveling through the shortest path.

Step4: The shortest path can be easily constructed by working backwards from the terminal node 'e' such that one moves to that predecessor from whom the current node has got its permanent label. Moving backwards, the minimum-cost path or the shortest path comes to be $s \rightarrow d \rightarrow e$.

5 Results and Discussions

The final result can be seen from the Table 5. Here it can be seen that $L(e) = (0.751,$ $0.150)$ is the weighted aggregated IFN for the path $s \rightarrow d \rightarrow e$ having the highest score of (0.751–0.150) = 0.601 in Iteration 4. This implies that the path $s \rightarrow d \rightarrow e$ is the most preferable path having the degree of acceptance 0.751 and the degree of rejection 0.150 for the weighted aggregated IF cost of this path. Hence the path $s \rightarrow d \rightarrow e$ is the most preferable path i.e., the minimum-cost path i.e., the shortest path w.r.t. the total intuitionistic fuzzy cost for traveling through the shortest path. Thus the discrete uncertain knowledge about the cost of traveling along the arcs in the form of IFV has been accumulated mathematically by the proposed method MIFDA resulting into a definite solution which is in terms of IFV and hence in terms of IFNs. For bigger problems, computer programs can be written for the proposed methodology. By a modification of the well known Dijkstra's Algorithm for incorporating the Intuitionistic Fuzzy arc parameters and applying the IFHG operator successfully, a new and efficient heuristic algorithm has been proposed in this paper which can take care of both optimistic and pessimistic opinion of the decision maker. Two types of weights have been considered in this algorithm. ω is the opinion of the expert regarding the IF cost $\tilde{c}_{ij}$ required for traveling along the arc $i - j$ which has been again normalized. Also, the associated weights $w_j > 0$ of the

IFHG operator are evaluated using successfully the Eqs. 8 and 9 where $\sum_{j=1}^{n} w_j = 1$. Uses of these two weights have helped the proposed algorithm to efficiently bring more accurate results. A numerical example shows the effectiveness of the proposed method. Since there is no other work on shortest path using Intuitionistic Fuzzy parameters for the arcs, numerical comparison of this work with other works could not be done. In future, this work can be extended for multi-criteria shortest path problem with data in the form of IFNs.

6 Conclusions

SPP is a very important area of study and are applied in various real life problems. In this paper, a new and innovative methodology has been proposed to solve SPP in an uncertain environment. In real life situations, precise values of costs or time or distances related to the arcs of a network may not be available. To incorporate the uncertainty, fuzzy numbers may be considered to represent the imprecise parameters. The most general type of fuzzy numbers i.e., IFNs have been considered here to represent the uncertain costs of traveling through each arc. IFNs represent both the optimistic and pessimistic opinion of the decision maker. IFHG operator, which is an important area of IFSs have been applied successfully to develop the proposed methodology MIFDA. This type of Shortest Path Problems (SPPs) with IF costs has never been posed or solved earlier to this work in the literature. This type of real life problem has been solved efficiently using the proposed MIFDA, applying successfully the different existing theories of IFSs. This signifies the major contribution of this paper. In future some other methods can be proposed to solve such problems and the results may be compared. Computer programs can be developed to implement the proposed methodology for large networks.

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